Controllability of Linear Time-Varying Systems with Quantized Controls and Finite Data-Rate

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Abstract

In this paper, we define a notion of controllability that is suitable for digital systems, i.e., with sampling, quantization, and operating with a finite data-rate. In particular, we study that notion for linear time-varying systems by proving a necessary condition and a sufficient condition for such systems to be controllable with quantized controls and finite data-rate.

1 Introduction

Digital controllers for continuous-time systems control are ubiquitous. Hence, it is natural to ask what constraints the connection between the digital and analog worlds imposes on capabilities of such controllers. For example, there are well-known constraints imposed on the controllability of periodic sampling of linear time-invariant (LTI) systems [11]. However, sampling is not the only characteristic of digital control that imposes constraints on the controller.

Another aspect of digital controllers is that they have a finite number of output and input values. Furthermore, these controllers can only receive and transmit information with a finite data-rate. This latter fact is a consequence of a digital clock and the circuit timing of the controller components [13]. These facts imply that, in digital control, we need to work not only with sampled data, but also with quantized controls and with a finite data-rate.

There exists an extensive literature on quantized control [14, 2, 5, 8]. Many of these works concern the control over communication channels with minimal data-rates [8], stabilization [2, 5], or containability [14]. However, controllability did not play a central role in those works. One of the goals of this paper is to propose a suitable notion of controllability for systems with quantized controls and finite data-rate. It is also worth mentioning that the literature on conditions for stabilization with quantized control of linear time-invariant systems is vast [5, 2, 4]. Also, there is some literature on linear time-varying (LTV) systems, mainly for switched linear systems [7, 12, 15]. Nonetheless, most of the results found in the literature only provide sufficient conditions for stabilizability of switched linear systems but not for general LTV systems. Furthermore, even for switched systems, necessary conditions for stabilization of linear time-varying systems with quantized controls are missing in the literature. In view of this, another goal of this paper is to present a necessary condition and a sufficient condition for controllability with quantized controls and finite data-rate for LTV systems. In this way, we hope to reduce the previously mentioned gap in the literature.

To address our goals, we borrow concepts from the paper [4]. In that article, the author defined a notion of stabilization with finite data-rate, which we strengthen to define what we mean by controllability for quantized LTV systems. We also note that [4] addressed the stabilization problem for LTI and autonomous nonlinear systems but not time-varying systems.

The structure of the paper is as follows: first, in section 1, we introduce the motivation and notations. Next, in section 2, we describe the problem and needed concepts. Further, we introduce the concept of controllability with finite data-rate and discuss why this concept is natural. Then, in section 3, we state some necessary results, recall the concept of complete controllability, and define persistent complete controllability. After that, in subsection 3.1, we prove that persistent complete controllability and another condition, the exponential energy-growth condition, are sufficient for an LTV system to be controllable in the sense we defined. Furthermore, in subsection 3.2, we prove that complete controllability is a necessary condition for an LTV system to be controllable with finite data-rate. Finally, in section 4, we conclude the paper and present some future research directions.

Notations: We denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of the positive (nonnegative) integers. We denote by $\mathbb{R}$ ($\mathbb{R}_{>a}$) the set of real numbers (larger than $a \in \mathbb{R}$). Given $n \in \mathbb{Z}_{>0}$, we denote $[n] := \{1, \dots, n\}$. Given a set S, we denote by $\#S$ its cardinality. Let $\mathcal{M}^d$ be the set of $d \times d$ real matrices. We denote the transpose of an element $A \in \mathcal{M}^d$ by A'. For every $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote by $|x| := \left(\sum_{i=1}^d x_i^2\right)^{1/2}$ the Euclidean norm. Also, if A is a $d \times d$ real matrix we denote by $\|A\| := \max\{|Ax| : |x| = 1, x \in \mathbb{R}^d\}$ the induced norm. For a matrix $A \in \mathcal{M}^d$, we denote by $\mathcal{N}(A)$ its null space. We denote by $L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$ the set of all integrable locally essentially bounded functions from $[t_0, \infty)$ to $\mathbb{R}^m$ where $t_0 \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$, i.e., the set of integrable functions $u(\cdot)$ such that for every compact set $L \subset [t_0, \infty)$, we have that $u(L) \subset \mathbb{R}^m$ is bounded. Also, we denote by $L^2([a, b], \mathbb{R}^m)$ the set of square-integrable functions on the interval $[a, b] \subset \mathbb{R}$ with image on $\mathbb{R}^m$. Let $u : A \to B$ and let $C \subset A$, then we denote by $u_{|C} : C \to B$ the restriction of the function u to the subset C of the domain A. Finally, we denote by $B(x, r) \subset \mathbb{R}^d$ the open ball of radius $r \in \mathbb{R}_{>0}$ and center $x \in \mathbb{R}^d$.

2 Preliminaries

In this section, we motivate the study of controllability of linear time-varying systems with quantized controls and finite data-rate. To do that, we first provide some necessary definitions. Next, we give an example that shows that the usual notion of controllability for linear time-varying systems is not enough to ensure controllability when we consider quantization and finite data-rate. Then, we provide a definition of controllability that makes sense when we consider the finite data-rate case. Finally, we motivate the study of our controllability notion through an example.

Our primary goal is to study the controllability with quantized controls and finite data-rate of systems described by equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

where the initial state is given by $x(t_0) = x_0 \in K \subset \mathbb{R}^d$ with K compact with nonempty interior, the initial time is given by $t_0 \in \mathbb{R}_{\geq 0}$, time is such that $t \in \mathbb{R}_{\geq t_0}$, $A(t)$ is a $d \times d$ real matrix, $B(t)$ is a $d \times m$ real matrix, and $u(t) \in \mathbb{R}^m$. Also, we assume that the functions $A(\cdot)$ and $B(\cdot)$ are bounded[1] and piecewise-continuous on $\mathbb{R}_{\geq t_0}$. Further, we define by $\Phi(t, \tau)$ for $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the *state-transition matrix* associated with the unforced response of system (1). Furthermore, we assume that $u(\cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$.

Now, we must define what we mean by controllability with finite data-rate. To do that, we need the following Definition 2.1, which is an adaptation from the definitions given in [4]. Also, we name some sets and properties that were not named in [4] to improve readability in later discussions.

Definition 2.1. We say that system (1) satisfies the *exponential decay condition* with rate $\mu \in \mathbb{R}_{>0}$, with $M \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$ if for each $x_0 \in K \subset \mathbb{R}^d$ there exists $u(\cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$ such that

$$|x(t)| \leq \big(M|x_0| + \epsilon\big)e^{-\mu(t-t_0)} \tag{2}$$

for all $t \in \mathbb{R}_{\geq t_0}$. For given $\mu \in \mathbb{R}_{>0}$, $M \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and $K \subset \mathbb{R}^d$ as above, we call the set $\mathscr{R}(\epsilon, M, K, \mu) \subset L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$ a *stabilizing control set* of system (1) if for every $x_0 \in K$, there exists a control function $u(\cdot) \in \mathscr{R}(\epsilon, M, K, \mu)$ such that (2) holds. Furthermore, we denote by

$$\mathscr{R}_T(\epsilon, M, K, \mu) := \{u_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0, T], \mathbb{R}^m) : u(\cdot) \in \mathscr{R}(\epsilon, M, K, \mu)\}$$

a *set of restrictions of stabilizing controls*, where $T > t_0$ is arbitrary. Moreover, we define the *data-rate* associated with system (1) in the following manner. First, given a stabilizing control set $\mathscr{R}(\epsilon, M, K, \mu)$, we define the quantity[2]

$$b(\mathscr{R}(\epsilon, M, K, \mu)) := \limsup_{T\to\infty} \frac{1}{T} \log(\#\mathscr{R}_T(\epsilon, M, K, \mu)).$$

Next, we define the data-rate as[3]

$$b(M, \mu) := \lim_{\epsilon\to 0} \inf\{b(\mathscr{R}(\epsilon, M, K, \mu)) : \mathscr{R}(\epsilon, M, K, \mu) \text{ is a stabilizing control set of (1)}\}.$$

Finally, we say that system (1) can be *stabilized with finite data-rate* with $M \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathbb{R}_{\geq 0}$ if $b(M, \mu) < \infty$.

