STUDIES ON STABILITY AND STABILIZATION OF RANDOMLY SWITCHED SYSTEMS

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Abstract

This thesis presents a study on stability analysis and stabilizing controller synthesis of randomly switched systems. These systems have two ingredients: a family of nonlinear subsystems and a random switching signal that specifies which subsystem is active at each time instant. In broad strokes, the approach pursued here consists of identifying key properties of the switching signal and the family of subsystems, and finding conditions to connect these two sets of properties such that the switched system has some desirable stability characteristics. The method of multiple Lyapunov functions is employed in conjunction with some statistical properties of the switching signal for the analysis. The results apply to situations where traditional methods involving infinitesimal generators are difficult to apply, either due to insufficient information about the properties of the switching signal, or due to nontrivial dependence on its past history. Some of the results have conceptual parallels in deterministic switched systems theory. Stability in the presence of exogenous deterministic inputs is also considered, properties analogous to input-to-state stability are proposed, and sufficient conditions are established under which a randomly switched system exhibits these properties. Stabilizing controllers are synthesized for randomly switched systems with control inputs; the analysis results are utilized in conjunction with multiple control-Lyapunov functions and universal formulas for feedback stabilization of nonlinear systems. This approach lends a modular structure to the synthesis stage and facilitates the usage of standard off-the-shelf controllers.

To My Parents

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My parents are my inspiration. To thank them in any way for their invisible contributions and sacrifices would amount to profanity. I offer them this thesis in obeisance. It contains some of my thoughts which fit together reasonably in a coherent framework.

Contents

CHAPTER 1

INTRODUCTION

§ 1.1. Randomly Switched Systems: Description and Examples

A *switched system* [**37**, Chapter 1] consists of a family of subsystems and a switching signal. The switching signal selects the active subsystem from the family at every instant of time, and its points of discontinuity are called switching instants. When the switching signal of such a system is a random process, and the dynamics are governed by an ordinary differential equation between successive switching instants, we say that it is a *randomly switched system*. They have been variously described as piecewise deterministic stochastic systems [**17**] or variable structure systems, and have been successfully used as models for systems affected by random structural changes. Randomly switched systems have become almost ubiquitous in engineering applications, as the following partial list shows.

Power Electronics: Power electronic devices are natural examples of switched systems. Random switching strategies have been utilized in DC-DC converters to attenuate the level and smoothen the switching noise spectrum, as in [**59**]. A Markov-chain based modulation logic for power electronic converters is proposed in [**58**]; the authors carry out analysis relating the spectral characteristics of the switching signal to probabilistic structures governing random disturbances, and designing random switching procedures which minimize suitable criteria for power spectra. Random switching and linear robust control methods have been combined with pulse width modulation techniques to reduce electromagnetic interference in power converters [**50**]. Performance of DC-DC converters is affected by random disturbances arising from components and input signals. Arguably the total effect on the overall performance due to such disturbances can be effectively captured by suitably lumping all disturbances in the switching signal of the model, as shown in [**54**]. The paper [**2**] discusses beneficial features of randomness in buck-boost converters which minimize the maximum power spectral density of noise waveforms in the output current.

Manufacturing: Randomly switched systems provide natural frameworks for modeling manufacturing systems; production processes may undergo sudden changes at random instants of time, leading to alterations in production rate. Inventories with multiple parts production, fixed quality deterioration rates of the parts, and random production capacity can be modeled as Markovian jump systems, as in [**12**]; the authors propose a control strategy using stochastic optimization techniques. The problem of controlling production rates in a failure-prone manufacturing system to minimize discounted inventory and cost, under constant demand for commodities, is investigated in [1]. The authors model the underlying dynamics as a Markovian jump system, and propose an optimal control strategy corresponding to an infinite horizon discounted cost criterion.

Communication Networks: The flow of traffic in data communication networks can be modeled as a stochastic hybrid system, as proposed in [6]. In the follow-up paper [7] the authors bridge the gap between purely packet-based models and purely fluid-based models in communication network literature by utilizing basic characteristics of hybrid dynamics, thereby capturing actual dynamical properties of data communications networks and retaining flexibility to model events such as congestion and different types of queuing policies. In analysis and simulation of wireless mobile ad hoc networks, the random waypoint mobility model is particularly useful. This model may be realized as a random switched system, with switching instants governed by a given (and possibly dynamic) distribution law, and the interswitching behavior governed by particularly simple ordinary differential equations; see [5].

Economics and Finance: Stochastic switched systems are frequently utilized to model uncertainties in stock prices and real exchange rates, as well as in prediction analysis. A simple random system model of a stylized equity market proposed in [24] can be used to assess the effects of uncertainty on the fundamentals of stock price dynamics. A Markovian switching vector error-correction nonlinear model is proposed in [16] to study short-term responses to permanent shocks and the effect of recessions on the long-run growth of an economy; the author demonstrates that this model shows lower one-step-ahead prediction mean square error compared to other linear models.

Biology: Microorganisms frequently exhibit different cellular structures under different environmental conditions. A dynamical population growth and diversity model of microorganisms conditioned on environmental and cellular states, where switches between different cellular states are triggered by noisy environmental data gathered by cellular sensors, is proposed in [63]. Certain light-sensitive microorganisms regularly reverse their swimming direction due to spontaneous switches in their flagellar motor rotations under normal environmental conditions. A stochastic model involving a "motor switch" running through a sequence of states and reversing upon completion of a cycle is proposed in [45]; this paper explains how the direction of motor rotations in these microorganisms is regulated.

§ 1.2. Main Contributions

Background on stochastic stability. Stability is one of the canonical problems in systems theory, and as such it is no surprise that stability of stochastic systems has a rich history. Even a brief literature review will take numerous pages, and the following attempt to isolate some key references whose contents and methods are most relevant to ours, is by no means complete. A particular class of randomly switched systems has received widespread attention, namely, Markovian jump linear systems (MJLS). They may be realized as a family of linear subsystems, together with a switching signal generated by the state of a continuous-time Markov chain. Stability and stabilization of MJLS have been extensively investigated, especially under the assumption that the parameters of the Markov chain are completely known; see, e.g., [8, 28, 20, 42] and the references therein. In particular, almost sure stabilization and stabilization in the mean of MJLS is discussed in [20], where the authors also establish precise equivalences between different stability notions for MJLS. A detailed exposition of MJLS may be found in the recent text [11].

A myriad of techniques have been employed to study stability and stabilization of piecewise deterministic stochastic systems. HJB-based optimal control schemes for piecewise deterministic stochastic systems are well-studied; see, e.g., [17] for a detailed account. Linear control systems admit analytically solvable linear quadratic optimal controllers, and such techniques have been effectively combined with the stochastic nature of structural variations in [28]; stabilization schemes based on Lyapunov exponents are studied in [20]. Game-theoretic techniques [3] in the presence of disturbance inputs, and spectral theory of Markov operators [27] have also been employed for analysis and control synthesis. Stabilization schemes using robust control methods are investigated in [66]; see also the references cited in it. Stochastic hybrid systems, where the switching signal and its transition probabilities are state-dependent, are studied in [15, 25], using an extended definition of the infinitesimal generator and optimal control strategies, respectively. A method of stabilization in probability of Markovian jump systems, with control and Brownian motion inputs for each subsystem (controlled switched diffusions), has been proposed in [4]. This method involves multiple Lyapunov functions, which are employed to iteratively find invariant sets, and the technique is similar in spirit to some of our results. Ergodic control techniques have been established for controlled switched diffusions in [22].

Our approach and contributions. A randomly switched system has two ingredients, namely, the family of subsystems and the random switching signal. Our approach consists of identifying key properties of the family of subsystems and the switching signal, and finding conditions to connect them such that the switched system has the desired characteristics. To borrow hybrid systems terminology, we extract properties of the components governing the continuous and discrete dynamics separately, and find conditions connecting them under which the desired characteristics of the switched system are obtained.

Thus, the basic structure of our main analysis results is as follows. The first step involves extracting properties which quantitatively express stability characteristics of the subsystems. This is carried out with the help of multiple Lyapunov functions. The method of multiple Lyapunov functions was developed originally in the context of deterministic switched systems, and is discussed in detail in, e.g., [37, Chapter 3]. This method is effective in quantitatively capturing the degree of stability (or instability) of the subsystems, and we employ this method for just that purpose. The second step involves extracting key properties of the switching signal. These properties are variously captured by the probability mass function of its rate of switching, the probability distribution of its jump destinations, distribution of holding times between switching instants, etc. Finally, the characteristics of the switched system generated by the switching signal from the family of subsystems are captured by inequalities which connect the above two sets of properties.

Our analysis results, particularly those in which each subsystem is required to be stable, have conceptual analogs in deterministic switched systems theory; see, e.g., [**37**, Chapter 3] for a detailed discussion. Stability of individual subsystems and a slow switching condition are the important features of these deterministic results. Our results involving unstable subsystems employ certain probabilistic characteristics of the switching signal in addition to slow switching; their conceptual analogs in deterministic switched systems literature are comparatively less known, with the exception of [**67**].

For systems with no external inputs, we concentrate on stability in almost sure and in $\boldsymbol{L}_1$ senses; since each of these implies stability in probability [**23, 33, 31**], our results also provide sufficient conditions for stability in probability of the systems under consideration. For switched systems with exogenous deterministic disturbance inputs, stability analysis and controller synthesis techniques are less known, although some results in this direction in the context of deterministic slow-switching systems with nonlinear subsystems have recently been reported in [**62, 65**]. We refer to stability in the presence of external disturbance inputs as external stability. In this thesis we report some partial results pertaining to external stability of randomly switched systems in the presence of deterministic disturbance inputs, and also some definitive results in the particular case of Markovian switching.

With our analysis results in hand, we turn to control synthesis and derive explicit controller formulas which ensure stability of the switched system in closed loop. In this context, there naturally arise two distinct cases: one in which the controller has full knowledge of the switching signal at each instant of time, and the other in which the controller is totally blind to the switching signal. We examine the distinctive features of each of these two cases and propose control synthesis strategies by employing *universal formulae* [**56, 39, 40, 41**] for nonlinear feedback stabilization, for systems with or without disturbance inputs. The advantages of our approach are evident here, for one does not need to design a controller separately for the switched system if there already exist *control-Lyapunov functions* for each individual subsystem; then, off-the-shelf controllers employing universal formulae are easily designed, and a modular organization of the controller synthesis stage is facilitated.

Comparison with more traditional methods. Traditionally, qualitative analysis of randomly switched systems has been centered around methods which consider the system state consisting of the continuous and discrete components together as a single random process, and the methods tend to rely on infinitesimal generators and stochastic Lyapunov functions. However these methods do not directly identify the features of the different components of the system responsible for the desired characteristics. A good example is furnished by the usual method of analysis of piecewise deterministic systems. In this case one identifies the state of an underlying typically finite-dimensional continuous-time Markov process with the switching signal, constructs the infinitesimal generator corresponding to the process, and proceeds to perform the analysis by employing this generator. The properties of the constituent subsystems are not directly involved in the analysis, but they do figure out indirectly couched inside the generator. One would expect that if the constituent subsystems are sufficiently stable and the switching signal is sufficiently lethargic on an average, then the switched system is also well-behaved. However, the infinitesimal generator approach appears not to support even such intuitive deductions.

In this thesis we do not follow the traditional approach; we dissect the randomly switched system into two principal components consisting of the family of subsystems and the switching signal. Classes of random switching signals and subsystems are identified which generate switched systems having desirable qualitative properties. For instance, we propose a class of switching signals characterized merely by a statistically slow switching condition, and prove that the switched systems generated by such signals are stable whenever every subsystem has sufficient stability margin. For stability of switched systems containing unstable subsystems, slow switching alone does not suffice and additional statistical properties must be imposed on the switching signal. We identify two classes of signals that possess such properties and study how their parameter variations affect stability of the switched systems that they generate. In particular, we deal with a class of semi-Markov [**10**, §20.4] switching signals with holding times being i.i.d uniform random variables. Our results apply to cases where writing down an infinitesimal generator is not possible, either due to insufficient information about the switching signal, as in the slow-switching case described above, or due to strong dependence on the past history, as in the semi-Markov case. Some of our results do specialize to Markovian jump systems. Although most of the analysis is carried out in the framework of general processes, we shall highlight at appropriate places throughout the thesis how the presence of additional probabilistic structures interact with the problems at hand.

§ 1.3. Preliminaries

In this section we establish some notations and conventions, define underlying probability spaces and some basic notions related to stochastic processes, discuss different stability notions at an abstract level, and briefly review the relationships between the notions.

Let the absolute value be denoted by $|\cdot|$, the Euclidean norm by $\|\cdot\|$, the interval $[0,\infty[$ by $\mathbb{R}_{\geqslant 0}$, and the set of natural numbers $\{1,2,\ldots\}$ by $\mathbb{N}$. As usual $\mathbb{R}$ stands for $]-\infty,\infty[$. A continuous function $\alpha : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if α is strictly increasing with $\alpha(0) = 0$, of class $\mathcal{K}_\infty$ if in addition $\alpha(r) \to \infty$ as $r \to \infty$; we write $\alpha \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$ respectively. A function $\beta : \mathbb{R}^2_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{KL}$ if for each s the function $\beta(\cdot, s)$ is of class $\mathcal{K}$, and for each r the function $\beta(r, \cdot)$ is monotone decreasing, with $\beta(r,s) \to 0$ as $s \to \infty$; we write $\beta \in \mathcal{KL}$. Standard notations on $\boldsymbol{L}_p$-spaces will be used sparsely.

§ 1.3.1. Basic facts from probability and stochastic processes. Our primary references for this subsection are [**52, 51**].

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a complete probability space, where Ω is the sample space, $\mathfrak{F}$ is a sigma-algebra on Ω, and P is a complete probability measure. Let $(\mathfrak{F}_t)_{t\geqslant 0}$ be an increasing system of sigma-subalgebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subseteq \mathfrak{F}_t$ whenever $0 \leqslant s \leqslant t < \infty$. Such a system is called a *filtration* or a *history*, and the probability space $\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, \mathsf{P}\big)$ equipped with this filtration is called a *filtered probability space*. We shall assume that the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ obeys the *usual conditions*, namely, $\mathfrak{F}_0$ contains all the P-null sets of $\mathfrak{F}$, and the filtration is right-continuous in the sense that $\bigcap_{s>0} \mathfrak{F}_{t+s} = \mathfrak{F}_t$ for each $t \geqslant 0$. We let $\mathsf{E}\big[\,\cdot\,\big]$ denote the expectation with respect to the measure P, and if $\mathfrak{G}$ is a sigma-subalgebra of $\mathfrak{F}$, then the conditional expectation with respect to $\mathfrak{G}$ is denoted by $\mathsf{E}^{\mathfrak{G}}\big[\,\cdot\,\big]$ or $\mathsf{E}\big[\,\cdot\,\big|\mathfrak{G}\big]$. Similarly, the conditional probability given a sigma-subalgebra $\mathfrak{G}$ is denoted by $\mathsf{P}^{\mathfrak{G}}\big(\,\cdot\,\big)$ or $\mathsf{P}\big(\,\cdot\,\big|\mathfrak{G}\big)$. We let $\mathfrak{B}(S)$ denote the Borel sigma-algebra on a subset S of $\mathbb{R}$, where we assume the presence of the usual topology on $\mathbb{R}$.

Let us recall some standard inequalities; see e.g., [**10**] for details. Let X and Y be real-valued random variables on the probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. *Chebyshev's inequality* states that if $s > 0$, then $\mathsf{P}\big(|X| \geqslant s\big) \leqslant \mathsf{E}\big[|X|\big]/s$. A function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is convex iff $\forall\, t \in [0,1]$ and $\forall\, x, y \in \mathbb{R}$, we have $\phi((1-t)x + ty) \leqslant (1-t)\phi(x) + t\phi(y)$. *Jensen's inequality* states that if X is integrable and ϕ is a convex function, then $\phi\big(\mathsf{E}\big[|X|\big]\big) \leqslant \mathsf{E}\big[\phi(|X|)\big]$. An exponential-($\lambda$) random variable X has the following probability distribution: $\mathsf{P}\big(X \leqslant s\big) = 1 - \mathrm{e}^{-\lambda s}$ if $s \geqslant 0$, and 0 otherwise. A uniform-(T) random variable Y has the probability distribution $\mathsf{P}\big(Y \leqslant s\big) = s/T$ if $s \in [0,T]$, 0 if $s < 0$, and 1 if $s > T$. If $\{A_i\}_{i\in\mathbb{N}}$ is a countable partition of Ω and ξ is an integrable random variable, then the *total probability formula* states that $\mathsf{E}\big[\xi\big] = \sum_{i\in\mathbb{N}} \mathsf{E}\big[\xi\big|A_i\big]\,\mathsf{P}\big(A_i\big)$.

Let I be a nonempty index set. A family of real-valued integrable random variables $\{\xi_i\}_{i\in I}$ is said to be *uniformly integrable* if

$$\lim_{c\to\infty} \sup_{i\in I} \mathsf{E}\big[\,|\xi_i|\,\mathbf{1}_{\{|\xi_i|>c\}}\big] = 0.$$

The following Hadamard-de la Vallée Poussin criterion [**14**, p. 286] for verifying uniform integrability of a family of random variables will be employed later.

1.1. Proposition. *A family of real-valued integrable random variables $\{\xi_i\}_{i\in I}$ is uniformly integrable if and only if there exists a convex function $\phi : \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$ with $\phi(0) = 0$ and $\lim_{r\to\infty} \phi(r)/r = \infty$, such that $\sup_{i\in I} \mathsf{E}\big[\phi(\xi_i)\big] < \infty$.*

Recall that a family of random variables $\{\xi_t\}_{t\geqslant 0}$ converges almost surely (a.s.) if it converges pointwise outside a P-null set. The following Proposition is standard and can be readily derived from, e.g., [**10**, Theorem 5, p. 113].

1.2. Proposition. *If $(\xi_t)_{t\geqslant 0}$ is a càdlàg (i.e., right-continuous and possessing limits from the left) random process on the filtered probability space above, $\{\xi_t\}_{t\geqslant 0}$ is uniformly integrable, and $(\xi_t)_{t\geqslant 0}$ converges to 0 a.s., then $\big(\mathsf{E}\big[\xi_t\big]\big)_{t\geqslant 0}$ converges to 0.*

A real-valued *stochastic process* $(\xi_t)_{t\geqslant 0}$ is a collection of real-valued random variables. A real-valued stochastic process $(\xi_t)_{t\geqslant 0}$ is said to be

- *measurable* if the map $\mathbb{R}_{\geqslant 0} \times \Omega \ni (t,\omega) \longmapsto \xi(t,\omega) \in \mathbb{R}$ is $\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{F}$-measurable, where $\mathfrak{B}(()\,\mathbb{R}_{\geqslant 0})$ is the Borel sigma-algebra on $\mathbb{R}_{\geqslant 0}$, and $\mathfrak{B}(\mathbb{R}_{\geqslant 0}) \otimes \mathfrak{F}$ is the product sigma-algebra on $\mathbb{R}_{\geqslant 0} \times \Omega$;
- $(\mathfrak{F}_t)_{t\geqslant 0}$-*adapted* if for each $t \geqslant 0$ the random variable ξ_t is $\mathfrak{F}_t$-measurable, i.e., $\xi_t^{-1}(\mathfrak{B}(\mathbb{R})) \subseteq \mathfrak{F}_t$ for each $t \geqslant 0$;
- $(\mathfrak{F}_t)_{t\geqslant 0}$-*progressively measurable* if for each $T \geqslant 0$ the restriction of ξ to $[0,T]\times\Omega$ is measurable, i.e., $\big(\xi_t \mathbf{1}_{[0,T]}\big)_{t\geqslant 0}$ is a measurable process.

Each of these concepts carries over to $\mathbb{R}^n$-valued stochastic processes with the obvious modifications.

We say that a real-valued stochastic process $(\xi_t)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-*martingale* (*supermartingale*) if $(\xi_t)_{t\geqslant 0}$ is $(\mathfrak{F}_t)_{t\geqslant 0}$-adapted, $\mathsf{E}\big[\,|\xi_t|\,\big] < \infty$ for each $t \geqslant 0$, and $\mathsf{E}^{\mathfrak{F}_s}\big[\xi_t\big] = \xi_s$ (resp. $\mathsf{E}^{\mathfrak{F}_s}\big[\xi_t\big] \leqslant \xi_s$) whenever $0 \leqslant s \leqslant t < \infty$. A nonnegative $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale $(\xi_t)_{t\geqslant 0}$ is called an $(\mathfrak{F}_t)_{t\geqslant 0}$-*potential* if $\lim_{t\to\infty} \mathsf{E}\big[\xi_t\big] = 0$.

An $(\mathfrak{F}_t)_{t\geqslant 0}$-*optional time* T is a $\mathbb{R}_{\geqslant 0}$-valued random variable such that for each $c \geqslant 0$ we have $\{T \leqslant c\} \in \mathfrak{F}_c$. Let us note that some authors prefer to call this an $(\mathfrak{F}_t)_{t\geqslant 0}$-stopping time, and a random time S is called an optional time if $\{S < c\} \in \mathfrak{F}_c$ for each $c > 0$; however, in the presence of the usual conditions on the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ the two definitions are equivalent.

A continuous-time version of a discrete-parameter supermartingale switching principle [**46**, Lemma II.2.8] will be employed later in the thesis; it states that if $(\xi^1_t)_{t\geqslant 0}$ and $(\xi^2_t)_{t\geqslant 0}$ are two càdlàg nonnegative $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingales and τ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time such that $\xi^1_\tau \geqslant \xi^2_\tau$ a.s., then the process $\xi_t := \xi^1_t \mathbf{1}_{\{t<\tau\}} + \xi^2_t \mathbf{1}_{\{t\geqslant\tau\}}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale.

A process $(\xi_t)_{t\geqslant 0}$ is said to a *Markov process* if it satisfies the *Markov property*: the conditional probability distribution of future states of the process, given the present state and all past states, depends only upon the present state; i.e., the process is conditionally independent of the past states given the present state. We shall later encounter switching signals given by the state of finite-dimensional *Markov chains*—continuous-time Markov processes on a finite state-space. A general *semi-Markov process* [**10**, §20.4] can be described as a continuous-time jump process $(\xi_t)_{t\geqslant 0}$ with jump instants $(\tau_i)_{i\in\mathbb{N}}$, for which the jump destination process $(\xi_{\tau_i})_{i\in\mathbb{N}}$ is a discrete-time Markov chain, and the holding-time process $(S_i)_{i\in\mathbb{N}}$, $S_i := \tau_i - \tau_{i-1}$, is a sequence of random variables, independent of the jump destination process, and the distribution of S_i may depend on both $\sigma(\tau_{i-1})$ and $\sigma(\tau_i)$.

Let $\mathcal{T}$ denote either $\mathbb{N}$ or $\mathbb{R}_{\geqslant 0}$. For the rest of this chapter we shall deal with $\mathbb{R}^n$-valued *stochastic processes*, a family of $\mathbb{R}^n$-valued random variables $x := (x_t)_{t\in\mathcal{T}}$ defined on $(\Omega, \mathfrak{F}, \mathsf{P})$ such that $(x_t)_{t\in\mathcal{T}}$ is $(\mathfrak{F}_t)_{t\in\mathcal{T}}$-adapted. At a point $t \in \mathcal{T}$, the random variable $x(t) = x_t$ represents the value taken by the process x. Corresponding to the sample point $\omega \in \Omega$, the value of the process at $t \in \mathcal{T}$ is denoted by $x(t, \omega)$, or $x_t(\omega)$. For fixed $\omega \in \Omega$, the set $(x_t(\omega))_{t\in\mathcal{T}}$ is called the *sample path* or trajectory of the process corresponding to ω.

Before embarking upon the study of specific types of stability of random switched systems, let us briefly recall some interesting definitions of stochastic stability. Stochastic stability concepts are generally described relative to a trivial realization $x^\star$ of the process considered. In what follows we shall assume that this realization $x^\star$ is identically 0, and denote the initial value of the process x by x°.

§ **1.3.2. Types of stochastic stability.** In this subsection we give a partial catalog of different types of stochastic stability; some relationships between them will be discussed in the following subsection. The concepts cataloged here are classical, discussed at length in, e.g., [**23**], and they are stated in terms of an abstract stochastic process. In Chapters 2 and 4 we shall view these concepts as instances of *internal stability* and differentiate these from a different class of stability concepts, which we shall call *external stability*. The latter class caters to systems which involve external (not necessarily random) disturbance inputs, and consequently their stability characteristics must involve these inputs in some form; see Chapter 3 for a discussion.

A stochastic process x is said to be *globally asymptotically stable in probability* if the following two conditions hold simultaneously:

wp′) (Lyapunov stability in probability) $\forall\, \eta' \in \,]0,1[\quad \forall\, \varepsilon > 0 \;\; \exists\, \delta(\eta', \varepsilon) > 0$ such that

$$\|x^\circ\| < \delta(\eta', \varepsilon) \implies \inf_{t\in\mathcal{T}} \mathsf{P}\big(\|x_t\| < \varepsilon\big) \geqslant 1 - \eta';$$

wp″) (global asymptotic convergence in probability) $\forall\, \eta'' \in \,]0,1[\quad \forall\, r, \varepsilon' > 0 \;\; \exists\, T(\eta'', r, \varepsilon') \in \mathcal{T}$ such that

$$\|x^\circ\| < r \implies \inf_{t \geqslant T(\eta'', r, \varepsilon')} \mathsf{P}\big(\|x_t\| < \varepsilon'\big) \geqslant 1 - \eta''.$$

A stronger notion is also widely employed in qualitative analysis of stochastic processes, namely, *strong global asymptotic stability in probability*. This notion is concerned with uniform behavior of a significant fraction of the trajectories. A stochastic process x possesses this property if the following two conditions hold simultaneously:

sp′) (strong Lyapunov stability in probability) $\forall\, \eta' \in \,]0,1[\quad \forall\, \varepsilon > 0 \quad \exists\, \delta(\eta', \varepsilon) > 0$ such that

$$\|x^\circ\| < \delta(\eta', \varepsilon) \implies \mathsf{P}\bigg(\sup_{t\in\mathcal{T}} \|x_t\| < \varepsilon\bigg) \geqslant 1 - \eta';$$

sp″) (strong global asymptotic convergence in probability) $\forall\,\eta'' \in \,]0,1[\quad \forall\, r, \varepsilon' > 0$ $\exists\, T(\eta'', r, \varepsilon') \in \mathcal{T}$ such that

$$\|x^\circ\| < r \implies \mathsf{P}\left(\sup_{t \geqslant T(\eta'', r, \varepsilon')} \|x_t\| < \varepsilon' \right) \geqslant 1 - \eta''.$$

It turns out that usually one gets stronger properties than global asymptotic stability in probability. Two such properties are global asymptotic stability in the mean and global asymptotic stability almost surely. The latter is of interest in applications because it allows us to conclude pathwise good behavior of a stochastic process; however, it does not give information about uniform behavior of trajectories. The strong global asymptotic stability in probability, on the other hand, does provide uniform estimates over a restricted set of events.

A stochastic process x is said to be *α-globally asymptotically stable in the mean* (α-GAS-M) for a function $\alpha \in \mathcal{K}$ if the following two conditions hold simultaneously:

m′) (Lyapunov stability in $\boldsymbol{L}_1$) $\forall\, \varepsilon > 0\;\; \exists\, \delta(\varepsilon) > 0$ such that $\|x^\circ\| < \delta(\varepsilon) \implies \sup_{t \in \mathcal{T}} \mathsf{E}\big[\alpha(\|x_t\|)\big] < \varepsilon$;

m″) (global asymptotic convergence in $\boldsymbol{L}_1$) $\forall\, r, \varepsilon' > 0\; \exists\, T(r, \varepsilon') \in \mathcal{T}$ such that $\|x^\circ\| < r \implies \sup_{t \geqslant T(r,\varepsilon')} \mathsf{E}\big[\alpha(\|x_t\|)\big] < \varepsilon'$.

Usually global asymptotic stability in the mean is stated without the function α; however, in our case we shall naturally get the notion of α-GAS-M in our results, and further specific properties can be derived thereafter. For instance, if α is convex, then an application of Jensen's inequality shows that α-GAS-M implies the usual GAS-M. Sometimes stability of the p-th mean of $\|x_t\|$ is considered; i.e., $\big(\mathsf{E}\big[\,\|x_t\|^p\,\big]\big)^{1/p}$ replaces $\mathsf{E}\big[\,\|x_t\|\,\big]$ in **m′) and m″)** above. This is known as *global asymptotic $\boldsymbol{L}_p$ stability* of x; GAS-M is therefore just global asymptotic $\boldsymbol{L}_1$ stability. Finally, x is said to be *globally asymptotically stable almost surely* (GAS a.s.) if the following two conditions hold simultaneously:

a′) (almost sure Lyapunov stability)

$$\mathsf{P}\left(\forall\, \varepsilon > 0\;\; \exists\, \delta(\varepsilon) > 0 \text{ such that } \|x^\circ\| < \delta(\varepsilon) \implies \sup_{t \in \mathcal{T}} \|x_t\| < \varepsilon \right) = 1;$$

a″) (almost sure global asymptotic convergence)

$$\mathsf{P}\left(\forall\, r, \varepsilon' > 0\;\; \exists\, T(r, \varepsilon') \in \mathcal{T} \text{ such that } \|x^\circ\| < r \implies \sup_{t \geqslant T(r,\varepsilon')} \|x_t\| < \varepsilon' \right) = 1.$$

Note that this notion of GAS a.s. does not claim any uniform properties of the sample paths. For instance, **a″)** states that the process converges for almost every event, but the rate of convergence expressed by the number T depends on the event. In other words, it says that almost every sample path is globally asymptotically stable. Also, we have to ensure that the various sets inside the probability measure P in the definitions are $\mathfrak{F}$-measurable. If $\mathcal{T} = \mathbb{R}_{\geqslant 0}$, then each of the sets above is measurable for processes with càdlàg sample paths.

§ 1.3.3. Some relationships between different stability concepts. GAS a.s. and GAS-M are perhaps the two most important notions as far as applications are concerned. Observe that while GAS a.s. states that almost every trajectory of the stochastic process x is globally asymptotically stable, GAS-M makes no claims about the behavior of individual trajectories of the process. In applications where sample-path behavior is of paramount importance, the former definition seems to be a natural choice (see also [**31**]).

We shall concentrate on global asymptotic stability almost surely and in the mean, with special emphasis placed on the former. While our choice to emphasize the former is dictated by the need to concretely describe the behavior of most, if not all, trajectories of the stochastic processes under consideration, the latter is the one that is more universally applied. This is partly due to the relative ease with which the latter property can be established for a large class of processes, and also because in a large number of applications almost sure global asymptotic stability is too restrictive. For our models, however, as we shall see, GAS a.s. is not a very stringent property, although establishing it is a nontrivial affair.

Let us look at some relationships between the various types of stability cataloged in §1.3.2. These are all standard (see, e.g., [**23**]), and later in the thesis we shall look at how some particular properties of the models that we study facilitate certain implications among the various stability notions.

GAS-M *implies global asymptotic stability in probability.* Suppose that x is GAS-M, and fix $\eta', \eta'' \in \,]0,1[$ and $r, \varepsilon, \varepsilon' > 0$. Since **m′)** holds, choose the δ corresponding to $\varepsilon\eta'$. Fix $t \in \mathcal{T}$ and choose x° such that $\|x^\circ\| < \delta$. In view of Chebyshev's inequality we get $\mathsf{P}\big(\,\|x_t\| \geqslant \varepsilon\big) \leqslant \mathsf{E}\big[\,\|x_t\|\,\big]/\varepsilon < \eta'$ by our choice of δ, which means $\mathsf{P}\big(\,\|x_t\| < \varepsilon\big) \geqslant 1-\eta'$. Since t is arbitrary, we conclude that **wp′)** holds. Also, since **m″)** holds, choose the T corresponding to r and $\varepsilon'\eta''$. Fix $t' \in \mathcal{T}$, $t \geqslant T$, and choose x° such that $\|x^\circ\| < r$. An application of Chebyshev's inequality once again shows that $\mathsf{P}\big(\,\|x_{t'}\| \geqslant \varepsilon'\big) \leqslant \mathsf{E}\big[\,\|x_{t'}\|\,\big]/\varepsilon' < \eta''$ by our choice of T, which in turn implies that $\mathsf{P}\big(\,\|x_{t'}\| < \varepsilon'\big) \geqslant 1-\eta''$. Since $t' \geqslant T$ is arbitrary, we conclude that **wp″)** holds. Therefore, GAS-M implies global asymptotic stability in probability.

Global asymptotic $\boldsymbol{L}_q$ *stability implies global asymptotic* $\boldsymbol{L}_p$ *stability for* $1 \leqslant p < q \leqslant \infty$. This is a standard exercise in measure-theory, relying on the Hölder inequality and finiteness of the measure P. Indeed, fix $t \in \mathcal{T}$. If $1 \leqslant p < q < \infty$, and x is globally asymptotically $\boldsymbol{L}_q$ stable, then let $p' = q/p$ and $q' = p'/(p'-1)$. Then,

$$\mathsf{E}\big[\,\|x_t\|^p\,\big] = \mathsf{E}\big[\,\|x_t\|^p \cdot \mathbf{1}_\Omega\big] \leqslant \mathsf{E}\Big[\,\|x_t\|^{p(p')}\,\Big]^{1/p'} \mathsf{E}\Big[\mathbf{1}_\Omega^{q'}\Big]^{1/q'} = \mathsf{E}\big[\,\|x_t\|^q\,\big]^{p/q},$$

which means that $\mathsf{E}\big[\,\|x_t\|^p\,\big]^{1/p} \leqslant \mathsf{E}\big[\,\|x_t\|^q\,\big]^{1/q}$. Since $t \in \mathcal{T}$ is arbitrary and the $\boldsymbol{L}_q$ norm of x_t majorizes its $\boldsymbol{L}_p$ norm, the assertion follows. An identical argument holds for $q = \infty$ as well, even without employing the Hölder inequality, since $\|x_t\| \leqslant \|x_t\|_{\boldsymbol{L}_\infty}$ P-a.s.

Strong global asymptotic stability in probability implies asymptotic stability in probability. Suppose that x satisfies **(sp′-sp″)**. Although the implication is immediate from the definitions, we present some straightforward details as follows. In terms of class-$\mathcal{KL}$ functions, for $\eta \in]0,1[$ there exists a function $\beta \in \mathcal{KL}$ such that $\mathsf{P}\big(\bigcap_{t\in\mathcal{T}} \{ \|x_t\| \leqslant \beta(\|x^\circ\|, t)\}\big) \geqslant 1-\eta$. Since $\bigcap_{t\in\mathcal{T}} \{ \|x_t\| \leqslant \beta(\|x^\circ\|, t)\} \subseteq \{ \|x_{t'}\| \leqslant \beta(\|x^\circ\|, t')\}$ for $t' \in \mathcal{T}$, it follows that $\mathsf{P}\big(\|x_t\| \leqslant \beta(\|x^\circ\|, t)\big) \geqslant 1-\eta \quad \forall\, t \in \mathcal{T}$. Converting this class-$\mathcal{KL}$ property to $\varepsilon - \delta$ form, we see that this is just global asymptotic stability in probability.

GAS-M *does not, in general, imply* GAS *a.s.* A counterexample is as follows. Let $n = 1$, $\mathcal{T} = \mathbb{R}_{\geqslant 0}$, and $(\xi_\nu)_{\nu\in\mathbb{N}}$ be a sequence of independent random variables such that ξ_ν has Uniform(1) distribution (i.e., $\mathsf{P}\big(\xi_\nu \leqslant t\big) = t$ if $t \in [0, 1[$, 0 if $t < 0$, and 1 if $t \geqslant 1$), and let us define the system

$$\dot{x}(t) = \begin{cases} \nu\xi_\nu & \text{if } t \in [2^\nu, 2^\nu + \xi_\nu/\nu[, \\ -\nu\xi_\nu & \text{if } t \in [2^\nu + \xi_\nu/\nu, 2^\nu + 2\xi_\nu/\nu[, \\ 0 & \text{otherwise.} \end{cases} \qquad \nu \in \mathbb{N}, \qquad x(0) = 0,$$

A typical solution of this system consists of triangular spikes of random length, and between two such triangles the solution attains the value 0. The height of the ν-th spike is ξ_ν^2, and its area is $2\xi_\nu^3/\nu$, and its bottom left edge is situated at $t = 2^\nu$, $\nu \in \mathbb{N}$. It immediately follows that $(x(t))_{t\geqslant 0}$ is GAS-M but not GAS a.s.

