On Stability of Switched Stochastic Systems

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Abstract

We propose a method of stability analysis of switched systems perturbed by a Wiener process. Our result utilizes the method of multiple Lyapunov functions, and is analogous to an existing result for deterministic switched systems.

1. Statement of the Problem

System:

$$\boxed{\mathrm{d}x = f_\sigma(x)\mathrm{d}t + G_\sigma(x)\mathrm{d}w, \qquad x(0) = x_0, \quad t \geqslant 0} \qquad (\star)$$

- $x \in \mathbb{R}^n$
- $\sigma : [0, \infty[\longrightarrow \mathcal{P}$—finite index set
- w a normalized Wiener process

Global asymptotic stability in probability (GAS-P):

System ($\star$) is GAS-P for a *given* σ, if:[1]

$$\forall\, \eta \in\,]0,1[\;\; \exists\, \beta \in \mathcal{KL} \qquad \mathsf{P}\Big[|x(t)| \geqslant \beta(|x_0|, t)\Big] < \eta \qquad \forall\, t \geqslant 0$$

[1] $\beta(\cdot, t)$ $\underbrace{\text{positive definite, increasing}}_{\text{class } \mathcal{K}}$, and $\beta(r, \cdot) \searrow 0$

2. Stability of deterministic switched systems

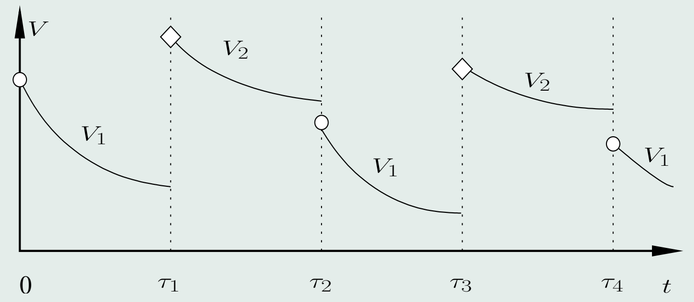
System:

$$\dot{x} = f_\sigma(x)$$

Methods of stability analysis:

- (Common Lyapunov function) $\exists\, V: \quad \dfrac{\partial V}{\partial x} f_p(x) \leqslant -W(|x|) \quad \forall\, p \in \mathcal{P}$
- (Multiple Lyapunov functions) $\exists\, V_p,\ p \in \mathcal{P}$:
  - $\dfrac{\partial V_p}{\partial x} f_p(x) \leqslant -W(|x|)$
  - $\forall\, \tau_i < \tau_j$: $\sigma(\tau_i) = \sigma(\tau_j) = p$,

$$V_p(x(\tau_j)) - V_p(x(\tau_i)) \leqslant -U(|x(\tau_i)|)$$



3. Stochastic stability of a single system

System:

$$\mathrm{d}x = f(x)\mathrm{d}t + G(x)\mathrm{d}w$$

- w normalized Wiener process

Stability analysis:

- (infinitesimal generator) $\mathscr{L}V = \dfrac{\partial V}{\partial x} f(x) + \dfrac{1}{2}\,\mathrm{tr}\left(\dfrac{\partial^2 V}{\partial x^2} G G^{\mathrm{T}}\right)$

- (stochastic stability) [Kushner'67, Haśminskii'80, Krstić&Deng'98]

$$\mathscr{L}V \leqslant -W(|x|), \;\; W \in \mathcal{K} \quad \Longrightarrow$$

$$\forall\, \eta \in \,]0,1[\quad \exists\, \beta \in \mathcal{KL} : \qquad \mathsf{P}\Big[|x(t)| \geqslant \beta(|x_0|, t)\Big] < \eta \;\; \forall\, t \geqslant 0$$

4. Our result

$$\boxed{\mathrm{d}x = f_\sigma(x)\mathrm{d}t + G_\sigma(x)\mathrm{d}w, \qquad x(0) = x_0, \quad t \geqslant 0} \qquad (\star)$$

Theorem: *Assume* $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}$, $\alpha_2 \circ \alpha_1^{-1}$ *concave,* V_p, $p \in \mathcal{P}$, $W, U \in \mathcal{K}$ *with* $U \circ \alpha_1^{-1}$ *convex:*

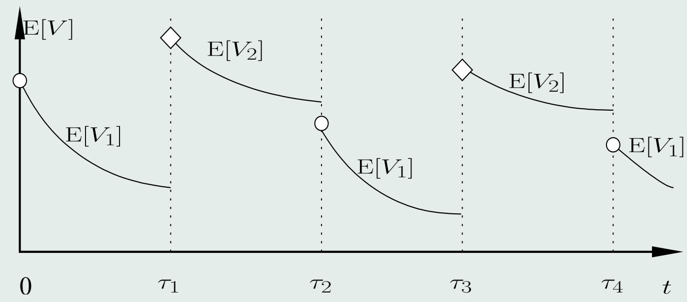
(i) $\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \quad \forall\, x, p$

(ii) $\mathscr{L}_p V_p(x) \leqslant -W(|x|) \quad \forall\, x, p$

(iii) $\forall\, \tau_i < \tau_j :\ \sigma(\tau_i) = \sigma(\tau_j) = p$

$$\mathsf{E}[V_p(x(\tau_j))] - \mathsf{E}[V_p(x(\tau_i))] \leqslant -\mathsf{E}[U(|x(\tau_i)|)]$$

Then the system $(\star)$ *is* GAS-P.