[1]That means that $A(\mathbb{R}_{\geq t_0})$ and $B(\mathbb{R}_{\geq t_0})$ are bounded subsets of $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively.

[2]The corresponding quantity in [4] uses the limit inferior instead of limit superior. Because of that, if the quantity given in [4] is also infinite, ours is also infinite.

[3]Note that $b(M, \mu)$ also depends on the set of initial conditions K. We drop that dependence to make the notation simpler.

Note that the limit $\epsilon \to 0$ could be substituted by[4] $\sup_{\epsilon>0}$. This latter fact implies that if we can stabilize system (1) with finite data-rate, then we can achieve (2) with an arbitrary $\epsilon \in \mathbb{R}_{>0}$. The reader might wonder if we can remove the ϵ term from inside equation (2) and still get a reasonable notion of stabilizability with finite data-rate. The answer is negative, and is proved in Proposition 2.2 of [4] where the author showed that LTI systems with poles with a nonnegative real part cannot satisfy (2) with $\epsilon = 0$ and have $b(M, \mu) < \infty$ for any choices of M and μ.

To continue our discussion, we recall the usual definition of controllability for LTV systems. See, e.g., Chapter 9 of [9].

Definition 2.2. We say that system (1) is *controllable in the usual sense* on $[t_0, T]$, where $T \geq t_0$, if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ there exists a function $u : [t_0, T] \to \mathbb{R}^m$ such that $x(T) = 0$.

Now, we are ready to define controllability with finite data-rate.

Definition 2.3. We say that system (1) is *controllable with finite data-rate* if for every $\mu \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{R}_{\geq 0}$ such that system (1) is stabilizable with finite data-rate $b(M, \mu) < \infty$.

It is important to remark that the previous definition is new and it differs from the definition of stabilization with finite data-rate, originally given in [4], in the sense that $\mu \in \mathbb{R}_{>0}$ is arbitrary. Now, the reader might wonder why we need a new definition of controllability when quantization is present. To answer that, consider the following example.

Example 1. Let $\dot{x}(t) = u(t)$ where $t \in \mathbb{R}$, $x_0 \in K \subset \mathbb{R}$ with K compact with a nonempty interior, and $u(t) \in C_t$ with $C_t \subset \mathbb{R}$ being a set of finite cardinality that may vary with t. We can easily solve this equation to get that $x(T) = x_0 + \int_{t_0}^{T} u(\tau)d\tau$. Note that, if $u(t) \in \mathbb{R}^m$, this system is controllable in the usual sense on the interval $[t_0, T]$. If we impose that the data-rate is finite, we have that the set of possible controls $u_{[t_0,T]}(\cdot)$ in any interval of time $t \in [t_0, T]$ has a finite cardinality. Therefore, the integral $\int_{t_0}^{T} u(\tau)d\tau$ attains at most finitely many values, but x_0 belongs to the set K, which has infinitely many points. Hence, it is not possible to make $x(T) = 0$ for an arbitrary initial condition in K. However, we prove in section 3 that this system is controllable with finite data-rate.

The previous example showed that we cannot have $x(T) = 0$ for an arbitrary initial condition in K, which proves that the usual controllability notion is unfit for the case where we have quantized controls. Thus, we relax that condition by saying that the norm of the state must converge to zero with an arbitrary exponential rate of decay. The idea behind this definition came from the fact that we can solve the pole placement problem for a linear time-invariant system if, and only if, it is controllable. Moreover, we can only make the norm of the state of an LTI system decay arbitrarily fast to zero if the system is controllable. Therefore, since definition 2.3 captures that property, we believe that it is a natural candidate for extending the concept of controllability to LTV systems with finite data-rate.

[4]See [4] for a discussion.

Before we continue our discussion, we recall the definition of controllability Gramian.

Definition 2.4. (Chapter 6 of [3]) Consider the system given by equation (1). We define the *controllability Gramian* from t_0 to t of system (1) as $W(t, t_0) := \int_{t_0}^{t} \Phi(t, \tau)B(\tau)B'(\tau)\Phi'(t, \tau)d\tau$.

At this point, we note that the system of Example 1 is controllable in the usual sense and we will see in section 3.1 that it is controllable with finite data-rate. Indeed, Theorem 3.1 ensures that this is true for LTI systems. Hence, a natural question is if the usual controllability condition for LTV systems based on the invertibility of the controllability Gramian is also enough to ensure that system (1) is controllable with finite data-rate. The next Example 2 shows that that is not the case.

Example 2. Consider system (1) with $A(t) = I_d$ and $B(t) = (1, 0)$ for $0 < t < 1$, and $A(t) = I_d$ and $B(t) = (0, 1)$ for $t \geq 1$. Also, let the initial time be $t_0 = 0$. It is easy to see that $W(2, 0)$ is invertible, which implies that system (1) is controllable in the usual sense. Nonetheless, we show in section 3 that it is not controllable with finite data-rate.

This example motivates us to provide necessary and sufficient conditions for system (1) to be controllable with finite data-rate. We do that in the next section.

3 Controllability with Finite Data-Rate

In this section, we present the main contribution of the paper. We give and prove a sufficient and a necessary condition for LTV systems to be controllable with finite data-rate. But first, we introduce needed definitions and state some technical lemmas.

The following definition 3.1 describes a controllability condition that is related to the concept of controllability of LTV systems with finite data-rate defined in the previous section. That connection will become clear in the statements of Theorems 3.1 and 3.2.

Definition 3.1. We say that system (1) is *completely controllable* if there exists an increasing sequence $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ with $s_0 = t_0$ and $s_n \to \infty$ such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$. If the sequence $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ also satisfies[5] $\limsup_{n \to \infty} \frac{s_{n+1}}{s_n} < \infty$, then we say that system (1) is *persistently completely controllable.*

Remark 3.1. A few remarks are in order. First, the notion of complete controllability was first stated in [6] in a different manner than in definition 3.1; we provide a proof that both statements are equivalent in the Appendix. Note, however, that the Definition of persistent complete controllability is new. Second, we notice that there exist necessary conditions and sufficient conditions for the complete controllability of LTV systems. For example, [10] gives several conditions[6] for complete

[5] This is equivalent to the statement: there exists $M \in \mathbb{R}_{>0}$ such that $\frac{s_{n+1}}{s_n} \leq M$ for all $n \in \mathbb{Z}_{\geq 0}$.

[6] Note that complete controllability is different from complete controllability on an interval.

controllability in the differentiable case[7]. Third, note that $s_{n+1} - s_n$ does not need to be bounded in neither statement from Definition 3.1.

Now, we state some technical results. One can find the proofs of all of the lemmas in the Appendix. First, we need the following technical Lemma 3.1 which will be useful in the proof of Theorem 3.1.

Lemma 3.1. Let system (1) be persistently completely controllable. Then, there exists a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$, that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, and that $\limsup_{n\to\infty} \frac{n}{s_n} < \infty$.

Next, let $\lambda^t := \sup\{\frac{1}{s}\log(\|\Phi(s,t_0)\|) : t \geq s \geq t_0\}$, $\xi := \sup\{\|A(t)\| : t \geq t_0\}$, and $\bar{\lambda} := \limsup_{t\to\infty} \lambda^t$. Further, consider Lemma 3.2, which collects some known facts about the state transition matrix. See, e.g., Chapter 4 of [9].

Lemma 3.2. Consider equation (1) and let $\xi < \infty$. Then, $e^{-\xi(t-t_0)} \leq |\Phi(t,t_0)v| \leq e^{\xi(t-t_0)}$ for all $t \geq t_0$ and all $v \in \mathbb{R}^d$ with $|v| = 1$. In particular, it is also true that $\|\Phi(t,t_0)\| \leq e^{\xi(t-t_0)}$.

Note that because $\xi < \infty$, we have that $\bar{\lambda}$ and λ^t are finite by Lemma 3.2. The next definition gives a bound for $\|W^{-1}(s_n, s_{n+1})\|$ as n goes to infinity, as fact that will be useful in the proof of Theorem 3.1. Furthermore, we need one more Lemma and an additional technical definition.

