On Stability of Switched Stochastic Systems

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Friday, December 17, 2004
CDC’04
FrA01: JUMP STOCHASTIC SYSTEMS

Abstract

We propose a method of stability analysis of switched systems perturbed by a Wiener process. Our result utilizes the method of multiple Lyapunov functions, and is analogous to an existing result for deterministic switched systems.
1. Statement of the Problem

System:

\[ dx = f_\sigma(x)dt + G_\sigma(x)dw, \quad x(0) = x_0, \quad t \geq 0 \]  
\[ (\star) \]

- \( x \in \mathbb{R}^n \)
- \( \sigma : [0, \infty[ \rightarrow \mathcal{P} \) — finite index set
- \( w \) a normalized Wiener process

Global asymptotic stability in probability (GAS-P):

System (\( \star \)) is GAS-P for a given \( \sigma \), if:

\[ \forall \eta \in ]0, 1[ \quad \exists \beta \in \mathcal{KL} \quad \mathbb{P} \left[ |x(t)| \geq \beta(|x_0|, t) \right] < \eta \quad \forall t \geq 0 \]

\[ ^1\beta(\cdot, t) \] positive definite, increasing, and \( \beta(r, \cdot) \searrow 0 \) class \( \mathcal{K} \)
2. Stability of deterministic switched systems

System:
\[ \dot{x} = f_\sigma(x) \]

Methods of stability analysis:

- (Common Lyapunov function) \( \exists V : \frac{\partial V}{\partial x} f_p(x) \leq -W(|x|) \quad \forall p \in \mathcal{P} \)
- (Multiple Lyapunov functions) \( \exists V_p, p \in \mathcal{P} : \)
  * \( \frac{\partial V_p}{\partial x} f_p(x) \leq -W(|x|) \)
  * \( \forall \tau_i < \tau_j : \sigma(\tau_i) = \sigma(\tau_j) = p, \)
  \[ V_p(x(\tau_j)) - V_p(x(\tau_i)) \leq -U(|x(\tau_i)|) \]
3. **Stochastic stability of a single system**

**System:**

\[ \text{dx} = f(x) \text{dt} + G(x) \text{dw} \]

- \( w \) normalized Wiener process

**Stability analysis:**

- (infinitesimal generator) \( \mathcal{L}V = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 V}{\partial x^2} GG^T \right) \)

- (stochastic stability) [Kushner’67, Haśminskii’80, Krstić&Deng’98]

\[ \mathcal{L}V \leq -W(|x|), \ W \in \mathcal{K} \implies \]

\[ \forall \eta \in ]0,1[ \quad \exists \beta \in \mathcal{KL} : \quad \mathbb{P} \left[ |x(t)| \geq \beta(|x_0|, t) \right] < \eta \quad \forall t \geq 0 \]
4. Our result

\[ dx = f_\sigma(x)dt + G_\sigma(x)dw, \quad x(0) = x_0, \quad t \geq 0 \]  

(\star)

**Theorem:** Assume \( \exists \alpha_1, \alpha_2 \in \mathcal{K}, \alpha_2 \circ \alpha_1^{-1} \) concave, \( V_p, p \in \mathcal{P}, W, U \in \mathcal{K} \) with \( U \circ \alpha_1^{-1} \) convex:

(i) \( \alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \quad \forall x, p \)

(ii) \( \mathcal{L}_pV_p(x) \leq -W(|x|) \quad \forall x, p \)

(iii) \( \forall \tau_i < \tau_j : \sigma(\tau_i) = \sigma(\tau_j) = p \)

\[ E[V_p(x(\tau_j))] - E[V_p(x(\tau_i))] \leq -E[U(|x(\tau_i)|)] \]

Then the system (\( \ast \)) is GAS-P.
4. Our result

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(\star)

**Theorem:** Assume \( \exists \alpha_1, \alpha_2 \in K, \alpha_2 \circ \alpha_1^{-1} \) concave, \( V_p, p \in P, W, U \in K \) with 
\( U \circ \alpha_1^{-1} \) convex:

(i) \( \alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \quad \forall x, p \)

(ii) \( \mathcal{L}_p V_p(x) \leq -W(|x|) \quad \forall x, p \)

(iii) \( \forall \tau_i < \tau_j : \sigma(\tau_i) = \sigma(\tau_j) = p \)

\[ E[V_p(x(\tau_j))] - E[V_p(x(\tau_i))] \leq -E[U(|x(\tau_i)|)] \]

Then the system (\( \star \)) is GAS-P.  