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(iii) $\forall\, \tau_i < \tau_j :\ \sigma(\tau_i) = \sigma(\tau_j) = p$

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TO BE VERIFIED!

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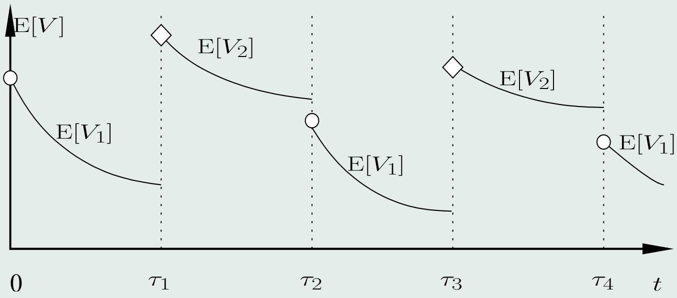

5. Sketch of proof

(i) $\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \quad \forall\, x, p$

(ii) $\mathscr{L}_p V_p(x) \leqslant -W(|x|) \quad \forall\, x, p$

(iii) $\forall\, \tau_i < \tau_j :\ \sigma(\tau_i) = \sigma(\tau_j) = p,\ \mathsf{E}[V_p(x(\tau_j))] - \mathsf{E}[V_p(x(\tau_i))] \leqslant -\mathsf{E}[U(|x(\tau_i)|)]$

1) Lyapunov stability:

a) (ii) & (i): $\mathsf{E}\big[V_{\sigma(t_0)}(x(t_1))\big] \leqslant \mathsf{E}\big[V_{\sigma(t_0)}(x_0)\big] \leqslant \alpha_2(|x_0|)$

b) Jensen's ineq. & (i): $\mathsf{E}[V_p(x(t_1))] \leqslant \mathsf{E}[\alpha_2(|x(t_1)|)] \leqslant \alpha_2 \circ \alpha_1^{-1}\left(\mathsf{E}\big[V_{\sigma(t_0)}(x(t_1))\big]\right)$

c) 1a) & 1b): $\mathsf{E}[V_p(x(t_1))] \leqslant \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2(|x_0|)$

2) Global asymptotic convergence:

a) $\exists\, p \in \mathcal{P}$: $\mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]_{\{\sigma(\tau_i)=p, i \geqslant 0\}}$ is infinite; by (iii) it is $\searrow$

b) $\mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]_{\{\sigma(\tau_i)=p, i \geqslant 0\}}$ attains a limit $c \geqslant 0$

c) Taking limits in (iii): $c - c \leqslant -\lim_{\substack{i \uparrow \infty \\ \sigma(t_i)=p}} \mathsf{E}[U(|x(t_i)|)]$

d) Jensen's ineq.: $\lim_{\substack{i \uparrow \infty \\ \sigma(t_i)=p}} \mathsf{E}[\alpha_1(|x(t_i)|)] = 0$

e) 1) & 2d): $\mathsf{E}[\alpha_1(|x(t)|)] \leqslant \beta(|x_0|, t)$

6. Example

System: $\mathrm{d}x = A_\sigma x \mathrm{d}t + B_\sigma x \mathrm{d}w$

* find functions $V_p(x) = x^{\mathrm{T}} P_p x$: $\quad P_p > 0,\ Q > 0$:

$$A_p^{\mathrm{T}} P_p + P_p A_p + B_p P_p B_p^{\mathrm{T}} \leqslant -Q$$

* $\alpha_2 \circ \alpha_1^{-1}$ automatically concave

Dwell-time: For the above system, let

* $\tau_{i+1} - \tau_i \geqslant \tau_{\mathrm{D}} \quad \forall\, i \geqslant 0$
* $\exists\, \mu > 1 : \quad \forall\, p \in \mathcal{P}, \quad V_p(x) \leqslant \mu V_q(x) \quad \forall\, x, p$
* $\exists\, \lambda > 0 : \quad \mathscr{L}_p V_p(x) \leqslant -\lambda V_p(x) \quad \forall\, x, p$
* $\tau_{\mathrm{D}} > \ln \mu / \lambda$

One can prove:

- $\mathsf{E}\big[\sigma(V_{t_{i+1}})(x(t_{i+1}))\big] \leqslant \mu \exp -\lambda \tau_{\mathrm{D}} \mathsf{E}\big[V_{\sigma(t_i)}(x(t_i))\big]$
- $\mathsf{E}[V_p(x(\tau_j))] - \mathsf{E}[V_p(x(\tau_i))] \leqslant -\mathsf{E}[U(|x(\tau_i)|)]$ with $U(r) = c\left(1 - \mathrm{e}^{\ln \mu - \lambda \tau_{\mathrm{D}}}\right) r^2$.

7. Extensions

Comparison principle and multiple Lyapunov functions:

- stochastic comparison principle + multiple Lyapunov functions
- provides greater flexibility, allows for overshoots and oscillations
- e.g., average dwell-time switching in switched stochastic systems
- stronger results: global asymptotic stability in the mean
- http://decision.csl.uiuc.edu/~liberzon/research/siam-swstab.pdf

Future Work:

- piecewise deterministic systems with random switching
- special case: switching signal is the state of a continuous time Markov chain