Lemma 3.3. For every sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ with $s_n \nearrow \infty$, the Gramian $W(s_{n+1}, s_n)$ associated with system (1) satisfies $\|W(s_{n+1}, s_n)\| \leq \sup\{\|B(t)\|^2 : t \geq t_0\}\frac{e^{2\xi(s_{n+1}-s_n)}-1}{2\xi}$.

The previous Lemma 3.3 states the fact that the Gramian can only grow exponentially fast with n if $A(\cdot)$ and $B(\cdot)$ are bounded.

Definition 3.2. Let $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ be an increasing sequence such that $\limsup_{n\to\infty} s_n = \infty$. Then, we say that system (1) satisfies the *exponential energy-growth condition* if there exists $\theta \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{R}_{>0}$ such that $\|W^{-1}(s_{n+1}, s_n)\| \leq Ne^{\theta s_{n+1}}$.

The intuition behind this definition is related to the minimum energy control on intervals of the form $[s_n, s_{n+1}]$. Note that the minimum-energy control, in the $L^2([s_n, s_{n+1}], \mathbb{R}^m)$ sense, that drives a state $x(s_n)$ at time s_n to the origin at time s_{n+1} is given by $x'(s_n)W^{-1}(s_n, s_{n+1})x(s_n)$. See, e.g., Theorem 1 in Chapter 22 from [1]. Thus, only if a system satisfies the exponential energy-growth condition, the energy needed to drive a given state to zero cannot grow faster than an exponential as n grows to infinity. Now, we are ready to prove necessary and sufficient conditions for system (1) to be controllable with finite data-rate.

3.1 Sufficient Condition

In this subsection, we prove Theorem 3.1, which gives a sufficient condition for system (1) to be controllable with finite data-rate.

Theorem 3.1. System (1) is controllable with finite data-rate if it is persistently completely controllable and satisfies the exponential energy-growth condition.

[7] When matrices A(t) and B(t) are differentiable functions of time.

Remark 3.2. Note that, for a controllable LTI system, if we choose $s_{n+1} - s_n = T$, the inverse of the Gramian exists and is constant. Thus, the exponential energy-growth condition is satisfied. Using the same argument, we see that such a system is persistently completely controllable. Thus, Theorem 3.1 shows that controllable LTI systems are controllable with finite data-rate.

Proof. Let $\{e_1, \cdots, e_d\} \subset \mathbb{R}^d$ be the canonical basis of $\mathbb{R}^d$. Pick an arbitrary $\tilde{\epsilon} \in \mathbb{R}_{>0}$ and an arbitrary $\mu \in \mathbb{R}_{>0}$. Also, let $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence that satisfies the conditions given in Definition 3.1 for system (1) to be persistently completely controllable. By Lemma 3.1, without loss of generality, we assume that $\limsup_{n\to\infty} \frac{n}{s_n} = Q < \infty$. Further, denote by $\alpha := 4\xi+\theta+\mu$ for simplicity. Finally, let $C = e^{\alpha(s_1-t_0)}$, $\epsilon = \frac{\sqrt{d}(2C+1)N\sup\{\|B(t)\|^2 : t\geq t_0\}}{2\xi}\tilde{\epsilon}$, and $M = \frac{\sqrt{d}CN\sup\{\|B(t)\|^2 : t\geq t_0\}}{\xi}$. Our proof can be divided into four parts: first, we construct a set of controls $\mathscr{U}(\epsilon, M, K, \mu)$, where each control corresponds to an initial condition in K. Second, we prove by induction that for every initial condition $x \in K$, there exists a control in $\mathscr{U}(\epsilon, M, K, \mu)$ such that $|x(s_n)| \leq C\big(|x(t_0)| + \tilde{\epsilon}\big)e^{-\alpha(s_{n+1}-t_0)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Third, we prove for any $n \in \mathbb{Z}_{\geq 0}$ and any $t \in [s_n, s_{n+1})$ we have a bound $|x(t)| \leq \big(M|x(t_0)|+\epsilon\big)e^{-\mu(t-t_0)}$, i.e., we show that $\mathscr{U}(\epsilon, M, K, \mu)$ is a stabilizing control set. Finally, we show that the data-rate $b(M, \mu)$ is finite for every possible $\mu \in \mathbb{R}_{>0}$ and our choice of $M \in \mathbb{R}_{>0}$ by proving an upper bound for $b(\mathscr{U}(\epsilon, M, K, \mu)) = \limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{U}_T(\epsilon, M, K, \mu))$ that is constant for every $\epsilon \in \mathbb{R}_{>0}$.
Part 1*:* Consider the following recursive definitions:
For $n \geq 0$ and for each $x \in K$, we define.

- For $n = 0$, define the constant function
$$\underline{\kappa}_i^0(x) := \min\{\langle x, e_i\rangle : x \in K\}$$
and
$$\overline{\kappa}_i^0(x) := \max\{\langle x, e_i\rangle : x \in K\}$$
for every $i \in [d]$. For $n \geq 1$, define the piecewise-constant functions
$$\underline{\kappa}_i^n(x) := \underline{\kappa}_i^{n-1}(x) + \Gamma_i^{n-1}(q_i^{n-1}(x) - 1)$$
and
$$\overline{\kappa}_i^n(x) := \underline{\kappa}_i^{n-1}(x) + \Gamma_i^{n-1}q_i^{n-1}(x)$$
for every $i \in [d]$;
- Define the constant $\Gamma_i^n := \frac{\tilde{\epsilon}}{d}e^{-(\lambda^{s_{n+1}}+\alpha)s_{n+1}}$ and the positive integer $\mathcal{C}_i^n := \Big\{1, \ldots, \Big\lceil \frac{\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x)}{\Gamma_i^n} \Big\rceil\Big\}$ for each $i \in \big\{1, \ldots, d\big\}$ and each $n \in \mathbb{Z}_{>0}$. Note that, by the defining equations of $\underline{\kappa}_i^n(x)$ and $\overline{\kappa}_i^n(x)$, $\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x) = \Gamma_i^{n-1}$. Thus, $\frac{\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x)}{\Gamma_i^n} = e^{(\lambda^{s_{n+1}}+\alpha)s_{n+1} - (\lambda^{s_n}+\alpha)s_n}$ for every $i \in [d]$, every $x \in K$, and every $n \in \mathbb{Z}_{\geq 0}$.
- Define the quantized value of the $i-$th projection of the initial state into the vector space span$\{e_i\}$ at time s_n by
$$q_i^n(x) := \{l \in \mathcal{C}_i^n : \underline{\kappa}_i^n(x) + \Gamma_i^n(l-1) \leq \langle x, e_i\rangle < \underline{\kappa}_i^n(x) + \Gamma_i^n l\}$$
for each $i \in [d]$;

- Define the quantized value of the $i-$th projection of the initial state into the vector space span$\{e_i\}$ at time s_n by
$$\hat{\beta}_i^n(x) := \underline{\kappa}_i^n(x) + \Gamma_i^n(q_i^n(x) - 1/2)$$
for each $i \in [d]$;

- Define the i-th projection of the initial state into the vector space span$\{e_i\}$ at time s_n by
$$\beta_i^n(x) := \langle x, e_i \rangle;$$

- With the notation $\sum_{i=1}^{b} c_i = 0$ for any $b \in \mathbb{Z}$ such that $b < 1$. Then, define the quantity[8]
$$\hat{x}(s_n) := \sum_{i=1}^{d} \hat{\beta}_i^n(x)\Phi(s_n, s_0)e_i + \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} \Phi(s_n, s)B(s)u(q^0(x), \cdots, q^k(x), s)ds;$$

- Define the control law in the interval $[s_n, s_{n+1})$ corresponding to the initial state x by
$$u(q^0(x), \cdots, q^n(x), t) := -B'(t)\Phi'(s_{n+1}, t)W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$
for $t \in [s_n, s_{n+1})$ where $q^n(x) := (q_1^n(x), \cdots, q_d^n(x))$. Further define $v(x, t) := u(q^0(x), \cdots, q^{n-1}(x), t)$, where n is the smallest integer such that $t < s_n$. Finally, define by $\mathscr{U}(\epsilon, M, K, \mu)$ the set of all such $v(x, \cdot)$. Also, denote by $\mathscr{U}_T(\epsilon, M, K, \mu)$ the set of restrictions of controls in $\mathscr{U}(\epsilon, M, K, \mu)$ from time t_0 to T. More explicitly $\mathscr{U}_T(\epsilon, M, K, \mu) := \{v_{|[t_0,T)}(x, \cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m) : v(x, \cdot) \in \mathscr{U}(\epsilon, M, K, \mu)\}$.