GAS *a.s. does not, in general, imply* GAS-M. A standard counterexample may be furnished as follows. Let $n = 1$, $\mathcal{T} = \mathbb{N}$, and the probability space be $\Omega = [0, 1[$ equipped with the Borel sigma-algebra on $[0, 1[$, with P being the Lebesgue measure restricted to the Borel sets of $[0, 1[$. Let us define the process $x_\nu := \nu \mathbf{1}_{[0,1/\nu[}$ for $\nu \in \mathcal{T}$. It follows that $x = (x_\nu)_{\nu\in\mathbb{N}}$ converges to the function $\mathbf{1}_{]0,1[}$ on $[0, 1[$ (i.e., almost everywhere on $[0, 1[$), but $\mathsf{E}\big[x_\nu\big] = 1$ for all ν, and therefore $\mathsf{E}\big[x_\nu\big]$ does not converge to $\mathsf{E}\big[\mathbf{1}_{]0,1[}\big] = 0$. Another counterexample may be furnished with $\Omega =]-1, 1[$ equipped with its Borel sigma-algebra, by a Dirac sequence with shrinking support; see [**36**, VIII, §3] for the definition of such a sequence.

§ 1.4. Thesis Organization

This thesis is organized as follows. In Chapter 2 we present our results on internal stability, i.e., stability when no external inputs are present. The system is defined together with precise technical conditions on the family of subsystems and the random switching signal in §2.2. The statements of the main results appear in §2.3, followed by a discussion and explanation of the results. Some of our results specialize to the case of Markovian switching signals, and these are described in §2.3.4. In §2.4 we examine how some special structures in the switching signal facilitate stability analysis of randomly switched systems. The proofs of the analysis results are given in §2.5. We conclude Chapter 2 in §2.6 with a summary of the chapter and a discussion of future directions.

Chapter 3 is concerned with external stability, i.e., stability in the presence of external inputs. Input-to-state stability of switched systems under average dwell-time switching signals is presented first in §3.1 as a motivation for the later results which deal with input-to-state stability type properties under random switching. §3.2 contains the definitions of the system, and the notions of external stability that are considered here. The statements of the main results are given in §§3.3 and 3.4, each section containing explanations of the statements of the results. The proofs of the main results are provided in §3.5. We conclude Chapter 3 in §3.6 with a summary of the chapter and a discussion of future work.

Chapter 4 is concerned with feedback controller synthesis. The synthesis problem is introduced in §4.1, and two distinct controller architectures are described. Stabilizing controllers are designed with the aid of multiple control-Lyapunov functions and universal formulas for feedback stabilization of nonlinear systems, first for systems with no external disturbance inputs in §4.2 and then for systems with external disturbance inputs in §4.3. We conclude in §4.4 by summarizing the results of the chapter.

Appendix A contains the proofs of an auxiliary result in Chapter 2, and the proof of input-to-state stability under average dwell-time switching using a comparison principle for systems with inputs. Appendix B contains an extended technical discussion of some steps that may be useful in proving Conjecture 3.17.

CHAPTER 2

INTERNAL STABILITY

§ 2.1. Introduction to the Analysis Problem

In this chapter we present our results on internal stability analysis of randomly switched systems. The setting, in broad strokes, is as follows. We are given a family of systems having a common equilibrium point at the origin, and a switching signal with some known statistical characteristics. Our objective is to determine sufficient conditions such that the switched system generated by the switching signal from the family of subsystems is globally asymptotically stable almost surely, or globally asymptotically stable in the mean. We refer the reader to §1.2 for a general motivation and discussion.

For deterministic switched systems, it is often possible to provide simple sufficient conditions for internal stability of a switched system in terms of the rate of switching. For instance, it is well-known [**37**, Chapter 3] that if there is a minimal difference between consecutive switching instants and the constituent subsystems are all stable, then so is the switched system. There also exist elegant sufficient conditions for stability under switching signals having a bound on their average rate of switching [**26**].

We shall see that the slow switching criteria for stability in the deterministic context have close analogs in the context of randomly switched systems. Of course slow switching is now quantitatively expressed in terms of the statistical parameters of the switching signal. In particular, we shall see that a switched system having unstable subsystems may still be stable if the switching signal possesses a suitable probability distribution on the family of subsystems. Three types of switching signals are considered, including one that is semi-Markov with history-dependent holding times, and one for which just an estimate of the probability distribution of the rate of switching is known. Some of the results specialize to Markovian switching signals and will also be discussed. We also look at how the presence of some additional probabilistic structures enable us to conclude some specific properties via alternate routes.

§ 2.2. System Model

We define the family of systems

$$\dot{x} = f_i(x), \qquad i \in \mathcal{P}, \tag{2.1}$$

where the state $x \in \mathbb{R}^n$, $\mathcal{P}$ is a finite index set of N elements: $\mathcal{P} = \{1, \dots, \mathrm{N}\}$, the vector fields $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are locally Lipschitz, $f_i(0) = 0$, $i \in \mathcal{P}$.

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a complete probability space as in §1.3. Let $\sigma := (\sigma(t))_{t\geqslant 0}$ be a càdlàg (i.e., right-continuous and possessing limits from the left) stochastic process taking values in $\mathcal{P}$, with $\sigma(0)$ completely known. Let the switching instants of σ be denoted by τ_i, $i \in \mathbb{N}$, and let $\tau_0 := 0$. We assume that for each $t \geqslant 0$ and each $\omega \in \Omega$ there exists a strictly positive number $\epsilon(t, \omega)$ such that $\sigma(t+s, \omega) = \sigma(t, \omega)$ on $[t, t+\epsilon(t,\omega)[$. Under this condition we know [**14**, Theorem T26, p. 304] that the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ generated by σ is right-continuous, and we augment $\mathfrak{F}_0$ with all P-null sets. As a consequence of the hypotheses of our results, the sequence $(\tau_i)_{i\in\mathbb{N}\cup\{0\}}$ is almost surely divergent, i.e., σ is nonexplosive. The *randomly switched system* generated by this *switching signal* σ from the family (2.1) is

$$\dot{x} = f_\sigma(x), \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0. \tag{2.2}$$

We assume that there are no jumps in the state x at the points of discontinuity of the switching signal; we say that these instants of time are the switching instants. Finally, we assume that for every compact $K \subseteq \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ there exists a Lebesgue-integrable function m_K on $\mathbb{R}_{\geqslant 0}$ satisfying $\sup_{i\in\mathcal{P}} \|f_i(x)\| \leqslant m_K(t)$ for all $(t, x) \in K$. Hence almost surely there exists a unique solution to (2.2) in the sense of Carathéodory [**21**] over a nontrivial time interval containing 0; existence and uniqueness of a global solution will follow from the hypotheses of our results. We let $x(\cdot)$ denote this solution. For $x_0 = 0$, the solution to (2.2) is identically 0 for every σ; we shall ignore this trivial case in the sequel. It is well-known [**52**] that every càdlàg process is progressively measurable with respect to the filtration it generates. Therefore, the solution process $x(\cdot)$ of (2.2) is an $(\mathfrak{F}_t)_{t\geqslant 0}$-progressively measurable continuous process, and is hence an $(\mathfrak{F}_t)_{t\geqslant 0}$-adapted measurable process on the aforementioned probability space.

Our analysis results below employ a family of Lyapunov functions, one for each subsystem. Our approach is motivated by the method of multiple Lyapunov functions developed in the context of deterministic switched systems; see, e.g., [**37**, Chapter 3] for an extensive discussion. The following assumption collects the properties we shall require from the members of this family of Lyapunov functions.[1]

2.3. Assumption. There exist a family of continuously differentiable real-valued functions $\{V_i\}_{i\in\mathcal{P}}$ on $\mathbb{R}^n$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, numbers $\mu > 1$ and $\lambda_i \in \Lambda \subseteq \mathbb{R}$, $i \in \mathcal{P}$, such that

(V1) $\alpha_1(\|x\|) \leqslant V_i(x) \leqslant \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, i \in \mathcal{P}$;

(V2) $\mathrm{L}_{f_i} V_i(x) \leqslant -\lambda_i V_i(x) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, i \in \mathcal{P}$;

(V3) $V_i(x) \leqslant \mu V_j(x) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, i, j \in \mathcal{P}$. ♢

2.4. Remark. (V1) is a fairly standard hypothesis, ensuring each V_i is positive definite and radially unbounded. (V2) furnishes a quantitative estimate of the degree of stability or instability, depending

[1] Strictly speaking we should call them "Lyapunov-like functions," because their gradients do not necessarily decrease along the corresponding system trajectories. However, we shall adhere to the term "Lyapunov functions" in the sequel.

on the sign of λ_i, of each subsystem of the family (2.1). The possible values that the λ_i's are allowed to take are specified by the set Λ. (To wit, if there are unstable subsystems, we allow Λ to contain negative real numbers so that the corresponding λ_i's may be negative; if there are no unstable subsystems, Λ is a subset of the positive real numbers.) The right-hand side of the inequality in (V2) being a linear function of V_i is no loss of generality; see, e.g., [**35**, Theorem 2.6.10] for details. (V3) certainly restricts the class of functions that the family $\{V_i\}_{i\in\mathcal{P}}$ can belong to; however, this hypothesis is commonly employed in the deterministic context [**37**, Chapter 3]. Quadratic Lyapunov functions universally utilized in the case of linear subsystems satisfy this hypothesis. $\triangleleft$

We focus on the two properties of GAS a.s. and GAS-M of (2.2); let us recall from §1.3.2 the precise statements.

2.5. Definition. The system (2.2) is said to be *globally asymptotically stable almost surely* (GAS a.s.) if the following two properties are simultaneously verified:

(AS1) $\mathsf{P}\left(\forall\,\varepsilon>0\quad \exists\,\delta(\varepsilon)>0 \text{ such that } \|x_0\|<\delta(\varepsilon) \implies \sup_{t\geqslant 0}\|x(t)\|<\varepsilon\right)=1;$

(AS2) $\mathsf{P}\left(\forall\,r,\varepsilon'>0\quad \exists\,T(r,\varepsilon')\geqslant 0 \text{ such that } \|x_0\|<r \implies \sup_{t\geqslant T(r,\varepsilon')}\|x(t)\|<\varepsilon'\right)=1.$ $\diamond$

Let us note that this property is well-defined because each of the sets appearing inside the measure P is $\mathfrak{F}$-measurable due to continuity of $x(\cdot)$.

2.6. Definition. The system (2.2) is said to be *α-globally asymptotically stable in the mean* (α-GAS-M) for a function $\alpha\in\mathcal{K}$ if the following two properties are simultaneously verified:

(SM1) $\forall\,\varepsilon>0\quad \exists\,\widetilde{\delta}(\varepsilon)>0 \text{ such that } \|x_0\|<\widetilde{\delta}(\varepsilon) \implies \sup_{t\geqslant 0}\mathsf{E}\big[\alpha(\|x(t)\|)\big]<\varepsilon;$

(SM2) $\forall\,r,\varepsilon'>0\quad \exists\,\widetilde{T}(r,\varepsilon')\geqslant 0 \text{ such that } \|x_0\|<r \implies \sup_{t\geqslant T(r,\varepsilon')}\mathsf{E}\big[\alpha(\|x(t)\|)\big]<\varepsilon'.$ $\diamond$

In deterministic systems literature, stability definitions usually involve just the norm of the state. The presence of the function α in the above definition allows some measure of flexibility in the sense that one need not worry about bounds for just the expectation of the norm of the state, i.e., $\boldsymbol{L}_1$-stability. Frequently one employs Lyapunov functions which are polynomial functions of the states, and with the aid of conditions such as in (V1), stronger bounds in terms of the $\boldsymbol{L}_p$ ($p>1$) norms of the state are obtained; see §1.3 for a discussion of $\boldsymbol{L}_p$-stability. For instance, quadratic Lyapunov functions yield bounds for mean-square or $\boldsymbol{L}_2$-stability, which is stronger than $\boldsymbol{L}_1$-stability.

§ 2.3. Classes of σ and Corresponding Results

In this section we present our main results providing sufficient conditions for GAS a.s. and GAS-M of randomly switched systems under three different classes of switching signals. These results also constitute the backbone of our stabilizing controller synthesis methodology to be presented in Chapter 4. The switching signals described here are fairly general and are quite natural to consider.

2.7. Definition.

(1) We say that the switching signal σ *belongs to class G* if there exist constants $M \in \mathbb{N} \cup \{0\}$ and $\overline{\lambda}, \widetilde{\lambda} > 0$, such that $\forall\, k \geqslant M$ we have $\mathsf{P}\big(N_\sigma(t,0) = k\big) \leqslant \big(\overline{\lambda} t\big)^k \, \mathrm{e}^{-\widetilde{\lambda} t}/k!$ for $t > 0$.

(2) We say that the switching signal σ *belongs to class EH* if:

- (EH1) the sequence $(S_i)_{i\in\mathbb{N}}$, $S_i := \tau_i - \tau_{i-1}$, of holding times is an independent identically distributed sequence of exponential-λ random variables;
- (EH2) $\exists\, q_i \in [0,1]$, $i \in \mathcal{P}$, such that $\forall\, j \in \mathbb{N}$, $\mathsf{P}\Big(\sigma(\tau_j) = i \big| (\sigma(\tau_k))_{k=0}^{j-1}\Big) = q_i$;
- (EH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$.

(3) We say that the switching signal σ *belongs to class UH* if:

- (UH1) the sequence $(S_i)_{i\in\mathbb{N}}$, $S_i := \tau_i - \tau_{i-1}$, of holding times is an independent identically distributed sequence of uniform-T random variables;
- (UH2) $\exists\, q_i \in [0,1]$, $i \in \mathcal{P}$, such that $\forall\, j \in \mathbb{N}$, $\mathsf{P}\Big(\sigma(\tau_j) = i \big| (\sigma(\tau_k))_{k=0}^{j-1}\Big) = q_i$;
- (UH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$. ◇

§ 2.3.1. Statements of the main results. The following are our main results; their proofs are provided in §2.5.

2.8. Theorem. *The system* (2.2) *is* GAS *a.s. if*

(G1) *Assumption 2.3 holds with* $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$*;*

(G2) σ *belongs to class G as defined in Definition 2.7;*

(G3) $\mu < \Big(\lambda_\circ + \widetilde{\lambda}\Big)/\overline{\lambda}$.

2.9. Theorem. *The system* (2.2) *is* α_1-GAS-M *under the hypotheses of Theorem 2.8, where* $\alpha_1 \in \mathcal{K}_\infty$ *is the function in (V1).*

2.10. Theorem. *The system* (2.2) *is* GAS *a.s. if*

(E1) *Assumption 2.3 holds with* $\Lambda = \mathbb{R}$*;*

(E2) σ *is of class EH as defined in Definition 2.7;*

(E3) $\lambda_i + \lambda > 0 \quad \forall\, i \in \mathcal{P}$*;*

(E4) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i}{(1 + \lambda_i/\lambda)} < 1$.

2.11. Corollary. *The system* (2.2) *is* α_1-GAS-M *under the hypotheses of Theorem 2.10, where* $\alpha_1 \in \mathcal{K}_\infty$ *is the function in (V1).*

2.12. Theorem. *The system* (2.2) *is* GAS *.a.s. if*

(U1) *Assumption 2.3 holds with* $\Lambda = \mathbb{R}$*;*

(U2) σ *is of class UH as defined in Definition 2.7;*

(U3) $\displaystyle\sum_{i\in\mathcal{P}} \left(\frac{\mu q_i \left(1 - \mathrm{e}^{-\lambda_i T}\right)}{\lambda_i T} \right) < 1.$

2.13. Corollary. *The system* (2.2) *is* α_1-GAS-M *under the hypotheses of Theorem 2.12, where* $\alpha_1 \in \mathcal{K}_\infty$ *is the function in (V1).*

§ 2.3.2. Discussion. In the sequel we shall use $N_\sigma(t', t)$ to denote the number of switches on the interval $]t, t'] \subseteq \mathbb{R}_{\geqslant 0}$, $t' > t$.

2.14. Remark. Intuitively, the requirement on a switching signal of class G is that statistically the number of switches on an interval $]0, t]$ does not grow arbitrarily large. As will be evident from the proofs of Lemma 2.31 and Lemma 2.32, the expected number of switches on the interval $]0, t]$ grows at most exponentially with t. If $\overline{\lambda} = \widetilde{\lambda} = \lambda$ and $M = 0$, then the probability mass function of $N_\sigma(t, 0)$ corresponds to that of a Poisson process. A switching signal of class G may therefore be regarded as a statistically slow switching random process. ◁

2.15. Remark. On the one hand, note that a switching signal of class G does not require any restrictions on the temporal probability distribution of σ on $\mathcal{P}$. Consequently, if one subsystem in the family $\{f_i\}_{i\in\mathcal{P}}$ is unstable, and the switching signal obeys the bound on the probability mass function in the definition but activates this particular unstable subsystem for most of the time, then the switched system may well become unstable. Therefore, this assumption is not strong enough for almost sure global asymptotic stability of the switched system, unless we further stipulate that each subsystem is stable. On the other hand, both the classes EH and UH require the existence of a memoryless and stationary probability distribution of the process $(\sigma(\tau_i))_{i\in\mathbb{N}\cup\{0\}}$ ((E2) and (U2), respectively), and are therefore better equipped to take into account instabilities of some subsystems. ◁

2.16. Remark. Theorem 2.8 is intuitively appealing; it states that if each subsystem has sufficient stability margin, and σ switches sufficiently slowly on an average, then the switched system is GAS a.s. By (G1) there is a uniform stability margin (expressed in terms of the Lyapunov functions) among the family of subsystems. (G3) connects the deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2.3, with the properties of the switching signal furnished by (G2). It is clear that the more stable the subsystems (the larger the $\lambda_\circ$), the faster can be the switching signal (the larger the $\overline{\lambda}$) that still ensures that (2.2) GAS a.s. This result is reminiscent of the well-known theorem [**37**, Theorem 3.2] on global asymptotic stability of deterministic switched systems under average dwell-time switching. Moreover, Theorem 2.8 applies to the case of σ being the state of a continuous-time Markov chain with a given generator matrix; further details on this important case will be furnished in §2.3.4. ◁

2.17. Remark. Let us examine the statement of Theorem 2.10 in some detail. Firstly, note that by (E1) not all subsystems are required to be stable, i.e., for some $i \in \mathcal{P}$, λ_i can be negative; then (V2) provides a measure of the rate of instability of the corresponding subsystems. Secondly, note that condition (E3) is always satisfied if each $\lambda_i > 0$. However, if $\lambda_i < 0$ for some $i \in \mathcal{P}$, then (E3) furnishes a maximum instability margin of the corresponding subsystems that can still lead to GAS a.s. of (2.2). Intuitively, in the latter case, the process $N_\sigma(t, 0)$ must switch fast enough (which corresponds to $\lambda > 0$ being large enough) so that the unstable subsystems are not active for too long. Potentially this fast switching may have a destabilizing effect. Indeed, it may so happen that for a given μ, a fixed probability distribution $\{q_i\}_{i\in\mathcal{P}}$, and a choice of functions $\{V_i\}_{i\in\mathcal{P}}$, (E3) and (E4) may be impossible to satisfy simultaneously, due to a very high degree of instability of even one subsystem for which the corresponding q_i is also large. Then we need to search for a different family of functions $\{V_i\}_{i\in\mathcal{P}}$ for which the hypotheses hold. Thirdly, (E4) connects the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2.3, with the properties of the switching signal. From (E4) it is clear that larger degrees of instability of a subsystem (small λ_i) can be compensated by a smaller probability of the switching signal activating the corresponding subsystem. $\triangleleft$

2.18. Remark. Let us make some observations about the statement of Theorem 2.12. Once again, just like Theorem 2.10, note that by (U1) not all subsystems are required to be stable; i.e., for some $i \in \mathcal{P}$, λ_i can be negative. (U3) connects the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2.3, with the properties of the switching signal. Also from (U3) it is clear that larger degrees of instability (larger λ_i) of a subsystem can be compensated by a smaller probability (smaller q_i) of the switching signal activating the corresponding subsystem. Notice that a switching signal of class UH is semi-Markov [**10**, §20.4]. There is a strong dependence on past history due to the uniform holding times. Indeed, at an arbitrary instant of time t we need to know how long ago the last jump occurred in order to compute the probability distribution of the next jump instant after t. $\triangleleft$

2.19. Remark. It may be observed that Theorem 2.10 requires a larger set of hypotheses compared to Theorem 2.12; however, this is only natural. Indeed, the switching signal in the latter case is constrained to switch at least once in T units of time, whereas no such hard constraint is present on the switching signal in the former case. We observed in Remark 2.17 that it is necessary for the switching signal to switch fast enough if there are unstable subsystems in the family (2.1), which necessitated the condition (E3). This fast switching is automatic if σ is of class UH, provided T is related to the instability margin of the subsystems in a particular way. (U3) captures this particular relationship, for, observe that if λ_i is negative and large in magnitude for some $i \in \mathcal{P}$, the ratio $\left(1 - \mathrm{e}^{-\lambda_i T}\right)/(\lambda_i T)$ is small provided T is small, and a smaller ratio is better for GAS a.s. of (2.2). Also for a given T, large and positive λ_i's (i.e., subsystems with high margins of stability) make the aforesaid ratio small. $\triangleleft$

Examples.

2.20. EXAMPLE. Let $\mathcal{P} = \{1, 2\}$, and consider the switched system constituted by the two planar subsystems

$$f_1(x) = \begin{bmatrix} -1.5x_1 + x_2 \\ (x_1 + x_2)\sin x_1 - 3x_2 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} -2x_1 - x_1^3 \\ x_1 - x_2 \end{bmatrix},$$

with the switching signal σ a jump stochastic process specified in terms of the holding times $S_k := \tau_{k+1} - \tau_k$ as follows: the sequence $(S_k)_{k\in\mathbb{N}\cup\{0\}}$ is an independent sequence of exponential random variables of parameter $\lambda = 0.2$. An easy calculation shows that σ satisfies (G2) of Theorem 2.8 with $\overline{\lambda} = \widetilde{\lambda} = \lambda$. Consider the two candidate Lyapunov functions

$$V_1(x) = 0.5\left(x_1^2 + x_2^2\right), \qquad V_2(x) = 0.5x_1^2 + x_2^2,$$

corresponding to the subsystems f_1 and f_2 above. Clearly, $V_1 \leqslant 2V_2$ and $V_2 \leqslant V_1$; therefore $V_i \leqslant 2V_j$ for $i, j \in \mathcal{P}$, which means $\mu = 2$. A quick calculation shows that

$$\begin{aligned} \mathrm{L}_{f_1}V_1(x) &= x_1(-1.5x_1 + x_2) + x_2\big((x_1 + x_2)\sin x_1 - 3x_2\big) \\ &= -1.5x_1^2 - 3x_2^2 + x_1x_2(1 - \sin x_1) + x_2^2 \sin x_1 \\ &\leqslant -1.5x_1^2 - 2x_2^2 + x_1x_2(1 + \sin x_1) \\ &\leqslant -x_1^2 - 1.5x_2^2 + x_1x_2 \sin x_1 \\ &\leqslant -0.5x_1^2 - x_2^2 \\ &\leqslant -V_1(x) \end{aligned}$$

and

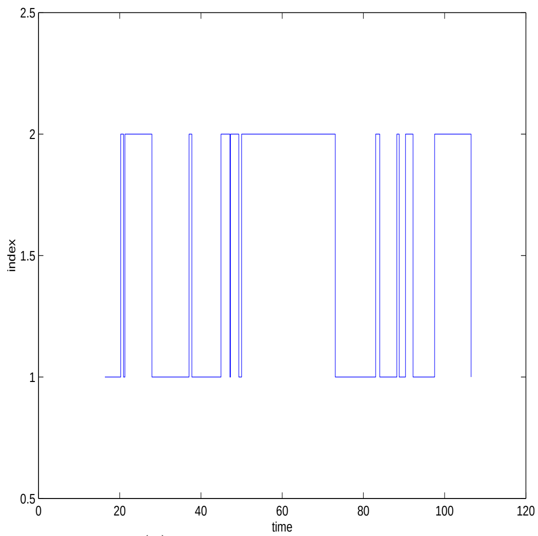
$$\mathrm{L}_{f_2}V_2(x) = x_1\left(-2x_1 - x_1^3\right) + 2x_2(x_1 - x_2) \leqslant -\left(x_1^2 + x_2^2\right) \leqslant -V_2(x),$$

which means $\lambda_0 = 1$. It follows easily that the (G3) of Theorem 2.8 holds, and hence the switched system under consideration is GAS a.s. Two typical execution fragments are given in Figures 1 and 2. The initial condition x_0 was taken to be $(15, 15)$ in each case. $\triangle$

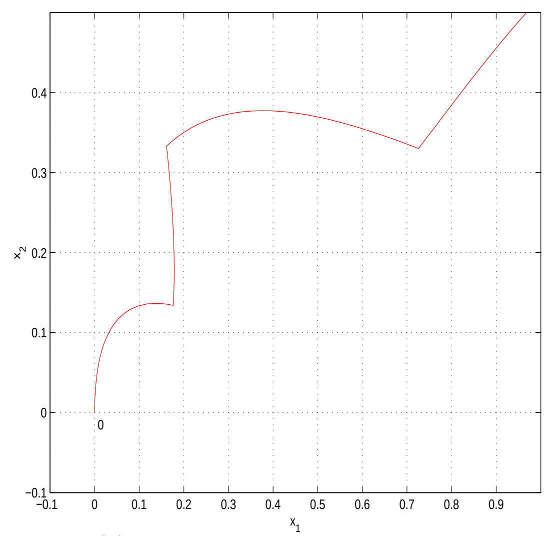
2.21. EXAMPLE. Let $\mathcal{P} = \{1, 2\}$ and consider the switched system constituted by the two planar subsystems

$$f_1(x) = \begin{bmatrix} -1.5x_1 + x_2 \\ (x_1 + x_2)\sin x_1 - 3x_2 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} 0.1x_1 + x_2 \\ -x_1 + 0.1x_2 \end{bmatrix},$$

and a switching signal σ of class EH with $\lambda = 0.5$ and $q_1 = 0.6$, $q_2 = 0.4$. Choosing Lyapunov functions $V_1(x) = V_2(x) = 0.5\big(x_1^2 + x_2^2\big)$, we immediately get that $\mu = 1$ and (keeping in mind the calculation in



(a) Plot of σ against time.



(b) A section of the state trajectory.

FIGURE 1. An execution of the switched system in Example 2.20.

Example 2.20)

$$\mathrm{L}_{f_1} V_1(x) \leqslant -V_1(x)$$
$$\mathrm{L}_{f_2} V_2(x) = 0.2 V_2(x).$$

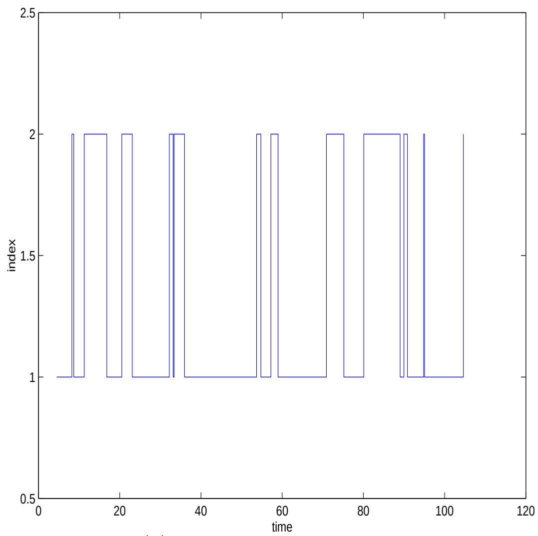
Therefore $\lambda_1 = 1$ and $\lambda_2 = -0.2$. Clearly (E1)-(E3) of Theorem 2.10 hold, and we verify that

$$\frac{q_1}{1 + 1/\lambda} + \frac{q_2}{1 - 0.2/\lambda} = 1/5 + 2/3 < 1,$$

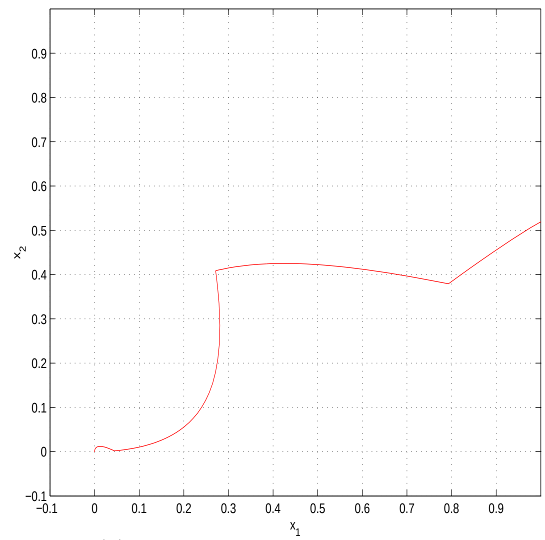
which shows that (E4) holds as well. Therefore, by Theorem 2.10 the switched system under consideration is GAS a.s. Two typical switching signals and the corresponding trajectories are shown in Figures 3 and 4. The initial condition x_0 was taken to be $(10, 10)$ in each case. $\triangle$

2.22. EXAMPLE. Let us consider an index set $\mathcal{P} = \{1, 2, 3\}$, a switching signal σ of class EH with $q_1 = 0.75$, $q_2 = 0.2$, $q_3 = 0.05$, and $\lambda = 1.5$, and the following two-dimensional vector fields:

$$f_1(x) = \begin{bmatrix} -3x_1 + x_2 \\ (x_1 + x_2)\sin(x_1) - 3x_2 \end{bmatrix},$$
$$f_2(x) = 2 \begin{bmatrix} -x_1 + x_1^2 - x_1^3 - x_2 \\ x_1 - x_2 + x_2^2 - x_2^3 \end{bmatrix},$$
$$f_3(x) = 1/4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

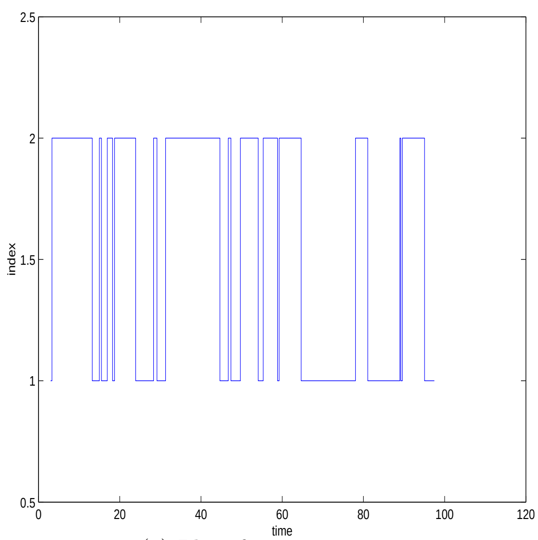


(a) Plot of σ against time.

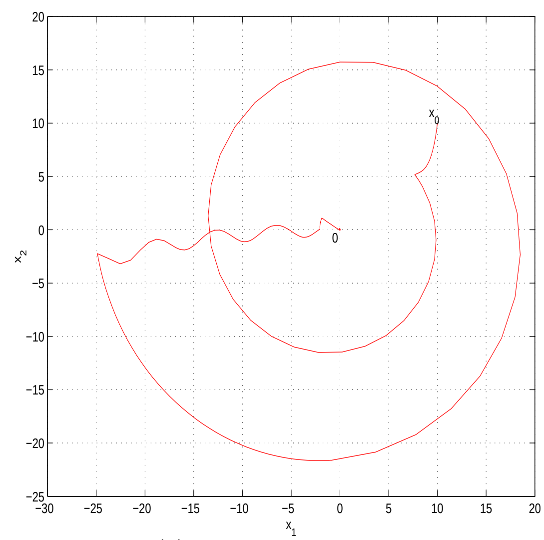


(b) A section of the state trajectory.

FIGURE 2. A second execution of the switched system in Example 2.20.



(a) Plot of σ against time.



(b) The state trajectory.

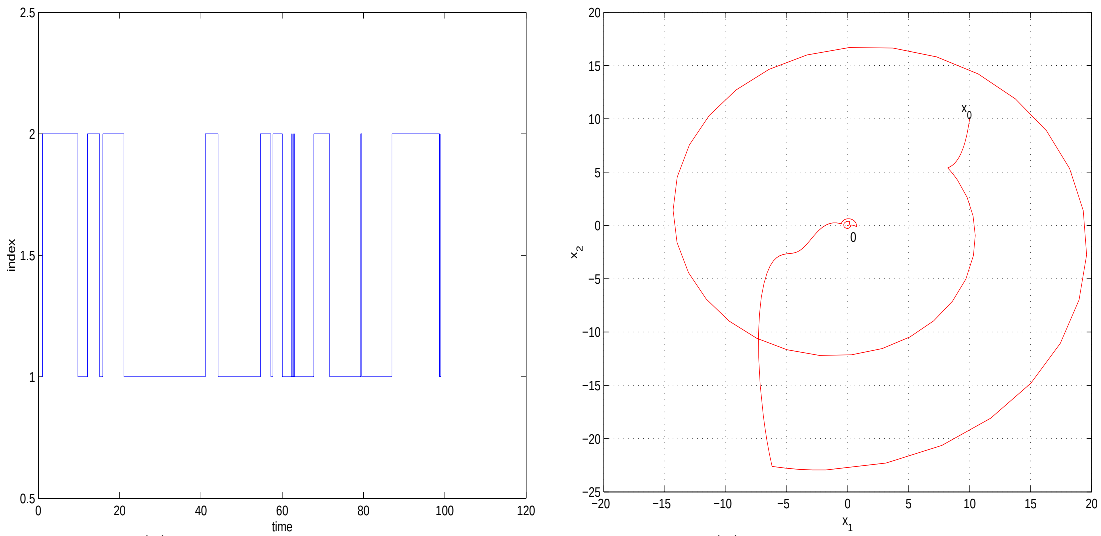
FIGURE 3. An execution of the switched system considered in Example 2.21.

Let us choose the candidate Lyapunov functions $V_1(x) = x_1^2/2 + x_2^2 + x_1^4/4 + x_2^4/2$, and $V_2(x) = V_3(x) = (x_1^2 + x_2^2)/2 + (x_1^4 + x_2^4)/4$. We have

$$\begin{aligned}\frac{\partial V_1}{\partial x}(x) f_1(x) &= (x_1 + x_1^3)(-3x_1 + x_2) + 2(x_2 + x_2^3)((x_1 + x_2)\sin(x_1) - 3x_2)\\ &\leqslant -3x_1^2 - 4x_2^2 - 3x_1^4 - 4x_2^4 + x_1 x_2(1 + 2\sin(x_1)) + x_1^3 x_2 + 2x_1 x_2^3 \sin(x_1),\end{aligned}$$

and with the aid of the estimates

$$|x_1 x_2(1 + 2\sin(x_1))| \leqslant 3x_1 x_2 \leqslant 3(x_1^2 + x_2^2)/2$$



(a) Plot of σ against time. (b) The state trajectory.