**TO BE VERIFIED!**
5. Sketch of proof

(i) \( \alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \quad \forall \, x, p \)

(ii) \( \mathcal{L}_p V_p(x) \leq -W(|x|) \quad \forall \, x, p \)

(iii) \( \forall \tau_i < \tau_j : \sigma(\tau_i) = \sigma(\tau_j) = p, \mathbb{E}[V_p(x(\tau_j))] - \mathbb{E}[V_p(x(\tau_i))] \leq -\mathbb{E}[U(|x(\tau_i)|)] \)

1) Lyapunov stability:

a) (ii) & (i): \( \mathbb{E}[V_{\sigma(t_0)}(x(t_1))] \leq \mathbb{E}[V_{\sigma(t_0)}(x_0)] \leq \alpha_2(|x_0|) \)

b) Jensen’s ineq. & (i): \( \mathbb{E}[V_p(x(t_1))] \leq \mathbb{E}[\alpha_2(|x(t_1)|)] \leq \alpha_2 \circ \alpha_1^{-1} \left( \mathbb{E}[V_{\sigma(t_0)}(x(t_1))] \right) \)

c) 1a) & 1b): \( \mathbb{E}[V_p(x(t_1))] \leq \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2(|x_0|) \)

2) Global asymptotic convergence:

a) \( \exists \, p \in \mathcal{P}: \mathbb{E}[V_{\sigma(\tau_i)}(x(\tau_i))]_{\{\sigma(\tau_i)=p, i \geq 0\}} \) is infinite; by (iii) it is \( \searrow \)

b) \( \mathbb{E}[V_{\sigma(\tau_i)}(x(\tau_i))]_{\{\sigma(\tau_i)=p, i \geq 0\}} \) attains a limit \( c \geq 0 \)

c) Taking limits in (iii): \( c - c \leq -\lim_{i \uparrow \infty} \mathbb{E}[U(|x(t_i)|)] \)

d) Jensen’s ineq.: \( \lim_{i \uparrow \infty} \mathbb{E}[\alpha_1(|x(t_i)|)] = 0 \)

e) 1) & 2d): \( \mathbb{E}[\alpha_1(|x(t)|)] \leq \beta(|x_0|, t) \)
6. Example

System: \( dx = A_\sigma x dt + B_\sigma x dw \)

* find functions \( V_p(x) = x^T P_p x \): \( P_p > 0, \ Q > 0 : \)

\[
A_p^T P_p + P_p A_p + B_p P_p B_p^T \leq -Q
\]

* \( \alpha_2 \circ \alpha_1^{-1} \) automatically concave

Dwell-time: For the above system, let

* \( \tau_{i+1} - \tau_i \geq \tau_D \ \forall \ i \geq 0 \)

* \( \exists \mu > 1 : \ \forall p \in \mathcal{P}, \ V_p(x) \leq \mu V_q(x) \ \forall x, p \)

* \( \exists \lambda > 0 : \ \mathcal{L}_p V_p(x) \leq -\lambda V_p(x) \ \forall x, p \)

* \( \tau_D > \ln \mu/\lambda \)

One can prove:

- \( \mathbb{E}[\sigma(V_{t_{i+1}}(x(t_{i+1})))] \leq \mu \exp -\lambda \tau_D \mathbb{E}[V_{\sigma(t_i)}(x(t_i))] \)
- \( \mathbb{E}[V_p(x(\tau_j))] - \mathbb{E}[V_p(x(\tau_i))] \leq -\mathbb{E}[U(|x(\tau_i)|)] \) with \( U(r) = c \left(1 - e^{\ln \mu - \lambda \tau_D} \right) r^2. \)
7. Extensions

Comparison principle and multiple Lyapunov functions:

• stochastic comparison principle + multiple Lyapunov functions
• provides greater flexibility, allows for overshoots and oscillations
• e.g., average dwell-time switching in switched stochastic systems
• stronger results: global asymptotic stability in the mean
• http://decision.csl.uiuc.edu/~liberzon/research/siam-swstab.pdf

Future Work:

• piecewise deterministic systems with random switching
• special case: switching signal is the state of a continuous time Markov chain