- *Part 2:*
Step 0: Trivially, we have that $|x(t_0)| \leq |x(t_0)| + \tilde{\epsilon} = C(|x(t_0)| + \tilde{\epsilon})e^{-\alpha(s_1 - t_0)}$ and we proved the base case, i.e., $|x(s_n)| \leq C(|x(t_0)| + \tilde{\epsilon})e^{-\alpha(s_{n+1} - t_0)}$ for $C \in \mathbb{R}_{>1}$ and for $n = 0$.
Step $n + 1$: Recall that for each $x \in K$ and for $t \in [s_n, s_{n+1})$ the control law we defined in the first part is given by
$$u(q^0(x), \cdots, q^n(x), t) = -B'(t)\Phi'(s_{n+1}, t)W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$
where
$$\hat{x}(s_n) = \sum_{i=1}^{d} \hat{\beta}_i^{s_{n+1}}(x)\Phi(s_n, s_0)e_i + \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} \Phi(s_n, s)B(s)u(q^0(x), \cdots, q^{k-1}(x), s)ds.$$
Now, writing down the variation of parameters formula at time s_{n+1} we get that
$$x(s_{n+1}) = \Phi(s_{n+1}, s_n)x(s_n) - \int_{s_n}^{s_{n+1}} \Phi(s_{n+1}, \tau)B(\tau)B'(\tau)\Phi'(s_{n+1}, \tau)d\tau \times$$
$$\times W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$

[8]This can be seen as an state estimate at time s_n.

from which we conclude that

$$x(s_{n+1}) = \Phi(s_{n+1}, s_n)\big(x(s_n) - \hat{x}(s_n)\big) = \sum_{i=1}^{d} (\beta_i^n(x) - \hat{\beta}_i^n(x))\Phi(s_{n+1}, s_0)e_i.$$

Then, by taking the norm on both sides and applying the triangle inequality, we conclude that

$$|x(s_{n+1})| \leq \sum_{i=1}^{d} |\beta_i^n(x) - \hat{\beta}_i^n(x)||\Phi(s_{n+1}, s_0)e_i|.$$

Now, by the definition of λ^t,[9] we get that $|\Phi(s_{n+1}, s_0)e_i| \leq e^{\lambda^{s_{n+1}} s_{n+1}}$ for all $i \in [d]$. Further, by recalling the expression of Γ_i^n and by the definitions of $\hat{\beta}_i^n$, β_i^n and $q_i^n(x)$, we conclude that $|\beta_i^n(x) - \hat{\beta}_i^n(x)| \leq \frac{\tilde{\epsilon}}{d} e^{-(\lambda^{s_{n+1}}+\alpha)s_{n+1}}$. Hence, we get that

$$|x(s_{n+1})| \leq \sum_{i=1}^{d} \frac{\tilde{\epsilon}}{d} e^{-\alpha s_{n+1}} = \tilde{\epsilon} e^{-\alpha s_{n+1}}.$$

Therefore, $|x(s_{n+1})| \leq \tilde{\epsilon} e^{-\alpha s_{n+1}} \leq C\big(|x(t_0)| + \tilde{\epsilon}\big)e^{-\alpha(s_{n+1}-t_0)}$ and we proved the case for step $n+1$.

- *Part* 3:

Now, pick any $n \in \mathbb{Z}_{\geq 0}$ and any $t \in [s_n, s_{n+1})$. Note that the variation of parameters formula gives us that

$$\begin{aligned} x(t) =& \Phi(t, s_n)x(s_n) - \\ & \int_{s_n}^{t} \Phi(t, s)B(s)B'(s)\Phi(s_{n+1}, s)ds W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n). \end{aligned}$$

Notice that

$$\int_{s_n}^{t} \Phi(t, s)B(s)B'(s)\Phi'(s_{n+1}, s)ds = \\ \Phi(t, s_{n+1}) \int_{s_n}^{t} \Phi'(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s)ds.$$

Next, let

$$\Omega(t, s_{n+1}, s_n) := \int_{s_n}^{t} \Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s)ds$$

and let

$$\Theta(t, s_{n+1}, s_n) := \int_{t}^{s_{n+1}} \Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s)ds.$$

Further, note that[10] $\Omega(t, s_{n+1}, s_n) \succcurlyeq 0$, $\Theta(t, s_{n+1}, s_n) \succcurlyeq 0$, and $W(s_{n+1}, s_n) \succ 0$. Also, the definitions imply that $W(s_{n+1}, s_n) = \Omega(t, s_{n+1}, s_n) + \Theta(t, s_{n+1}, s_n)$ for any $t \in [s_n, s_{n+1})$. The two latter facts imply that $\|\Omega(t, s_{n+1}, s_n)\| \leq \sqrt{d}\|W(s_{n+1}, s_n)\|$ and $\|\Theta(t, s_{n+1}, s_n)\| \leq \sqrt{d}\|W(s_{n+1}, s_n)\|$.

[9] Recall that $\lambda^{s_{n+1}} = \sup\{\frac{1}{t}\log(\|\Phi(t, t_0)\|) : s_{n+1} \geq t \geq t_0\}$.

[10] Since $\Phi(s_{n+1}, s)B(s)B'(s)\Phi'(s_{n+1}, s) \succcurlyeq 0$ for all $s \in [s_n, s_{n+1})$.

Recall the semigroup property for the transition matrix, i.e.,
$\Phi(t,z)=\Phi(t,r)\Phi(r,z)$ for any $z\geq t_0$, $t\geq t_0$ and any $z\geq t_0$. So, we get

$$\begin{aligned}
x(t) &= \Phi(t,s_n)x(s_n) - \\
&\quad \Phi(t,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)\hat{x}(s_n) = \\
&\quad \Phi(t,s_n)\Big(x(s_n)-\Phi(s_n,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)\hat{x}(s)\Big).
\end{aligned}$$

By rewritting $\hat{x}(s_n)=\hat{x}(s_n)-x(s_n)+x(s_n)$ and using the fact that $\Phi(t,r)\Phi(r,t)=I_d$ for every $t\geq t_0$ and every $r\geq t_0$, we get

$$\begin{aligned}
x(t) &= \Phi(t,s_n)\Big(\Phi(s_n,s_{n+1})(I-\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n))\Phi(s_{n+1},s_n)x(s) \\
&\quad -\Phi(s_n,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)(\hat{x}(s_n)-x(s_n))\Big) = \\
&\quad \Phi(t,s_{n+1})\Theta(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)x(s) \\
&\quad -\Phi(t,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)(\hat{x}(s_n)-x(s_n)).
\end{aligned}$$

Taking the norm on both sides and using the triangle inequality yields

$$\begin{aligned}
|x(t)| &\leq \|\Phi(t,s_{n+1})\|\|\Theta(t,s_{n+1},s_n)\|\|W^{-1}(s_{n+1},s_n)\|\times \\
&\times\|\Phi(s_{n+1},s_n)\||x(s)|+\|\Phi(s_n,s_{n+1})\|\|\Omega(t,s_{n+1},s_n)\|\times \\
&\times\|W^{-1}(s_{n+1},s_n)\|\|\Phi(s_{n+1},s_n)\||\hat{x}(s_n)-x(s_n)|.
\end{aligned}$$

We invoke Lemma 3.3 and notice that it implies that

$$\|W(s_{n+1},s_n)\|\leq\frac{\sup\{\|B(t)\|^2:t\geq t_0\}}{2\xi}e^{2\xi s_{n+1}}.$$