FIGURE 4. A second execution of the switched system considered in Example 2.21.

$$\left|x_1^3 x_2\right| \leqslant x_1^2(x_1^2 + x_2^2)/2$$
$$\left|x_1 x_2^3 \sin(x_1)\right| \leqslant x_2^2(x_1^2 + x_2^2)/2,$$

we arrive at

$$\frac{\partial V_1}{\partial x}(x) f_1(x) \leqslant -5V_1(x)/2.$$

Similarly,

$$\begin{aligned}\frac{1}{2}\frac{\partial V_2}{\partial x}(x) f_2(x) &= -x_1^2 + x_1^3 - 2x_1^4 + x_1^5 - x_1^6 - x_1^3 x_2 - x_2^2 + x_2^3 - 2x_2^4 + x_2^5 - x_2^6 + x_1 x_2^3 \\ &\leqslant (-x_1^2/3 + x_1^3 - x_1^4/4) + (-3x_1^4/4 + x_1^5 - x_1^6/2) \\ &\quad + (-x_2^2/3 + x_2^3 - x_2^4/4) + (-3x_2^4/4 + x_2^5 - x_2^6/2) \\ &\leqslant -(x_1^2 + x_2^2)/6 - (x_1^4 + x_2^4)/4 \\ &\leqslant -V_2(x)/3,\end{aligned}$$

and

$$\frac{\partial V_3}{\partial x}(x) f_3(x) \leqslant V_3(x).$$

Therefore, $\lambda_1 = 5/2$, $\lambda_2 = 2/3$, and $\lambda_3 = -1$. It may be readily verified that

$$\sum_{p\in\mathcal{P}} \frac{q_p}{1 + \lambda_p/\lambda} < \frac{1}{2},$$

whence by Theorem 2.10 we conclude that the switched system generated by the given σ and the above subsystems is GAS a.s. $\triangle$

§ 2.3.3. An excursion into global asymptotic stability in probability. Among the several notions of stochastic stability in the literature, one particular notion that encodes uniform behavior of system trajectories is strong GAS-P. Recall that

2.23. DEFINITION. The system (2.2) is *strongly globally asymptotically stable in probability* if the following two properties are simultaneously verified:

(i) $\forall\, \eta \in \,]0,1[\ \ \forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\|x_0\| < \delta \implies \mathsf{P}\big(\sup_{t\geqslant 0}\|x(t)\| > \varepsilon\big) \leqslant \eta$;

(ii) $\forall\, \eta' \in \,]0,1[\ \ \forall\, r, \varepsilon' > 0\ \exists\, T > 0$ such that $\|x_0\| < r \implies \mathsf{P}\big(\sup_{t\geqslant T}\|x(t)\| > \varepsilon'\big) \leqslant \eta'$. ◇

Let us note that each of the sets inside the measure P in (i) and (ii) above is $\mathfrak{F}$-measurable due to continuity of $x(\cdot)$; the notion is therefore well-defined. An equivalent statement may be made up in terms of class-$\mathcal{KL}$ functions: the system (2.2) satisfies the strong global asymptotic stability in probability property (S-GAS-P) if for every $\eta \in \,]0,1[$ there exists a function $\beta \in \mathcal{KL}$ such that $\mathsf{P}\big(\|x(t)\| \leqslant \beta(\|x_0\|, t)\ \ \forall\, t \geqslant 0\big) \geqslant 1 - \eta$.[2] In the context of randomly switched systems it can be derived from GAS a.s. with the aid of the local Lipschitz property of the vector fields. We state this in the following proposition, whose proof is provided in Appendix A.

2.24. PROPOSITION. *If* (2.2) *is* GAS *a.s., then it is s*-GAS-P.

In particular, the hypotheses of Theorems 2.8, 2.10, and 2.12 all imply S-GAS-P of (2.2).

§ 2.3.4. Switching signals coming from Markov chains. We note that (G2) of Theorem 2.8 stipulates that for all $t \in \mathbb{R}_{\geqslant 0}$ the tail of the probability mass function of the random variable $N_\sigma(t,0)$ is majorized by the probability mass function of a "maximally" switching jump-stochastic process. This hypothesis can be verified, in particular, if σ is the state of a continuous-time Markov chain, with a given generator matrix $Q = [q_{ij}]_{\mathrm{N}\times\mathrm{N}}$ and a given initial probability distribution π° (recall that N is the number of elements of $\mathcal{P}$); we denote this by $\sigma \sim (\pi^\circ, Q)$. Corollary 2.26 below makes this statement precise.

Let us recall some basic facts about continuous-time Markov chains; see, e.g., [**47**] for further details. Associated with the Markov chain $\sigma \sim (\pi^\circ, Q)$ is the Kolmogorov forward equation

$$\dot{P}(t) = P(t)Q, \qquad P(0) = I_{\mathrm{N}\times\mathrm{N}}, \quad t \geqslant 0,$$

where $I_{\mathrm{N}\times\mathrm{N}}$ is the N-dimensional identity matrix; the probability (row) vector at time $t \geqslant 0$ is given by $\pi(t) = \pi^\circ P(t)$. We need the following two facts:

(MC1) For the generator matrix Q we have $q_{ij} \geqslant 0 \quad \forall\, i,j \in \mathcal{P}, i \neq j$, and $\sum_{j\in\mathcal{P}\smallsetminus\{i\}} q_{ij} = -q_{ii} \quad \forall\, i,j \in \mathcal{P}$.

[2] A sketch of this equivalence is given in Appendix A.

(MC2) For $h > 0$, $\mathsf{P}\Big(\sigma(t+h) = j \;\Big|\; \sigma(t) = i\Big) = \delta_{ij} + q_{ij}h + o(h)$, where δ_{ij} is the Kronecker delta. This is known as the infinitesimal description of a continuous-time Markov chain.

Let us define

$$\overline{q} := \max\big\{|q_{ii}| \,\big|\, i \in \mathcal{P}\big\}, \qquad \widetilde{q} := \max\big\{q_{ij} \,\big|\, i, j \in \mathcal{P}\big\}. \tag{2.25}$$

We have the following Corollary, whose proof is provided in §2.5.

2.26. Corollary. *Consider the system* (2.2), *and let* $\overline{q}, \widetilde{q}$ *be defined by* (2.25). *Suppose that* $\sigma \sim (\pi^\circ, Q)$ *is an irreducible Markov chain, Assumption 2.3 holds with* $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$, *and* $\mu < (\lambda_\circ + \widetilde{q})/\overline{q}$. *Then* (2.2) *is* GAS *a.s. and* α_1-GAS-M.

§ 2.4. Additional Probabilistic Structures

Since the beginning of research on stability of stochastic systems, the theory of martingales played an essential role. The seminal works [**23, 33**] pretty much defined the basic structural aspects of the analysis of stochastic systems, particularly for stochastic differential equations which have Markovian properties. Studies on the martingale problem [**19**, Chapter 4] of stochastic processes gave a further boost to this approach. In particular, sophisticated methods for solving the martingale problem in the case of Markov processes are available, and it ultimately comes down to writing down an extended generator and constructing a martingale via Dynkin's formula thereafter. This method has been widely successful in applications; see, e.g., [**17**] for details. The case of Markovian switching signals has been widely investigated, some references are given in §1.2, and here we shall pursue this no further. Instead, we look at some martingale structures induced by the switching signal under appropriate hypotheses.

As we mentioned above, it is quite usual in stability analysis of stochastic systems to impose conditions such that the process $\big(V_{\sigma(t)}(x(t))\big)_{t\geqslant 0}$ is a supermartingale or a potential (with respect to an appropriate filtration), and thereafter derive asymptotic convergence or Lyapunov stability properties with the aid of the supermartingale structure. Our results on stability of (2.2) given in §2.3 do not appeal to supermartingale-related arguments. Indeed, the hypotheses of Theorem 2.8 are too general for such properties to exist. However, if the characteristics of the switching signal are a little more specialized, then such a route is feasible. Let us consider the following strengthened version of class G switching signals:

- There exist numbers $\overline{\lambda}, \widetilde{\lambda} > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$ and all $0 \leqslant s < t < \infty$ we have $\mathsf{P}^{\mathfrak{F}_s}\big(N_\sigma(t,s) = k\big) \leqslant \mathrm{e}^{-\widetilde{\lambda}(t-s)} \frac{\left(\overline{\lambda}(t-s)\right)^k}{k!}$.

We shall require a further strengthening of this property in Chapter 3, but this is enough for the present discussion.

Let us see how to prove the assertion of Theorem 2.8 if (G2) is replaced by the above property of σ, while (G1) and (G3) are retained. Fixing $0 \leqslant s < t < \infty$, a calculation similar to Lemma 2.31 yields

$$\begin{aligned} \mathsf{E}^{\mathfrak{F}_s}\left[V_{\sigma(t)}(x(t))\right] &= \mathsf{E}^{\mathfrak{F}_s}\left[V_{\sigma(s)}(x(s))\,\mathrm{e}^{-\lambda_\circ(t-s)}\mu^{N_\sigma(t,s)}\right] \\ &= V_{\sigma(s)}(x(s))\,\mathrm{e}^{-\lambda_\circ(t-s)}\sum_{k=0}^{\infty} k\mathsf{P}^{\mathfrak{F}_s}\left(N_\sigma(t,s)=k\right) \\ &\leqslant V_{\sigma(s)}(x(s))\,\mathrm{e}^{-\lambda_\circ(t-s)}\,\mathrm{e}^{-\widetilde{\lambda}(t-s)+\mu\overline{\lambda}(t-s)} \\ &= V_{\sigma(s)}(x(s))\,\mathrm{e}^{-\lambda(t-s)}, \end{aligned} \tag{2.27}$$

where $\lambda := \lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda} > 0$ by (G3). Taking expectations on both sides at $s = 0$ we get $\mathsf{E}\left[V_{\sigma(t)}(x(t))\right] \leqslant \alpha_2(\|x_0\|)\,\mathrm{e}^{-\lambda t} \leqslant \alpha_2(\|x_0\|) < \infty$. This shows that $V_{\sigma(t)}(x(t))$ is integrable for each $t \geqslant 0$. The calculation in (2.27) also shows that $\mathsf{E}^{\mathfrak{F}_s}\left[V_{\sigma(t)}(x(t))\right] < V_{\sigma(s)}(x(s))$, which proves that $\left(V_{\sigma(t)}(x(t))\right)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale. Further, from (2.27) it follows that $\lim_{t\to\infty}\mathsf{E}\left[V_{\sigma(t)}(x(t))\right] = 0$, and therefore the process is a càdlàg nonnegative $(\mathfrak{F}_t)_{t\geqslant 0}$-potential. Almost sure convergence to 0 of $\left(V_{\sigma(t)}(x(t))\right)_{t\geqslant 0}$ follows from [**52**, Corollary 2.11, p. 65] which gives the (AS2) property of (2.2) via an application of (V1), and as in the proof of Theorem 2.8 one can prove the (AS1) property of (2.2) using the locally Lipschitz property of the family of vector-fields $\{f_i\}_{i\in\mathcal{P}}$. The GAS a.s. property of (2.2) follows.

Let us now see how to go about proving the s-GAS-P property of (2.2). At this stage there are at least two possible routes to establish the GAS-P property. One route proceeds along the lines of the proof of Proposition 2.24 as follows. By a standard supermartingale convergence result [**29**, Problem 3.16, p. 18] we know that $\lim_{t\to\infty} V_{\sigma(t)}(x(t)) = 0$ a.s. Applying (V1) we have $\lim_{t\to\infty}\alpha_1(\|x(t)\|) = 0$ a.s., and since $\alpha_1 \in \mathcal{K}_\infty$, $\lim_{t\to\infty}\|x(t)\| = 0$ a.s. Hereafter appealing to Lemma 2.37 one proves the almost uniform global asymptotic convergence property (ii) in Definition 2.23. A second route is to employ a supermartingale inequality [**52**, Exercise 1.13, Chapter 2] which states: if $(\xi_t)_{t\geqslant 0}$ is a càdlàg positive $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale, then $\mathsf{P}^{\mathfrak{F}_s}\left(\sup_{t\geqslant s}\xi_t > a\right) \leqslant \frac{\xi_s}{a} \wedge 1$. To prove the property (ii) in Definition 2.23 let $\eta' \in \,]0,1[$ and $r, \varepsilon' > 0$ be given, and let $\|x_0\| < r$. From (2.27) with $s = 0$ and an application of (V1) it follows that

$$\mathsf{E}\left[V_{\sigma(t)}(x(t))\right] \leqslant \alpha_2(\|x_0\|)\,\mathrm{e}^{-\lambda t} < \alpha_2(r)\,\mathrm{e}^{-\lambda t} < \eta'\alpha_2(\varepsilon') \qquad \forall\, t \geqslant \frac{1}{\lambda}\ln\frac{\alpha_2(r)}{\eta'\alpha_2(\varepsilon')} \wedge 0.$$

Let $T := \frac{1}{\lambda}\ln\frac{\alpha_2(r)}{\eta'\alpha_2(\varepsilon')} \wedge 0$. We compute

$$\begin{aligned} \mathsf{P}\left(\sup_{t\geqslant T}\|x(t)\| > \varepsilon'\right) &= \mathsf{P}\left(\sup_{t\geqslant T}\alpha_2(\|x_0\|) > \alpha_2(\varepsilon')\right) && \text{since } \alpha_2 \in \mathcal{K}_\infty \\ &\leqslant \mathsf{P}\left(\sup_{t\geqslant T} V_{\sigma(t)}(x(t)) > \alpha_2(\varepsilon')\right) && \text{by (V1)} \\ &= \mathsf{E}\left[\mathsf{P}^{\mathfrak{F}_S}\left(\sup_{t\geqslant T} V_{\sigma(t)}(x(t)) > \alpha_2(\varepsilon')\right)\right]. \end{aligned}$$

But

$$\mathsf{E}\Big[\mathsf{P}^{\mathfrak{F}_S}\Big(\sup_{t\geqslant T} V_{\sigma(t)}(x(t)) > \alpha_2(\varepsilon')\Big)\Big] \leqslant \frac{\mathsf{E}\big[V_{\sigma(T)}(x(T))\big]}{\alpha_2(\varepsilon')} < \eta',$$

where the first inequality follows from the supermartingale inequality and the second from the definition of T. Since T depends only on η', r and ε', the property (ii) follows. The supermartingale inequality also gives us a way to establish the property (i) of Definition 2.23. Indeed, given $\eta \in]0,1[$ and $\varepsilon > 0$, we choose $\delta = \alpha_2^{-1}(\eta\alpha_2(\varepsilon))$ to see that

$$\mathsf{P}\Big(\sup_{t\geqslant 0}\|x(t)\| > \varepsilon\Big) \leqslant \mathsf{P}\Big(\sup_{t\geqslant 0} V_{\sigma(t)}(x(t)) > \alpha_2(\varepsilon)\Big) \leqslant \frac{\mathsf{E}\big[V_{\sigma(0)}(x_0)\big]}{\alpha_2(\varepsilon)} \leqslant \frac{\alpha_2(\|x_0\|)}{\alpha_2(\varepsilon)} < \eta$$

whenever $\|x_0\| < \delta$, which is the desired property (i).

Let us note that the first route via Proposition 2.24 is not in general equivalent to the second route involving the supermartingale structure. The reason is that the measurable set obtained by appealing to Egorov's theorem (as in the proof of Proposition 2.24) is $\mathfrak{F}$-measurable; therefore, $V_{\sigma(t)}(x(t))$ restricted to A_η is not necessarily $(\mathfrak{F}_t)_{t\geqslant 0}$-adapted, which precludes applicability of the supermartingale property.

§ 2.5. Proofs

The proofs of the results of §2.3 are presented in this section. In order to simplify the presentation, a number of auxiliary lemmas are established first, followed by the actual proofs of the main results

Recall that the random variable $N_\sigma(t',t)$ is the number of switches of σ on the interval $]t,t']$ for $t' > t$, $N_\sigma(0,0) = 0$, and $(\tau_i)_{i\in\mathbb{N}}$ is the set of switching instants. The extended real-valued random variable $\zeta := \sup_{\nu\in\mathbb{N}} \tau_\nu$ is the *explosion time* [**49**] of the process $(N_\sigma(t,0))_{t\geqslant 0}$. If $\zeta = \infty$ a.s., then the process $(N_\sigma(t,0))_{t\geqslant 0}$ is said to have *no explosions*; we shall also say that under this condition σ has no explosions.

§ 2.5.1. Auxiliary lemmas.

2.28. Lemma. *Suppose σ is of class G. Then $\mathsf{P}\big(N_\sigma(t,0) < \infty \quad \forall\, t \geqslant 0\big) = 1$; i.e., almost surely σ has no explosion.*

Proof. Suppose σ is of class G. If $t' > 0$, the event that there is an explosion at $t = t'$ is given by $\bigcap_{\nu\in\mathbb{N}}\big\{N_\sigma(t',0) \geqslant \nu\big\}$. But

$$\begin{aligned}\mathsf{P}\Big(\bigcap_{\nu\in\mathbb{N}}\big\{N_\sigma(t',0)\geqslant\nu\big\}\Big) &\leqslant \limsup_{\nu\to\infty}\mathsf{P}\Big(\bigcup_{k=\nu}^{\infty}\big\{N_\sigma(t',0)=k\big\}\Big)\\ &\leqslant \limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathsf{P}\big(N_\sigma(t',0)=k\big),\end{aligned}$$

and from the hypothesis of our assumption we get

$$\limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathsf{P}\big(N_\sigma(t',0)=k\big) \leqslant \limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathrm{e}^{-\widetilde{\lambda}t'}\frac{(\overline{\lambda}t')^k}{k!}.$$

Since $\sum_{k=\nu}^{\infty}(\overline{\lambda}t')^k/k!$ is the tail of $\mathrm{e}^{\overline{\lambda}t'}$, it vanishes as $\nu \to \infty$. Therefore, since $t' \in \mathbb{R}_{\geqslant 0}$ is arbitrary, we conclude that σ has no explosion almost surely. □

2.29. LEMMA. *Suppose σ is of class EH. Then*

$$\mathsf{P}\Big(\lim_{t\to\infty} N_\sigma(t,0) = \infty\Big) = 1 \qquad \textit{and} \qquad \mathsf{P}\big(N_\sigma(t,0) < \infty \quad \forall\, t \geqslant 0\big) = 1.$$

PROOF. To see the first assertion, consider the event $\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0}$ such that $\forall\, t \geqslant t' \quad N_\sigma(0,t) = N_\sigma(t',0)\big\}$. But this event is $\big\{\forall\, \nu \in \mathbb{N} \quad S_{N_\sigma(t',0)+1} > \nu\big\} = \bigcap_{\nu\in\mathbb{N}}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}$. In light of (EH1), the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}\Bigg) &\leqslant \limsup_{\nu\to\infty} \mathsf{P}\Bigg(\bigcup_{k=\nu}^{\infty}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}\Bigg)\\ &\leqslant \limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathsf{P}\big(S_{N_\sigma(t',0)+1} > \nu\big)\\ &= \limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathrm{e}^{-\lambda k}\\ &= 0.\end{aligned}$$

Therefore, almost surely $N_\sigma(t,0) \to \infty$ as $t \to \infty$. To see the second assertion, suppose that the process $(N_\sigma(t,0))_{t\geqslant 0}$ explodes at $t' > 0$. In view of (EH1) the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}}\big\{N_\sigma(t',0) \geqslant \nu\big\}\Bigg) &\leqslant \mathsf{P}\Big(\exists\, M \in \mathbb{N} \text{ such that } \forall\, i \geqslant M \;\; S_i < 1\Big)\\ &\leqslant \sum_{M\in\mathbb{N}}\prod_{i\geqslant M}\mathsf{P}\big(S_i < 1\big)\\ &= 0\end{aligned}$$

by (EH1). Since t' is arbitrary, it follows that almost surely σ has no explosion. □

2.30. LEMMA. *Suppose σ is of class UH. Then*

$$\mathsf{P}\Big(\lim_{t\to\infty} N_\sigma(t,0) = \infty\Big) = 1 \qquad \textit{and} \qquad \mathsf{P}\big(N_\sigma(t,0) < \infty \quad \forall\, t \geqslant 0\big) = 1.$$

PROOF. The proof mimics the proof of Lemma 2.29; for completeness we provide it below. To see the first assertion, consider the event $\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0}$ such that $\forall\, t \geqslant t' \quad N_\sigma(t,0) = N_\sigma(t',0)\big\}$. But this event is $\big\{\forall\, \nu \in \mathbb{N} \quad S_{N_\sigma(t',0)+1} > \nu\big\} = \bigcap_{\nu\in\mathbb{N}}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}$. In light of (UH1), $\exists\, \nu \in \mathbb{N}$ such that $T < \nu$; therefore, the probability of this event can be estimated as

$$\begin{aligned}\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}\Bigg) &\leqslant \limsup_{\nu\to\infty} \mathsf{P}\Bigg(\bigcup_{k=\nu}^{\infty}\big\{S_{N_\sigma(t',0)+1} > \nu\big\}\Bigg)\\ &\leqslant \limsup_{\nu\to\infty}\sum_{k=\nu}^{\infty}\mathsf{P}\big(S_{N_\sigma(t',0)+1} > \nu\big)\\ &= 0.\end{aligned}$$

Therefore, almost surely $N_\sigma(t,0) \to \infty$ as $t \to \infty$. To see the second assertion, consider the event of an explosion, i.e.,

$$\big\{\exists\, t' \in \mathbb{R}_{\geqslant 0} \text{ such that } \forall\, \nu \in \mathbb{N}\ \ N_\sigma(t',0) \geqslant \nu\big\}.$$

But in view of (UH1) the probability of this event can be estimated as

$$\begin{aligned}
\mathsf{P}\Bigg(\bigcap_{\nu\in\mathbb{N}} \big\{N_\sigma(t',0) \geqslant \nu\big\}\Bigg) &\leqslant \mathsf{P}\Big(\exists\, M \in \mathbb{N} \text{ such that } \forall\, i \geqslant M\ \ S_i < T/2\Big)\\
&\leqslant \sum_{M\in\mathbb{N}} \prod_{i\geqslant M} \mathsf{P}\big(S_i < T/2\big)\\
&= \sum_{M\in\mathbb{N}} \prod_{i\geqslant M} \left(\frac{1}{2}\right)^i\\
&= 0
\end{aligned}$$

by (UH1). Since t' is arbitrary, it follows that almost surely σ has no explosion. □

Lemmas 2.29 and 2.30 can also be established by appealing to the Strong Law of Large Numbers.

2.31. Lemma. *Suppose that the hypotheses of Theorem 2.8 hold. Then for every $t \geqslant 0$ we have* $\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \mathsf{E}\big[\mathrm{e}^{(\ln\mu)N_\sigma(t)}\big]\, V_{\sigma(0)}(x_0)\, \mathrm{e}^{-\lambda_\circ t}$.

Proof. Recall that $(\tau_i)_{i\in\mathbb{N}}$ are the switching instants of σ. On the set $\big\{t \in [\tau_i, \tau_{i+1}[\big\}$ an application of (V2) gives

$$V_{\sigma(\tau_i)}(x(t)) \leqslant V_{\sigma(\tau_i)}(x(\tau_i))\, \mathrm{e}^{-\lambda_\circ (t-\tau_i)}.$$

In conjunction with (V3) this yields

$$V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) \leqslant \mu V_{\sigma(\tau_i)}(x(\tau_i))\, \mathrm{e}^{-\lambda_\circ(\tau_{i+1}-\tau_i)}.$$

Iterating the last inequality from $i = 0$ to $i = N_\sigma(t,0)$ for an arbitrary time $t > 0$, we arrive at

$$V_{\sigma(t)}(x(t)) \leqslant \mu^{N_\sigma(t,0)}\, \mathrm{e}^{-\lambda_\circ t} V_{\sigma(0)}(x_0).$$

Since the initial condition is deterministic, taking expectations on both sides of the above inequality we get

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \mathsf{E}\Big[\mu^{N_\sigma(t,0)}\Big]\ \mathrm{e}^{-\lambda_\circ t} V_{\sigma(0)}(x_0),$$

which proves the claim. □

2.32. Lemma. *Suppose that hypothesis (G1) of Theorem 2.8 holds. Then there exists $S \geqslant 0$ such that for every $t \in \mathbb{R}_{\geqslant 0}$ the moment generating function $\mathsf{E}\big[\mathrm{e}^{sN_\sigma(t,0)}\big]$ of $N_\sigma(t,0)$ satisfies*

$$\mathsf{E}\Big[\mathrm{e}^{sN_\sigma(t,0)}\Big] \leqslant S + \mathrm{e}^{\left(\mathrm{e}^s\overline{\lambda}-\widetilde{\lambda}\right)t} \quad \forall\, s \geqslant 0. \tag{2.33}$$

PROOF. In view of (G1), for $s \geqslant 0$ we have

$$
\begin{aligned}
\mathsf{E}\Big[\mathrm{e}^{sN_\sigma(t,0)}\Big] &= \sum_{k=0}^{\infty} \mathrm{e}^{sk}\ \mathsf{P}\big(N_\sigma(t,0)=k\big)\\
&\leqslant \sum_{k=0}^{M-1} \mathrm{e}^{sk}\mathsf{P}\big(N_\sigma(t,0)=k\big) + \sum_{k=M}^{\infty} \mathrm{e}^{sk}\,\frac{(\overline{\lambda}t)^k\,\mathrm{e}^{-\widetilde{\lambda}t}}{k!}\\
&\leqslant \sum_{k=0}^{M-1} \mathrm{e}^{sk} + \sum_{k=M}^{\infty} \mathrm{e}^{sk}\,\frac{(\overline{\lambda}t)^k\,\mathrm{e}^{-\widetilde{\lambda}t}}{k!} \leqslant S + \mathrm{e}^{\left(\mathrm{e}^{s}\overline{\lambda}-\widetilde{\lambda}\right)t},
\end{aligned}
$$

where $S := \sum_{k=0}^{M-1} \mathrm{e}^{sk} \geqslant 0$. It is clear that $\mathsf{E}\big[\mathrm{e}^{sN_\sigma(t,0)}\big]$ is well defined for $t \geqslant 0$. □

2.34. LEMMA. *If $\alpha \in \mathcal{K}$ and $\displaystyle\int_0^\infty \alpha(\|x(t)\|)\,\mathrm{d}t < \infty$ a.s., then $\displaystyle\lim_{t\to\infty}\|x(t)\| = 0$ a.s.*

PROOF. Suppose that the claim is false. Then for every event in a set of positive probability there exist some $\varepsilon' > 0$ and a monotone increasing divergent sequence $(s_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_{\geqslant 0}$ such that $\alpha(\|x(s_i)\|) > \varepsilon'$ for all i. Since $\{f_i\}_{i\in\mathcal{P}}$ is a finite family of locally Lipschitz vector fields, there exist some $\varepsilon'' > 0$ and $L_{\varepsilon''} > 0$ such that

$$\sup_{\substack{i\in\mathcal{P},\\ \|x\|\in[0,\varepsilon''[}} \|f_i(x)\| \leqslant L_{\varepsilon''}\,\|x\|\,.$$

Let $\varepsilon := \varepsilon' \wedge \varepsilon''$. Note that $\forall\, x \in \mathbb{R}^n \smallsetminus \{0\}$ we have

$$\left|\frac{\mathrm{d}\,\|x\|^2}{\mathrm{d}t}\right| = \left\|2x^{\mathrm{T}}\,\frac{\mathrm{d}x}{\mathrm{d}t}\right\| \leqslant 2\,\|x\|\,\left\|\frac{\mathrm{d}x}{\mathrm{d}t}\right\|,$$

and

$$\left|\frac{\mathrm{d}\,\|x\|^2}{\mathrm{d}t}\right| = 2\,\|x\|\,\left\|\frac{\mathrm{d}\,\|x\|}{\mathrm{d}t}\right\|.$$

These two inequalities lead to $\left|\frac{\mathrm{d}\|x\|}{\mathrm{d}t}\right| \leqslant \left\|\frac{\mathrm{d}x}{\mathrm{d}t}\right\|$. It follows that

$$\mathbf{1}_{]0,\varepsilon[}(x(t))\left|\frac{\mathrm{d}\,\|x(t)\|}{\mathrm{d}t}\right| \leqslant L_\varepsilon\,\|x(t)\|\,. \tag{2.35}$$

By the finiteness condition on the integral in the hypothesis, almost surely there exists $T(\varepsilon) > 0$ such that

$$\int_{T(\varepsilon)}^{\infty} \alpha(\|x(t)\|)\,\mathrm{d}t < \frac{1}{2}\int_0^{\frac{\ln 2}{L_\varepsilon}} \alpha\left(\frac{\varepsilon}{2}\,\mathrm{e}^{-L_\varepsilon s}\right)\mathrm{d}s, \tag{2.36}$$

where the right-hand side is a strictly positive quantity since $\alpha \in \mathcal{K}$. For every event on a set of positive probability we have assumed that $(s_i)_{i\in\mathbb{N}}$ is a monotone increasing divergent sequence with $y(s_i) > \varepsilon$, and therefore there exists $i(\varepsilon) \in \mathbb{N}$ such that $s_{i(\varepsilon)} > T(\varepsilon)$ with strictly positive probability. By continuity of $\|\cdot\|$ and $x(\cdot)$, there exists an instant $t' > s_{i(\varepsilon)}$ such that $\|x(t')\| = \varepsilon/2$, also with positive probability. But by (2.35) we have $\|x(t)\| \in\,]0,\varepsilon[$ for all $t \in\,]t', t' + \frac{\ln 2}{L_\varepsilon}[$, and therefore

$$\int_{t'}^{t'+\frac{\ln 2}{L_\varepsilon}} \alpha(\|x(t)\|)\,\mathrm{d}t \geqslant \int_{t'}^{t'+\frac{\ln 2}{L_\varepsilon}} \alpha\left(\frac{\varepsilon}{2}\,\mathrm{e}^{-L_\varepsilon(t-t')}\right)\mathrm{d}t$$

with positive probability, which is a contradiction in view of (2.36). The assertion of the Lemma follows. □

2.37. LEMMA. *Let* $y : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ *be a monotone nondecreasing Borel measurable function with* $y(0) = 0$. *If for some* $\lambda_\circ > 0$ *we have* $\displaystyle\int_0^\infty \mathrm{e}^{-\lambda_\circ t} y(t)\,\mathrm{d}t < \infty$, *then* $\lim_{t\to\infty} \mathrm{e}^{-\lambda_\circ t} y(t) = 0$.

PROOF. Arguing by contradiction, let us suppose that the thesis is false. Then there exists $\varepsilon > 0$ and a monotone increasing divergent sequence $(s_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}_{\geqslant 0}$ such that $\mathrm{e}^{-\lambda_\circ s_i} y(s_i) \geqslant \varepsilon$ for each i. Since the integral in the hypothesis is finite, there exists $T > 0$ such that $\int_T^\infty \mathrm{e}^{-\lambda_\circ s} y(s)\,\mathrm{d}s < \frac{\varepsilon}{2\lambda_\circ}$. Since $(s_i)_{i\in\mathbb{N}}$ is divergent, there exists $i' \in \mathbb{N}$ such that $s_{i'} \geqslant T$. Then

$$\frac{\varepsilon}{2\lambda_\circ} > \int_T^\infty \mathrm{e}^{-\lambda_\circ t} y(t)\,\mathrm{d}t \geqslant \int_{s_{i'}}^\infty \mathrm{e}^{-\lambda_\circ t} y(t)\,\mathrm{d}t, \tag{2.38}$$

which by monotonicity of y shows that

$$\begin{aligned}\int_{s_{i'}}^\infty \mathrm{e}^{-\lambda_\circ t} y(t)\,\mathrm{d}t &\geqslant \int_{s_{i'}}^\infty \mathrm{e}^{-\lambda_\circ t} y(s_{i'})\,\mathrm{d}t \\ &\geqslant \int_{s_{i'}}^\infty \mathrm{e}^{-\lambda_\circ t} \varepsilon\, \mathrm{e}^{\lambda_\circ s_{i'}}\,\mathrm{d}t \\ &= \varepsilon\, \mathrm{e}^{\lambda_\circ s_{i'}} \frac{\mathrm{e}^{-\lambda_\circ s_{i'}}}{\lambda_\circ} \\ &= \frac{\varepsilon}{\lambda_\circ},\end{aligned}$$

which contradicts the first inequality in (2.38). The thesis follows. □

2.39. LEMMA. *The system* (2.2) *has the following property: for every* $\varepsilon > 0$ *there exists* $L_\varepsilon > 0$ *such that* $\mathbf{1}_{]0,\varepsilon[}(x(t))\,\|x(t)\| \leqslant \|x_0\|\,\mathrm{e}^{L_\varepsilon t}\ \ \forall\, t \geqslant 0$.

PROOF. Since the vector field of each individual subsystem of the family (2.1) is locally Lipschitz and $\mathcal{P}$ is a finite set, there exists a constant $L_\varepsilon > 0$ such that

$$\sup_{\substack{i\in\mathcal{P},\\ \|x\|\in[0,\varepsilon[}} \|f_i(x)\| \leqslant L_\varepsilon \|x\|. \tag{2.40}$$

A calculation similar to those leading to (2.35) in the proof of Lemma 2.34 shows that

$$\frac{\mathrm{d}\,\|x\|}{\mathrm{d}t} \leqslant L_\varepsilon \|x\| \qquad \forall\, x \in \bigl\{x \in \mathbb{R}^n \bigm| \|x\| < \varepsilon\bigr\} \smallsetminus \{0\}. \tag{2.41}$$

An application of a standard differential inequality [**35**, Theorem 1.2.1] indicates that every solution $x(\cdot)$ of (2.2) satisfies

$$\|x(t)\| \leqslant \|x_0\|\,\mathrm{e}^{L_\varepsilon t}$$

so long as $\|x(t)\| < \varepsilon$. This proves the claim. □

2.42. LEMMA. *Under the hypotheses of Theorem 2.10, for each* $j \in \mathbb{N}$ *we have*

$$\mathsf{E}\Bigl[V_{\sigma(\tau_j)}^{1+\kappa}(x(\tau_j))\Bigr] \leqslant \alpha_2^{1+\kappa}(\|x_0\|)\eta^j(\kappa) \quad \textit{whenever } (1+\kappa)\lambda_i + \lambda > 0 \textit{ for all } i \in \mathcal{P},$$

where $\displaystyle\eta(\kappa) := \sum_{j\in\mathcal{P}} \frac{\mu^{1+\kappa} q_j}{1 + \lambda_j(1+\kappa)/\lambda}$.

Proof. Indeed, from the independence hypothesis (EH3) and (V1), for a fixed $j \in \mathbb{N}$ we have

$$\begin{aligned} \mathsf{E}\Big[V^{1+\kappa}_{\sigma(\tau_j)}(x(\tau_j))\Big] &\leqslant \alpha_2^{1+\kappa}(\|x_0\|)\mathsf{E}\left[\left(\prod_{i=0}^{j-1} \mu\, \mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\right)^{1+\kappa}\right] \\ &= \alpha_2^{1+\kappa}(\|x_0\|)\prod_{i=0}^{j-1} \mu^{1+\kappa}\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big]. \end{aligned} \tag{2.43}$$

Since by assumption $(1+\kappa)\lambda_j + \lambda > 0$ for all $j \in \mathcal{P}$, this leads to

$$\begin{aligned} \mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big]\Big] \\ &= \mathsf{E}\Big[\int_0^\infty \lambda\, \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)s-\lambda s}\, \mathrm{d}s\Big] \\ &= \mathsf{E}\Big[\frac{\lambda}{\lambda_{\sigma(\tau_i)}(1+\kappa)+\lambda}\Big] \\ &= \sum_{j\in\mathcal{P}} \frac{q_j}{1+(1+\kappa)\lambda_j/\lambda}. \end{aligned} \tag{2.44}$$

Substituting the right-hand side of (2.44) in (2.43) leads to

$$\mathsf{E}\Big[V^{1+\kappa}_{\sigma(\tau_j)}(x(\tau_j))\Big] \leqslant \alpha_2^{1+\kappa}(\|x_0\|)\left(\sum_{i\in\mathcal{P}} \frac{\mu^{1+\kappa}q_i}{1+(1+\kappa)\lambda_i/\lambda}\right)^j,$$

and considering the definition of $\eta(\kappa)$ the thesis follows. □

2.45. Lemma. *Under the hypotheses of Theorem 2.12, for each $j \in \mathbb{N}$ we have*

$$\mathsf{E}\Big[V^{1+\kappa}_{\sigma(\tau_j)}(x(\tau_j))\Big] \leqslant \alpha_2^{1+\kappa}(\|x_0\|)\eta^j(\kappa),$$

where $\eta(\kappa) := \sum_{j\in\mathcal{P}} \dfrac{\mu^{1+\kappa}q_j\left(1-\mathrm{e}^{-\lambda_j(1+\kappa)T}\right)}{\lambda_j(1+\kappa)T}$.