Then, we combine that with the fact that

$$\max\{\|\Omega(t,s_{n+1},s_n)\|,\|\Theta(t,s_{n+1},s_n)\|\}\leq\sqrt{d}\|W(s_n,s_{n+1})\|,$$

to conclude that

$$\begin{aligned}
\max\{\|\Omega(t,s_{n+1},s_n)\|,&\|\Theta(t,s_{n+1},s_n)\|\}\leq \\
&\frac{\sqrt{d}N\sup\{\|B(t)\|^2:t\geq t_0\}}{2\xi}e^{(2\xi+\theta)(s_{n+1})}.
\end{aligned}$$

By the exponential energy-growth condition, we know that there exist $\theta\in\mathbb{R}_{\geq 0}$ and $N\in\mathbb{R}_{>0}$ such that $\|W^{-1}(s_{n_1},s_n)\|\leq Ne^{\theta s_{n+1}}$. So, we have that

$$\begin{aligned}
|x(t)|\leq&\frac{\sqrt{d}N\sup\{\|B(t)\|^2:t\geq t_0\}}{2\xi}e^{(2\xi+\theta)(s_{n+1})}\|\Phi(s_{n+1},s_n)\|\times \\
&\times\Big(\|\Phi(t,s_{n+1})\||x(s)|+\|\Phi(s_n,s_{n+1})\||\hat{x}(s_n)-x(s_n)|\Big).
\end{aligned}$$

By Lemma 3.2, for any $t\in[s_n,s_{n+1})$, we get $\|\Phi(t,s_n)\|\leq e^{\xi(t-t_0)}$, which implies that

$$\begin{aligned}
|x(t)|\leq&\frac{\sqrt{d}N\sup\{\|B(t)\|^2:t\geq t_0\}}{2\xi}e^{(4\xi+\theta)s_{n+1}}\big(|x(s_n)|+|\hat{x}(s_n)|\big)\leq \\
&\frac{\sqrt{d}N\sup\{\|B(t)\|^2:t\geq t_0\}}{2\xi}e^{(4\xi+\theta)(s_{n+1}-t_0)}\big(|x(s_n)|+|\hat{x}(s_n)|\big),
\end{aligned}$$

where the last inequality follows from the fact that $t_0 \geq 0$. Note that $|x(s_n) - \hat{x}(s_n)| \leq \sum_{i=1}^{d} \|\Phi(s_n, s_0)e_i\| |\beta_i^n(x) - \hat{\beta}_i^n(x)| \leq \tilde{\epsilon} e^{-\alpha(s_{n+1}-t_0)}$ by the defining equations of β_i^n, $\hat{\beta}_i^n$, and $\hat{x}(s_n)$ presented in part 1 of the proof, from which we conclude that

$$|\hat{x}(s_n)| \leq \tilde{\epsilon} e^{-\alpha s_{n+1}} + |x(s_n)|.$$

So, we can write

$$|x(t)| \leq \frac{\sqrt{d} N \sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)(s_{n+1}-t_0)} \big(2|x(s_n)| + \tilde{\epsilon} e^{-\alpha(s_{n+1}-t_0)}\big).$$

Thus, by the conclusion of the proof of part 2, we get

$$\begin{aligned} &|x(t)| \leq \\ &\frac{\sqrt{d} N \sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)(s_{n+1}-t_0)} \big(2C(|x(t_0)| + \tilde{\epsilon}) + \tilde{\epsilon}\big) e^{-\alpha(s_{n+1}-t_0)} \leq \\ &\frac{\sqrt{d} N \sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} \big(2C|x(t_0)| + (2C+1)\tilde{\epsilon}\big) e^{-\mu(s_{n+1}-t_0)}. \end{aligned}$$

Since $\alpha = (4\xi + \theta + \mu)$. Finally, recall that $\epsilon = \frac{\sqrt{d}(2C+1)N \sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} \tilde{\epsilon}$ and $M = \frac{\sqrt{d} C N \sup\{\|B(t)\|^2 : t \geq t_0\}}{\xi}$. Hence, we conclude that

$$|x(t)| \leq (M|x(t_0)| + \epsilon) e^{-\mu(s_{n+1}-t_0)} \leq (M|x(t_0)| + \epsilon) e^{-\mu(t-t_0)}$$

for all $t \geq t_0$. Therefore, we proved that $\mathscr{U}_T(\epsilon, M, K, \mu)$ is a stabilizing control set, concluding the proof of part 3.

- *Part 4*: Note that there is a bijection between the elements of $\prod_{j=0}^{n} \prod_{i=1}^{d} \mathcal{C}_i^j$ and those of $\mathscr{U}_T(\epsilon, M, K, \mu)$ by the definition of $v(x,t)$. So, $\#\mathscr{U}_T(\epsilon, M, K, \mu) = \prod_{j=0}^{n} \prod_{i=1}^{d} \#\mathcal{C}_i^j$. Also, by the same equations, we have that $\#\mathscr{U}_T(\epsilon, M, K, \mu)$ is constant for $T \in [s_n, s_{n+1})$ for each $n \in \mathbb{Z}_{\geq 0}$. Thus,

$$\frac{1}{T} \log\big(\#\mathscr{U}_T(\epsilon, M, K, \mu)\big) \leq \frac{1}{s_n} \log\big(\#\mathscr{U}_T(\epsilon, M, K, \mu)\big)$$

for $T \in [s_n, s_{n+1})$. Also, note that

$$\#\mathcal{C}_i^n = \left\lceil e^{(\lambda^{s_{n+1}}+\alpha)s_{n+1} - (\lambda^{s_n}+\alpha)s_n} \right\rceil$$

for every $i \in [d]$ and $n \in \mathbb{Z}_{\geq 1}$. Therefore,

$$\log\left(\prod_{j=1}^{n} \prod_{i=1}^{d} \mathcal{C}_i^j\right) \leq d\Big((\lambda^{s_{n+1}} + \alpha)s_{n+1} - (\lambda^{s_1} + \alpha)s_1 + n\Big),$$

where the inequality comes from the facts that $\log(\lceil e^y \rceil) \leq y + 1$ for $y \in \mathbb{R}_{\geq 1}$ and from the property of telescoping series. Combining our previous results, we arrive at $\frac{1}{T} \log(\#\mathscr{U}_T(\epsilon, M, K, \mu)) \leq \frac{d}{s_n}\Big((\lambda^{s_{n+1}} + \alpha)s_{n+1} - (\lambda^{s_1} + \alpha)s_1 + n\Big) + \frac{\sum_{i=1}^{d} \log\left(\#\mathcal{C}_i^0\right)}{s_n}$.

Taking the limit superior on the left hand side with T going to infinity implies that we are taking the limit superior on the right hand sided with n going to infinity because $n = \inf\{l \in \mathbb{Z}_{\geq 0} : s_l \leq T \text{ and } s_{l+1} > T\}$. Hence, we get $\limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{U}_T(\epsilon, M, K, \mu)) \leq \limsup_{n\to\infty} \frac{d(\lambda^{s_{n+1}}+\alpha)s_{n+1}}{s_n} + \frac{n}{s_n} \leq d(\bar{\lambda}+\alpha)R + Q$. The first inequality follows from the fact that $\sum_{i=1}^{d}\log(\#\mathcal{C}_i^0)$ and $(\lambda^{s_1}+\alpha)s_1$ are finite. The last inequality follows because $\limsup_{n\to\infty} \frac{n}{s_n} = Q$ and because given two sequences of positive numbers $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ and $(b_n)_{\mathbb{Z}_{\geq 0}}$, then $\limsup_{n\to\infty} a_n b_n \leq \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$ and we have that

$$\limsup_{n\to\infty} \lambda^{s_{n+1}} = \bar{\lambda}$$

and $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = R$ by persistent complete controllability. Since our bound does not depend on ϵ, we have that the previous inequality gives an upper bound for $b(M, \mu)$. In this way, we proved that

$$b(M, \mu) < \lim_{\epsilon\to 0} b(\mathscr{U}(\epsilon, M, K, \mu)) < d(\bar{\lambda}+\alpha)R + Q < \infty$$

for every $\mu \in \mathbb{R}_{>0}$ and our chosen M. Thus, we conclude the proof of the theorem.