Proof. Indeed, from the independence hypothesis (UH3) and (V1), for a fixed $j \in \mathbb{N}$ we have

$$\begin{aligned} \mathsf{E}\Big[V^{1+\kappa}_{\sigma(\tau_j)}(x(\tau_j))\Big] &\leqslant \alpha_2^{1+\kappa}(\|x_0\|)\mathsf{E}\left[\left(\prod_{i=0}^{j-1} \mu\, \mathrm{e}^{-\lambda_{\sigma(\tau_i)} S_{i+1}}\right)^{1+\kappa}\right] \\ &= \alpha_2^{1+\kappa}(\|x_0\|)\prod_{i=0}^{j-1} \mu^{1+\kappa}\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big]. \end{aligned} \tag{2.46}$$

But

$$\begin{aligned} \mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)S_{i+1}}\Big]\Big] \\ &= \mathsf{E}\Big[\int_0^T \frac{1}{T}\, \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)s}\, \mathrm{d}s\Big] \\ &= \mathsf{E}\left[\frac{1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\kappa)T}}{\lambda_{\sigma(\tau_i)}(1+\kappa)T}\right] \\ &= \sum_{j\in\mathcal{P}} \frac{q_j\left(1-\mathrm{e}^{-\lambda_j(1+\kappa)T}\right)}{\lambda_j(1+\kappa)T}. \end{aligned} \tag{2.47}$$

Substituting the right-hand side of (2.47) in (2.46) leads to

$$\mathsf{E}\Big[V_{\sigma(\tau_j)}^{1+\kappa}(x(\tau_j))\Big] \leqslant \alpha_2^{1+\kappa}(\|x_0\|)\left(\sum_{i\in\mathcal{P}} \frac{\mu^{1+\kappa} q_i\left(1-\,\mathrm{e}^{-\lambda_i(1+\kappa)T}\right)}{\lambda_i(1+\kappa)T}\right)^j,$$

and considering the definition of $\eta(\kappa)$ the assertion follows. □

2.48. Lemma. *Under the hypotheses of Theorem 2.10 we have* $\mathsf{P}\bigg(\displaystyle\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t < \infty\bigg) = 1.$

Proof. For a fixed $t \in \mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] &= \mathsf{E}\left[\sum_{i=0}^{\infty} V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\right]\\ &= \sum_{i=0}^{\infty}\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big],\end{aligned} \tag{2.49}$$

where we have employed the monotone convergence theorem [**53**, p. 265] to get the second equality. An application of (V1) and Tonelli's theorem [**53**, p. 309] gives us

$$\mathsf{E}\bigg[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigg] \leqslant \mathsf{E}\bigg[\int_0^\infty V_{\sigma(t)}(x(t))\,\mathrm{d}t\bigg] = \int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\;\mathrm{d}t. \tag{2.50}$$

From (2.49) and (2.50) we obtain

$$\begin{aligned}\mathsf{E}\bigg[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigg] &\leqslant \int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\;\mathrm{d}t\\ &= \int_0^\infty \sum_{i=0}^{\infty}\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big]\;\mathrm{d}t.\end{aligned}$$

A second application of the monotone convergence theorem on the right-hand side of the above leads to

$$\mathsf{E}\bigg[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigg] = \sum_{i=0}^{\infty}\int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big]\;\mathrm{d}t,$$

and another application of Tonelli's theorem gives

$$\begin{aligned}\mathsf{E}\bigg[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigg] &= \sum_{i=0}^{\infty}\int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big]\;\mathrm{d}t,\\ &= \sum_{i=0}^{\infty}\mathsf{E}\bigg[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\bigg].\end{aligned} \tag{2.51}$$

Each term in the series on the right-hand side of (2.51) may be estimated as follows:

$$\mathsf{E}\bigg[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\bigg] \leqslant \mathsf{E}\bigg[\int_0^\infty V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(t-\tau_i)}\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\bigg]$$

by (V2), and therefore

$$\begin{aligned}
\mathsf{E}\Big[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\Big] &= \mathsf{E}\Big[\int_{\tau_i}^{\tau_{i+1}} V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(t-\tau_i)}\,\mathrm{d}t\Big]\\
&= \mathsf{E}\Big[V_{\sigma(\tau_i)}(x(\tau_i))\Big(\frac{1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\Big)\Big]\\
&= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[V_{\sigma(\tau_i)}(x(\tau_i))\Big(\frac{1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\Big)\Big]\Big]\\
&= \mathsf{E}\Big[V_{\sigma(\tau_i)}(x(\tau_i))\Big(\frac{1-\mathsf{E}^{\mathfrak{F}_{\tau_i}}\big[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\big]}{\lambda_{\sigma(\tau_i)}}\Big)\Big]\\
&= \mathsf{E}\Big[\frac{V_{\sigma(\tau_i)}(x(\tau_i))}{\lambda_{\sigma(\tau_i)}}\Big(1-\int_0^\infty \lambda\,\mathrm{e}^{-(\lambda_{\sigma(\tau_i)}+\lambda)s}\,\mathrm{d}s\Big)\Big],
\end{aligned}$$

which shows that

$$\begin{aligned}
\mathsf{E}\Big[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\Big] &= \mathsf{E}\Big[\frac{V_{\sigma(\tau_i)}(x(\tau_i))}{\lambda_{\sigma(\tau_i)}}\cdot\frac{\lambda_{\sigma(\tau_i)}}{\lambda_{\sigma(\tau_i)}+\lambda}\Big]\\
&\leqslant \mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]\,\frac{1}{\min_{j\in\mathcal{P}}\lambda_j+\lambda}. \qquad (2.52)
\end{aligned}$$

By (E3) and the finiteness of $\mathcal{P}$ we have $\min_{j\in\mathcal{P}}\lambda_j+\lambda>0$. From (2.51) and (2.52) we get

$$\begin{aligned}
\mathsf{E}\Big[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\Big] &\leqslant \sum_{i=0}^\infty \mathsf{E}\Big[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\Big]\\
&\leqslant \frac{\alpha_2(\|x_0\|)}{\min_{j\in\mathcal{P}}\lambda_j+\lambda}\sum_{i=0}^\infty \mathsf{E}\big[V_{\sigma(\tau_i)}(x(\tau_i))\big]\\
&\leqslant \frac{\alpha_2(\|x_0\|)}{\min_{j\in\mathcal{P}}\lambda_j+\lambda}\sum_{i=0}^\infty \eta^j(0)\\
&< \infty,
\end{aligned}$$

where η is as defined in Lemma 2.42, and $\eta(0)\in\,]0,1[$ by (E4). This establishes the claim. □

2.53. Lemma. *Under the hypotheses of Theorem 2.12 we have* $\mathsf{P}\Big(\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t<\infty\Big)=1.$

Proof. Working just as in the proof of Lemma 2.48, for a fixed $t\in\mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}
\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] &= \mathsf{E}\Big[\sum_{i=0}^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\\
&= \sum_{i=0}^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big], \qquad (2.54)
\end{aligned}$$

where we have employed the monotone convergence theorem to get the second equality. An application of (V1) and Tonelli's theorem gives us

$$\mathsf{E}\Big[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\Big]\leqslant \mathsf{E}\Big[\int_0^\infty V_{\sigma(t)}(x(t))\,\mathrm{d}t\Big]=\int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\,\mathrm{d}t, \qquad (2.55)$$

and in conjunction with (2.54) we obtain

$$\mathsf{E}\Big[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\Big]\leqslant=\int_0^\infty\sum_{i=0}^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big]\,\mathrm{d}t.$$

A second application of the monotone convergence theorem on the right-hand side of the above leads to

$$\mathsf{E}\biggl[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\biggr] = \sum_{i=0}^\infty \int_0^\infty \mathsf{E}\bigl[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\bigr]\,\mathrm{d}t,$$

and a further application of Tonelli's theorem gives

$$\sum_{i=0}^\infty \int_0^\infty \mathsf{E}\bigl[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\bigr]\,\mathrm{d}t = \sum_{i=0}^\infty \mathsf{E}\biggl[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr]. \tag{2.56}$$

Each term in the series on the right-hand side of (2.56) may be estimated as follows:

$$\mathsf{E}\biggl[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr] \leqslant \mathsf{E}\biggl[\int_0^\infty V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(t-\tau_i)}\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr]$$

by (V2), and therefore

$$\begin{aligned}
\mathsf{E}\biggl[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr] &= \mathsf{E}\biggl[\int_{\tau_i}^{\tau_{i+1}} V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(t-\tau_i)}\,\mathrm{d}t\biggr]\\
&= \mathsf{E}\biggl[V_{\sigma(\tau_i)}(x(\tau_i))\biggl(\frac{1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\biggr)\biggr]\\
&= \mathsf{E}\biggl[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\biggl[V_{\sigma(\tau_i)}(x(\tau_i))\biggl(\frac{1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\biggr)\biggr]\biggr]\\
&= \mathsf{E}\biggl[V_{\sigma(\tau_i)}(x(\tau_i))\biggl(\frac{1-\mathsf{E}^{\mathfrak{F}_{\tau_i}}\bigl[\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}\bigr]}{\lambda_{\sigma(\tau_i)}}\biggr)\biggr]\\
&= \mathsf{E}\biggl[\frac{V_{\sigma(\tau_i)}(x(\tau_i))}{\lambda_{\sigma(\tau_i)}}\biggl(1-\int_0^T \frac{1}{T}\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}s}\,\mathrm{d}s\biggr)\biggr]\\
&= \mathsf{E}\biggl[\frac{V_{\sigma(\tau_i)}(x(\tau_i))}{\lambda_{\sigma(\tau_i)}}\biggl(1-\frac{1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}T}}{\lambda_{\sigma(\tau_i)}T}\biggr)\biggr]\\
&\leqslant M\mathsf{E}\bigl[V_{\sigma(\tau_i)}(x(\tau_i))\bigr], \qquad (2.57)
\end{aligned}$$

where $M := \max_{i\in\mathcal{P}}\Bigl(\frac{1}{\lambda_i}-\frac{1-\mathrm{e}^{-\lambda_i T}}{\lambda_i^2 T}\Bigr)$ is a well-defined positive real number because of the finiteness of $\mathcal{P}$. From (2.56) we get

$$\mathsf{E}\biggl[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\biggr] \leqslant \sum_{i=0}^\infty \mathsf{E}\biggl[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr],$$

and (2.57) shows that

$$\begin{aligned}
\sum_{i=0}^\infty \mathsf{E}\biggl[\int_0^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\,\mathrm{d}t\biggr] &\leqslant M\alpha_2(\|x_0\|)\sum_{i=0}^\infty \mathsf{E}\bigl[V_{\sigma(\tau_i)}(x(\tau_i))\bigr]\\
&\leqslant M\alpha_2(\|x_0\|)\sum_{i=0}^\infty \eta^i(0)\\
&< \infty,
\end{aligned}$$

where η is as defined in Lemma 2.42, and $\eta(0)\in\,]0,1[$ by (U3). Hence $\mathsf{E}\bigl[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigr] < \infty$, which proves the claim. □

Recall from §1.3 that a family $\{\xi_i\}_{i\in I}$ of real-valued integrable random variables on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ is *uniformly integrable* if ξ_i is integrable for each $i \in I$ and

$$\lim_{L\to\infty} \sup_{i\in I} \mathsf{E}\big[\,|\xi_i|\,\mathbf{1}_{\{|\xi_i|>L\}}\big] = 0.$$

2.58. Lemma. *The family of random variables* $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ *is uniformly integrable under the hypotheses of Corollary 2.11.*

Proof. To establish uniform integrability of the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ we appeal to the Hadamard-de la Vallée Poussin criterion in Proposition 1.1. Since the function $]-1,\infty[\ \ni r \longmapsto (1+r)\lambda_i + \lambda \in \mathbb{R}$ is continuous for each $i \in \mathcal{P}$ and $\mathcal{P}$ is a finite set, by (E3) there exists $\delta' > 0$ such that $(1+\delta')\lambda_i + \lambda > 0$ for all $i \in \mathcal{P}$. Also, since the function

$$]-1,\infty[\ \ni r \longmapsto \sum_{j\in\mathcal{P}} \frac{\mu^{1+r} q_j}{1+(1+r)\lambda_j/\lambda} \in \mathbb{R}$$

is continuous, by (E4) there exists $\delta'' > 0$ such that $\sum_{j\in\mathcal{P}} \frac{\mu^{1+\delta''} q_j}{1+(1+\delta'')\lambda_j/\lambda} < 1$. Let $\delta := \delta' \wedge \delta''$. The function $\phi(r) := r^{1+\delta}$ clearly is convex on $\mathbb{R}_{\geqslant 0}$, and $\lim_{r\to\infty} \phi(r)/r = \infty$. Let us prove that $\sup_{t\geqslant 0} \mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] < \infty$.

First let us note that for each $i \in \mathbb{N}\cup\{0\}$ the function $V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}$ is integrable for arbitrary $t \in \mathbb{R}_{\geqslant 0}$. Indeed,

$$\begin{aligned}
\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\Big] \\
&\leqslant \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\delta)(t-\tau_i)}\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\Big] \\
&= \mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\delta)(t-\tau_i)}\mathsf{E}^{\mathfrak{F}_{\tau_i}}\big[\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big]\Big],
\end{aligned}$$

and since $\mathsf{E}^{\mathfrak{F}_{\tau_i}}\big[\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big] = \mathbf{1}_{\{t\in[\tau_i,\infty[\}}\mathsf{P}^{\mathfrak{F}_{\tau_i}}\big(S_{i+1} > t-\tau_i\big) = \mathbf{1}_{\{t\in[\tau_i,\infty[\}}\,\mathrm{e}^{-\lambda(t-\tau_i)}$ because S_{i+1} is exp-λ and independent of $\mathfrak{F}_{\tau_i}$, we arrive at

$$\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] \leqslant \mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-(\lambda_{\sigma(\tau_i)}(1+\delta)+\lambda)(t-\tau_i)}\mathbf{1}_{\{t\in[\tau_i,\infty[\}}\Big]. \tag{2.59}$$

By definition of δ, the right-hand side of (2.59) is at most $\mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\Big]$. Lemma 2.42 with $\kappa = \delta$ shows that

$$\mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\Big] \leqslant \alpha_2^{1+\delta}(\|x_0\|)\eta(\delta)^i, \tag{2.60}$$

where $\eta(\delta) \in]0,1[$ by construction. But since the random variable $V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}$ is integrable for each i by (2.59), we can apply the monotone convergence theorem to get

$$\begin{aligned}
\mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] &= \mathsf{E}\left[\left(\sum_{i=0}^{\infty} V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\right)^{1+\delta}\right]\\
&= \mathsf{E}\left[\sum_{i=0}^{\infty} V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\right]\\
&= \sum_{i=0}^{\infty}\mathsf{E}\Big[V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]. \qquad (2.61)
\end{aligned}$$

We know from (2.60) that $\mathsf{E}\Big[V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] \leqslant \alpha_2^{1+\delta}(\|x_0\|)\eta^i(\delta)$ for each $i \in \mathbb{N}\cup\{0\}$. Substitution in (2.61) leads to

$$\begin{aligned}
\sup_{t\geqslant 0}\mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] &= \sup_{t\geqslant 0}\sum_{i=0}^{\infty}\mathsf{E}\Big[V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\\
&\leqslant \sup_{t\geqslant 0}\alpha_2^{1+\delta}(\|x_0\|)\sum_{i=0}^{\infty}\eta^i(\delta) \qquad (2.62)\\
&< \infty.
\end{aligned}$$

This shows that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable. □

2.63. Lemma. *The family of random variables $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable under the hypotheses of Corollary 2.13.*

Proof. To establish uniform integrability of the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ we again appeal to the Hadamard-de la Vallée Poussin criterion in Proposition 1.1. Since the function

$$]-1,\infty[\ \ni r \longmapsto \sum_{j\in\mathcal{P}}\frac{\mu^{1+r}q_j\left(1-\mathrm{e}^{-\lambda_j(1+r)T}\right)}{\lambda_j(1+r)T} \in \mathbb{R}$$

is continuous, by (U3) there exists $\delta>0$ such that $\sum_{j\in\mathcal{P}}\frac{\mu^{1+\delta}q_j\left(1-\mathrm{e}^{-\lambda_j(1+\delta)T}\right)}{\lambda_j(1+\delta)T} < 1$. The function $\phi(r) := r^{1+\delta}$ clearly is convex on $\mathbb{R}_{\geqslant 0}$, and $\lim_{r\to\infty}\phi(r)/r = \infty$. Let us prove that $\sup_{t\geqslant 0}\mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] < \infty$.

First let us note that for each $i \in \mathbb{N}\cup\{0\}$ the function $V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}$ is integrable for arbitrary $t\in\mathbb{R}_{\geqslant 0}$. Indeed,

$$\begin{aligned}
\mathsf{E}\Big[V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[V_{\sigma(t)}^{1+\delta}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\Big]\\
&\leqslant \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[V_{\sigma(\tau_i)}^{1+\delta}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\delta)(t-\tau_i)}\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\Big]\\
&= \mathsf{E}\Big[V_{\sigma(\tau_i)}^{1+\delta}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\delta)(t-\tau_i)}\mathsf{E}^{\mathfrak{F}_{\tau_i}}\Big[\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\Big],
\end{aligned}$$

and since S_{i+1} is exp-λ and independent of $\mathfrak{F}_{\tau_i}$, we have

$$\begin{aligned}
\mathsf{E}^{\mathfrak{F}_{\tau_i}}\big[\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\big] &= \mathbf{1}_{\{t\in[\tau_i,\infty[\}}\mathsf{P}^{\mathfrak{F}_{\tau_i}}\big(S_{i+1} > t-\tau_i\big)\\
&= \left(\left(1-\frac{t-\tau_i}{T}\right)\vee 0\right).
\end{aligned}$$

Therefore,

$$\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] \leqslant \mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(1+\delta)(t-\tau_i)}\left(\left(1-\frac{t-\tau_i}{T}\right)\vee 0\right)\mathbf{1}_{\{t\in[\tau_i,\infty[\}}\Big]. \tag{2.64}$$

By definition of δ, the right-hand side of (2.64) is at most $M\mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\Big]$, where $M := \exp\big(\min_{j\in\mathcal{P}}\lambda_j\cdot(1+\delta)T\big)$. Lemma 2.45 with $\kappa=\delta$ shows that

$$\mathsf{E}\Big[V^{1+\delta}_{\sigma(\tau_i)}(x(\tau_i))\Big] \leqslant \alpha_2^{1+\delta}(\|x_0\|)\eta(\delta)^i, \tag{2.65}$$

where $\eta(\delta)\in\,]0,1[$ by construction. But since the random variable $V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}$ is integrable for each i by (2.64), we can apply the monotone convergence theorem to get

$$\begin{aligned}\mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] &= \mathsf{E}\left[\left(\sum_{i=0}^{\infty}V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\right)^{1+\delta}\right]\\ &= \mathsf{E}\left[\sum_{i=0}^{\infty}V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\right]\\ &= \sum_{i=0}^{\infty}\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big].\end{aligned} \tag{2.66}$$

We know from (2.65) that $\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big] \leqslant M\alpha_2^{1+\delta}(\|x_0\|)\eta^i(\delta)$ for each $i\in\mathbb{N}\cup\{0\}$. Substitution in (2.66) leads to

$$\begin{aligned}\sup_{t\geqslant 0}\mathsf{E}\Big[\big(V_{\sigma(t)}(x(t))\big)^{1+\delta}\Big] &= \sup_{t\geqslant 0}\sum_{i=0}^{\infty}\mathsf{E}\Big[V^{1+\delta}_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\tau_i,\tau_{i+1}[\}}\Big]\\ &\leqslant \sup_{t\geqslant 0}M\alpha_2^{1+\delta}(\|x_0\|)\sum_{i=0}^{\infty}\eta^i(\delta)\\ &< \infty.\end{aligned} \tag{2.67}$$

This shows that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable. □

We need the following Lemma for the proof of Corollary 2.26.

2.68. Lemma. *Suppose that $\sigma\sim(\pi^\circ,Q)$ is a Markov chain. Then for every $t\in\mathbb{R}_{\geqslant 0}$ and every $k\in\mathbb{N}\cup\{0\}$ we have $\mathsf{P}\big(N_\sigma(0,t)=k\big)\leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^k/k!$.*

Proof. For $t\in\mathbb{R}_{\geqslant 0}$ and $k\in\mathbb{N}\cup\{0\}$, define $\eta_k(t) := \mathsf{P}\big(N_\sigma(0,t)=k\big)$. For $h>0$ sufficiently small, $\forall\, k\in\mathbb{N}\cup\{0\}$,

$$\eta_k(t+h) = \sum_{i=0}^{k}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=i\big)\,\mathsf{P}\big(N_\sigma(0,t)=k-i\big). \tag{2.69}$$

By the infinitesimal description of a Markov chain (MC2),

$$\begin{aligned}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=0\big) &= \sum_{i\in\mathcal{P}}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=0 \mid \sigma(t)=i\big)\,\mathsf{P}\big(\sigma(t)=i\big)\\ &\leqslant \sum_{i\in\mathcal{P}}\pi_i(t)\left(\max_{j\in\mathcal{P}}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=0 \mid \sigma(t)=j\big)\right)\\ &\leqslant 1-\widetilde{q}h+o(h),\end{aligned}\tag{2.70}$$

and

$$\begin{aligned}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=1\big) &= \sum_{i\in\mathcal{P}}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=1 \mid \sigma(t)=i\big)\,\mathsf{P}\big(\sigma(t)=i\big)\\ &\leqslant \sum_{i\in\mathcal{P}}\pi_i(t)\left(\max_{j\in\mathcal{P}}\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=1 \mid \sigma(t)=j\big)\right)\\ &\leqslant \overline{q}h+o(h).\end{aligned}\tag{2.71}$$

For all natural numbers $k\geqslant 2$, (MC2) shows that

$$\mathsf{P}\big(N_\sigma(0,t+h)-N_\sigma(0,t)=k\big)=o(h).\tag{2.72}$$

Using (2.70)-(2.72), we continue the calculation in (2.69):

$$\eta_k(t+h)\leqslant(1-\widetilde{q}h+o(h))\eta_k(t)+(\overline{q}h+o(h))\eta_{k-1}(t)+o(h),$$

which leads to

$$\frac{\eta_k(t+h)-\eta_k(t)}{h}\leqslant -\widetilde{q}\eta_k(t)+\overline{q}\eta_{k-1}(t)+O(h).$$

Taking limits with $h\downarrow 0$, the following differential inequality is obtained:

$$\dot{\eta}_k(t)\leqslant -\widetilde{q}\eta_k(t)+\overline{q}\eta_{k-1}(t),\qquad \eta_k(0)=0,\quad \forall\, k\in\mathbb{N}.$$

(We have identical differential inequalities starting with $t>0$ and $h<0$ sufficiently small.) A similar analysis yields

$$\dot{\eta}_0(t)\leqslant -\widetilde{q}\eta_0(t),\qquad \eta_0(0)=1.$$

In matrix notation, the set of differential inequalities involving $\dot{\eta}_k,\ k\in\mathbb{N}\cup\{0\}$, stands as

$$\begin{bmatrix}\dot{\eta}_0\\ \dot{\eta}_1\\ \dot{\eta}_2\\ \vdots\end{bmatrix}\leqslant\begin{bmatrix}-\widetilde{q} & 0 & 0 & \cdots\\ \overline{q} & -\widetilde{q} & 0 & \cdots\\ 0 & \overline{q} & -\widetilde{q} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\\ \vdots\end{bmatrix},\qquad \begin{bmatrix}\eta_0(0)\\ \eta_1(0)\\ \eta_2(0)\\ \vdots\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\\ \vdots\end{bmatrix},\tag{2.73}$$

where the "$\leqslant$" is interpreted componentwise. Clearly, $\eta_0(t)\leqslant \mathrm{e}^{-\widetilde{q}t},\ t\geqslant 0$, satisfies the first differential inequality. We claim that

$$\eta_k(t)\leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^k/k!\qquad \forall\, t\geqslant 0\quad \forall\, k\in\mathbb{N}\tag{2.74}$$

is a solution to (2.73). Indeed, for $k=1$ we have $\dot{\eta}_1\leqslant\overline{q}\eta_0-\widetilde{q}\eta_1\leqslant\overline{q}\,\mathrm{e}^{-\widetilde{q}t}-\widetilde{q}\eta_1$, which leads to

$$\mathrm{e}^{\widetilde{q}t}\eta_1(t)\leqslant \mathrm{e}^{\widetilde{q}t}\eta_1(0)+\overline{q}\int_0^t \mathrm{d}s;$$

hence $\eta_1 \,\mathrm{e}^{\widetilde{q}t} \leqslant (\overline{q}t)$ (in view of $\eta_1(0) = 0$), yielding $\eta_1(t) \leqslant (\overline{q}t)\,\mathrm{e}^{-\widetilde{q}t},\ t \geqslant 0$. Having verified the claim for $k = 1$, we suppose that it is true for $k = j > 1$. Then the $(j+2)$th differential inequality is $\dot{\eta}_{j+1} \leqslant \overline{q}\eta_j - \widetilde{q}\eta_{j+1} \leqslant \overline{q}\,\mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^j/j! - \widetilde{q}\eta_{j+1}$, which leads to

$$\mathrm{e}^{\widetilde{q}t}\eta_{j+1}(t) \leqslant \mathrm{e}^{\widetilde{q}t}\eta_{j+1}(0) + \overline{q}\int_0^t \frac{(\overline{q}s)^j}{j!}\,\mathrm{d}s.$$

Hence $\eta_{j+1}\,\mathrm{e}^{\widetilde{q}t} \leqslant (\overline{q}t)^{j+1}/(j+1)!$ (in view of $\eta_{j+1}(0) = 0$), which yields

$$\eta_{j+1}(t) \leqslant \mathrm{e}^{-\widetilde{q}t}(\overline{q}t)^{j+1}/(j+1)!, \quad t \geqslant 0.$$

By induction we conclude that our claim (2.74) holds. In view of the definition of $\eta_k(t)$, the thesis of the Lemma follows. □

§ 2.5.2. Proofs of the main results.

Proof of Theorem 2.8. To prove that (2.2) is GAS a.s. we need to verify the (AS1) and (AS2) properties in Definition 2.5.

From Lemmas 2.31 and 2.32 with $s = \ln \mu$ it follows that

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \alpha_2(\|x_0\|)\left(S\,\mathrm{e}^{-\lambda_\circ t} + \mathrm{e}^{-(\lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda})t}\right), \tag{2.75}$$

where S is a constant. Since $\lambda_\circ + \widetilde{\lambda} - \mu\widetilde{\lambda} > 0$ by (G3), we have $\int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\,\mathrm{d}t < \infty$, and an application of Tonelli's theorem on the left-hand side of (2.75) gives

$$\begin{aligned}\mathsf{E}\bigg[\int_0^\infty V_{\sigma(t)}(x(t))\,\mathrm{d}t\bigg] &= \int_0^\infty \mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\,\mathrm{d}t\\ &\leqslant \int_0^\infty \alpha_2(\|x_0\|)\left(S\,\mathrm{e}^{-\lambda_\circ t} + \mathrm{e}^{-(\lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda})t}\right)\mathrm{d}t.\end{aligned} \tag{2.76}$$

Since S is a constant, an application of (V1) on the left-hand side of (2.76) yields

$$\begin{aligned}\mathsf{E}\bigg[\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t\bigg] &\leqslant \int_0^\infty \alpha_2(\|x_0\|)\left(S\,\mathrm{e}^{-\lambda_\circ t} + \mathrm{e}^{-(\lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda})t}\right)\mathrm{d}t\\ &< \infty,\end{aligned}$$

which shows that $\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t < \infty$ a.s. Lemma 2.34 now shows that $\lim_{t\to\infty}\|x(t)\| = 0$ a.s. This proves (AS2) because the only dependence on the initial condition is through $\alpha_2(\|x_0\|)$ and x_0 is arbitrary. Alternatively, (AS2) can also be established via Lemma 2.37, where the dependence on x_0 becomes especially transparent.

Now we verify (AS1). Fix $\varepsilon > 0$. We know from the (AS2) property proved above that almost surely there exists $T(1,\varepsilon) > 0$ such that $\|x_0\| < 1$ implies $\sup_{t \geqslant T(1,\varepsilon)}\|x(t)\| < \varepsilon$. Select $\delta(\varepsilon) = \min\left\{\varepsilon\,\mathrm{e}^{-L_\varepsilon T(1,\varepsilon)}, 1\right\}$. By Lemma 2.39, $\|x_0\| < \delta(\varepsilon)$ implies

$$\|x(t)\| \leqslant \|x_0\|\;\mathrm{e}^{L_\varepsilon t} < \delta(\varepsilon)\,\mathrm{e}^{L_\varepsilon T(1,\varepsilon)} < \varepsilon \quad \forall\, t \in [0, T(1,\varepsilon)].$$

Further, the (AS2) property guarantees that with the above choice of δ and x_0, we have $\sup_{t\geqslant T(1,\varepsilon)}\|x(t)\| < \varepsilon$ for events in a set of full measure. Thus, $\|x_0\| < \delta(\varepsilon)$ implies that $\sup_{t\geqslant 0}\|x(t)\| < \varepsilon$ a.s. Since ε is arbitrary, the (AS1) property of (2.2) follows.

We conclude that (2.2) is GAS a.s. □

PROOF OF THEOREM 2.9. Let us prove global asymptotic convergence of $\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]$ to 0. In view of (V1), taking limits in (2.75) we get

$$\begin{aligned}\lim_{t\to\infty}\mathsf{E}\big[\alpha_1(\|x(t)\|)\big] &\leqslant \lim_{t\to\infty}\mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\\ &\leqslant \lim_{t\to\infty}\alpha_2(\|x_0\|)\left(S\,\mathrm{e}^{-\lambda_\circ t}+\mathrm{e}^{-\left(\lambda_\circ+\widetilde{\lambda}-\mu\overline{\lambda}\right)t}\right)\\ &=0.\end{aligned}$$

It is clear that the only dependence on the initial condition is through $\alpha_2(\|x_0\|)$ and x_0 can be arbitrary. The (SM2) property of (2.2) follows. For an arbitrary fixed $\varepsilon>0$ it suffices to choose $\widetilde{\delta}>\alpha_2^{-1}\big(\varepsilon/(S+1)\big)$ in order to verify the property (SM1). The thesis follows. □

PROOF OF THEOREM 2.10. To see the property (AS2) of (2.2) we note that by Lemma 2.48

$$\mathsf{P}\left(\int_0^\infty \alpha_1(\|x(t)\|)\,\mathrm{d}t<\infty\right)=1.$$

Lemma 2.34 now shows that $\lim_{t\to\infty}\|x(t)\|=0$ a.s. since $\alpha_1\in\mathcal{K}_\infty$. This proves (AS2) because the only dependence on the initial condition is through $\alpha_2(\|x_0\|)$ and x_0 is arbitrary.

The procedure for verification of (AS1) is identical to that followed in the above proof of Theorem 2.8, and is omitted. The assertion of the Theorem follows immediately. □

PROOF OF COROLLARY 2.11. Our first objective is to prove asymptotic convergence to 0 of the net $\big\{\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]\big\}_{t\geqslant 0}$. The global asymptotic convergence a.s. of the process $(x(t))_{t\geqslant 0}$ to 0 in view of Theorem 2.10 shows via hypothesis (V1) that the process $\big(V_{\sigma(t)}(x(t))\big)_{t\geqslant 0}$ also converges a.s. to 0 since $\alpha_2\in\mathcal{K}_\infty$. Therefore, it suffices to show that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable to conclude that $\lim_{t\to\infty}\mathsf{E}\big[V_{\sigma(t)}(x(t))\big]=0$. This will imply global asymptotic convergence of $\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]$ to 0 in light of (V1). But from Lemma 2.58 we know that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable; therefore, in view of Theorem 2.10 and Proposition 1.2 we conclude that $\lim_{t\to\infty}\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]=0$. This verifies the (SM2) property with $\alpha=\alpha_1$.

It remains to establish (SM1). Following the notation of Lemma 2.58, note that $\eta(0)\in\,]0,1[$ by (E4). To establish (SM1) we only need to observe that with $\delta=0$ in (2.62) we have

$$\sup_{t\geqslant 0}\mathsf{E}\big[V_{\sigma(t)}(x(t))\big]\leqslant \alpha_2(\|x_0\|)\frac{1}{1-\eta(0)}.$$

For $\varepsilon>0$ preassigned, we choose $\widetilde{\delta}<\alpha_2^{-1}\big(\varepsilon(1+\eta(0))\big)$ to see that $\sup_{t\geqslant 0}\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]<\varepsilon$ whenever $\|x_0\|<\widetilde{\delta}$. The (SM1) property with $\alpha=\alpha_1$ follows, thereby completing the proof. □

PROOF OF THEOREM 2.12. The proof mimics the proof of Theorem 2.10 above; we merely need to substitute Lemma 2.53 in place of Lemma 2.48. □

PROOF OF COROLLARY 2.13. Following the structure of the proof of Corollary 2.11, first we prove asymptotic convergence of the net $\big\{\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]\big\}_{t\geqslant 0}$ to 0. We have proved global asymptotic convergence a.s. of the process $(x(t))_{t\geqslant 0}$ to 0 in Theorem 2.10, and via hypothesis (V1) this shows that the process $\big(V_{\sigma(t)}(x(t))\big)_{t\geqslant 0}$ also converges a.s. to 0 since $\alpha_2 \in \mathcal{K}_\infty$. Therefore, it suffices to show that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable to conclude that $\lim_{t\to\infty} \mathsf{E}\big[V_{\sigma(t)}(x(t))\big] = 0$. This will in turn imply global asymptotic convergence of $\mathsf{E}\big[\alpha_1(\|x(t)\|)\big]$ to 0 in light of (V1). But from Lemma 2.63 we know that the family $\big\{V_{\sigma(t)}(x(t))\big\}_{t\geqslant 0}$ is uniformly integrable; therefore, in view of Theorem 2.12 and Proposition 1.2 we conclude that $\lim_{t\to\infty} \mathsf{E}\big[\alpha_1(\|x(t)\|)\big] = 0$. This verifies the (SM2) property with $\alpha = \alpha_1$.

It remains to prove (SM1). Following the notation of the proof of Lemma 2.63, we note that $\eta(0) \in \,]0,1[$ by (U3). To establish (SM1) we only need to note that with $\delta = 0$ in (2.62) we have

$$\sup_{t\geqslant 0} \mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant M\alpha_2(\|x_0\|)\frac{1}{1-\eta(0)}.$$

For $\varepsilon > 0$ preassigned, we choose $\widetilde{\delta} < \alpha_2^{-1}\big(\varepsilon(1+\eta(0))/M\big)$ to see that $\sup_{t\geqslant 0} \mathsf{E}\big[\alpha_1(\|x(t)\|)\big] < \varepsilon$ whenever $\|x_0\| < \widetilde{\delta}$. The (SM1) property with $\alpha = \alpha_1$ follows, thereby completing the proof. □

Proof of Corollary 2.26. The assertion follows directly from Lemma 2.68, and Theorem 2.8 with $M = 0$ in (G2). □

§ 2.6. Concluding Remarks and Future Work

We have established sufficient conditions for several types of stability of the randomly switched system (2.2) under some general classes of switching signals. Most of the results relied on no special underlying structure of the switching signal σ; in particular, we treated general stochastically slow and semi-Markov switching signals in our framework. In §2.4 we discussed how to apply standard methods involving martingales under additional probabilistic structures.

In Theorem 2.12 we treated the case where the jump destination process is memoryless; one important future direction is to get conditions for stability when the jump destination process is a discrete-time Markov chain without strengthening the rest of the hypotheses any further. We believe that it may be possible to prove stability results similar to Theorem 2.12 for holding times that are not just uniform but more general random variables with compactly supported density. It would also be interesting to relax the hypothesis (V3) in Assumption 2.3, since this would enlarge the class from which the Lyapunov functions can be chosen.

CHAPTER 3

EXTERNAL STABILITY

In this chapter we present our results on external stability of switched systems; i.e., stability in the presence of external disturbance inputs. The setting changes to the following. We are given a family of subsystems such that each of its members possesses a disturbance input, and a switching signal with some known characteristics. Our objective consists of finding sufficient conditions such that the switched system generated by the given switching signal satisfies some input-to-state stability estimate. We refer the reader to §1.2 for a general discussion and motivation.

This chapter is organized as follows. In §3.1 we consider the case of deterministic average dwell-time switching as a motivation for applying techniques similar to those developed for randomly switched systems without inputs in Chapter 2. Such techniques are applied to establish some input-to-state stability estimates in §3.2. Some of the technical issues associated with our approach are presented in §3.3.2. We also consider external stability under Markovian switching signals in §3.4. Finally, the proofs of the results are provided in §3.5.

There have been some previous works on stochastic stability in the presence of external inputs. For instance, in the context of differential equations perturbed by Brownian motion, the concept of *exponential input-to-state stability* [**60**] has been utilized in constructive ways for stability analysis and feedback stabilizing controller design. However, this is the first time that external stability under random switching is being investigated.

§ 3.1. Input-to-State Stability Under a Class of Deterministic Switching Signals

Determining conditions for external stability of switched systems under different classes of switching signals is a subject of ongoing research; some recent results on ISS-type properties of switched systems may be found in [**62**]. One of the key results in [**62**] (Theorem 3.3 below) establishes ISS of a switched system when each subsystem is ISS, and the switching signal possesses a finite upper bound on the rate of switching. Theorem 3.3 may be considered to be the analog of the well-known result [**37**, Theorem 3.2] on global asymptotic stability in the absence of external inputs under deterministic average dwell-time switching signals. As pointed out in Remark 2.16, our Theorem 2.8 on internal stability of (2.2) employed hypotheses similar to [**37**, Theorem 3.2], and our approach to internal stability for randomly switched systems pursued in Chapter 2 was largely motivated by such deterministic results. As we have seen, we did obtain strong results in Chapter 2 by following this approach. We look at Theorem 3.3

as our motivation to extend the input-to-state stability property to randomly switched systems under statistically slow switching, as an analog of Theorem 2.8 in the case of systems with inputs; this extension will be taken up in §3.2.

Suppose that each subsystem of the family (2.1) is disturbed by a k-dimensional input d, so that the family stands as

$$\dot{x} = f_i(x, d), \qquad i \in \mathcal{P}, \tag{3.1}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is continuously differentiable for every $i \in \mathcal{P}$ with $f_i(0, 0) = 0$. We allow $d : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}^k$ to be a measurable and essentially bounded function of time [**57**]. We let $\mathcal{P}$ be a finite set and the switching signal $\sigma : \mathbb{R}_{\geqslant 0} \longrightarrow \mathcal{P}$ be a piecewise constant right-continuous deterministic function of time. The switched system that σ generates from the family (3.1) is given by

$$\dot{x} = f_\sigma(x, d), \qquad x(0) = x_0, \quad t \geqslant 0. \tag{3.2}$$

We say that σ satisfies the *average dwell-time condition* [**37**, Chapter 3] if

- there exist $N_\circ > 0$ and $\tau_a > 0$ such that $N_\sigma(t', t) \leqslant N_\circ + (t' - t)/\tau_a$ for all $0 \leqslant t \leqslant t' < \infty$,

where as before, $N_\sigma(t_2, t_1)$ denotes the number of switches on the interval $]t_1, t_2] \subseteq \mathbb{R}_{\geqslant 0}$.

We are interested in finding sufficient conditions for input-to-state stability of (3.2) with respect to the input d. Recall that (3.2) is said to be *input-to-state stable* (ISS) [**55**] if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}^n$, all essentially bounded measurable inputs d, and all $t \geqslant 0$, we have $\|x(t)\| \leqslant \beta(\|x_0\|, t) + \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)$. Here the notation $\|d\|_{\mathbb{R}_{\geqslant 0}}$ stands for the essential supremum norm of d over the domain $\mathbb{R}_{\geqslant 0}$.

We have the following theorem.

3.3. Theorem. *Consider the system* (3.2). *Suppose that there exist continuously differentiable functions $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$, class-$\mathcal{K}_\infty$ functions $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\gamma}$, and numbers $\lambda_\circ > 0$ and $\mu > 1$ such that for all $(x, d) \in \mathbb{R}^n \times \mathbb{R}^k$ and $i, j \in \mathcal{P}$,*

$$\overline{\alpha}_1(\|x\|) \leqslant V_i(x) \leqslant \overline{\alpha}_2(\|x\|), \tag{3.4}$$

$$\frac{\partial V_i}{\partial x}(x) f_i(x, d) \leqslant -\lambda_\circ V_i(x) + \overline{\gamma}(\|d\|), \tag{3.5}$$

$$V_i(x) \leqslant \mu V_j(x). \tag{3.6}$$

If σ satisfies the average dwell-time condition and $\tau_a > \dfrac{\ln \mu}{\lambda_\circ}$, then (3.2) *is* ISS.

A proof is provided in Appendix A, where a comparison framework for switched systems with inputs is also established.

§ 3.2. System Model

Following the definition in §3.1 we let

$$\dot{x} = f_i(x, d), \qquad i \in \mathcal{P}, \tag{3.7}$$

be a system with an input d, where $f_i : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is continuously differentiable for every $i \in \mathcal{P}$, with $f_i(0,0) = 0$. We allow $d : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}^k$ to be a measurable and essentially bounded function of time [**57**]. As in §2.2 we assume that $\mathcal{P}$ is finite. Let the switching signal σ be a càdlàg process on the probability space described in §2.2; the randomly switched system generated by σ from the family (3.7) stands as

$$\dot{x} = f_\sigma(x, d), \qquad x(0) = x_0, \quad t \geqslant 0. \tag{3.8}$$

The various measurability properties of the solution of (2.2) discussed in §2.2 carry over to the solution $x(\cdot)$ of (3.8). In what follows we ignore the trivial case of $x_0 = 0$.

3.9. Assumption. There exist a family of continuously differentiable real-valued functions $\{V_i\}_{i\in\mathcal{P}}$ on $\mathbb{R}^n$, functions $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$, constants $\mu > 1$ and $\lambda_i \in \Lambda \subseteq \mathbb{R}$, $i \in \mathcal{P}$, such that

(Vd1) $\alpha_1(\|x\|) \leqslant V_i(x) \leqslant \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, i \in \mathcal{P}$;
(Vd2) $\dfrac{\partial V_i}{\partial x}(x) f_i(x, d) \leqslant -\lambda_i V_i(x) + \chi(\|d\|) \quad \forall\, i \in \mathcal{P}$;
(Vd3) $V_i(x) \leqslant \mu V_j(x) \quad \forall\, x \in \mathbb{R}^n \;\; \forall\, i, j \in \mathcal{P}$. ♢

As the following proposition shows, the pair of assumptions ((Vd1),(Vd2)) is equivalent to requiring that each subsystem (3.7) is ISS.

3.10. Proposition ([**48**]). *The following are equivalent:*

(i) *The i-th subsystem of the family* (3.7) *is* ISS.
(ii) *There exist a continuously differentiable function $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, functions $\alpha_1, \alpha_2, \chi \in \mathcal{K}_\infty$, and $\lambda_i > 0$, such that $\forall\, (x, d) \in \mathbb{R}^n \times \mathbb{R}^k$*

$$\alpha_1(\|x\|) \leqslant V_i(x) \leqslant \alpha_2(\|x\|),$$
$$\frac{\partial V_i}{\partial x}(x) f_i(x, d) \leqslant -\lambda_i V_i(x) + \chi(\|d\|).$$

(iii) *There exist a continuously differentiable function $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, functions $\alpha_1, \alpha_2, \rho, \chi \in \mathcal{K}_\infty$, and $\lambda_i > 0$, such that $\forall\, (x, d) \in \mathbb{R}^n \times \mathbb{R}^k$*

$$\alpha_1(\|x\|) \leqslant V_i(x) \leqslant \alpha_2(\|x\|),$$
$$\|x\| \geqslant \rho(\|d\|) \implies \frac{\partial V_i}{\partial x}(x) f_i(x, d) \leqslant -\lambda_i V_i(x).$$

The function V_i in (ii) and (iii) above is called an ISS-*Lyapunov function*. Let us note that conventionally ISS-Lyapunov functions are defined in a somewhat different way; namely, the right-hand side of the second displayed inequality in (ii) is taken to be $-\alpha'(\|x\|) + \chi'(\|d\|)$, or the right-hand side of the second displayed inequality in (iii) is taken to be $-\alpha'(\|x\|)$, for $\alpha', \chi' \in \mathcal{K}_\infty$. The functions χ and ρ in (ii) and (iii), respectively, are related to each other; for further details see [**48**, §7].

We shall focus on the following two types of external stability. Let us note that the external input signal d is not modeled as a random process.

3.11. Definition. The system (3.8) is said to satisfy an *input-to-state stability in $\boldsymbol{L}_1$ estimate at switching instants* if there exist functions $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$, such that for all initial conditions $x_0 \in \mathbb{R}^n$ and for all essentially bounded inputs d, the estimate

$$\mathsf{E}\big[\alpha(\|x(\tau_i)\|)\big] \leqslant \beta(\|x_0\|, i) + \sup_{s \geqslant 0} \gamma(\|d(s)\|) \tag{3.12}$$

holds for all $i \in \mathbb{N}$. ◇

3.13. Definition. The system (3.8) is said to be *input-to-state stable in $\boldsymbol{L}_1$* (ISS-M) if there exist functions $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$, such that for all initial conditions $x_0 \in \mathbb{R}^n$ and for all essentially bounded inputs d,

$$\mathsf{E}\big[\alpha(\|x(t)\|)\big] \leqslant \beta(\|x_0\|, t) + \sup_{s \in \mathbb{R}_{\geqslant 0}} \gamma(\|d(s)\|) \qquad \forall\, t \geqslant 0. \tag{ISS-M}$$
◇

In deterministic systems literature, ISS inequalities involve just the norm of the state. The presence of the function α in the above two definitions allows some measure of flexibility in the sense that one need not worry about bounds for just the expectation of the norm of the state, i.e., $\boldsymbol{L}_1$-stability. Frequently one employs Lyapunov functions which are polynomial functions of the states, and with the aid of conditions such as in (Vd1), stronger bounds in terms of the $\boldsymbol{L}_p$ $(p > 1)$ norms of the state may be obtained; see §1.3 for a discussion of $\boldsymbol{L}_p$-stability. For instance, quadratic Lyapunov functions yield bounds for mean-square or $\boldsymbol{L}_2$-stability, which is stronger than $\boldsymbol{L}_1$-stability.

§ 3.3. Classes of σ and Corresponding Results

§ 3.3.1. Statements of the main results.

3.14. Definition. We say that a switching signal σ belongs to class GS if there exist constants $\overline{\lambda}, \widetilde{\lambda} > 0$, such that for every $k \in \mathbb{N} \cup \{0\}$, every $(\mathfrak{F}_t)_{t \geqslant 0}$-optional time t', and every $s \geqslant 0$ we have $\mathsf{P}^{\mathfrak{F}_{t'}}\big(N_\sigma(t'+s, t') = k\big) \leqslant \big(\overline{\lambda}s\big)^k \,\mathrm{e}^{-\widetilde{\lambda}s}/k!$. ◇

3.15. Proposition. *Consider the system (3.8), and suppose that*

(Gd1) *Assumption 3.9 holds with $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$;*

(Gd2) *σ is of class GS as defined in Definition 3.14;*

(Gd3) $\mu < \Big(\lambda_\circ + \widetilde{\lambda}\Big)/\overline{\lambda}$.

Then there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, and a sequence $(T_i)_{i\in\mathbb{N}}$ of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times such that the following estimate holds

$$\mathsf{E}\big[\alpha_1(\|x(t)\|)\mathbf{1}_{\{t\in[T_i,T_{i+1}[\}\cap\{T_i<\infty\}}\big] \leqslant \beta(\|x_0\|, t) \vee \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \qquad \forall\, i \in \mathbb{N} \quad \forall\, t \geqslant 0. \tag{3.16}$$

3.17. Conjecture. The system (3.8) is ISS-M under the hypotheses of Proposition 3.15.

3.18. Proposition. *Consider the system (3.8) and suppose that*

(Ed1) *Assumption 3.9 holds with $\Lambda = \mathbb{R}$;*

(Ed2) *σ is of class EH as defined in Definition 2.7;*

(Ed3) $\lambda_i + \lambda > 0 \quad \forall\, i \in \mathcal{P}$;

(Ed4) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i}{1 + \lambda_i/\lambda} < 1.$

Then (3.8) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants.

3.19. Proposition. *Consider the system (3.8), and suppose that*

(Ud1) *Assumption 3.9 holds with $\Lambda = \mathbb{R}$;*

(Ud2) *σ is of class UH as defined in Definition 2.7;*

(Ud3) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i\left(1 - \mathrm{e}^{-\lambda_i T}\right)}{\lambda_i T} < 1.$

Then (3.8) satisfies an ISS in $\boldsymbol{L}_1$ estimate at switching instants.

§ 3.3.2. Discussion.

3.20. Remark. Let us note the difference between the classes G and GH. The conditional probability distribution in the latter assumption will be needed in the proof of Proposition 3.15. In the proof of Theorem 2.8 all that was required was the moment generating function of the random process $(N_\sigma(t,0))_{t\geqslant 0}$. However, as we shall see, the proof of Proposition 3.15 crucially relies on the assumption that the conditional probability distribution of the process $(N_\sigma(s+h,s))_{h\geqslant 0}$, given the history $\mathfrak{F}_s$ generated by $(\sigma(t))_{t\geqslant 0}$ up to the time s, depends only on the difference h, where s is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time. See also Remark 3.51 following the proof of Proposition 3.15 for further information. A similar situation occurs in the deterministic case of average dwell-time switching signals discussed in §3.1 above (see also [**26**]), where we require that the number of switches between t' and $t'+h$ grows as a linear function of h for every $t' \geqslant 0$. ◁

3.21. Remark. The assertions of Propositions 3.18 and 3.19 are concerned with ISS in $\boldsymbol{L}_1$ estimates defined in Definition 3.11. We believe that these properties closely reflect the behavior of the continuous-time process $(x(t))_{t\geqslant 0}$, particularly considering the facts that d is an essentially bounded signal, and that the process $(N_\sigma(t,0))_{t\geqslant 0}$ corresponding to switching signals of class EH or class UH is almost surely of finite variation, i.e., $(N_\sigma(t,0))_{t\geqslant 0}$ does not explode (see Lemmas 2.29 and 2.30 for detailed proofs). ◁

3.22. REMARK. It may appear that if we impose further structure on the switching signals than those of class GS, then it may be possible to obtain the ISS-M property under the hypotheses of Proposition 3.18. As an example, suppose that the counting process $(N_\sigma(t,0))_{t\geqslant 0}$ corresponding to σ is a stationary Poisson process of intensity λ, and let us retain the rest of the hypotheses of Proposition 3.15. We know that conditioned on the event $\{N_\sigma(t,0)=\nu\}$, $\nu\in\mathbb{N}$, the probability distribution of the jump instants $\{\tau_1,\ldots,\tau_\nu\}$ are i.i.d unif(t) random variables, with the instants reordered to be increasing. This corresponds to the density

$$f_{\tau_1,\cdots,\tau_\nu}(t_1,\cdots t_\nu)=\begin{cases}\frac{\nu!}{t^\nu} & \text{if } 0\leqslant t_1\leqslant\cdots\leqslant t_\nu\leqslant t,\\ 0 & \text{otherwise.}\end{cases}$$

For a fixed $t\geqslant 0$, applying the total probability formula in §1.3.1, we arrive at

$$\begin{aligned}\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] &= \sum_{\nu=0}^{\infty}\mathsf{P}\big(N_\sigma(t,0)=\nu\big)\,\mathsf{E}\big[V_{\sigma(t)}(x(t))\,\big|\,N_\sigma(t,0)=\nu\big]\\ &\leqslant \sum_{\nu=0}^{\infty}\mathsf{P}\big(N_\sigma(t,0)=\nu\big)\,\mathsf{E}\Bigg[\alpha_2(\|x_0\|)\mu^\nu\,\mathrm{e}^{-\lambda_\circ t}+\\ &\quad\frac{\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\mu^\nu\,\mathrm{e}^{-\lambda_\circ t}}{\lambda_\circ}\Bigg(\mu^{-\nu}\left(\mathrm{e}^{\lambda_\circ t}-\mathrm{e}^{\lambda_\circ\tau_\nu}\right)+\sum_{i=0}^{\nu-1}\mu^{-i}\left(\mathrm{e}^{\lambda_\circ\tau_{i+1}}-\mathrm{e}^{\lambda_\circ\tau_i}\right)\Bigg)\,\Bigg|\,N_\sigma(t,0)=\nu\Bigg].\end{aligned}\tag{3.23}$$

An exact evaluation of the second term on the right-hand side of the inequality above can theoretically be carried out with the aid of the formula

$$\begin{aligned}&\mathsf{E}\Bigg[\mu^{-\nu}\left(\mathrm{e}^{\lambda_\circ t}-\mathrm{e}^{\lambda_\circ\tau_\nu}\right)+\sum_{i=0}^{\nu-1}\mu^{-i}\left(\mathrm{e}^{\lambda_\circ\tau_{i+1}}-\mathrm{e}^{\lambda_\circ\tau_i}\right)\,\Bigg|\,N_\sigma(t,0)=\nu\Bigg]\\ &=\int_0^t\mathrm{d}t_\nu\int_0^{t_\nu}\mathrm{d}t_{\nu-1}\cdots\int_0^{t_2}\mathrm{d}t_1\,\frac{\nu!}{t^\nu}\Bigg[\mu^{-\nu}\left(\mathrm{e}^{\lambda_\circ t}-\mathrm{e}^{\lambda_\circ\tau_\nu}\right)+\sum_{i=0}^{\nu-1}\mu^{-i}\left(\mathrm{e}^{\lambda_\circ\tau_{i+1}}-\mathrm{e}^{\lambda_\circ\tau_i}\right)\Bigg].\end{aligned}$$

However, the evaluation of the first few terms of this integration shows no special structure, so we resort to maximizing the integrand (i.e., the term inside the conditional expectation). This problem can be formulated as

$$\begin{aligned}&\text{maximize} && J_\nu:=\mu^{-\nu}\left(\mathrm{e}^{\lambda_\circ t}-\mathrm{e}^{\lambda_\circ\tau_\nu}\right)+\sum_{i=0}^{\nu-1}\mu^{-i}\left(\mathrm{e}^{\lambda_\circ\tau_{i+1}}-\mathrm{e}^{\lambda_\circ\tau_i}\right)\\ &\text{subject to the constraints} && 0\leqslant\tau_1\leqslant\tau_2\leqslant\cdots\leqslant\tau_\nu\leqslant t.\end{aligned}$$

We rewrite the cost J_ν as

$$J_\nu=\mu^{-\nu}\,\mathrm{e}^{\lambda_\circ t}-1+\sum_{i=1}^{\nu}\mu^{-(i-1)}(1-\mu^{-1})\,\mathrm{e}^{\lambda_\circ\tau_i},$$

and since $\mu>1$ and the weights $\mu^{-(i-1)}(1-\mu^{-1})$ are between 0 and 1, it follows that this constrained optimization problem is well-posed. Since the weights decrease with i, the term with the largest weight gives us the desired maximum, and the corresponding J_ν becomes $\mathrm{e}^{\lambda_\circ t}-1$. Employing this upper bound in (3.23) we get

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \sum_{\nu=0}^{\infty}\left(\mathrm{e}^{-\lambda t}\frac{(\lambda t)^{\nu}}{\nu!}\right)\left(\alpha_2(\|x_0\|)\mu^{\nu}\,\mathrm{e}^{-\lambda_\circ t} + \frac{\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\mu^{\nu}\,\mathrm{e}^{-\lambda_\circ t}}{\lambda_\circ}\left(\mathrm{e}^{\lambda_\circ t}-1\right)\right)$$

$$\leqslant \alpha_2(\|x_0\|)\,\mathrm{e}^{-(\lambda_\circ-(\mu-1)\lambda)t} + \frac{\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_\circ}\,\mathrm{e}^{(\mu-1)\lambda t}. \tag{3.24}$$

If $\mu < 1 + \frac{\lambda_\circ}{\lambda}$, then the first term on the right-hand side of (3.24) is a class-$\mathcal{KL}$ function. It follows immediately that for each $T > 0$ we get a uniform upper bound on $\mathsf{E}\big[V_{\sigma(t)}(x(t))\big]$ for all $t \in [0, T]$. $\triangleleft$

3.25. Remark. The results above fall short of being satisfactory. Indeed, perhaps the most natural adaptation of the ISS concept to the stochastic case would involve bounds of the type

$$\mathsf{E}\big[\alpha(\|x(t)\|)\big] \leqslant \beta(\|x_0\|, t) + \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \tag{3.26}$$

for all $x_0 \in \mathbb{R}^n$, $t \geqslant 0$, and essentially bounded inputs d. We shall obtain this estimate in §3.4, but under the additional assumption that σ is Markovian. However, the technical difficulties, in the absence of Markovian assumption on σ, are formidable. For illustrative purposes, on the one hand let us consider switching signals belonging to class G. If $B(r)$ denotes the open ball around the origin of radius $r > 0$, then the solution trajectory $x(\cdot)$ can make indefinitely many excursions outside $B(r)$ with positive probability. Here $r > 0$ can be arbitrary. There is no further structure which prevents the number of these excursions from increasing at least linearly with time t (the linearity follows at once from the observation that the set of vector fields $\{f_i\}_{i\in\mathcal{P}}$ is locally Lipschitz, and taking $r = \|d\|_{\mathbb{R}_{\geqslant 0}} < \infty$). Further, since estimates for the probability distribution of the holding times are not available, "gain-margin" type arguments appear to be the only mode of attack, as we pursue in §3.5.

On the other hand, in the case of switching signals of class UH, the holding times are explicitly characterized, but the chief difficulty lies in obtaining an estimate for $\mathsf{E}\big[\alpha(\|x(t)\|)\big]$ from an ISS estimate in $\boldsymbol{L}_1$ at switching instants. To wit, there can potentially be indefinitely many jumps of σ before and after a given time t; therefore countably many simultaneous interpolations are needed to get an estimate of $\mathsf{E}\big[\alpha(\|x(t)\|)\big]$, and such an interpolation is again a difficult problem. Even under simplier assumptions for class EH switching signals this state of affairs persists, as we observed in Remark 3.22 above. Unlike in the deterministic case, one is necessarily forced to work with random intervals.

Let us note that ISS-type estimates "in probability" for diffusion processes have appeared in the literature, for instance, in [**32**, Theorem 4.2], and more recently in [**64**, §2]. Although the system models in the above references differ from ours, the essential technical difficulties remain the same. In particular, with positive probability the state of the system can make indefinitely many excursions outside any prespecified closed ball around the origin (which is why we need to define the corresponding optional times in (3.47) below.) Unfortunately, these difficulties were not realized in the aforesaid references, and the claims made in both of them are still open. $\triangleleft$

§ 3.4. Additional Probabilistic Structures

In this section we let $\sigma \sim (\pi^\circ, Q)$. Recall from §2.3.4 the notation $\sigma \sim (\pi^\circ, Q)$ which means that π° is the initial probability vector of σ and $Q = [q_{ij}]_{\mathrm{N}\times\mathrm{N}}$ is its generator matrix. Then

(MC1) for the generator matrix Q we have $q_{ij} \geqslant 0 \quad \forall\, i, j \in \mathcal{P}, i \neq j$, and $\sum_{j\in\mathcal{P}\smallsetminus\{i\}} q_{ij} = -q_{ii} \quad \forall\, i, j \in \mathcal{P}$; and

(MC2) for $h > 0$, $\mathsf{P}\Big(\sigma(t+h) = j \;\Big|\; \sigma(t) = i\Big) = \delta_{ij} + q_{ij}h + o(h)$, where δ_{ij} is the Kronecker delta.

Recall that a process $(\xi_t)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-local martingale [**52**, p. 123] if there exists a "localizing sequence" $(T_i)_{i\in\mathbb{N}}$ of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times with $T_i \uparrow \infty$ a.s., such that the stopped process $\big(\xi_{t\wedge T_i}\big)_{t\geqslant 0}$ is an uniformly integrable $(\mathfrak{F}_t)_{t\geqslant 0}$-martingale. The following definition [**17**] concerns the extended generator of the Markov process $(\sigma(t), x(t))_{t\geqslant 0}$ where $\sigma \sim (\pi^\circ, Q)$.

3.27. Definition. Let $D(\mathcal{L})$ denote the set of all functions $V : \mathbb{R}_{\geqslant 0} \times \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ for which there exists a measurable function $U : \mathbb{R}_{\geqslant 0} \times \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that for every initial condition $(\sigma(0), x(0)) \in \mathcal{P} \times \mathbb{R}^n$ the process

$$\left(V(t, \sigma(t), x(t)) - V(0, \sigma(0), x(0)) - \int_0^t U(s, \sigma(s), x(s))\, \mathrm{d}s\right)_{t\geqslant 0}$$

is an $(\mathfrak{F}_t)_{t\geqslant 0}$-local martingale. The *extended generator* of the process $(\sigma(t), x(t))_{t\geqslant 0}$ is defined as $\mathcal{L}V := U$. ◇

It is more usual to define the extended generator via a differentiation operation, and the following proposition characterizes the extended generator of the process $(\sigma(t), x(t))_{t\geqslant 0}$ in such terms.

3.28. Proposition. *Let $g : \mathbb{R}_{\geqslant 0} \times \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function such that $g(t, i, x)$ is continuously differentiable in t and x for each $i \in \mathcal{P}$. The extended generator of the Markov process $(\sigma(t), x(t))_{t\geqslant 0}$, where $x(\cdot)$ is the solution of (3.8) corresponding to the switching signal $\sigma(\cdot)$, acting on g is given by the expression*

$$\begin{aligned}\mathcal{L}g(t, i, x) &= \lim_{h\downarrow 0} \frac{\mathsf{E}\big[g(t+h, \sigma(t+h), x(t+h)) - g(t, i, x) \big| (\sigma(t), x(t)) = (i, x)\big]}{h}\\ &= \frac{\partial g}{\partial t}(t, i, x) + \frac{\partial g}{\partial x}(t, i, x) f_i(x, d) + \sum_{j\in\mathcal{P}} q_{ij} g(t, j, x).\end{aligned}$$

The proof is standard (see, e.g., [**17**]) and is omitted; it follows readily from the infinitesimal description and càdlàg property of the Markov chain σ, and continuity of x.

It is clear from Proposition 3.28 and Definition 3.27 that the domain $D(\mathcal{L})$ of the extended generator $\mathcal{L}$ in our context contains the set of functions $g : \mathbb{R}_{\geqslant 0}\times\mathcal{P}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ which are continuously differentiable functions on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ pointwise on $\mathcal{P}$. We shall only be concerned with such functions here.

3.29. THEOREM. *Consider the system* (3.8). *Suppose that there exist a function* $V : \mathcal{P}\times\mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, *continuously differentiable in the second argument pointwise on* $\mathcal{P}$, *functions* $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$, *and a constant* $\lambda_\circ > 0$, *such that*

(ISS1) $\alpha_1(\|x\|) \leqslant V(i,x) \leqslant \alpha_2(\|x\|)$ *for all* $(i,x) \in \mathcal{P}\times\mathbb{R}^n$;

(ISS2) $\mathcal{L}V(i,x) \leqslant -\lambda_\circ V(i,x)$ *whenever* $\|x\| > \rho(\|d\|)$ *and* $i \in \mathcal{P}$.

Then (3.8) *is* ISS-M.

3.30. REMARK. We have made a sudden notational change in this section by writing a single Lyapunov function V on the state-space $\mathcal{P}\times\mathbb{R}^n$ instead of the collection of Lyapunov functions $\{V_i\}_{i\in\mathcal{P}}$ as in Assumption 3.9. This is motivated by the fact that we are now looking at the Markov process $(\sigma(t), x(t))_{t\geqslant 0}$, and V is a Lyapunov function on the state-space of this process. The process $(x(t))_{t\geqslant 0}$ by itself is clearly not Markovian, since the future evolution of x depends on σ. ◁

3.31. REMARK. Let us examine the hypotheses of Theorem 3.29 in some detail. Note that from the hypothesis (ISS2) it is not in general possible to get an intuitive understanding of whether the individual subsystems are themselves ISS or not. This is in contrast to Propositions 3.15, 3.18, and 3.19, where we followed a bottom-up approach and formulated the requirements on the subsystems and the switching signal separately from the very beginning. However, this latter approach led to an impasse so far as establishing the exact property ISS-M under the hypotheses of the aforesaid propositions.

It is interesting to note that the condition (ISS2) of Theorem 3.29 closely resembles the positive Harris recurrence condition (CD2) in [**44**, p. 529], which is called *Foster-Lyapunov drift condition* there. However, (ISS2) is weaker because we do not require the function V to be greater than 1. Foster-Lyapunov conditions are useful in the theory of stability of Markov chains; see, e.g., [**43**] for further information. The idea of employing a stochastic Lyapunov condition such as (ISS2) of Theorem 3.29 appears in [**9**, §8.3], where the author establishes a type of uniform stability in distribution of controlled diffusions. ◁

§ 3.5. Proofs

In this section we present proofs of the results in §§ 3.3 and 3.4.

§ 3.5.1. Auxiliary lemmas. The following is a simple switching lemma for nonnegative $(\mathfrak{F}_t)_{t\geqslant 0}$ potentials. Recall that a process $(\xi_t)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-potential if it is a nonnegative $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale that almost surely converges to 0.

3.32. LEMMA. *Let* $(\xi_t)_{t\geqslant 0}$ *be a nonnegative càdlàg* $(\mathfrak{F}_t)_{t\geqslant 0}$*-potential and* τ *be an* $(\mathfrak{F}_t)_{t\geqslant 0}$ *optional time. Then the process* $\big(\xi_t \mathbf{1}_{\{t<\tau\}}\big)_{t\geqslant 0}$ *is a càdlàg* $(\mathfrak{F}_t)_{t\geqslant 0}$*-potential.*

PROOF. It is clear that since $\mathsf{E}\big[\xi_t\big] < \infty$ for each t, we have $\mathsf{E}\big[\xi_t \mathbf{1}_{\{t<\tau\}}\big] < \infty$. Let us fix $0 \leqslant s < t < \infty$ and $A \in \mathfrak{F}_s$. Then

$$\begin{aligned}
\int_A \xi_s \mathbf{1}_{\{s<\tau\}}\,\mathrm{d}P &= \int_{A\cap\{s<\tau\}} \xi_s\,\mathrm{d}P \quad \text{since } \{s<\tau\}\in\mathfrak{F}_s\\
&\geqslant \int_{A\cap\{s<\tau\}} \xi_t\,\mathrm{d}P \quad \text{since } (\xi_t)_{t\geqslant 0} \text{ is an } (\mathfrak{F}_t)_{t\geqslant 0}\text{-supermartingale}\\
&= \int_{A\cap\{t<\tau\}} \xi_t\,\mathrm{d}P \quad \text{since } \mathfrak{F}\ni\{t<\tau\}\subseteq\{s<\tau\} \text{ and } \xi_t\geqslant 0\\
&= \int_A \xi_t \mathbf{1}_{\{t<\tau\}}\,\mathrm{d}P\\
&= \int_A \mathsf{E}^{\mathfrak{F}_s}\big[\xi_t \mathbf{1}_{\{t<\tau\}}\big]\,\mathrm{d}P \quad \text{by definition of conditional expectation.}
\end{aligned}$$

Since A is arbitrary and the extreme integrands are integrable and $\mathfrak{F}_s$-measurable, this shows that $\big(\xi_t \mathbf{1}_{\{t<\tau\}}\big)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale. The càdlàg property follows from the definition. It remains to show that $\lim_{t\to\infty} \xi_t \mathbf{1}_{\{t<\tau\}} = 0$ a.s. But $\xi_t \mathbf{1}_{\{t<\tau\}} = \xi_t \mathbf{1}_{\{t<\tau<\infty\}} + \xi_t \mathbf{1}_{\{t<\tau\}\cap\{\tau=\infty\}}$ and both the terms on the right-hand side vanish a.s. as $t\to\infty$ because $(\xi_t)_{t\geqslant 0}$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-supermartingale. The thesis follows. □

3.33. LEMMA. *Under the hypotheses of Proposition 3.18, for every $\nu \in \mathbb{N}$, the following estimate holds:*

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big] \leqslant \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}} \frac{\mu q_j}{1+\lambda_j/\lambda}\right)^{\nu} + k\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(1 - \left(\sum_{j\in\mathcal{P}} \frac{\mu q_j}{1+\lambda_j/\lambda}\right)^{\nu}\right),$$

where $k := \Big(\sum_{j\in\mathcal{P}} \frac{\mu q_j}{\lambda_j+\lambda}\Big) \Big/ \Big(1 - \Big(\sum_{j\in\mathcal{P}} \frac{\mu q_j}{1+\lambda_j/\lambda}\Big)\Big)$.

PROOF. Fix $\nu\in\mathbb{N}$. In view of (Vd2), pointwise on $\big\{s\in[\tau_i,\tau_{i+1}[\big\}$, $i\in\mathbb{N}$, we get

$$V_{\sigma(\tau_i)}(x(s)) \leqslant V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(s-\tau_i)} + \frac{\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_{\sigma(\tau_i)}}\Big(1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(s-\tau_i)}\Big).$$

From (Vd3) at $t=\tau_{i+1}$ we have

$$V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) \leqslant \mu V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)} + \frac{\mu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_{\sigma(\tau_i)}}\Big(1-\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\Big).$$

Iterating the above inequality from $i=0$ through $i=\nu-1$, we get

$$\begin{aligned}
(3.34)\quad V_{\sigma(\tau_\nu)}(x(\tau_\nu)) \leqslant \mu^\nu V_{\sigma(0)}(x_0)\prod_{i=0}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)} +&\\
\mu^\nu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\sum_{i=0}^{\nu-1}\frac{\mu^{-i}}{\lambda_{\sigma(\tau_i)}}&\left(\prod_{j=i+1}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right).
\end{aligned}$$

In view of (Vd1), the expectation of the first term on the right-hand side of (3.34) can be written as

$$\mathsf{E}\left[\mu^\nu V_{\sigma(0)}(x_0)\prod_{i=0}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\right] \leqslant \alpha_2(\|x_0\|)\prod_{i=0}^{\nu-1}\mathsf{E}\Big[\mu\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\Big],$$

and utilizing (EH1)-(EH3) we arrive at

$$\mathsf{E}\left[\mu^\nu V_{\sigma(0)}(x_0)\prod_{i=0}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\right] = \alpha_2(\|x_0\|)\prod_{i=0}^{\nu-1}\left(\sum_{j\in\mathcal{P}}\int_0^\infty \lambda\mu q_j\,\mathrm{e}^{-\lambda_j s}\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right)$$

$$= \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}}\frac{\mu\lambda q_j}{\lambda+\lambda_j}\right)^\nu. \tag{3.35}$$

Also, from (Ed2) we have

$$\mathsf{E}\left[\frac{1}{\lambda_{\sigma(\tau_i)}}\left(\prod_{j=i+1}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right)\right]$$

$$= \mathsf{E}\left[\prod_{j=i+1}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\left(\frac{1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}}{\lambda_{\sigma(\tau_i)}}\right)\right]$$

$$= \prod_{j=1+1}^{\nu-1}\mathsf{E}\left[\mathrm{e}^{-\lambda_{\sigma(\tau_{j+1})}S_{j+1}}\right]\mathsf{E}\left[\frac{1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\right]. \tag{3.36}$$

Now for each $j\in\mathbb{N}$ we have

$$\mathsf{E}\left[\mathrm{e}^{-\lambda_{\sigma(\tau_j)}S_{j+1}}\right] = \int_0^\infty \sum_{k\in\mathcal{P}}\lambda q_k\,\mathrm{e}^{-(\lambda_k+\lambda)s}\,\mathrm{d}s = \sum_{k\in\mathcal{P}}\frac{\lambda q_k}{\lambda_k+\lambda}, \tag{3.37}$$

and for each $i\in\mathbb{N}$,

$$\mathsf{E}\left[\frac{1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\right] = \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k} - \sum_{k\in\mathcal{P}}\frac{1}{\lambda_k}\int_0^\infty q_k\lambda\,\mathrm{e}^{-(\lambda_k+\lambda)s}\,\mathrm{d}s$$

$$= \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k} - \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\cdot\frac{\lambda}{\lambda_k+\lambda}$$

$$= \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k+\lambda}. \tag{3.38}$$

Substituting the right-hand sides of (3.38) and (3.37) back into (3.36) we get

$$\mathsf{E}\left[\frac{1}{\lambda_{\sigma(\tau_i)}}\left(\prod_{j=i+1}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right)\right]$$

$$= \left(\sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k+\lambda}\right)\left(\sum_{k\in\mathcal{P}}\frac{\lambda q_k}{\lambda_k+\lambda}\right)^{\nu-i-1}.$$

Therefore, the expectation of the second term on the right-hand side of (3.34) is given by

$$\mu^\nu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(\sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k+\lambda}\right)\sum_{i=0}^{\nu-1}\mu^{-i}\left(\sum_{j\in\mathcal{P}}\frac{\lambda q_j}{\lambda_j+\lambda}\right)^{\nu-i-1},$$

which after simplification stands as

$$\mu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(\sum_{j\in\mathcal{P}}\frac{q_j}{\lambda_j+\lambda}\right)\frac{1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j}{1+\lambda_j/\lambda}\right)^\nu}{1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j}{1+\lambda_j/\lambda}\right)}.$$

Considering the definition of k in the hypothesis, we finally arrive at

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big] \leqslant \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j}{1+\lambda_j/\lambda}\right)^{\nu} + k\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j}{1+\lambda_j/\lambda}\right)^{\nu}\right), \tag{3.39}$$

as asserted. □

The following Lemma is needed for the proof of Proposition 3.19.