As mentioned in Remark 3.2, controllable LTI systems are controllable with finite data-rate. Thus, the system from Example 1 is controllable with finite data-rate as mentioned earlier. In the next subsection, we finally show why Example 2 cannot by controllable with finite data-rate.

3.2 Necessary Condition

In this subsection, we show a necessary condition for system (1) to be controllable with finite data-rate.

Theorem 3.2. System (1) is controllable with finite data-rate only if it is completely controllable.

Remark 3.3. It is important to notice the gap between the hypothesis of the necessary condition and the sufficient condition, i.e., the exponential energy-growth rate and the persistency of complete controllability. The former condition is only used in the part 3 of the proof of Theorem 3.1 to bound the growth of the state between times s_n and s_{n+1} for $n \in \mathbb{Z}_{\geq 0}$. Informally, this condition ensures that the state does not grow too much on the interval $[s_n, s_{n+1})$. At the moment, it is not clear if this condition is necessary or if it is a consequence of our choice of stabilizing control set $\mathscr{U}(\epsilon, M, K, \mu)$ in the proof of Theorem 3.1. Note, however, that the exponential energy-growth rate is a reasonable assumption since requiring the boundedness of the control energy, a stronger assumption, is normally desirable in practice. The latter fact, if the sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ that appears in Definition 3.1 satisfies $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, appears in the last part of the proof of Theorem 3.1 to bound the data-rate. Nonetheless, at the moment, it is not clear if we can remove it from the statement of Theorem 3.1.

Proof. We prove this theorem by contradiction. Assume that there exists $s \geq t_0$ such that for all $t \geq s$ we have that the Gramian of system (1) $W(t, s)$ is not

invertible[11], but system (1) can be stabilized with finite data-rate for arbitrary $\alpha \in \mathbb{R}_{\geq 0}$ and arbitrary $\epsilon \in \mathbb{R}_{>0}$. Thus, there exists $w(t) \in \mathbb{R}^d$ for every $t \geq t_0$ such that $w(t) \in \mathcal{N}(W(t,s))$ for all $t \geq s$ and that $|w(t)| = 1$ for all $t \geq s$.
First, note that $w'(t)\int_s^t \Phi(t,\tau)B(\tau)u(\tau)d\tau = 0$ for all $u(\cdot) \in L^{\infty}_{\text{loc}}([t_0,\infty),\mathbb{R}^m)$. To see that, recall that since $w(t) \in \mathcal{N}(W(t,s))$ for all $t \geq s$, we have that $w'(t)W(t,s)w(t) = 0$. That implies that

$$w'(t)\int_s^t \Phi(t,\tau)B(\tau)B'(\tau)\Phi(t,\tau)d\tau w(t) = \int_s^t |w'(t)\Phi(t,\tau)B(\tau)|^2 d\tau = 0,$$

which implies that $w'(t)\Phi(t,\tau)B(\tau) = 0$ for almost all $\tau \in [s,t]$. By its turn, this implies the claim $w'(t)\int_s^t \Phi(t,\tau)B(\tau)u(\tau)d\tau = 0$. Second, we pick $\alpha > \xi$ and pick some arbitrary $\epsilon \in \mathbb{R}_{>0}$. Since the data rate is finite, we know that there exists a stabilizing control set $\mathscr{R}(\epsilon,M,K,\alpha)$ such that the cardinality of a set of restrictions of stabilizing controls $N_s = \#\mathscr{R}_s(\epsilon,M,K,\alpha)$ is finite. Thus, if we choose N_s+1 distinct initial conditions $x(t_0)$ we have that at least two of them have the same associated control restriction $u_{|[t_0,s]}(t)$ for all $t \in [t_0,s]$. Now, let $\bar{x} \in K$ be some interior point to K. Pick an open ball $B(\bar{x},r)$ that is contained in the interior of K. Thus, for each $i \in [d]$, we can pick N_s+1 colinear points that lie on a line that is parallel to e_i. More precisely, define $y_{j,i} = \bar{x} + r\left(\frac{j-1}{N_s+1} - \frac{1}{2}\right)e_i$ for every $j \in [N_s+1]$ and every $i \in [d]$. Note that all of such points belong to $B(\bar{x},r)$. Denote by $u_{j,i}(t) \in \mathbb{R}^m$ the control function from the stabilizing control-set corresponding to the initial condition $y_{j,i}$ at time $t \geq t_0$ for each $i \in [d]$ and $j \in [N_s+1]$, and denote by $x_{j,i}(t)$ the corresponding state trajectory at time $t \geq t_0$ for each $i \in [d]$ and $j \in [N_s+1]$. Then, we can use the variation of constants formula to get

$$x_{j,i}(t) = \Phi(t,t_0)y_{j,i} + \int_{t_0}^t \Phi(t,\tau)B(\tau)u_{j,i}(\tau)d\tau$$

for all $t \geq t_0$. Now, by the pigeonhole principle, for each $i \in [d]$, there exists at least two distinct indices $j_i^* \in [N_s+1]$ and $k_i^* \in [N_s+1]$ such that the restriction of their corresponding controls $(u_{j,i})_{[t_0,s]}(t)$ is the same for $t \in [t_0,s]$. Let $z_i = y_{j_i^*,i} - y_{k_i^*,i} = e_i\frac{r(j_i^*-k_i^*)}{N_s+1}$ for each $i \in [d]$ and notice that $\{z_1,\cdots,z_d\}$ form an orthogonal basis[12] for $\mathbb{R}^d$. Further note that $|z_i| \geq \frac{r}{N_s+1}$ since $j_i^* - k_i^*$ is a nonzero integer. Also, let $\phi_i(t) \coloneqq x_{j_i^*,i}(t) - x_{k_i^*,i}(t)$ for every $i \in [d]$ and all $t \geq t_0$. Therefore, again by the variation of parameters formula, we get that $\phi_i(t) = \Phi(t,t_0)z_i$ for $t \in [t_0,s]$ and for $i \in [d]$ and

$$\phi_i(t) = \Phi(t,t_0)z_i + \int_{t_0}^t \Phi(t,\tau)B(\tau)(u_{j_i^*,i}(\tau) - u_{k_i^*,i}(\tau))d\tau$$

for $t \geq s$ and for $i \in [d]$. Now, for each $i \in [d]$ multiply $\phi_i(t)$ on the left by $w'(t)$ and note that $w'(t)\phi_i(t) = w'(t)\Phi(t,t_0)z_i$ for all $t \geq t_0$ by the fact that $w'(t)\int_s^t \Phi(t,\tau)B(\tau)u(\tau)d\tau = 0$ for all $t \geq s$ and all integrable $u(\cdot)$. Next,

[11] By the remark following Definition 3.1, we know that this implies that system (1) is not completely controllable.

[12] We have that z_i is parallel to e_i for each $i \in [d]$.

for every fixed time $t \geq t_0$, define coefficients $a_i(t) \in \mathbb{R}$ for all $i \in [d]$ such that $\sum_{i=1}^d |a_i(t)| = 1$ and $\Phi(t,t_0)z(t) \in \text{span}\{w(t)\}$, where $z(t) \coloneqq \sum_{i=1}^d a_i(t)z_i$. First, note that such coefficients always exist since $\{z_1, \cdots, z_d\}$ forms a basis for $\mathbb{R}^d$ and $\Phi(t,t_0)$ is invertible for every $t \geq t_0$. Hence, we can define $a_i(t)$ as $c\langle \Phi^{-1}(t,t_0)w(t), z_i\rangle/|z_i|^2$ for $c = \frac{1}{|\sum_i^d \langle \Phi^{-1}(t,t_0)w(t),z_i\rangle/|z_i|^2|}$. This follows from the fact that $\sum_{i=1}^d |a_i(t)| = |c||\sum_i^d \langle \Phi^{-1}(t,t_0)w(t), z_i\rangle/|z_i|^2| = 1$ and that $z(t) = c\sum_{i=1}^d \langle \Phi^{-1}(t,t_0)w(t), \frac{z_i}{|z_i|}\rangle \frac{z_i}{|z_i|} = c\sum_{i=1}^d \langle \Phi^{-1}(t,t_0)w(t), e_i\rangle e_i = c\Phi^{-1}(t,t_0)w(t)$. Further, note that $|z(t)| = \sum_{i=1}^d |a_i(t)||z_i| \geq \frac{r}{N_s+1}$, where the equality follows from the fact that $\{z_1, \ldots, z_d\}$ is an orthogonal basis and the inequality follows since $\sum_{i=1}^d |a_i(t)| = 1$ and the fact that $|z_i| \geq \frac{r}{N_s+1}$ for each $i \in [d]$. Let $\phi(t) \coloneqq \sum_{i=1}^d a_i(t)\phi_i(t)$ for every $t \geq t_0$. Thus, for every $t \geq t_0$ we have $w'(t)\phi(t) = w'(t)\Phi(t,t_0)z(t)$ for every $t \geq t_0$. Taking the norm on both sides and using the Cauchy-Schwarz inequality, we see that $|w'(t)\phi(t)| = |\Phi(t,t_0)z(t)|$ because $|w'(t)\Phi(t,t_0)z(t)| = |\Phi(t,t_0)z(t)|$ since $\Phi(t,t_0)z(t) \in \text{span}\{w(t)\}$ and $|w(t)| = 1$. Now, recall that, by definition of controllability with finite data-rate, for every $\alpha \geq 0$ and every initial condition $x(t_0)$, we have that