3.40. Lemma. *Under the hypotheses of Proposition 3.19, for every $\nu \in \mathbb{N}$, the following estimate holds:*

$$\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big] \leqslant \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu} + k'\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu}\right),$$

where $k' := \mu\left(\sum_{j\in\mathcal{P}}\frac{q_j}{\lambda_j}\left(1-\frac{1-\mathrm{e}^{-\lambda_j T}}{\lambda_j T}\right)\right)\Big/\left(1-\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)$.

Proof. Fix $\nu \in \mathbb{N}$. In view of (Vd2), pointwise on $\big\{s \in [\tau_i, \tau_{i+1}[\big\}$, $i \in \mathbb{N}$, we get

$$V_{\sigma(\tau_i)}(x(s)) \leqslant V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(s-\tau_i)} + \frac{\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_{\sigma(\tau_i)}}\left(1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(s-\tau_i)}\right).$$

From (Vd3) at $t = \tau_{i+1}$ we have

$$V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) \leqslant \mu V_{\sigma(\tau_i)}(x(\tau_i))\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)} + \frac{\mu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)}{\lambda_{\sigma(\tau_i)}}\left(1-\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\right).$$

Iterating the above inequality from $i = 0$ through $i = \nu - 1$, we get

$$V_{\sigma(\tau_\nu)}(x(\tau_\nu)) \leqslant \mu^\nu V_{\sigma(0)}(x_0)\prod_{i=0}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)} + \mu^\nu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\sum_{i=0}^{\nu-1}\frac{\mu^{-i}}{\lambda_{\sigma(\tau_i)}}\left(\prod_{j=i+1}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right). \tag{3.41}$$

The expectation of the first term on the right-hand side of (3.41) can be evaluated as

$$\begin{aligned}\mathsf{E}\left[\mu^\nu V_{\sigma(0)}(x_0)\prod_{i=0}^{\nu-1}\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\right] &\leqslant \alpha_2(\|x_0\|)\prod_{i=0}^{\nu-1}\mathsf{E}\Big[\mu\,\mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}\Big]\\ &= \alpha_2(\|x_0\|)\prod_{i=0}^{\nu-1}\left(\sum_{j\in\mathcal{P}}\int_0^T \mu q_j\,\mathrm{e}^{-\lambda_j s}\frac{1}{T}\,\mathrm{d}s\right) = \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu},\end{aligned} \tag{3.42}$$

where we have utilized (Vd1) and (UH1)-(UH3) in successive steps. Also, from (Ud2) we have

$$\begin{aligned}
&\mathsf{E}\left[\frac{1}{\lambda_{\sigma(\tau_i)}}\left(\prod_{j=i+1}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right)\right] \\
&\qquad = \mathsf{E}\left[\prod_{j=i+1}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} \left(\frac{1- \mathrm{e}^{-\lambda_{\sigma(\tau_i)}(\tau_{i+1}-\tau_i)}}{\lambda_{\sigma(\tau_i)}}\right)\right] \\
&\qquad = \prod_{j=1+1}^{\nu-1} \mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_{j+1})}S_{j+1}}\Big]\, \mathsf{E}\Big[\frac{1- \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\Big]. \qquad (3.43)
\end{aligned}$$

Now for each $j \in \mathbb{N}$ we have

$$\mathsf{E}\Big[\mathrm{e}^{-\lambda_{\sigma(\tau_j)}S_{j+1}}\Big] = \int_0^T \sum_{k\in\mathcal{P}} q_k\, \mathrm{e}^{-\lambda_k s}\frac{1}{T}\,\mathrm{d}s = \sum_{k\in\mathcal{P}} \frac{q_k\left(1- \mathrm{e}^{-\lambda_k T}\right)}{\lambda_k T}, \qquad (3.44)$$

and for each $i \in \mathbb{N}$,

$$\begin{aligned}
\mathsf{E}\Big[\frac{1- \mathrm{e}^{-\lambda_{\sigma(\tau_i)}S_{i+1}}}{\lambda_{\sigma(\tau_i)}}\Big] &= \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k} - \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\int_0^T \frac{1}{T}\,\mathrm{e}^{-\lambda_k s}\,\mathrm{d}s \\
&= \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k} - \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\cdot\frac{1- \mathrm{e}^{-\lambda_k T}}{\lambda_k T} \\
&= \sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\left(1-\frac{1- \mathrm{e}^{-\lambda_k T}}{\lambda_k T}\right). \qquad (3.45)
\end{aligned}$$

Substituting the right-hand sides of (3.45) and (3.44) back into (3.43) we get

$$\begin{aligned}
&\mathsf{E}\left[\frac{1}{\lambda_{\sigma(\tau_i)}}\left(\prod_{j=i+1}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)} - \prod_{j=i}^{\nu-1} \mathrm{e}^{-\lambda_{\sigma(\tau_j)}(\tau_{j+1}-\tau_j)}\right)\right] \\
&\qquad = \left(\sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\left(1-\frac{1- \mathrm{e}^{-\lambda_k T}}{\lambda_k T}\right)\right)\left(\sum_{k\in\mathcal{P}}\frac{q_k\left(1- \mathrm{e}^{-\lambda_k T}\right)}{\lambda_k T}\right)^{\nu-i-1}.
\end{aligned}$$

Therefore, the expectation of the second term on the right-hand side of (3.41) is given by

$$\mu^\nu \chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(\sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\left(1-\frac{1- \mathrm{e}^{-\lambda_k T}}{\lambda_k T}\right)\right)\sum_{i=0}^{\nu-1}\mu^{-i}\left(\sum_{j\in\mathcal{P}}\frac{q_j\left(1- \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu-i-1},$$

which after simplification stands as

$$\mu\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(\sum_{k\in\mathcal{P}}\frac{q_k}{\lambda_k}\left(1-\frac{1- \mathrm{e}^{-\lambda_k T}}{\lambda_k T}\right)\right)\frac{1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1- \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu}}{1-\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1- \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}}.$$

Considering the definition of k' in the hypothesis, we finally arrive at

$$\begin{aligned}
\mathsf{E}\big[V_{\sigma(\tau_\nu)}(x(\tau_\nu))\big] &\leqslant \alpha_2(\|x_0\|)\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1- \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu} \\
&\quad + k'\chi\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\left(1-\left(\sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1- \mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}\right)^{\nu}\right), \qquad (3.46)
\end{aligned}$$

as asserted. ∎

§ 3.5.2. Proofs of the main results.

Proof of Proposition 3.15. The argument is divided into five steps for convenience.

Step 1. We first recall the equivalence between the hypotheses ((Vd1)-(Vd2)) and the condition (iii) in Proposition 3.10; we shall employ the condition in (iii) in the aforesaid proposition in the arguments that follow, and in particular, the function ρ appearing below is one is obtained from this particular condition. Let us fix an essentially bounded disturbance input signal d with $\|d\|_{\mathbb{R}_{\geqslant 0}} > 0$, an initial condition $x_0 \in \mathbb{R}^n$, and define the open sets $C_1 := \bigl\{z \in \mathbb{R}^n \bigm| \|z\| < \rho\bigl(\|d\|_{\mathbb{R}_{\geqslant 0}}\bigr)\bigr\}$ and $C_2 := \bigl\{z \in \mathbb{R}^n \bigm| \|z\| < \eta\rho\bigl(\|d\|_{\mathbb{R}_{\geqslant 0}}\bigr)\bigr\}$, where $\eta > 0$ is chosen such that $\alpha_1\bigl(\eta\rho\bigl(\|d\|_{\mathbb{R}_{\geqslant 0}}\bigr)\bigr) > 2\alpha_2\bigl(\rho\bigl(\|d\|_{\mathbb{R}_{\geqslant 0}}\bigr)\bigr)$. First let us suppose that $x_0 \notin C_1$, the other case being similar. Let us define the following sequence of random times taking values in $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$:

$$
\begin{aligned}
&\check{t}_1 := \inf\{t > 0 \mid x(t) \in C_1\},\\
&\hat{t}_1 := \inf\{t > \check{t}_1 \mid x(t) \in \mathbb{R}^n \smallsetminus C_2\},\\
&\dots\\
&\check{t}_{i+1} := \inf\{t > \hat{t}_i \mid x(t) \in C_1\} \quad \text{for } i \in \mathbb{N},\\
&\hat{t}_{i+1} := \inf\{t > \check{t}_{i+1} \mid x(t) \in \mathbb{R}^n \smallsetminus C_2\} \quad \text{for } i \in \mathbb{N},
\end{aligned} \tag{3.47}
$$

where it is understood that if any $\check{t}_i$ or $\hat{t}_i$ is ∞, then each of the definitions which follow it in the above sequence is set to ∞. We note that both $\check{t}_i$ and $\hat{t}_i$ are $[0, \infty]$-valued $(\mathfrak{F}_t)_{t \geqslant 0}$-optional times.

Step 2. Pointwise on $\bigl\{t, \tau_i \in [0, \check{t}_1[\bigr\}$ we have $x(t), x(\tau_i) \in \mathbb{R}^n \smallsetminus C_1$, and from (Vd2)-(Vd3) we get

$$
\begin{aligned}
&\frac{\partial V_{\sigma(t)}}{\partial x}(x(t)) f_{\sigma(t)}(x(t), d(t)) \leqslant -\lambda_\circ V_{\sigma(t)}(x(t)),\\
&\forall\, k \in \mathcal{P} \quad V_{\sigma(\tau_i)}(x(\tau_i)) \leqslant \mu V_k(x(\tau_i)).
\end{aligned} \tag{3.48}
$$

This yields $V_{\sigma(\tau_j)}(x(\tau_j)) \leqslant \mu V_{\sigma(\tau_{j-1})}(x(\tau_{j-1}))\, \mathrm{e}^{-\lambda_\circ(\tau_j - \tau_{j-1})}$, and iterating this inequality for $j = 1$ through $j = N_\sigma(0, t)$ we get

$$
\forall\, t \in [0, \check{t}_1[\quad V_{\sigma(t)}(x(t)) \leqslant V_{\sigma(0)}(x_0) \mu^{N_\sigma(0,t)}\, \mathrm{e}^{-\lambda_\circ t}.
$$

Taking expectations on both sides and applying (Vd1) we get

$$
\begin{aligned}
\mathsf{E}\bigl[V_{\sigma(t)}(x(t)) \mathbf{1}_{\{t \in [0, \check{t}_1[\}}\bigr] &\leqslant \alpha_2(\|x_0\|) \mathsf{E}\Bigl[\mathrm{e}^{-\lambda_\circ t} \mu^{N_\sigma(0,t)} \mathbf{1}_{\{t \in [0, \check{t}_1[\}}\Bigr]\\
&\leqslant \alpha_2(\|x_0\|) \mathsf{E}\Bigl[\mathrm{e}^{-\lambda_\circ t} \mu^{N_\sigma(0,t)}\Bigr]\\
&= \alpha_2(\|x_0\|)\, \mathrm{e}^{-\lambda_\circ t} \left(\sum_{k=0}^{\infty} \mu^k \mathsf{P}\bigl(N_\sigma(0,t) = k\bigr)\right)\\
&= \alpha_2(\|x_0\|)\, \mathrm{e}^{-\left(\lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda}\right)t}.
\end{aligned}
$$

Therefore,

$$
\mathsf{E}\bigl[V_{\sigma(t)}(x(t)) \mathbf{1}_{\{t \in [0, \check{t}_1[\}}\bigr] \leqslant \beta(\|x_0\|, t) \quad \forall\, t \geqslant 0,
$$

where $\beta(r, s) := \alpha_2(r)\, \mathrm{e}^{-\lambda s}$, $\lambda := \lambda_\circ + \widetilde{\lambda} - \mu\overline{\lambda} > 0$ by (Gd3).

Step 3. Pointwise on $\{t,\tau_i \in [\check{t}_j, \hat{t}_j[\} \cap \{\check{t}_j < \infty\}$ for $i,j \in \mathbb{N}$ we have $x(t), x(\tau_i) \in C_2$ by (3.47) and continuity of $x(\cdot)$. Employing (Vd1) leads to

$$\forall\, t \in [\check{t}_j, \hat{t}_j[\quad V_{\sigma(t)}(x(t)) \leqslant \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big) \quad \text{whenever } \hat{t}_j < \infty.$$

Taking expectations we arrive at

$$\begin{aligned}\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{\hat{t}_j<\infty\}\cap\{t\in[\check{t}_j,\hat{t}_j[\}}\Big] &\leqslant \mathsf{E}\Big[\alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\mathbf{1}_{\{\hat{t}_j<\infty\}\cap\{t\in[\check{t}_j,\hat{t}_j[\}}\Big] \\ &= \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\mathsf{P}\big(\{t \in [\check{t}_j,\hat{t}_j[\} \cap \{\hat{t}_j<\infty\}\big).\end{aligned}$$

Step 4. Pointwise on $\{t,\tau_i \in [\hat{t}_j, \check{t}_{j+1}[\} \cap \{\hat{t}_j < \infty\}$ for $i,j \in \mathbb{N}$ we have

$$\begin{aligned}&\frac{\partial V_{\sigma(t)}}{\partial x}(x(t)) f_{\sigma(t)}(x(t), d(t)) \leqslant -\lambda_\circ V_{\sigma(t)}(x(t)), \\ &\forall\, k \in \mathcal{P} \quad V_{\sigma(\tau_i)}(x(\tau_i)) \leqslant \mu V_k(x(\tau_i))\end{aligned} \tag{3.49}$$

in view of (Vd2)-(Vd3). Therefore,

$$\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\hat{t}_j<\infty\}}\Big] \leqslant \mathsf{E}\Big[\sup_{s\geqslant 0} V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big]. \tag{3.50}$$

From Lemma 3.32 we know that the process $\big(V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\big)_{s\geqslant 0}$ is a nonnegative $\big(\mathfrak{F}_{\hat{t}_j+s}\big)_{s\geqslant 0}$-potential. The facts that $\check{t}_{j+1}$ is an $\big(\mathfrak{F}_{\hat{t}_j+s}\big)_{s\geqslant 0}$-optional time and that $\big(\mathfrak{F}_{\hat{t}_j+s}\big)_{s\geqslant 0}$ is right-continuous are needed to justify the application of Lemma 3.32; these technical details are given in in Step 4a-f in Appendix B. Since the function $r \longmapsto \mu^{1+r}\overline{\lambda} - \widetilde{\lambda} - (1+r)\lambda_\circ$ is a continuous function, by (Gd3) there exists a $\delta > 0$ such that $\mu^{1+\delta}\overline{\lambda} - \widetilde{\lambda} - (1+\delta)\lambda_\circ < 0$. It can be proved that the process $\big(V^{1+\delta}_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\big)_{s\geqslant 0}$ is also an $\big(\mathfrak{F}_{\hat{t}_j+s}\big)_{s\geqslant 0}$-potential; see, for instance, Appendix B where we attempt to obtain better estimates. Therefore,

$$\begin{aligned}&\mathsf{E}\Big[\sup_{s\geqslant 0}\Big(V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big)\Big] \\ &\leqslant \mathsf{E}\Big[\sup_{s\geqslant 0}\Big(\alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\mu^{N_\sigma(\hat{t}_i+s)}\,\mathrm{e}^{-\lambda_\circ s}\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big)\Big] \\ &= \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\mathsf{E}\Big[\sup_{s\geqslant 0}\Big(\mu^{N_\sigma(\hat{t}_i+s)}\,\mathrm{e}^{-\lambda_\circ s}\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big)\Big] \\ &= \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\mathsf{E}\Big[\int_0^\infty \mathsf{P}^{\mathfrak{F}_{\hat{t}_j}}\Big(\sup_{s\geqslant 0}\Big(\mu^{N_\sigma(\hat{t}_i+s)}\,\mathrm{e}^{-\lambda_\circ s}\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big) > u\Big)\,\mathrm{d}u\Big] \\ &= \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big) \\ &\quad\cdot \mathsf{E}\Big[\int_0^\infty \mathsf{P}^{\mathfrak{F}_{\hat{t}_j}}\Big(\sup_{s\geqslant 0}\Big(\mu^{N_\sigma(\hat{t}_i+s)(1+\delta)}\,\mathrm{e}^{-\lambda_\circ(1+\delta)s}\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big) > u^{1+\delta}\Big)\,\mathrm{d}u\Big].\end{aligned}$$

We know that $\mu^{N_\sigma(\hat{t}_j,\hat{t}_j)(1+\delta)} = 1$ a.s.; therefore, from [52, Exercise 1.15, Chapter 2] it follows that

$$\begin{aligned}&\mathsf{E}\Big[\int_0^\infty \mathsf{P}^{\mathfrak{F}_{\hat{t}_j}}\Big(\sup_{s\geqslant 0}\Big(\mu^{N_\sigma(\hat{t}_i+s)(1+\delta)}\,\mathrm{e}^{-\lambda_\circ(1+\delta)s}\mathbf{1}_{\{\hat{t}_j+s<\check{t}_{j+1}\}\cap\{\hat{t}_j<\infty\}}\Big) > u^{1+\delta}\Big)\,\mathrm{d}u\Big] \\ &\leqslant \int_0^\infty \Big(\frac{1}{u^{1+\delta}} \wedge 1\Big)\,\mathrm{d}u \\ &\leqslant (1+1/\delta) < \infty.\end{aligned}$$

Therefore, from (3.50) it follows that

$$\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\hat{t}_j<\infty\}}\Big] \leqslant \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big),$$

where we let $\gamma(r) := (1+1/\delta)\alpha_2(\eta\rho(r))$.

Step 5. It remains to define the sequence $(T_i)_{i\in\mathbb{N}}$ of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times. Letting $T_{2k-1} := \check{t}_k$ and $T_{2k} := \hat{t}_k$, $k\in\mathbb{N}$, we see from Steps 2 through 4 that

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[T_{i-1},T_i[\}\cap\{T_{i-1}<\infty\}}\big] \leqslant \beta(\|x_0\|,t)\vee\gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big),$$

which proves the claim. □

3.51. Remark. We believe that the system (3.8) is ISS-M under the hypotheses of Proposition 3.15; this is the statement of Conjecture 3.17. The basic problem with establishing ISS-M along the lines of the above proof of Proposition 3.15 lies with Step 4; we have not pulled out the probability of the event $\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\hat{t}_j<\infty\}$, and therefore, the summation $\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] = \sum_{i=1}^{\infty}\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_j,\check{t}_{j+1}[\}\cap\{\check{t}_{j+1}<\infty\}}\Big]$ (as given in (B.9) in Appendix B) diverges. And even if we employ the total probability formula and pull out the aforesaid probability, the conditional expectation obtained is difficult to compute because of the dependence of the supremum of the process $\big(V_{\sigma(\hat{t}_j+s)}(x(\hat{t}_j+s))\big)_{s\geqslant 0}$ on $\check{t}_{j+1}$. We have attempted in Appendix B to get improved bounds by initializing a supermartingale at the instant that $x(\cdot)$ exits the outer ball C_2, which only depends on the number of switches after that instant. This procedure weakens the dependence of the supermartingale constructed on the instant $\check{t}_{j+1}$, but since $(N_\sigma(t,0))_{t\geqslant 0}$ affects both instant $\check{t}_{j+1}$ and the process, they are not completely independent. An explicit characterization of the dependence relationship would enable one to finish the proof. ◁

Proof of Proposition 3.18. This follows directly from Lemma 3.33. Indeed, let $\beta(r,s) := \alpha_2(r)\eta^s$, where $\eta := \sum_{j\in\mathcal{P}}\frac{\mu q_j}{1+\lambda_j/\lambda}$, $\gamma(r) := k\chi(r)$, and k is as defined in Lemma 3.33. An application of (Vd1) on the left-hand side of (3.39) immediately proves the assertion. □

Proof of Proposition 3.19. This follows directly from Lemma 3.40. Indeed, letting $\beta(r,s) := \alpha_2(r)\eta^s$, where $\eta := \sum_{j\in\mathcal{P}}\frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T}$, $\gamma(r) := k'\chi(r)$, and k' is as defined in Lemma 3.40. An application of (Vd1) on the left-hand side of (3.46) immediately proves the assertion. □

Proof of Theorem 3.29. Fix a measurable and essentially bounded input signal d with $\|d\|_{\mathbb{R}_{\geqslant 0}} > 0$, and let $\overline{d} := \rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)$. Let us define the closed ball of radius $r>0$ to be $\overline{B}(r) := \big\{y\in\mathbb{R}^n \bigm| \|y\|\leqslant r\big\}$. Since $\overline{B}(r)$ is a compact subset of $\mathbb{R}^n$ and the function $V(i,\cdot)$ is continuous for each $i\in\mathcal{P}$, we have

$$c_1(r) := \sup_{(i,x)\in\mathcal{P}\times\overline{B}(r)} \mathcal{L}V(i,x) < \infty$$

and

$$c_2(r) := \lambda_\circ\left(\sup_{(i,x)\in\mathcal{P}\times\overline{B}(r)} V(i,x)\right) \leqslant \lambda_\circ\alpha_2(r) < \infty.$$

If $x \in \overline{B}(\rho(\|d\|))$, then from (ISS2) it follows that $\mathcal{L}V(i,x) \leqslant c_1(\rho(\|d\|)) + c_2(\rho(\|d\|))$; therefore,

$$\mathcal{L}V(i,x)\mathbf{1}_{\overline{B}(\rho(\|d\|))}(x) \leqslant \big(c_1(\rho(\|d\|)) + c_2(\rho(\|d\|))\big)\mathbf{1}_{\overline{B}(\rho(\|d\|))}(x).$$

And $\mathcal{L}V(i,x)\left(1 - \mathbf{1}_{\overline{B}(\rho(\|d\|))}(x)\right) \leqslant -\lambda_\circ V(i,x)$ directly from (ISS2). Hence,

$$\mathcal{L}V(i,x) \leqslant -\lambda_\circ V(i,x) + C(\rho(\|d\|))\mathbf{1}_{\overline{B}(\rho(\|d\|))}(x), \tag{3.52}$$

where $C(r) := c_1(r) + c_2(r)$. We define $T_m := \inf\big\{t > 0 \,\big|\, x(t) \notin \overline{B}(m)\big\}$ for $m \in \mathbb{N}$. Since x is a continuous $\mathbb{R}^n$-valued process and the set $\overline{B}(m)$ is a compact subset of $\mathbb{R}^n$, we know that T_m is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time. Also, since $V(i,\cdot)$ is continuously differentiable for each $i \in \mathcal{P}$, for each $t \geqslant 0$ we have

$$\mathsf{E}\left[\int_0^{t\wedge m\wedge T_m} |\mathcal{L}V(\sigma(s),x(s))|\,\mathrm{d}s\right] \leqslant m\left(\sup_{(i,x)\in\mathcal{P}\times\overline{B}(m)} |\mathcal{L}V(i,x)|\right) < \infty.$$

Consider the function $g : \mathbb{R}_{\geqslant 0} \times \mathcal{P} \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ defined by $g(t,i,x) := \mathrm{e}^{\lambda_\circ t}V(i,x)$. Applying the extended generator $\mathcal{L}$ to this function, Itô's formula and the estimate in (3.52) give us

$$\begin{aligned}\mathcal{L}g(t,i,x) &= \lambda_\circ\,\mathrm{e}^{\lambda_\circ t}V(i,x) + \mathrm{e}^{\lambda_\circ t}\mathcal{L}V(i,x)\\ &\leqslant \lambda_\circ\,\mathrm{e}^{\lambda_\circ t}V(i,x) + \mathrm{e}^{\lambda_\circ t}\big(-\lambda_\circ V(i,x) + C(\overline{d})\mathbf{1}_{\overline{B}(\overline{d})}(x)\big)\\ &= C(\overline{d})\,\mathrm{e}^{\lambda_\circ t}\mathbf{1}_{\overline{B}(\overline{d})}(x). \end{aligned}\tag{3.53}$$

We define the process $\xi(t) := \mathrm{e}^{\lambda_\circ t}V(\sigma(t),x(t))$ for $t \geqslant 0$; note that $\xi(t)$ is a function of the Markov process $\big(\mathrm{e}^{\lambda_\circ t},\sigma(t),x(t)\big)_{t\geqslant 0}$. Since $\sigma(0)$ and $x(0)$ are completely known, $\xi(0) = V(\sigma(0),x(0))$ is a nonnegative deterministic number. By definition of T_m and Doob's optional sampling theorem [**29**, Chapter 1, Theorem 3.22], it follows that the stopped process

$$\left(\xi(t\wedge m\wedge T_m) - \int_0^{t\wedge m\wedge T_m} \mathcal{L}g(s,\sigma(s),x(s))\,\mathrm{d}s\right)_{t\geqslant 0}$$

is a uniformly integrable $(\mathfrak{F}_t)_{t\geqslant 0}$-martingale for each $m \in \mathbb{N}$, and therefore

$$\mathsf{E}\left[\xi(t\wedge m\wedge T_m) - \int_0^{t\wedge m\wedge T_m} \mathcal{L}g(s,\sigma(s),x(s))\,\mathrm{d}s\right] = \xi(0).$$

By definition of T_m we have $\mathsf{E}\Big[\int_0^{t\wedge m\wedge T_m} |\mathcal{L}g(s,\sigma(s),x(s))|\,\mathrm{d}s\Big] < \infty$ for each $t > 0$. Therefore, transposing the expected value of the integral to the right-hand side is justified, and in view of the definition of the process ξ and the inequality (3.53) we get

$$\begin{aligned}&\mathsf{E}\Big[\mathrm{e}^{\lambda_\circ(t\wedge m\wedge T_m)}V\big(\sigma(t\wedge m\wedge T_m),x(t\wedge m\wedge T_m)\big)\Big]\\ &\qquad= \xi(0) + \mathsf{E}\left[\int_0^{t\wedge m\wedge T_m} \mathcal{L}g(s,\sigma(s),x(s))\,\mathrm{d}s\right]\\ &\qquad\leqslant \xi(0) + C(\overline{d})\mathsf{E}\left[\int_0^{t\wedge m\wedge T_m} \mathrm{e}^{\lambda_\circ s}\mathbf{1}_{\overline{B}(\overline{d})}(x(s))\,\mathrm{d}s\right]\\ &\qquad\leqslant V(\sigma(0),x(0)) + C(\overline{d})\mathsf{E}\left[\int_0^{t\wedge m\wedge T_m} \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s\right]. \end{aligned}\tag{3.54}$$

By continuity of x it follows that $\lim_{m\to\infty} T_m = \infty$; therefore $\int_0^{t\wedge m\wedge T_m} \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s$ monotonically increases to $\int_0^t \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s$ a.s. as $m \to \infty$. Also, for each m we have $\mathsf{E}\Big[\int_0^{t\wedge m\wedge T_m} \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s\Big] < \infty$. Therefore, the monotone convergence theorem applies and we get

$$\lim_{m\to\infty} \mathsf{E}\left[\int_0^{t\wedge m\wedge T_m} \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s\right] = \mathsf{E}\left[\int_0^t \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s\right]. \tag{3.55}$$

For each $t > 0$, the function $\mathrm{e}^{\lambda_\circ(t\wedge m\wedge T_m)} V\big(\sigma(t\wedge m\wedge T_m), x(t\wedge m\wedge T_m)\big)$ has finite expectation; therefore, applying Fatou's Lemma keeping in mind that $(T_m)_{m\in\mathbb{N}}$ is a monotonically increasing sequence, we get

$$\begin{aligned}\mathsf{E}\big[\mathrm{e}^{\lambda_\circ t} V(\sigma(t), x(t))\big] &= \mathsf{E}\Big[\liminf_{m\to\infty} \mathrm{e}^{\lambda_\circ(t\wedge m\wedge T_m)} V\big(\sigma(t\wedge m\wedge T_m), x(t\wedge m\wedge T_m)\big)\Big]\\ &\leqslant \liminf_{m\to\infty} \mathsf{E}\Big[\mathrm{e}^{\lambda_\circ(t\wedge m\wedge T_m)} V\big(\sigma(t\wedge m\wedge T_m), x(t\wedge m\wedge T_m)\big)\Big]. \end{aligned}\tag{3.56}$$

Letting $m \to \infty$ and applying the monotone convergence theorem on the right-hand and Fatou's lemma on the left-hand side of (3.54), from (3.55) and (3.56) we obtain

$$\begin{aligned}\mathsf{E}\big[\mathrm{e}^{\lambda_\circ t} V(\sigma(t), x(t))\big] &\leqslant V(\sigma(0), x(0)) + C(\overline{d}) \int_0^t \mathrm{e}^{\lambda_\circ s}\,\mathrm{d}s\\ &\leqslant V(\sigma(0), x(0)) + \frac{C(\overline{d})}{\lambda_\circ}\big(\mathrm{e}^{\lambda_\circ t} - 1\big).\end{aligned}$$

By definition of $C(\overline{d})$ and (ISS1) the last inequality gives us

$$\begin{aligned}\mathsf{E}\big[V(\sigma(t), x(t))\big] &\leqslant \mathrm{e}^{-\lambda_\circ t} V(\sigma(0), x(0)) + \frac{C(\overline{d})}{\lambda_\circ}\big(1 - \mathrm{e}^{-\lambda_\circ t}\big)\\ &\leqslant \mathrm{e}^{-\lambda_\circ t}\alpha_2(\|x_0\|) + \frac{1}{\lambda_\circ} C(\overline{d}).\end{aligned}$$

From the definition of the extended generator and the hypothesis on the function V it is clear that $\mathcal{L}V(i, 0) = 0$, and hence the function c_1 can be upper-bounded by a class-$\mathcal{K}_\infty$ function, say c_1', by employing standard arguments. Since $c_2(r) = \lambda_\circ \alpha_2(r)$, it follows that $C(r) \leqslant c_1'(r) + c_2(r)$, and $c_1' + c_2$ is a class-$\mathcal{K}_\infty$ function. Since $\beta(r, s) := \alpha_2(r)\,\mathrm{e}^{-\lambda_\circ s}$ is a class-$\mathcal{KL}$ function, the ISS-M property of (3.8) follows. □

§ 3.6. Concluding Remarks and Future Work

We have provided several sets of sufficient conditions for ISS in $\boldsymbol{L}_1$ estimates at switching instants of (3.8), and sufficient conditions for ISS-M of (3.8) under Markovian assumptions on the switching signal σ. The estimates obtained at switching instants may be viewed, due to continuity of $x(\cdot)$, as randomly sampled estimates which closely resemble the behavior of the continuous process $(x(t))_{t\geqslant 0}$. Considering the locally Lipschitz property of the family $\{f_i\}_{i\in\mathcal{P}}$ and essential boundedness of the disturbance input, we believe that Conjecture 3.17 is true, although there seem to be nontrivial issues associated with the method of solution we have pursued here. An effort in this direction is carried out in Appendix B. The absence of any special structure, however, precludes the possibility of applying standard procedures.

We also believe that (3.8) is ISS-M under the hypotheses of Proposition 3.18 or Proposition 3.19. It is quite evident that the crux lies in interpolating the ISS in $\boldsymbol{L}_1$ estimates at switching instants to an arbitrary time t, and since the switching instants are random, countably many interpolations would be needed for each t. A complete development of the approach pursued here will definitely contribute much to the field. Also, one of the difficulties lies with the way the total probability formula is applied; for instance, in (3.23), the sum on the right-hand side is not in general finite. One clear option is to get tighter bounds of the conditional expectation of $V_{\sigma(t)}(x(t))$ given the event $\{N_\sigma(t, 0) = \nu\}$ for $\nu \in \mathbb{N}$, compared to what we obtained in Remark 3.22. However, the relevant conditional density is in general difficult to compute; for instance, in the case of switching signals belonging to class UH, it is the ν-fold convolution of the uniform-(T) density which is difficult to handle analytically.

A second alternative is to develop arguments in the spirit of *Brownian motion excursions straddling a given time* [**52**, p. 488] for general processes. These arguments allow one to get estimates of a Brownian motion between two random times, the first of which is not optional. The proof of Proposition 3.15 can be carried out under the hypotheses of Proposition 3.18 or Proposition 3.19, and one can clearly envision that what we actually need are methods that yield estimates of the Lyapunov functions during excursions of the process x between the two balls C_1 and C_2 constructed there. Such arguments for Markov processes are rather delicate and demanding; even if we assume that σ is Markovian, they would not apply to our case because the process $\left(V_{\sigma(t)}(x(t))\right)_{t \geqslant 0}$ is not Markovian.

CHAPTER 4

CONTROL SYNTHESIS

§ 4.1. Introduction to the Synthesis Problem

Our goal in this chapter is to synthesize feedback control functions for stabilization (in a suitable stochastic sense) of randomly switched systems with control inputs. There are two distinct ingredients which govern the dynamical behavior of such a system, namely, the feedback control function and the switching signal σ. If the statistical characteristics of σ cannot be changed, then stabilization must be achieved with the aid of the control function alone. The feedback control function itself may be of two different kinds, namely, one that depends on each subsystem at each instant of time, or one that has no knowledge of σ at any instant of time.

The setting of this chapter is as follows. Throughout §4.2 we assume that the randomly switched system under consideration consists of a finite number N of subsystems, and each subsystem possesses a control input. A random càdlàg switching signal σ which selects the index of the active subsystem at each instant of time generates the randomly switched system from the family. First we consider the case in which the controller is subsystem-dependent, i.e., the controller has information about the state of σ at each time instant. The analysis results in Chapter 2 and universal formulas for feedback stabilization of nonlinear systems play important roles in our synthesis methodology. We give a control design methodology which provides the required control functions whenever there is a universal formula available for feedback stabilization of the individual subsystems. Second we consider the case in which the controller is subsystem-independent; i.e., the controller has no information about the state of σ at any time instant. Once again the analysis results in Chapter 2 and universal formulas for feedback stabilization play important roles. We mention that the feature that Theorem 2.10 and Theorem 2.12 do not require each subsystem to be stable is of importance here, for designing a single controller which simultaneously stabilizes every subsystem is difficult in general. In §4.3 we consider external disturbance input signals affecting the subsystems as well.

We shall restrict ourselves to controllers which render the closed loop switched system GAS a.s. The required modifications to be made to the results if one is interested in GAS-M instead of GAS a.s. are straightforward, in view of the close relationship between the sufficient conditions for GAS a.s. and GAS-M that we established in Chapter 2.

It is also possible to look at the switching signal as an extra control input, if its statistical characteristics can be modified either offline or online. We shall show that for some interesting and general classes of random switching signals, it is always possible to achieve desired stability properties of the closed loop system if the statistical characteristics of the switching signals can be arbitrarily altered. In this context, we may recall Remarks 2.17 and 2.18, where we observed that even if some (but not all) subsystems are unstable, for two interesting classes of switching signals it is always possible to attain GAS a.s. of the switched system by suitably manipulating the (stationary) temporal probability distribution of σ on the index set $\mathcal{P}$. This last case is presumably of lesser importance, and will not be pursued here.

On the external stability front, we shall follow the same procedure as in the case of internal stability, and the corresponding results are provided in §4.3.

§ 4.2. Internal Stability in Closed Loop

In this section we present our results dealing with internal stabilization of randomly switched systems with control inputs. We distinguish two cases, namely, the first in which the controller is mode-dependent, and the second in which the controller is mode-independent. This distinction may also be viewed as being based on whether the controller has complete knowledge of the switching signal σ at every instant of time or not. From an implementation point of view, the first case corresponds to the situation in which N controllers are designed, one for each of the N subsystems, and each controller is placed in closed loop with its respective subsystem. The second case corresponds to the situation in which one controller is designed for the entire family of N subsystems, and this controller is placed in closed loop with the complete switched system.

Pictorially the two different architectures are as shown in Figures 1 and 2. In the first case the subsystems and the corresponding controllers are represented as P_i and C_i $(i = 1, 2)$, respectively, while in the second case there is a central controller C for each of the subsystems P_i. It is important to keep in mind that the switched system in either case is *not* generated from subsystems evolving separately over time by the switching signal. Figures 1 and 2 are purely pictorial depictions.

§ 4.2.1. Mode-dependent controllers. In this subsection we design feedback controllers for the system (4.1) that possess complete knowledge of the active subsystem (mode) at each instant of time. In other words, the controller depends on the value of σ.