$$|x(t)| \leq \big(M|x(t_0)| + \epsilon\big)e^{-\alpha(t-t_0)}$$

for some $M \in \mathbb{R}_{>0}$, some $\epsilon > 0$, and all $t \geq t_0$. In particular, this must hold for our choice of $\alpha > \xi$ and our arbitrary choice of ϵ. This implies that

$$|\phi(t)| = |\sum_{i=1}^d a_i(t)\phi_i(t)| \leq \sum_{i=1}^d a_i(t)(|x_{j_i^*,i}(t)| + |x_{k_i^*,i}(t)|) \leq 2\big(MR_0 + \epsilon\big)e^{-\alpha(t-t_0)}$$

where the first inequality comes from the triangle inequality. The second inequality follows from the facts that $\sum_{i=1}^d |a_i(t)| = 1$, by construction, that

$$\max\{|x_{j_i^*,i}(t)|, |x_{k_i^*,i}(t)|\} \leq \big(M|x(t_0)| + \epsilon\big)e^{-\alpha(t-t_0)},$$

by controllability with finite data-rate, and that $|x(t_0)| \leq R_0$. Now, by the Cauchy-Schwarz inequality, we have that $|\Phi(t,t_0)z(t)| = |w'(t)\phi(t)| \leq |\phi(t)|$ since $|w(t)| = 1$. Hence, we arrive at $2\big(MR_0 + \epsilon\big)e^{-\alpha(t-t_0)} \geq |\Phi(t,t_0)z(t)|$. Finally, note that

$$|\Phi(t,t_0)z(t)| \geq \frac{r}{N_s+1}e^{-\xi(t-t_0)}.$$

To see this latter fact note that $|\Phi(t,t_0)v| \geq e^{-\xi(t-t_0)}$ for all $v \in \mathbb{R}^d$ with $|v| = 1$ by the lower bound in Lemma 3.2. That implies that $|\Phi(t,t_0)\frac{z(t)}{|z(t)|}| \geq e^{-\xi(t-t_0)}$. Thus, $|\Phi(t,t_0)z(t)| \geq e^{-\xi(t-t_0)}|z(t)| \geq \frac{r}{N_s+1}e^{-\xi(t-t_0)}$, where the last equality comes from the construction of $z(t)$. Since this must hold for each $t \geq t_0$ and we picked $\alpha > \xi$, we arrived at a contradiction. Therefore, system (1) must be completely controllable. □

Now, we can see why Example 2 cannot be controllable with finite data-rate. Note that every increasing sequence (s_n) with $\lim_{n\to\infty} s_n = \infty$ will have an $n_0 \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq n_0$, we have that $s_n > 1$. So, for all $n \geq n_0$, we have that $W(s_{n+1}, s_n)$ is not invertible. Thus, this proves that such a system is not controllable with finite data-rate.

4 Conclusion

In this paper, we discussed the problem of controlling LTV systems using quantized controls and finite data-rate. We motivated the study of this concept by showing that the usual controllability notion is not suitable for systems under finite data-rate constraints, such as systems that use digital controllers. Then, we presented a definition for controllability with finite data-rate for LTV systems that is consistent with properties of controllable LTV systems when no data-rate constraints are present. Next, we introduced a controllability notion, namely persistent complete controllability, which is related to the concept of controllability with finite data-rate that we defined. Finally, we presented a necessary condition and a sufficient condition relating the controllability notion to controllability with finite data-rate.

In future work, we want to study the concept of controllability with finite data-rate for switched linear systems. Also, we will propose a related notion of stabilizability with finite data-rate which is consistent with the controllability notion of this paper and with the usual concept used in the literature of systems without data-rate constraints.

5 Appendix

Proof. [Proof of equivalence between the two definitions of complete controllability] The definition of complete controllability in [6] can be understood as follows: For every[13] $t \in \mathbb{R}_{\geq t_0}$, there exists $\bar{t} \geq t$, such that $W(\bar{t}, t)$ is positive definite. First, we prove that this definition implies the complete controllability definition given in Definition 3.1. We prove this fact by induction. For our base step, pick $s_0 = t_0$. By the defintion of complete controllability from [6], we know that there exists $s_1 > s_0$ such that $W(s_1, s_0)$ is positive definite, which implies that it is invertible. Now we consider the step $n \in \mathbb{Z}_{\geq 1}$. Note that there exists $s_{n+1} > s_n$ such that $W(s_{n+1}, s_n)$ is positive definite. Hence, we proved that there exists an increasing sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $W(s_{n_1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. Therefore, we proved the first part of the claim.
Now, we assume Definition 3.1 and we show that this implies the definition given in [6]. For any $t \in \mathbb{R}_{\geq t_0}$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $t \leq s_n$. Consider $W(s_{n+1}, t)$. Note that $W(s_{n+1}, t) = W(s_{n+1}, s_n) + \Phi(s_{n+1}, s_n)W(s_n, t)\Phi'(s_{n+1}, s_n)$. By hypothesis, we know that $W(s_{n+1}, s_n)$ is positive definite[14] and we know that $\Phi(s_{n+1}, s_n)W(s_n, t)\Phi'(s_{n+1}, s_n)$ is positive semi-definite. Therefore, $W(s_{n+1}, t)$ is positive definite and we proved the claim.

□

Proof. [Proof of Lemma 3.1] Let $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ be such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$ and that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = R$. Recursively define $\bar{s}_0 := s_0$ and $\bar{s}_n := \min\{s \in (s_n)_{n\in\mathbb{Z}_{\geq 0}} : s \geq \bar{s}_{n-1}+1\}$ for every $n \in \mathbb{Z}_{\geq 1}$. First, notice that $W(\bar{s}_{n+1}, \bar{s}_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$ because there exists at least two distinct elements from $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ in the interval $[\bar{s}_n, \bar{s}_{n+1}]$. Next, note that, for every $n \geq 1$, we have

[13] Here we are imposing that the initial time is $t_0 \in \mathbb{R}$, which was not required in [6].
[14] This is equivalent to invertibility of the Gramian.