Consider the affine-in-control switched system:

$$\dot{x} = f_\sigma(x) + \sum_{j=1}^{m} g_{\sigma,j}(x)u_j, \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0, \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state, u_j, $j = 1, \ldots, m$, are the (scalar) control inputs, f_i and $g_{i,j}$ are smooth vector fields on $\mathbb{R}^n$, with $f_i(0) = 0, g_{i,j}(0) = 0$, for each $i \in \mathcal{P}, j \in \{1, \ldots, m\}$. Let $\mathcal{U}$ be the set where

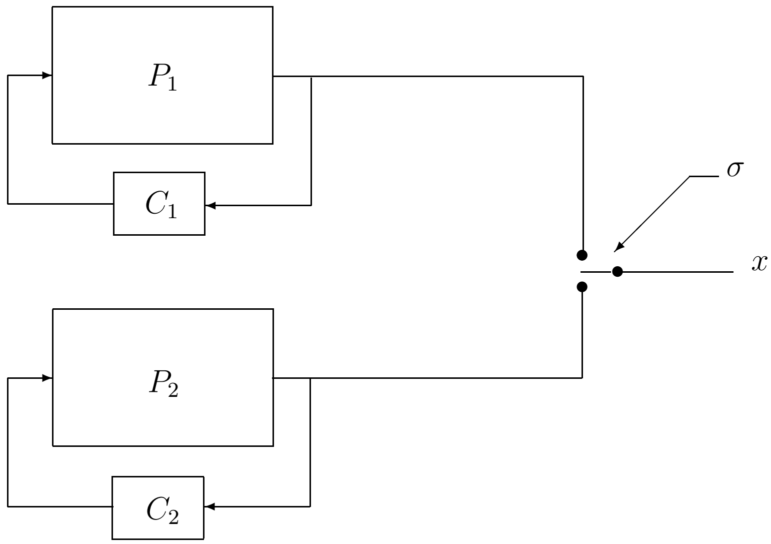


FIGURE 1. Controller architecture: At each t the value of $\sigma(t)$ is completely known.

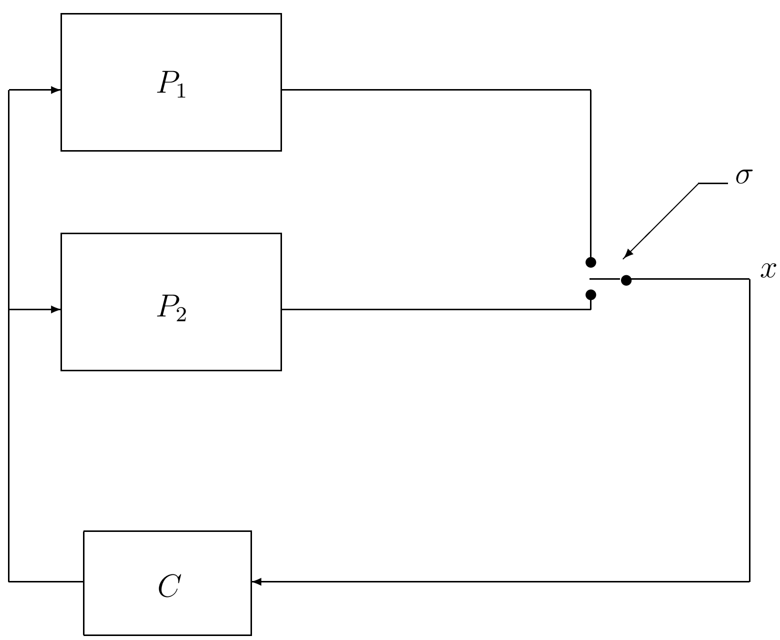


FIGURE 2. Controller architecture: The Value of σ is completely unknown.

the control $u := [u_1, \ldots, u_m]^{\mathrm{T}}$ takes its values. For the moment, we let $\mathcal{U}$ be a subset of $\mathbb{R}^m$ containing the origin; later we shall consider the case when $\mathcal{U}$ is a more general set, e.g., a Minkowski ball in $\mathbb{R}^m$. With a feedback control function $\overline{u}_\sigma(x) = [u_{\sigma,1}(x), \ldots, u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed loop system stands as

$$\dot{x} = f_\sigma(x) + \sum_{j=1}^{m} g_{\sigma,j}(x)\overline{u}_{\sigma,j}(x), \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0. \tag{4.2}$$

We let the switching signal σ be a stochastic process as defined at the beginning of Chapter 2, and let $x_0 \neq 0$. Our objective is to choose a control function $\overline{u}_\sigma$ such that (4.2) is GAS a.s. We shall appeal to our analysis results in Chapter 2 to attain this objective.

For a differentiable function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a vector field $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we let $\mathrm{L}_f h$ denote the Lie derivative of the function h along the vector field f, i.e., $\mathrm{L}_f h(x) := \frac{\partial h}{\partial x}(x)f(x)$.

Let us describe the controller design methodology. A universal formula for stabilization of control-affine nonlinear systems was first constructed in [**56**], for the control taking values in $\mathcal{U} = \mathbb{R}^m$. The articles [**39**],[**40**], and [**41**] provide universal formulas for bounded controls, positive controls, and controls restricted to Minkowski balls, respectively. Combining the analysis results in §2.3 and universal formulas established in the aforementioned articles, it is possible to synthesize controllers $\overline{u}_\sigma$ for (4.1), such that the closed loop system (4.2) is GAS a.s. Recall that three different types of switching signals were considered in §2.3; the corresponding hypotheses appear in Definition 2.7. We obtain one synthesis scheme for each type of control set $\mathcal{U}$ and switching signal σ; Theorem 4.3 below provides a typical illustration of such a result. A complete recipe to obtain other such results is provided in Remark 4.5. We note that as the hypotheses of Theorem 4.3 indicate, our controller design methodology presupposes the existence of a suitable control-Lyapunov function for each subsystem. Recall that a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ is a control-Lyapunov function [**38**, §3] for the i-th subsystem of (4.1) if it is continuously differentiable, positive definite, and $\inf_{u\in\mathcal{U}}\big(\mathrm{L}_{f_i}V(x) + \mathrm{L}_{g_i}V(x)u\big) < 0$ for all $x \neq 0$.

4.3. Theorem. *Consider the system (4.1), with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ is of class EH, and there exists a family $\{V_i\}_{i\in\mathcal{P}}$ of continuously differentiable nonnegative functions on $\mathbb{R}^n$, such that*

(C1) *(V1) of Assumption 2.3 holds;*

(C2) *(V3) of Assumption 2.3 holds;*

(C3) $\exists\, \lambda_i \in \Lambda = \mathbb{R},\ i \in \mathcal{P}$, *such that* $\forall\, x \in \mathbb{R}^n \smallsetminus \{0\}$ *and* $\forall\, i \in \mathcal{P}$

$$\inf_{u\in\mathcal{U}}\left\{\mathrm{L}_{f_i}V_i(x)+\lambda_i V_i(x)+\sum_{j=1}^{m}u_j \mathrm{L}_{g_{i,j}}V_i(x)\right\} < 0;$$

(C4) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ *such that if* $x(\neq 0)$ *satisfies* $\|x\| < \delta$, *then* $\exists\, u \in \mathbb{R}^m,\ \|u\| < \varepsilon$, *such that* $\forall\, i \in \mathcal{P}$

$$\mathrm{L}_{f_i}V_i + \sum_{j=1}^{m} u_j \cdot \mathrm{L}_{g_{i,j}}V_i \leqslant -\lambda_i V_i;$$

(C5) *((E3), (E4)) holds.*

Then the feedback control function

$$\overline{u}_\sigma(x) = [k_{\sigma,1}(x), \ldots, k_{\sigma,m}(x)]^{\mathrm{T}},$$

where

$$k_{i,j}(x) := -\mathrm{L}_{g_{i,j}} V_i(x) \cdot \varphi\Big(\overline{W}_i(x), \widetilde{W}_i(x)\Big) \tag{4.4a}$$

$$\overline{W}_i(x) := \mathrm{L}_{f_i} V_i(x) + \lambda_i V_i(x), \tag{4.4b}$$

$$\widetilde{W}_i(x) := \sum_{j=1}^{m} \big(\mathrm{L}_{g_{i,j}} V_i(x)\big)^2, \tag{4.4c}$$

and

$$\varphi(a,b) := \begin{cases} \dfrac{a + \sqrt{a^2+b^2}}{b} & \text{if } b \neq 0,\\ 0 & \text{otherwise,} \end{cases} \tag{4.4d}$$

renders (4.2) GAS *a.s.*

The condition (C3) above means that V_i is a control-Lyapunov function for the i-th subsystem.

PROOF. The proof relies on the construction of the universal formula in [**56**]. Fix $t \in \mathbb{R}_{\geqslant 0}$. If $x \neq 0$, applying the definition of φ, we get

$$\begin{aligned} &\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \sum_{j=1}^{m} k_{\sigma(t),j}(x) \mathrm{L}_{g_{\sigma(t),j}} V_{\sigma(t)}(x) \\ &= \mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) - \widetilde{W}_{\sigma(t)}(x) \cdot \varphi\Big(\overline{W}_{\sigma(t)}(x), \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^2\Big) \\ &= -\lambda_{\sigma(t)} V_{\sigma(t)}(x) - \sqrt{\big(\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x)\big)^2 + \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^4} \\ &< -\lambda_{\sigma(t)} V_{\sigma(t)}(x). \end{aligned}$$

Since t is arbitrary, we conclude that the above inequality holds for all $t \in \mathbb{R}_{\geqslant 0}$. Note that by (C3), if $x \in \bigcap_{j=1}^{m} \ker\big(\mathrm{L}_{g_{i,j}} V_i\big)$ for any $i \in \mathcal{P}$, we automatically have $\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \lambda_{\sigma(t)} V_{\sigma(t)}(x) < 0$.

The above arguments, in conjunction with (C1) and (C2) enable us to conclude that the family $\{V_i\}_{i\in\mathcal{P}}$ satisfies Assumption 2.3 for the closed loop system (4.2) and $\Lambda = \mathbb{R}$. (C5) ensures that (E3) and (E4) hold, for the closed-loop system (4.2). Since σ is of class EH, (E2) holds as well. Hence, it follows from Theorem 2.10 that (4.2) is GAS a.s. as asserted. □

4.5. REMARK. Theorem 4.3 can be modified to suit a different control set $\mathcal{U}$ and a different type of switching signal σ using the following simple recipe. First, recall from the discussion preceding Theorem 4.3 that $\mathcal{U}$ may be the entire $\mathbb{R}^m$, or the nonnegative orthant of $\mathbb{R}^m$, or a bounded subset of $\mathbb{R}^m$, or a Minkowski ball in $\mathbb{R}^m$; σ may belong to any of the classes G, EH or UH. Now suppose that a control set $\mathcal{U}$ and a switching signal σ among these possibilities is given to us. Then:

(R1) (C1) and (C2) remain unchanged;

(R2) the given $\mathcal{U}$ replaces the $\mathcal{U} = \mathbb{R}^m$ in Theorem 4.3;

(R3) if the given σ is of class G, then the slow switching condition in (1) of Definition 2.7 replaces the hypotheses on a switching signal of class EH, the pair ((E3), (E4)) appearing in hypothesis (C5) is replaced by (G3), and $\Lambda = \mathbb{R}$ appearing in (C3) is replaced by the set $\Lambda = \{\lambda_\circ\}$;

(R4) if the given σ is of class UH, then the condition in (3) of Definition 2.7 replaces the hypotheses on a switching signal of class EH, and the pair ((E3), (E4)) appearing in hypothesis (C5) is replaced by (U3);

(R5) the universal formula corresponding to the given $\mathcal{U}$ replaces the one given in (4.4). ◁

4.6. Remark. For linear systems it is possible to design controllers in a simpler fashion, still using the analysis results in Chapter 2. For illustration purposes, suppose that σ is of class G. Consider the following linear version of (4.1):

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0, \tag{4.7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^n \times \mathbb{R}^m$. Let us try to find a control $\overline{u}_\sigma(x) = K_{\sigma(t)}x$, where K_i is a $(m\times n)$ matrix for each $i \in \mathcal{P}$ that achieves GAS a.s. of (4.7) in closed loop. For a square matrix A of dimension n, with eigenvalues $\{\lambda_i\}_{i=1}^n$, let $\rho_1(A) := \min_i |\Re(\lambda_i)|$ and $\rho_2(A) := \max_i |\Re(\lambda_i)|$. Suppose that there exists a set of $(m \times n)$ matrices $\{K_i\}_{i\in\mathcal{P}}$ and a number $\lambda_\circ > 0$, such that the symmetric positive definite solution set $\{M_i\}_{i\in\mathcal{P}}$ to the linear matrix inequalities

$$(A_i + B_iK_i)^{\mathrm{T}} M_i + M_i (A_i + B_iK_i) \leqslant -\lambda_\circ M_i \tag{4.8}$$

satisfies the following estimate:

$$\mu := \frac{\max_{i\in\mathcal{P}} \rho_2(M_i)}{\min_{i\in\mathcal{P}} \rho_1(M_i)} < \frac{\lambda_\circ + \widetilde{\lambda}}{\overline{\lambda}}. \tag{4.9}$$

Standard and efficient computational tools for solving the linear matrix inequalities like (4.8) exist, see e.g., [13]; therefore, finding the set $\{K_i\}_{i\in\mathcal{P}}$ is not difficult. It is clear that we have found a family of Lyapunov functions $V_i(x) = x^{\mathrm{T}}M_ix$, $i \in \mathcal{P}$, for which (V1) and (V3) hold by the definitions of the V_i's, and (V2) holds due to (4.8). Also, observe that the set Λ is $\{\lambda_\circ\}$, and (4.9) is just (G3). It follows by Theorem 2.8 that the control function $\overline{u}_\sigma$ defined above renders (4.7) GAS a.s. in closed loop. ◁

§ 4.2.2. Mode-independent controllers. In this subsection we design feedback controllers for the system (4.1) that possess no knowledge of the active subsystem (mode) at any instant of time. In other words, the controller is independent of the value of σ. In contrast to Subsection 4.2.1, now the controllers depend only on the state of the system.

Consider the affine in control switched system (4.1). Let $\overline{u}(x) = [u_1(x), \ldots, u_m(x)]^{\mathrm{T}}$ be a feedback control function, with which the closed loop system stands as

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{\sigma,i}(x)\overline{u}_i(x), \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0. \tag{4.10}$$

We let the switching signal σ be a stochastic process as defined in §2.2, and let $x_0 \neq 0$. Our objective is to choose a control function $\overline{u}$ such that (4.10) is GAS a.s. Once again we shall appeal to our analysis results in Chapter 2.

It is possible to draw some connections between our objective above and *uniform stabilization* of a switched system in the deterministic context. Recall that a (deterministic) switched system (2.2) is said to be *globally uniformly asymptotically stable* (GUAS) [**37**, Chapter 2] (uniformity is with respect to switching signals) if the solution of (2.2) corresponding to every deterministic switching signal σ is globally asymptotically stable. Two representative theorems providing conditions for GUAS of (2.2) are as follows.

- [**37**, Theorem 2.1] *Suppose that there exist a continuously differentiable function* $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ *and a function* $\alpha \in \mathcal{K}$, *such that* $\mathrm{L}_{f_i} V(x) \leqslant -\alpha(\|x\|)$ *for every* $i \in \mathcal{P}$ *and* $x \in \mathbb{R}^n$. *Then* (2.2) *is* GUAS.
- [**37**, Theorem 2.6] *If every subsystem in the family* (2.1) *is* GAS, *each* f_i *is continuously differentiable, and* $[f_i, f_j] = 0 \quad \forall\, i, j \in \mathcal{P}$,[1] *then the switched system* (2.2) *is* GUAS.

If the subsystems possess control inputs, *uniformly stabilizing controllers* seek to render the closed loop switched system GUAS. Clearly, a uniformly stabilizing controller must achieve *simultaneous stabilization* of the family of subsystems. Now, if we can design a uniformly stabilizing controller, then the switching signal does not affect stability of the closed loop system, which, therefore, is *always* globally asymptotically stable. But uniform stabilization is difficult and involves restrictive conditions for a general family of nonlinear subsystems; in particular, satisfying the hypotheses of the two conditions mentioned above is clearly nontrivial. Our results given below, however, do not propose uniformly stabilizing controllers; they provide control functions under weaker conditions, which achieve GAS a.s. of the closed loop switched system. Once again we assume that the switching signal belongs to any one of the class G, EH, or UH.

In order to avoid constructing uniformly stabilizing controllers, we need to utilize the available statistical characteristics of the switching signal. To this end, suppose first that σ belongs to class G. From Remark 2.15 it follows that (for a fixed chosen controller) if even one subsystem is unstable in closed loop, then it is not possible to ensure GAS a.s. of the closed loop switched system. Therefore, to attain our objective for a class G switching signal σ, it is necessary to find a controller such that every subsystem is GAS with a sufficiently large stability margin in closed loop with this controller. This is evidently quite a restrictive design specification, similar to uniform stabilization discussed in the preceding paragraph. One possible way to relax this restrictive state of affairs is to make allowances for possibly unstable subsystems in closed loop with a controller. In view of Remark 2.15, now we need to impose additional structure on the switching signal, for instance, by considering switching

[1] Here $[\cdot,\cdot]$ denotes the Lie bracket.

signals of class EH or UH. Let us assume that σ is of class EH. As mentioned in §4.1, we shall assume that the stationary probability distribution $\{q_i\}_{i\in\mathcal{P}}$ is fixed. Now our objective is to find a family of control-Lyapunov functions $\{V_i\}_{i\in\mathcal{P}}$ and a control function $\overline{u}$, such that for every $i \in \mathcal{P}$ we have $\mathrm{L}_{(f_i+g_i\overline{u})}V_i(x) \leqslant -\lambda_i V_i(x)$, $\lambda_i \in \mathbb{R}$, and moreover the conditions (E3) and (E4) of Theorem 2.10 are satisfied. We noted in Remark 2.17 that not all λ_i need to be positive; satisfying ((E3),(E4)) would be enough to ensure that the switched system is GAS a.s. in closed loop. Similarly, if σ is of class UH, for GAS a.s. of the closed loop switched system it suffices to find a controller $\overline{u}$ such that for every $i \in \mathcal{P}$, $\mathrm{L}_{(f_i+g_i\overline{u})}V_i(x) \leqslant -\lambda_i V_i(x)$, where the family $\{\lambda_i\}_{i\in\mathcal{P}}$ satisfies (U3) of Theorem 2.12. This strategy leads us to the following results.

4.11. Theorem. *Consider the system* (4.1) *with* $\mathcal{U} = \mathbb{R}^m$. *Suppose that* σ *is of class EH, and there exists a family* $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$, *of continuously differentiable functions such that*

(CP1) *(V1) and (V3) of Assumption 2.3 holds;*

(CP2) *there exists a control function* $\overline{u} : \mathbb{R}^n \longrightarrow \mathcal{U}$ *and* $\lambda_i \in \mathbb{R}$, $i \in \mathcal{P}$, *such that* $\mathrm{L}_{f_i+g_i\overline{u}}V_i(x) \leqslant -\lambda_i V_i(x)$ *for every* $i \in \mathcal{P}$, $x \in \mathbb{R}^n$;

(CP3) $\lambda_i + \lambda > 0 \quad \forall\, i \in \mathcal{P}$;

(CP4) $\displaystyle\sum_{j\in\mathcal{P}} \frac{\mu q_j}{(1+\lambda_j/\lambda)} < 1.$

Then $\overline{u}$ *renders* (4.1) GAS *a.s. in closed loop.*

Proof. The assertion follows immediately by first observing that the closed loop system is (4.10), and then applying Theorem 2.10 to (4.10). Indeed, note that (E2) holds by our assumption on σ, ((CP3),(CP4)) is identical to ((E3),(E4)), and the pair ((CP1),(CP2)) ensures that (E1) holds for (4.10). □

4.12. Theorem. *Consider the system* (4.1) *with* $\mathcal{U} = \mathbb{R}^m$. *Suppose that* σ *is of class UH, and there exists a family* $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$ *of continuously differentiable functions such that*

(CU1) *(V1) and (V3) of Assumption 2.3 holds;*

(CU2) *there exists a control function* $\overline{u} : \mathbb{R}^n \longrightarrow \mathcal{U}$, *such that* $\mathrm{L}_{f_i+g_i\overline{u}}V_i(x) \leqslant -\lambda_i V_i(x)$ *for every* $i \in \mathcal{P}$, $x \in \mathbb{R}^n$;

(CU3) $\displaystyle\sum_{j\in\mathcal{P}} \frac{\mu q_j\left(1-\mathrm{e}^{-\lambda_j T}\right)}{\lambda_j T} < 1.$

Then $\overline{u}$ *renders* (4.1) GAS *a.s. in closed loop.*

Proof. The assertion follows immediately by first observing that the closed loop system is (4.10), and then applying Theorem 2.12 to (4.10). Indeed, note that (U2) holds by our assumption on σ, (CU3) is identical to (U3), and the pair ((CU1),(CU2)) ensures that (U1) holds for (4.10). □

4.13. EXAMPLE. Let $\mathcal{P} = \{1,2\}$ and consider the family of planar control systems

$$f_1(x) + g_1(x)u = \begin{bmatrix} x_1 - x_1^3 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} u, \qquad f_2(x) + g_2(x)u = \begin{bmatrix} x_2 \\ -x_1/2 + x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

Let σ be a switching signal of class UH with $T = 0.5$, $q_1 = 0.8$, and $q_2 = 0.25$, and let us assume that we do not have information about σ at any instant of time t. Let us further suppose that the only state available for feedback is x_1, and our objective is to find a control function $\overline{u}(x) = kx_1$, where k is a constant, such that the closed loop switched system is GAS a.s.

We observe that the first subsystem has multiple equilibrium points for zero input, but by choosing an appropriate k it is possible to render the origin the unique equilibrium point of the closed loop subsystem. Note also that the first system is zero-input unstable at the origin, and no matter what k is, the second subsystem is always unstable. (The latter fact follows immediately from the fact that if we choose a Lyapunov function $V(x) = 0.5(x_1^2 + x_2^2)$, then $\mathrm{L}_{f_2+g_2\overline{u}}V(x) = kx_1^2 + x_2^2$, from which we see that the conditions of Chetaev's theorem [**61**, Chapter 5, Theorem 99] are fulfilled for every $k \in \mathbb{R}$; this implies instability of the origin.) Therefore, without a control input the switched system is unstable at the origin.

With $k = -3$ the closed loop first subsystem becomes identical to the second subsystem of Example 2.20. Let us choose $V_1(x) = V_2(x) = 0.5x_1^2 + x_2^2$, which gives us $\mu = 1$. We immediately see that

$$\mathrm{L}_{f_1+g_1\overline{u}}V_1(x) \leqslant -V_1(x),$$
$$\mathrm{L}_{f_2+g_2\overline{u}}V_2(x) \leqslant 2V_2(x),$$

which means $\lambda_1 = 1$ and $\lambda_2 = -2$. We see that (CU1)-(CU2) of Theorem 4.12 are satisfied. It is also easy to see that

$$\frac{q_1(1 - \mathrm{e}^{-T})}{T} + \frac{q_2(1 - \mathrm{e}^{2T})}{-2T} < 1 = \frac{1}{\mu},$$

which implies that (CU3) holds. We conclude that with $k = -3$ the switched control system under consideration is GAS a.s. by Theorem 4.12. △

§ 4.3. External Stability in Closed Loop

In this section we present our results dealing with external stabilization of randomly switched systems with exogenous disturbance and control inputs. Once again we shall distinguish two different controller architectures: one in which the controller is mode-dependent, and the other in which the controller is mode-independent.

§ 4.3.1. Mode-dependent controllers. Consider the affine in control switched system perturbed by a disturbance signal

$$\dot{x} = f_\sigma(x, d) + \sum_{j=1}^{m} g_{\sigma,j}(x)u_j, \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0, \tag{4.14}$$

where $x \in \mathbb{R}^n$ is the state, u_j, $j = 1, \ldots, m$, are the (scalar) control inputs, $f_i : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $g_{i,j} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are smooth maps for each $i \in \mathcal{P}$, $j \in \{1, \ldots, m\}$. Let $\mathcal{U}$ be the set where the control $u := [u_1, \ldots, u_m]^{\mathrm{T}}$ takes its values. For the moment we let $\mathcal{U}$ be a subset of $\mathbb{R}^m$ containing the origin. With a feedback control function $\overline{u}_\sigma(x) := [u_{\sigma,1}(x), \ldots, u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed loop system stands as

$$\dot{x} = f_\sigma(x, d) + \sum_{j=1}^{m} g_{\sigma,j}(x)\overline{u}_{\sigma,j}(x), \qquad x(0) = x_0, \quad t \geqslant 0. \tag{4.15}$$

We let the switching signal σ be a stochastic process as defined in §2.2, and let $x_0 \neq 0$. Our goal is to choose a control function $\overline{u}_\sigma$ so that (4.15) is input-to-state stable in a suitable stochastic sense. We shall appeal to our analysis results of Chapter 3 and universal formulas for ISS disturbance attenuation to achieve this objective.

Universal feedback control functions attaining ISS disturbance attenuation for nonlinear systems affected by disturbances and possessing control inputs were constructed in [**38**]. The results in [**38**] rely on universal formulas for asymptotic feedback stabilization of nonlinear systems; the results apply to systems in which the control takes values in various restricted control sets (e.g., the control sets considered in §4.2) and a universal formula is available. In our results below we utilize off-the-shelf universal feedback control functions for ISS disturbance attenuation from [**38**], and in doing so we observe that it is possible to construct one synthesis scheme for each type of $\mathcal{U}$ for which a universal formula is available, and each type of σ for which an external stability result is available from Chapter 3. Proposition 4.16 below is a typical illustration of such a result. Note that this controller design methodology presupposes the existence of a suitable ISS control-Lyapunov function for each subsystem of (4.14). Recall that a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$ is an ISS control-Lyapunov function [**38**, Definition 1] for the i-th subsystem of (4.14) if it is continuously differentiable, positive definite, and there exist class-$\mathcal{K}_\infty$ functions α and χ such that $\inf_{u \in \mathcal{U}} \left(\frac{\partial V}{\partial x}(x) f_i(x, d) + \mathrm{L}_{g_i} V(x) u \right) \leqslant -\alpha(\|x\|) + \chi(\|d\|)$ for all x, d.

4.16. Proposition. *Consider the system (4.14) with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ is of class EH and there exists a family $\{V_i\}_{i \in \mathcal{P}}$ of continuously differentiable functions on $\mathbb{R}^n$ such that*

(Cd1) *(Vd1) of Assumption 3.9 holds;*

(Cd2) *(Vd3) of Assumption 3.9 holds;*

(Cd3) *$\exists\, \alpha, \chi \in \mathcal{K}_\infty$, $\exists\, \lambda_i \in \Lambda = \mathbb{R}$, $i \in \mathcal{P}$, such that $\forall\, x \in \mathbb{R}^n \smallsetminus \{0\}$, $\forall\, d \in \mathbb{R}^k$ and $\forall\, i \in \mathcal{P}$*

$$\inf_{u \in \mathcal{U}} \left\{ \frac{\partial V_i}{\partial x}(x) f_i(x, d) + 3\lambda_i V_i(x) + \sum_{j=1}^{m} \mathrm{L}_{g_{i,j}} V_i(x) u_j \right\} \leqslant \chi(\|d\|);$$

(Cd4) *$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that if $x (\neq 0)$ satisfies $\|x\| < \delta$, then $\exists\, u \in \mathbb{R}^m$, $\|u\| < \varepsilon$, such that $\forall\, i \in \mathcal{P}$*

$$\max_{d \in \mathbb{R}^k} \left\{ \frac{\partial V_i}{\partial x}(x) f_i(x, d) - \chi(\|d\|) \right\} + \sum_{j=1}^{m} \mathrm{L}_{g_{i,j}} V_i(x) u_j \leqslant -\lambda_i V_i(x);$$

(Cd5) *((Ed3)-(Ed4)) holds.*

Then under the feedback control function

$$\overline{u}_\sigma(x) = \varphi\left(\overline{W}_\sigma(x), \widetilde{W}_\sigma^{\mathrm{T}}(x)\right) \tag{4.17}$$

the system (4.15) *satisfies an* ISS *in* $\boldsymbol{L}_1$ *estimate at switching instants, where the map* $\varphi : \mathbb{R}\times\mathbb{R}^m \longrightarrow \mathbb{R}$ *is defined by*

$$\varphi(a,b) := \begin{cases} -\dfrac{a+\sqrt{a^2+\|b\|^4}}{\|b\|^2}\, b & \text{if } b \neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

$\widetilde{W}_i(x) := \left[\mathrm{L}_{g_{i,1}}V_i(x), \ldots, \mathrm{L}_{g_{i,m}}V_i(x)\right]$, *and* $\overline{W}_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a map smooth away from* 0 *and continuous at* 0, *satisfying*

$$\begin{aligned} &\max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_i}{\partial x}(x) f_i(x,d) - \chi(\|d\|)\right\} + \lambda_i V_i(x) \leqslant \overline{W}_i(x)\\ &\qquad \leqslant \max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_i}{\partial x}(x) f_i(x,d) - \chi(\|d\|)\right\} + 2\lambda_i V_i(x)\end{aligned} \tag{4.18}$$

for all $x \in \mathbb{R}^n$, $i \in \mathcal{P}$.

Note that (Cd3) says that there exists a suitable ISS control-Lyapunov function for each subsystem.

PROOF. The proof relies heavily on the proof of [38, Theorem 3]. Fix $t \in \mathbb{R}_{\geqslant 0}$. If $x \neq 0$, then

$$\begin{aligned} &\frac{\partial V_{\sigma(t)}}{\partial x}(x) f_{\sigma(t)}(x,d) + \widetilde{W}_{\sigma(t)}(x)\overline{u}_{\sigma(t)}(x)\\ &\leqslant \max_{d\in\mathbb{R}^k}\left\{\frac{\partial V_{\sigma(t)}}{\partial x}(x) f_{\sigma(t)}(x,d) - \chi(\|d\|)\right\} + \widetilde{W}_{\sigma(t)}(x)\overline{u}_{\sigma(t)}(x) + \chi(\|d\|)\\ &\leqslant -\overline{W}_{\sigma(t)}(x) - \lambda_{\sigma(t)}V_{\sigma(t)}(x) + \widetilde{W}_{\sigma(t)}(x)\overline{u}_{\sigma(t)}(x) + \chi(\|d\|)\\ &\leqslant -\lambda_{\sigma(t)}V_{\sigma(t)}(x) + \chi(\|d\|)\end{aligned}$$

in view of (4.18) and since we have $\overline{W}_p(x) + \widetilde{W}_p(x)\overline{u}_p(x) < 0$ for all $x \neq 0$ by definition of φ. Since t is arbitrary, the above inequality holds for all $t \in \mathbb{R}_{\geqslant 0}$.

The above arguments in conjunction with ((Cd1),(Cd2)), and (Cd5) show that the hypotheses ((Ed1),(Ed4)) hold for (4.15). It follows from Proposition 3.18 that (4.15) satisfies (3.12) as claimed. ☐

4.19. REMARK. Proposition 4.16 can be modified to suit a different control set $\mathcal{U}$ and a different type of σ using the following recipe. First recall that $\mathcal{U}$ may be any one among $\mathbb{R}^m$, the nonnegative orthant of $\mathbb{R}^m$, a bounded subset of $\mathbb{R}^m$, and a Minkowski ball in $\mathbb{R}^m$; σ may belong to class EH or UH. Suppose that a $\mathcal{U}$ and a σ from among these possibilities is given to us. Then:

(Rd1) (Cd1) and (Cd2) remain unchanged;

(Rd2) the given $\mathcal{U}$ replaces the $\mathcal{U} = \mathbb{R}^m$ in Proposition 4.16;

(Rd3) if the given σ is of class UH, then the hypotheses in (3) of Definition 2.7 replace the hypotheses on a switching signal of class EH, the pair ((Ed3),(Ed4)) appearing in hypothesis (Cd5) is replaced by (Ud3);

(Rd4) the universal formula corresponding to the given $\mathcal{U}$ replaces the one given in (4.17). ◁

§ 4.3.2. Mode-independent controllers. Consider the affine-in-control switched system (4.14). Let $\overline{u}(x) = [u_1(x), \ldots, u_m(x)]^{\mathrm{T}}$ be a feedback control function, with which the closed loop system stands as

$$\dot{x} = f_\sigma(x, d) + \sum_{j=1}^{m} g_{\sigma,j}(x)\overline{u}_j(x), \qquad (x(0), \sigma(0)) = (x_0, \sigma_0), \quad t \geqslant 0. \tag{4.20}$$

We let the switching signal σ be a stochastic process as defined in §2.2, and let $x_0 \neq 0$. Our objective is to choose a control function $\overline{u}$ such that (4.20) is input-to-state stable in a suitable stochastic sense.

4.21. Proposition. *Consider the system (4.14) with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ is of class EH, and there exists a family $\{V_i\}_{i\in\mathcal{P}}$ of continuously differentiable functions such that*

(CEd1) *(Vd1) and (Vd3) of Assumption 3.9 holds;*

(CEd2) *there exists a control function $\overline{u} : \mathbb{R}^n \longrightarrow \mathcal{U}$, such that*

$$\frac{\partial V_i}{\partial x}(x)\big(f_i(x, d) + g_i(x)\overline{u}(x)\big) \leqslant -\lambda_i V_i(x) + \chi(\|d\|) \quad \forall\, i \in \mathcal{P}, \ \ \forall\, x \in \mathbb{R}^n;$$

(CEd3) $\lambda_i + \lambda > 0 \quad \forall\, i \in \mathcal{P}$;

(CEd4) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i}{(1 + \lambda_i/\lambda)} < 1.$

Then under the control $\overline{u}$ the system (4.14) satisfies an ISS *in $\boldsymbol{L}_1$ estimate at switching instants.*

Proof. The assertion follows immediately by first observing that the closed loop system is (4.20), and then applying Proposition 3.18 to (4.20). Indeed, note that (Ed2) holds by our assumption on σ, ((CEd3),(CEd4)) are identical to ((Ed3),(Ed4)), and ((CEd1),(CEd2)) together ensure that (Ed1) holds for (4.20). □

4.22. Theorem. *Consider the system (4.14) with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ is of class UH, and there exists a family $\{V_i\}_{i\in\mathcal{P}}$ of continuously differentiable functions such that*

(CUd1) *(Vd1) and (Vd3) of Assumption 3.9 holds;*

(CUd2) *there exists a control function $\overline{u} : \mathbb{R}^n \longrightarrow \mathcal{U}$, such that*

$$\frac{\partial V_i}{\partial x}(x)\big(f_i(x, d) + g_i(x)\overline{u}(x)\big) \leqslant -\lambda_i V_i(x) + \chi(\|d\|) \quad \forall\, i \in \mathcal{P}, \ \ \forall\, x \in \mathbb{R}^n;$$

(CUd3) $\displaystyle\sum_{i\in\mathcal{P}} \frac{\mu q_i\big(1 - \mathrm{e}^{-\lambda_i T}\big)}{\lambda_i T} < 1.$

Then under the control $\overline{u}$ the system (4.1) satisfies an ISS *in $\boldsymbol{L}_1$ estimate at switching instants.*

Proof. The assertion follows immediately by first observing that the closed loop system is (4.20), and then applying Proposition 3.19 to (4.20). Indeed, note that (Ud2) holds by our assumption on σ, (CUd3) is identical to (Ud3), and ((CUd1),(CUd2)) together ensure that (Ud1) holds for (4.20). □

§ 4.4. Concluding Remarks and Future Work

In this chapter we have derived explicit controller formulas for feedback stabilization of randomly switched systems. Standard off-the-shelf universal formulas for nonlinear feedback stabilization and the analysis results of Chapters 2 and 3 were the two basic ingredients of our controller design methodology. Two controller architectures were considered, one in which the controller has perfect information of the state of the switching signal, and the other in which it has no such information. In case of the former architecture, our approach has the additional advantage that designing a separate controller for the switched system is not necessary as long as suitable control-Lyapunov functions for each individual subsystem exist. Two directions of future work appear to be particularly interesting, namely, controller synthesis for attaining ISS-M in closed loop under Markovian switching, the corresponding analysis result being Theorem 3.29, and the design of controllers under partial information of σ, for instance, when an estimate $\hat{\sigma}$ is available at each instant of time.

APPENDIX A

MISCELLANEOUS PROOFS

§ A.1. A Proof of the Equivalence Between the Two Definitions of s-GAS-P

We provide a sketch of the equivalence between the two definitions of GAS-P mentioned in §2.3.3. The two definitions were:

- The system (2.2) is GAS-P if for every $\eta \in]0,1[$ there exists a function $\beta \in \mathcal{KL}$ such that $\mathsf{P}\big(\|x(t)\| \leqslant \beta(\|x_0\|, t)\ \ \forall\, t \geqslant 0\big) \geqslant 1-\eta$.
- The system (2.2) is GAS-P if the following two properties are simultaneously verified:
  (i) $\forall\, \eta \in]0,1[\ \ \forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\|x_0\| < \delta \implies \mathsf{P}\big(\sup_{t\geqslant 0} \|x(t)\| > \varepsilon\big) \leqslant \eta$;
  (ii) $\forall\, \eta' \in]0,1[\ \ \forall\, r, \varepsilon' > 0\ \exists\, T > 0$ such that $\|x_0\| < r \implies \mathsf{P}\big(\sup_{t\geqslant T} \|x(t)\| > \varepsilon'\big) \leqslant \eta'$.