that $\bar{s}_n \geq 1$ because $s_0 \geq 0$ and the fact that $\bar{s}_n \geq \bar{s}_1 \geq 1$. Now, for every $n \in \mathbb{Z}_{\geq 0}$ we have that $\bar{s}_n \in (s_n)_{n\in\mathbb{Z}_{\geq 0}}$. Thus, there exists $m_n \in \mathbb{Z}_{\geq 0}$ such that $\bar{s}_n = s_{m_n}$. Write $\frac{s_{m_{n+1}}}{s_{m_n}}$. By the definition of $\bar{s}_{n+1}$, we have that $s_{m_{n+1}-1} < s_{m_n} + 1$. Hence, $\frac{s_{m_{n+1}-1}}{s_{m_n}} < \frac{s_{m_n}+1}{s_{m_n}} \leq 2$, where the last inequality comes from the fact that $s_{m_n} \geq 1$. With this, we conclude that $\limsup_{n\to\infty} \frac{\bar{s}_{n+1}}{\bar{s}_n} = \limsup_{n\to\infty} \frac{s_{m_{n+1}}}{s_{m_n}} \frac{s_{m_{n+1}-1}}{s_{m_{n+1}-1}} \leq \limsup_{n\to\infty} 2\frac{s_{m_{n+1}}}{s_{m_{n+1}-1}} = 2R$. Further, note that $\bar{s}_{i+1} - \bar{s}_i \geq 1$ for every $i \in \mathbb{Z}_{\geq 0}$. Thus, $\bar{s}_n - \bar{s}_0 = \sum_{i=0}^{n} \bar{s}_{i+1} - \bar{s}_i \geq n$, where the first equality comes from the equality for telescoping sums. Hence, $\frac{\bar{s}_{n+1}-\bar{s}_0}{\bar{s}_n} \geq \frac{n}{\bar{s}_n}$. Taking the limit superior when n goes to infinity, we get that $2R \geq \limsup_{n\to\infty} \frac{\bar{s}_{n+1}-\bar{s}_0}{\bar{s}_n} \geq \limsup_{n\to\infty} \frac{n}{\bar{s}_n}$.
Therefore, we proved that given a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ we can build a subsequence $(\bar{s}_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $W(\bar{s}_{n+1}, \bar{s}_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$, that $\limsup_{n\to\infty} \frac{\bar{s}_{n+1}}{\bar{s}_n} < \infty$, and that $\limsup_{n\to\infty} \frac{n}{\bar{s}_n} < \infty$.

□

Proof. [Proof of Lemma 3.3] Note that, for every $v \neq 0$ in $\mathbb{R}^d$, we have that

$$\begin{aligned} v'\Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s)v &= |v'\Phi(s_{n+1}, s)B(s)|^2 \\ &\leq \|\Phi(s_{n+1}, s)\|^2 \|B(s)\|^2 |v|^2. \end{aligned}$$

Also, because $\|\Phi(s_{n+1}, s)\| \leq e^{\xi(s_{n+1}-s)}$ for every $s \in [s_n, s_{n+1})$ by Lemma 3.2, we get

$$\Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s) \preccurlyeq \sup\{\|B(t)\|^2 : t \geq t_0\} e^{2\xi(s_{n+1}-s)} I_d.$$

Now, integrating both sides from s_n to s_{n+1}, we conclude that

$$W(s_{n+1}, s_n) \preccurlyeq \sup\{\|B(t)\|^2 : t \geq t_0\} \frac{e^{2\xi(s_{n+1}-s_n)} - 1}{2\xi} I_d.$$

Finally, taking the norm and noticing that $e^{-2\xi s_n} < 1$, we get

$$\|W(s_{n+1}, s_n)\| \leq \sup\{\|B(t)\|^2 : t \geq t_0\} \frac{e^{2\xi(s_{n+1}-s_n)} - 1}{2\xi}.$$

and we conclude the proof. □

Proof of Lemma 3.2. Recall that $X(t) = \Phi(t, t_0)$ is the solution to the matrix differential equation

$$\frac{dX(t)}{dt} = A(t)X(t)$$

with $X(t_0) = I_d$ and that $X(\cdot)$ is given by the Peano-Baker series[15]. More explicitly, consider the recursively defined matrices $M_k(t, t_0)$ for $t \geq t_0$ and all $k \in \mathbb{Z}_{\geq 0}$ by

$$M_0(t, t_0) := I_d$$

[15]See, e.g. Chapter 4 of [9] or Chapter 3 of [1].

and

$$M_k(t,t_0) := I_d + \int_{t_0}^{t} A(\tau)M_{k-1}(\tau,t_0)d\tau$$

for $k \in \mathbb{Z}_{>0}$. Now, pick an arbitrary $t_1 > t_0$. It is a well-known fact that $M_k(\cdot,t_0)$ converges uniformly[16] to $X(\cdot) = \Phi(\cdot,t_0)$ on the interval $[t_0,t_1]$.

Our goal now is to prove that $\|\Phi(t,t_0)\| \le e^{\xi(t-t_0)}$ for all $t \in [t_1,t_0]$. We do that by proving that $\|M_k(t,t_0)\| \le \sum_{i=0}^{k} \xi^i \frac{(t-t_0)^i}{i!}$ holds for every $k \in \mathbb{Z}_{\ge 0}$ using induction. The base case $\|M_0(t,t_0)\| \le 1$ is trivially true[17]. Now, assume that $\|M_{k-1}(t,t_0)\| \le \sum_{i=0}^{k-1} \xi^i \frac{(t-t_0)^i}{i!}$ is true. Then,

$$\|M_k(t,t_0)\| \le 1 + \int_{t_0}^{t} \xi\|M_{k-1}(\tau,t_0)d\tau\| \le 1 + \sum_{i=0}^{k-1} \xi^{i+1}\frac{(t-t_0)^{i+1}}{(i+1)!} = \sum_{j=0}^{k} \xi^j \frac{(t-t_0)^j}{j!}$$

where $j = i+1$ and the inequality holds for all $t \in [t_0,t_1]$. Thus,

$$\|\Phi(t,t_0)\| = \|\lim_{N\to\infty} M_N(t,t_0)\| = \lim_{N\to\infty} \|M_N(t,t_0)\| \le e^{\xi(t-t_0)}.$$

for all $t \in [t_0,t_1]$. Since $t_1 > t_0$ was arbitrary, $\|\Phi(t,t_0)\| \le e^{\xi(t-t_0)}$ holds for every $t \ge t_0$. Moreover, by definition of norm, we have that $\|\Phi(t,t_0)\| \ge |\Phi(t,t_0)v|$ for any $v \in \mathbb{R}^d$ with $|v| = 1$. Thus, we get $|\Phi(t,t_0)v| \le e^{\xi(t-t_0)}$ for all $t \ge t_0$ and all $|v| = 1$, which proves the upper bound.

For the lower bound, let $Z(t) = \Phi'(t_0,t)$. It is a well-known that[18]

$$\frac{dZ(t)}{dt} = -A'(t)Z(t)$$

with $Z(t_0) = I_d$. Thus, we can apply an analogous reasoning to get that $\|\Phi'(t_0,t)\| \le e^{\xi(t-t_0)}$ since $\xi = \sup\{\|-A'(t)\| : t \ge t_0\}$ as well. Finally, pick any $v \in \mathbb{R}^d$ with $|v| = 1$ and note that

$$1 = |v'v| = |v'I_d v| = |v'\Phi(t_0,t)\Phi(t,t_0)v| \le |v'\Phi(t_0,t)||\Phi(t,t_0)v|.$$

Now, divide by[19] $|v'\Phi(t_0,t)|$ to get

$$|\Phi(t,t_0)v| \ge |v'\Phi(t_0,t)|^{-1}.$$

Next, note that

$$\begin{aligned}|\Phi(t,t_0)v| &\ge \min\{|\Phi(t,t_0)v| : |v|=1\} \ge \min\{|v'\Phi(t_0,t)|^{-1} : |v|=1\}\\ &= \Big(\max\{|v'\Phi(t_0,t)| : |v|=1\}\Big)^{-1} = \|\Phi(t_0,t)\|^{-1},\end{aligned}$$

where the last equality follows from the definition of norm of a matrix. Finally, recall that $\|\Phi'(t_0,t)\| = \|\Phi(t_0,t)\|$. So, we have, for any $v \in \mathbb{R}^d$ and $|v| = 1$, that

[16]See, e.g., Theorem 1 of Chapter 3 of [1].

[17]We are using the convention that, for $t = t_0$, $(t-t_0)^0 = \lim_{t\to t_0}(t-t_0)^0 = 1$.

[18]See, e.g., Chapter 4 of [9].

[19]$\Phi(\cdot,\cdot)$ is always invertible, so $|v'\Phi(t_0,t)|$ cannot be zero.

$$|\Phi(t,t_0)v| \geq \|\Phi(t_0,t)\|^{-1} \geq e^{-\xi(t-t_0)}.$$

Therefore, we concluded the proof. □

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