Assuming the class-$\mathcal{KL}$ definition, one fixes $\eta \in]0,1[$ and gets a class-$\mathcal{KL}$ function β such that $\mathsf{P}\big(\|x(t)\| \leqslant \beta(\|x_0\|, t)\ \ \forall\, t \geqslant 0\big) \geqslant 1-\eta$. Then for every $\varepsilon > 0$ we take $\delta = \beta(\cdot, 0)^{-1}(\varepsilon)$ to get the property (i) (if $\beta(\cdot, 0) \in \mathcal{K}$, then we simply take a smaller ε for which the inverse is defined). To get the property (ii) for fixed $\eta' \in]0,1[$ and $r, \varepsilon' > 0$ it suffices to take T such that $\beta(r, T) < \varepsilon'$. Conversely, fixing an $\eta \in]0,1[$, we get a measurable set of measure at most $\eta/2$ such that (i) holds on its complement, and a measurable set of measure at most $\eta/2$ such that (ii) holds on its complement. Then (i) and (ii) hold simultaneously on a measurable set of measure at least $1-\eta$. Now the equivalence of global asymptotic stability in $\varepsilon-\delta$ form and class-$\mathcal{KL}$ form in the deterministic setting [**30**, Lemma 5.3] immediately shows that GAS-P holds.

§ A.2. Proof of Proposition 2.24

We need Egorov's theorem on almost uniform convergence of a sequence of measurable functions (see, e.g., [**18**, p. 243] for a proof), and an auxiliary Lemma on convergence of a product of a decaying exponential and a nonnegative monotone nondecreasing function.

A.1. THEOREM (Egorov). *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $(\Omega, \mathfrak{F}, \mathsf{P})$ and $g_n \to g$ a.s. Then for every $\varepsilon > 0$ there exists a measurable set A_ε with $\mathsf{P}\big(\Omega \smallsetminus A_\varepsilon\big) < \varepsilon$ such that $\big(g_n \mathbf{1}_{A_\varepsilon}\big)_{n\in\mathbb{N}}$ converges to $g\mathbf{1}_{A_\varepsilon}$ uniformly.*

PROOF OF PROPOSITION 2.24. Let us verify property (ii) assuming that (2.2) is GAS a.s. Fix $\eta, r, \varepsilon' > 0$ and $x_0 \in \mathbb{R}^n$ with $\|x_0\| < r$. Since $\{f_i\}_{i\in\mathcal{P}}$ is a finite set of locally Lipschitz vector fields, there exists $L_{\varepsilon'} > 0$ such that $\sup_{i\in\mathcal{P},\|x\|<\varepsilon'} \|f_i(x)\| \leqslant L_{\varepsilon'} \|x\|$. Let $c := \frac{\ln 2}{L_{\varepsilon'}}$, and define the sequence of time instants $(s_j)_{j\in\mathbb{N}\cup\{0\}}$ such that $s_0 := 0$ and $s_j - s_{j-1} = c$ for every $j \in \mathbb{N}$. By the (AS1) property of (2.2) we have $\mathsf{P}\big(\lim_{t\to\infty} \|x(t)\| = 0\big) = 1$, which also implies that $\mathsf{P}\big(\lim_{i\to\infty} \|x(s_i)\| = 0\big) = 1$. By Egorov's theorem there exists a measurable set A_η such that $\mathsf{P}\big(\Omega\smallsetminus A_\eta\big) < \eta$ and $\big(x(s_i)\mathbf{1}_{A_\eta}\big)_{i\in\mathbb{N}}$ uniformly converges to 0. The uniform convergence condition by definition implies that there exists $i_0 \in \mathbb{N}$ such that $\sup_{i\geqslant i_0}\big(\|x(s_i)\|\,\mathbf{1}_{A_\eta}\big) < \frac{\varepsilon'}{2}$. By construction of the sequence $(s_i)_{i\in\mathbb{N}}$ we must have $\|x(t)\|\,\mathbf{1}_{A_\eta} < \varepsilon'$ for all $t \geqslant s_{i_0}$ in view of continuity of $x(\cdot)$. To see this, fix a time $t' > s_{i_0}$. The construction of the sequence $(s_i)_{i\in\mathbb{N}}$ shows that there exists a $j(t') \in \mathbb{N}$ such that $t' \in [s_{j(t')-1}, s_{j(t')}[$. The local Lipschitz condition on the set of vector fields $\{f_i\}_{i\in\mathcal{P}}$ implies that

$$\|x(t')\|\,\mathbf{1}_{A_\eta} \leqslant \sup_{s\in[s_{j(t')-1},s_{j(t')}[} \|x(s)\|\,\mathbf{1}_{A_\eta} < \frac{\varepsilon'}{2}\,\mathrm{e}^{L_{\varepsilon'}\left(s-s_{j(t')}\right)} < \frac{\varepsilon'}{2}\,\mathrm{e}^{L_{\varepsilon'}c} = \varepsilon',$$

where the last equality follows from the definition of c. Since t' was arbitrary, the assertion follows. Since x_0 was arbitrary, to establish the property (ii) it only remains to show that the solutions restricted to A_η corresponding to all initial conditions x_0' with $\|x_0'\| < \|x_0\|$ are also asymptotically convergent. Since x_0 was arbitrary, (ii) follows. To establish (i), let us fix $\eta \in\,]0,1[$ and $\varepsilon > 0$. By (ii) there exists a $T > 0$ corresponding to $\eta' = \eta$, $r = 1$ and $\varepsilon' = \eta$ such that $\|x_0\| < 1$ implies that $\sup_{t\geqslant T} \|x(t)\|\,\mathbf{1}_{A_\eta} < \varepsilon$. The local Lipschitz condition on the set of vector fields $\{f_i\}_{i\in\mathbb{N}}$ guarantees the existence of a positive $\delta' > 0$ such that $\sup_{t\in[0,T]} \|x(t)\| < \varepsilon$ whenever $\|x_0\| < \delta$. Picking $\delta = 1 \wedge \delta'$ we see that $\|x_0\| < \delta$ implies that $\sup_{t\geqslant 0} \|x(t)\|\,\mathbf{1}_{A_\eta} < \varepsilon$. The implication is now completely established. □

§ A.3. Proof of Theorem 3.3

§ A.3.1. A comparison principle for switched systems with inputs. Let the switched system (3.8) be given. We consider scalar impulsive differential systems of the type

$$\text{(A.2)}\qquad \begin{cases} \dot\xi = \phi(t,\xi,\eta), \quad \xi(0) = \xi_0, \quad t \neq \tau_i, \\ \xi(\tau_i) = \psi_i(\xi(\tau_i^-)), \end{cases} \qquad i \geqslant 0, \quad t \geqslant 0;$$

where $\phi : \mathbb{R}^3_{\geqslant 0} \longrightarrow \mathbb{R}$ is a continuous function with $\phi(\cdot,0,0) \equiv 0$, $\psi_i : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ is continuous, nondecreasing and $\psi_i(0) = 0$ for all $i \geqslant 1$, τ_i are the instants of the impulses, $\tau_0 := 0$, and the sequence $(\tau_i)_{i\in\mathbb{N}}$ is identified with the sequence of jump instants of σ. The *solution of this system initialized at* $(0,\xi_0)$ *driven by an input* η is denoted by $\xi(t)$. We shall tacitly assume that this solution is unique; sufficient conditions for existence and uniqueness of solutions may be found in [**34**].

A.3. Lemma. *Suppose that there exists a system*

$$\text{(A.4)}\qquad \dot\xi = \phi(t,\xi,\eta), \qquad \xi(\tau) = \xi_\tau, \quad t \geqslant \tau \geqslant 0,$$

and an absolutely continuous function $\zeta : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}_{\geqslant 0}$ *satisfying*

$$\text{(A.5)}\qquad \dot\zeta \leqslant \phi(t,\zeta,\eta), \qquad \zeta(\tau) = \zeta_\tau, \quad \textit{for a.e. } t \geqslant \tau,$$

where ϕ is as in (A.2), η is the common input to (A.4) and (A.5). Then $\zeta_\tau \leqslant \xi_\tau \implies \zeta(t) \leqslant \xi(t) \quad \forall\, t \geqslant \tau$, where $\xi(\cdot)$ is the solution of (A.4).

Proof. If the assertion of the lemma is false, then the set $\mathcal{T} := \{t \geqslant \tau \mid \zeta(t) > \xi(t)\}$ is nonempty. Let $t_\star := \inf \mathcal{T}$. Then for $h > 0$ sufficiently small, we have $\zeta(t_\star + h) > \xi(t_\star + h)$ by continuity of $\zeta(t)$ and $\xi(t)$. It follows that

$$\frac{\zeta(t_\star + h) - \zeta(t_\star)}{h} > \frac{\xi(t_\star + h) - \xi(t_\star)}{h}.$$

Letting $h \downarrow 0$, we have $\dot{\zeta}(t_\star) > \dot{\xi}(t_\star)$ (η being the common input to (A.4) and (A.5)), which contradicts the differential inequality (A.5). The assertion follows. □

Suppose that there exist continuously differentiable functions $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that the condition (Vd1) of Assumption 3.9 holds. Suppose also that there exist a continuous function $U : \mathbb{R}^m \longrightarrow \mathbb{R}_{\geqslant 0}$, which serves as a Lyapunov-like function over the set of inputs, and a function $\alpha_3 \in \mathcal{K}_\infty$, such that $\forall\, u \in \mathbb{R}^m$,

(A.6) $$U(u) \leqslant \alpha_3(\|u\|).$$

These functions are utilized simultaneously below to arrive at a comparison principle for systems with inputs.

A.7. Lemma. *Consider the ith subsystem of the family (3.7). Suppose that there exist continuously differentiable functions $V_i : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{P}$, a continuous function $U : \mathbb{R}^m \longrightarrow \mathbb{R}_{\geqslant 0}$, and a system Σ of the type (A.2), such that $\forall\,(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$, the estimate*

(A.8) $$\frac{\partial V_i(x)}{\partial x} f_i(x) \leqslant \phi(t, V_i(x), U(u)), \qquad \text{for a.e. } t \geqslant \tau \geqslant 0,$$

holds, where $U(u)$ is the common input to Σ and (A.8). Then $V_i(\tau, x(\tau)) \leqslant \xi(\tau) \implies V_i(t, x(t)) \leqslant \xi(t)$ for all $t \geqslant \tau$, where $x(t)$ solves the system with index i in the family (3.7).

The following Lemma provides a comparison framework for switched systems with inputs, and the proof follows from Lemma A.3.

A.9. Lemma. *Consider the switched system (3.8). Suppose that there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, a family of continuously differentiable functions $\{V_i\}_{i\in\mathcal{P}}$, a continuous nonnegative function U, and a system Σ of the type (A.2), such that*

- (i) *the estimate (Vd1) of Assumption 3.9 and (A.6) hold for all $(x,d) \in \mathbb{R}^n \times \mathbb{R}^m$;*
- (ii) *$\frac{\partial V_i}{\partial x}(x) f_i(x,d) \leqslant \phi(t, V_i(x), U(u))$ for all $(i,x,d) \in \mathcal{P} \times \mathbb{R}^n \times \mathbb{R}^m$;*
- (iii) *$V_{\sigma(\tau_i)}(x(\tau_i)) \leqslant \xi(\tau_i) \quad \forall\, i \geqslant 0$;*
- (iv) *Σ is ISS with respect to η.*

Then (3.8) is ISS with respect to d.

§ **A.3.2. Proof of Theorem 3.3.** Consider the impulsive system

$$\text{(A.10)}\qquad \begin{cases} \dot{\xi} = -\lambda_\circ \xi + \rho, \quad t \neq \tau_i, \\ \xi(\tau_i) = \mu \xi(\tau_i^-), \end{cases} \qquad i \geqslant 0, \quad t \geqslant 0,$$

where $\xi, \rho \in \mathbb{R}_{\geqslant 0}$, $\tau_0 := 0$, $\lambda_\circ > 0$ and $\rho_{\mathcal{I}}$ denotes the supremum of ρ over the time interval $\mathcal{I} \subseteq \mathbb{R}_{\geqslant 0}$.

For $t \in]\tau_\ell, \tau_{\ell+1}[$ where ℓ is a nonnegative integer, we have the solution of (A.10) as

$$\xi(t) \leqslant \xi(\tau_\ell)\,\mathrm{e}^{-\lambda_\circ(t-\tau_\ell)} + \frac{\rho_{[\tau_\ell, t[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(t-\tau_\ell)}\right).$$

In the particular case of $t = \tau_{\ell+1}^-$, we have

$$\xi(\tau_{\ell+1}^-) \leqslant \xi(\tau_\ell)\,\mathrm{e}^{-\lambda_\circ(\tau_{\ell+1}-\tau_\ell)} + \frac{\rho_{[\tau_\ell, \tau_{\ell+1}[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(\tau_{\ell+1}-\tau_\ell)}\right).$$

Combining with the jump in the state ξ at τ_ℓ, we have

$$\xi(\tau_{\ell+1}) \leqslant \mu\xi(\tau_\ell)\,\mathrm{e}^{-\lambda_\circ(\tau_{\ell+1}-\tau_\ell)} + \mu\frac{\rho_{[\tau_\ell, \tau_{\ell+1}[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(\tau_{\ell+1}-\tau_\ell)}\right).$$

Let $\nu := N_\sigma(T, 0)$. Iterating this inequality from $\ell = 0$ to $\ell = \nu$, we have

$$\xi(\tau_\nu) \leqslant \mu^\nu \xi_0\,\mathrm{e}^{-\lambda_\circ \tau_\nu} + \frac{\rho_{[0,\tau_\nu[}}{\lambda_\circ}\,\mathrm{e}^{-\lambda_\circ\tau_\nu}\sum_{j=0}^{\nu-1}\mu^{\nu-j}\left(\mathrm{e}^{\lambda_\circ\tau_{j+1}} - \mathrm{e}^{\lambda_\circ\tau_j}\right)$$

and

$$\text{(A.11)}\qquad \xi(T) \leqslant \mu^\nu \xi_0\,\mathrm{e}^{-\lambda_\circ T} + \frac{\rho_{[0,T[}}{\lambda_\circ}\,\mathrm{e}^{-\lambda_\circ T}\left(\sum_{j=0}^{\nu-1}\mu^{\nu-j}\left(\mathrm{e}^{\lambda_\circ\tau_{j+1}} - \mathrm{e}^{\lambda_\circ\tau_j}\right) + \mathrm{e}^{\lambda_\circ T} - \mathrm{e}^{\lambda_\circ\tau_\nu}\right).$$

In conjunction with the estimate

$$T - \tau_{j+1} \geqslant -N_\circ\tau_a + (\nu - (j+1))\tau_a,$$

which comes from the definition of the average dwell-time, we see that the second term on the right-hand side of the inequality (A.11) is upper-bounded by

$$\begin{aligned} &\frac{\rho_{[0,T[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(T-\tau_\nu)} + \mu^\nu\,\mathrm{e}^{-\nu\lambda_\circ\tau_a}\,\mathrm{e}^{\lambda_\circ N_\circ\tau_a}\,\mathrm{e}^{\lambda_\circ\tau_a}\sum_{j=0}^{\nu-1}\mu^{-j}\,\mathrm{e}^{\lambda_\circ\tau_a j}\right) \\ =&\frac{\rho_{[0,T[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(T-\tau_\nu)} + \mathrm{e}^{\lambda_\circ\tau_a(N_\circ+1)}\mu^\nu\,\mathrm{e}^{-\lambda_\circ\tau_a\nu}\frac{\mu^{-\nu}\,\mathrm{e}^{\lambda_\circ\tau_a\nu} - 1}{\mu^{-1}\,\mathrm{e}^{\lambda_\circ\tau_a} - 1}\right) \\ \leqslant&\frac{\rho_{[0,T[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(T-\tau_\nu)} + \frac{\mathrm{e}^{\lambda_\circ\tau_a(N_\circ+1)}}{\mu^{-1}\,\mathrm{e}^{\lambda_\circ\tau_a} - 1}\right), \end{aligned}$$

since $\tau_a > \frac{\ln\mu}{\tau_a}$. Considering this upper bound in the estimate (A.11), we have

$$\xi(T) \leqslant \xi_0\,\mathrm{e}^{-\lambda_\circ T + \left(N_\circ + \frac{T}{\tau_a}\right)\ln\mu} + \frac{\rho_{[0,T[}}{\lambda_\circ}\left(1 - \mathrm{e}^{-\lambda_\circ(T-\tau_\nu)} + \frac{\mathrm{e}^{\lambda_\circ\tau_a(N_\circ+1)}}{\mathrm{e}^{\lambda_\circ\tau_a - \ln\mu} - 1}\right)$$

which shows that

$$\xi(T) \leqslant \xi_0\,\mathrm{e}^{-\lambda_\circ T + \left(N_\circ + \frac{T}{\tau_a}\right)\ln\mu} + \frac{\rho_{[0,T[}}{\lambda_\circ}\left(1 + \frac{\mathrm{e}^{\lambda_\circ\tau_a(N_\circ+1)}}{\mathrm{e}^{\lambda_\circ\tau_a - \ln\mu} - 1}\right).$$

In the light of the assumption $\tau_a > \frac{\ln\mu}{\lambda_\circ}$, for any $t \geqslant 0$, this leads to

$$\text{(A.12)}\qquad \xi(t) \leqslant \kappa_\xi\xi_0\,\mathrm{e}^{-\lambda t} + \kappa_\rho\rho_{[0,t[},$$

for positive constants $\lambda := \lambda_\circ - \frac{\ln \mu}{\tau_a}$, $\kappa_\xi := \mu^{N_\circ}$, and $\kappa_\rho := \frac{1}{\lambda_\circ}\left(1 + \frac{e^{\lambda_\circ \tau_a (N_\circ + 1)}}{e^{\lambda_\circ \tau_a - \ln \mu} - 1}\right)$.

From (A.12) it follows that (A.10) is ISS with respect to ρ. Replacing ρ with $\gamma(\|d\|)$, the ISS property of (3.8) with respect to d follows from Lemma A.9.

APPENDIX B

SOME DISCUSSION OF CONJECTURE 3.17

In this appendix we furnish some calculations which may be helpful to conclusively prove Conjecture 3.17. We need the following proposition which collects the properties of optional times that we need in the sequel. The proof is standard, and may be found in, e.g., [51, Chapter 4].

B.1. PROPOSITION. *Let* $(\mathfrak{F}_t)_{t\geqslant 0}$ *be a filtration on* $(\Omega, \mathfrak{F}, \mathsf{P})$ *satisfying the usual conditions, and let* τ, τ', τ_i *be* $(\mathfrak{F}_t)_{t\geqslant 0}$*-optional times. Then*

(i) $\mathfrak{F}_\tau := \big\{A \in \mathfrak{F} \,\big|\, A \cap \{\tau \leqslant t\} \in \mathfrak{F}_t$ *for all* $t \geqslant 0\big\}$ *is a sigma-subalgebra of* $\mathfrak{F}$.

(ii) τ *is* $\mathfrak{F}_\tau$*-measurable.*

(iii) $\tau + s$ *is* $(\mathfrak{F}_t)_{t\geqslant 0}$*-optional for all* $s \geqslant 0$.

(iv) *If* $\tau_i \searrow \tau$ *pointwise for* $i \in \mathbb{N}$, *then* $\mathfrak{F}_\tau = \bigcap_{i\in\mathbb{N}} \mathfrak{F}_{\tau_i}$.

(v) $\mathfrak{F}_{\tau\wedge\tau'} = \mathfrak{F}_\tau \cap \mathfrak{F}_{\tau'}$.

(vi) $\{\tau \leqslant \tau'\}, \{\tau = \tau'\} \in \mathfrak{F}_{\tau\wedge\tau'}$.

We resume from Step 4 in the proof of Proposition 3.15. Fixing an arbitrary essentially bounded measurable input d, we constructed two concentric balls around the origin whose radii depend on $\|d\|_{\mathbb{R}_{\geqslant 0}}$. The instants that $x(\cdot)$ hits the boundaries of these balls were defined in Step 1. In Step 2 we obtained a bound on the expected value of $V_{\sigma(t)}(x(t))$ restricted to the set on which $x(t)$ is yet to hit the inner ball for the first time; this bound turned out to be in terms of a class-$\mathcal{KL}$ function of the initial state and time. In Step 3 we bounded the expected value of $V_{\sigma(t)}(x(t))$ restricted to the set on which $x(t)$ is inside the outer ball. In Step 4 below we attempt to obtain a bound on the expected value of $V_{\sigma(t)}(x(t))$ restricted to the set on which $x(t)$ has evolved out of the outer ball but is yet to hit the inner ball. This step involves some technical constructions and is further subdivided into six substeps for convenience. The bounds in Step 3 turn out to be in terms of a $\mathcal{K}_\infty$ function of $\|d\|_{\mathbb{R}_{\geqslant 0}}$. Once a reasonable bound is available from Step 4, the procedure in Step 5 of Proposition 3.15 can be carried out to obtain the ISS-M property.

Step 4. For each $i \in \mathbb{N}$ consider the process $\big(\xi_i(s)\big)_{s\geqslant 0}$ defined pointwise on the set $\{\hat{t}_i < \infty\}$ given by the solution to the impulsive differential equation

$$
\text{(B.2)}\qquad \begin{cases} \xi_i(0) = V_{\sigma(\hat{t}_i)}(x(\hat{t}_i)), & \\ \xi_i(s) = \mu\xi_i(s^-) & \text{if } \hat{t}_i + s = \tau_j \text{ for } j \in \mathbb{N}, s > 0, \\ \dfrac{\mathrm{d}\xi_i}{\mathrm{d}t}(s) = -\lambda_\circ \xi_i(s) & \text{if } \hat{t}_i + s \neq \tau_j \text{ for } j \in \mathbb{N}, s > 0. \end{cases}
$$

We shall demonstrate the following properties below.

- A solution process $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ of (B.2) is a càdlàg $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-supermartingale, and satisfies

(B.3) $$\xi_i(s)\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}} \geqslant V_{\sigma(\hat{t}_i+s)}(x(\hat{t}_i+s))\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}} \quad \forall\, s\geqslant 0.$$

  This will guarantee that (B.2) is well-defined on $\{\hat{t}_i<\infty\}$, and standard conditions [34, Chapter 2] can be invoked to infer that (B.2) has pathwise unique solutions by definition.
- $\big(\xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is an $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-supermartingale for some $\delta>0$.

Substep 4a. Claim: The filtration $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$ is right-continuous.

Proof of claim: Indeed, since $\hat{t}_i$ is $(\mathfrak{F}_t)_{t\geqslant 0}$-optional, by Proposition B.1 we know that $\hat{t}_i+s$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time for every $s\geqslant 0$. Therefore for fixed $s\geqslant 0$, $\hat{t}_i+s+1/k$ is an $(\mathfrak{F}_t)_{t\geqslant 0}$-optional time for every $k\in\mathbb{N}$, and the sequence $\big(\hat{t}_i+s+1/k\big)_{k\in\mathbb{N}}$ decreases monotonically and pointwise (on Ω) to $\hat{t}_i+s$. By Proposition B.1 we have $\mathfrak{F}_{\hat{t}_i+s}=\bigcap_{k\in\mathbb{N}}\mathfrak{F}_{\hat{t}_i+s+1/k}$ which proves the right continuity of $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$.

Substep 4b. Claim: $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is an $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-adapted càdlàg process.

Proof of claim: We note that the sequence of $(\mathfrak{F}_t)_{t\geqslant 0}$-optional times $(\tau_j)_{j\in\mathbb{N}}$ is also adapted to $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$ for $i\in\mathbb{N}$. Indeed, since τ_j and $\hat{t}_i+s$ are $(\mathfrak{F}_t)_{t\geqslant 0}$-optional for $s\geqslant 0$ and $i,j\in\mathbb{N}$, by Proposition B.1 we have $\{\tau_j\leqslant\hat{t}_i+s\}\in\mathfrak{F}_{\tau_j\wedge(\hat{t}_i+s)}=\mathfrak{F}_{\tau_j}\cap\mathfrak{F}_{\hat{t}_i+s}\subseteq\mathfrak{F}_{\hat{t}_i+s}$. This shows that $\{\tau_j\leqslant\hat{t}_i+s\}\in\mathfrak{F}_{\hat{t}_i+s}$, which proves that τ_j is $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-optional for each $j\in\mathbb{N}$. An identical argument with the aid of Proposition B.1 shows that $\{\hat{t}_i<\tau_j\}\in\mathfrak{F}_{\hat{t}_i}\cap\mathfrak{F}_{\tau_j}\subseteq\mathfrak{F}_{\hat{t}_i+s}$. By definition of N_σ we have $N_\sigma(\hat{t}_i,\hat{t}_i+s)=\sum_{j\in\mathbb{N}}\mathbf{1}_{\{\hat{t}_i<\tau_j\leqslant\hat{t}_i+s\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}=\sum_{j\in\mathbb{N}}\mathbf{1}_{\{\hat{t}_i<\tau_j\}}\mathbf{1}_{\{\tau_j\leqslant\hat{t}_i+s\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}$; since each of the indicator functions is $\mathfrak{F}_{\hat{t}_i+s}$-measurable, we conclude that $N_\sigma(\hat{t}_i,\hat{t}_i+s)$ is $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-adapted. On $\{\hat{t}_i<\infty\}$ the pointwise solution of the impulsive differential equation (B.2) is given by

$$\xi_i(s)=\xi_i(0)\,\mathrm{e}^{-\lambda_\circ s}\mu^{N_\sigma(\hat{t}_i,\hat{t}_i+s)}=V_{\sigma(\hat{t}_i)}(x(\hat{t}_i))\,\mathrm{e}^{-\lambda_\circ s}\mu^{N_\sigma(\hat{t}_i,\hat{t}_i+s)},$$

and from the above arguments it is clear that $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-adapted. By its definition the solution $\xi(\cdot)$ of the differential equation in (B.2) is a monotone decreasing function on each holding time, which ensures finite limits from the left, and this holds pointwise on Ω. Also, $\xi(\cdot)$ is continuous from the right pointwise on $\{\hat{t}_i<\infty\}$ by definition at the jump instants. The càdlàg property of the solution process $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ follows.

Substep 4c. Claim: $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is an $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-supermartingale.

Proof of claim: By Substep 4b, the family $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-adapted. Fixing $s\geqslant 0$

and $h > 0$, we get

$$\begin{aligned}
\mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\xi_i(s+h)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big] &= \mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\xi_i(s)\,\mathrm{e}^{-\lambda_\circ h}\mu^{N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)}\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big] \quad \text{by (B.2)}\\
&= \xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ h}\mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\mu^{N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)}\Big] \quad \text{by Proposition B.1}\\
&= \xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ h}\sum_{k=0}^{\infty}\mu^k\mathsf{P}^{\mathfrak{F}_{\hat{t}_i+s}}\big(N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)=k\big)\\
&\leqslant \xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ h}\sum_{k=0}^{\infty}\mathrm{e}^{-\widetilde{\lambda}h}\frac{\big(\mu\overline{\lambda}h\big)^k}{k!} \quad \text{by (Gd2)}\\
&= \xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\big(\lambda_\circ+\widetilde{\lambda}-\mu\overline{\lambda}\big)h}\\
&< \xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}} \quad \text{by (Gd3).} \qquad \text{(B.4)}
\end{aligned}$$

The calculation in (B.4) is valid provided the family $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is integrable. But a calculation analogous to (B.4) shows that for $s \geqslant 0$

$$\begin{aligned}
\mathsf{E}\Big[\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\hat{t}_i}}\Big[\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]\Big]\\
&= \mathrm{e}^{-\big(\lambda_\circ+\widetilde{\lambda}-\mu\overline{\lambda}\big)s}\mathsf{E}\Big[\mathbf{1}_{\{\hat{t}_i<\infty\}}\mathsf{E}^{\mathfrak{F}_{\hat{t}_i}}\big[\xi_i(0)\big]\Big]\\
&= \mathrm{e}^{-\big(\lambda_\circ+\widetilde{\lambda}-\mu\overline{\lambda}\big)s}\mathsf{E}\Big[\mathbf{1}_{\{\hat{t}_i<\infty\}}V_{\sigma(\hat{t}_i)}(x(\hat{t}_i))\Big]\\
&\leqslant \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big).
\end{aligned}$$

(In fact, what we get from the above equation is that $\big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is an $\big(\mathfrak{F}_s\big)_{s\geqslant 0}$-potential.) The claim follows.

Substep 4d. Claim: $\big(\xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is an $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-supermartingale for some $\delta > 0$.

Proof of claim: Since the map $\mathbb{R}_{\geqslant 0} \ni r \longmapsto (1+r)\lambda_\circ + \widetilde{\lambda} - (1+r)\mu\overline{\lambda} \in \mathbb{R}$ is continuous, (Gd3) shows that there exists $\delta > 0$ such that $(1+\delta)\lambda_\circ + \widetilde{\lambda} - (1+\delta)\mu\overline{\lambda} > 0$. For each $s \geqslant 0$, $\xi_i(s) \longmapsto \xi_i^{1+\delta}(s)$ is a continuous function; therefore, the family $\big(\xi^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\big)_{s\geqslant 0}$ is $\big(\mathfrak{F}_{\hat{t}_i+s}\big)_{s\geqslant 0}$-adapted by Substep 4b. Computing as in Substep 4c, we have

$$\begin{aligned}
&\mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\xi_i^{1+\delta}(s+h)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]\\
&\quad= \mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\xi_i^{1+\delta}(\hat{t}_i+s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ(1+\delta)h}\mu^{(1+\delta)N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)}\Big]\\
&\quad= \xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ(1+\delta)h}\mathsf{E}^{\mathfrak{F}_{\hat{t}_i+s}}\Big[\mu^{(1+\delta)N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)}\Big]\\
&\quad= \xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ(1+\delta)h}\sum_{k=0}^{\infty}\mu^{k(1+\delta)}\mathsf{P}^{\mathfrak{F}_{\hat{t}_i+s}}\big(N_\sigma(\hat{t}_i+s,\hat{t}_i+s+h)=k\big)\\
&\quad\leqslant \xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\lambda_\circ(1+\delta)h}\sum_{k=0}^{\infty}\mathrm{e}^{-\widetilde{\lambda}h}\frac{\big(\mu^{1+\delta}\overline{\lambda}h\big)^k}{k!}\\
&\quad= \xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\mathrm{e}^{-\big(\lambda_\circ(1+\delta)+\widetilde{\lambda}-\mu\overline{\lambda}(1+\delta)\big)h}\\
&\quad< \xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}.
\end{aligned}$$

Integrability of the family $\left(\xi_i^{1+\delta}(s)\right)_{s\geqslant 0}$ is obtained as

$$\begin{aligned}\mathsf{E}\Big[\xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big] &= \mathsf{E}\Big[\mathsf{E}^{\mathfrak{F}_{\hat{t}_i}}\Big[\xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]\Big]\\ &= \mathrm{e}^{-\left(\lambda_\circ(1+\delta)+\widetilde{\lambda}-\mu\overline{\lambda}(1+\delta)\right)s}\mathsf{E}\Big[\mathbf{1}_{\{\hat{t}_i<\infty\}}\mathsf{E}^{\mathfrak{F}_{\hat{t}_i}}\left[\xi_i^{1+\delta}(0)\right]\Big]\\ &= \mathrm{e}^{-\left(\lambda_\circ(1+\delta)+\widetilde{\lambda}-\mu\overline{\lambda}(1+\delta)\right)s}\mathsf{E}\Big[\mathbf{1}_{\{\hat{t}_i<\infty\}}V_{\sigma(\hat{t}_i)}^{1+\delta}(x(\hat{t}_i))\Big]\\ &\leqslant \mathsf{E}\Big[\mathbf{1}_{\{\hat{t}_i<\infty\}}V_{\sigma(\hat{t}_i)}^{1+\delta}(x(\hat{t}_i))\Big]\\ &\leqslant \alpha_2^{1+\delta}\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big).\end{aligned}$$

The claim follows.

Substep 4e. Claim: (B.3) holds.

Proof of claim: Indeed, by (Gd1) and (3.47) we get

$$V_{\sigma(\hat{t}_i+s)}(x(\hat{t}_i+s))\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}} \leqslant V_{\sigma(\hat{t}_i)}(x(\hat{t}_i))\mu^{N_\sigma(\hat{t}_i,\hat{t}_i+s)}\,\mathrm{e}^{-\lambda_\circ s}\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}$$

for every $i\in\mathbb{N}$, and from Substep 4b,

$$V_{\sigma(\hat{t}_i)}(x(\hat{t}_i))\mu^{N_\sigma(\hat{t}_i,\hat{t}_i+s)}\,\mathrm{e}^{-\lambda_\circ s}\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}} \leqslant \xi_i(s)\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}.$$

Combining the two inequalities we get (B.3) which proves the claim.

Substep 4f. For a positive $(\mathfrak{F}_s)_{s\geqslant 0}$-supermartingale $(y_s)_{s\geqslant 0}$ there is a bound on the distribution function of $\sup_{s\geqslant 0} y(s)$ given by

$$\mathsf{P}^{\mathfrak{F}_0}\left(\sup_{s\geqslant 0} y_s \geqslant a\right) \leqslant \frac{y_0}{a}\wedge 1.$$

This can be obtained directly from [**52**, Exercise 1.13, Chapter 2].

We compute

$$\begin{aligned}&\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]\\ &\leqslant \mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}}\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]\,\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)\\ &\leqslant \mathsf{E}\Big[\sup_{s\geqslant 0}\Big(V_{\sigma(\hat{t}_i+s)}(x(\hat{t}_i+s))\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\Big)\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]\,\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)\\ &\leqslant \mathsf{E}\Big[\sup_{s\geqslant 0}\Big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big)\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]\,\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)\end{aligned}$$

by (B.3) and monotonicity of conditional expectations. Therefore,

$$\begin{aligned}&\mathsf{E}\Big[\sup_{s\geqslant 0}\Big(\xi_i(s)\mathbf{1}_{\{\hat{t}_i+s<\check{t}_{i+1}\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big)\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]\,\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)\\ &\leqslant \mathsf{E}\Big[\mathsf{E}\Big[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\Big|\,\{\hat{t}_i<\infty\}\textstyle\bigcap\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\bigcap\mathfrak{F}_{\hat{t}_i}\Big]\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]\\ &\qquad\cdot\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)\,.\end{aligned}$$

If it is possible to get a bound on

$$\mathsf{E}\Big[\mathsf{E}\Big[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\Big|\,\{\hat{t}_i<\infty\}\textstyle\bigcap\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\bigcap\mathfrak{F}_{\hat{t}_i}\Big]\,\Big|\, t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Big]$$

in terms of, say, $\mathsf{E}\Big[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]$, then we can complete the proof. Indeed, then we have

$$\begin{aligned}
&\mathsf{E}\Bigg[\mathsf{E}\Bigg[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\,\Bigg|\,\{\hat{t}_i<\infty\}\bigcap\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\bigcap\mathfrak{F}_{\hat{t}_i}\Bigg]\,\Bigg|\,t\in[\hat{t}_i,\check{t}_{i+1}[,\hat{t}_i<\infty\Bigg] \\
&\leqslant M\mathsf{E}\Bigg[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Bigg]\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big)
\end{aligned} \tag{B.5}$$

for some $M>0$. But

$$\begin{aligned}
\mathsf{E}\Bigg[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Bigg] &= \mathsf{E}\Bigg[\mathsf{E}^{\mathfrak{F}_{\hat{t}_i}}\Bigg[\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}\Bigg]\Bigg] \\
&\leqslant \mathsf{E}\Bigg[\int_0^\infty \mathsf{P}^{\mathfrak{F}_{\hat{t}_i}}\Bigg(\sup_{s\geqslant 0}\xi_i(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}>a\Bigg)\,\mathrm{d}a\Bigg] \\
&= \mathsf{E}\Bigg[\int_0^\infty \mathsf{P}^{\mathfrak{F}_{\hat{t}_i}}\Bigg(\sup_{s\geqslant 0}\xi_i^{1+\delta}(s)\mathbf{1}_{\{\hat{t}_i<\infty\}}>a^{1+\delta}\Bigg)\,\mathrm{d}a\Bigg] \\
&\leqslant \mathsf{E}\Bigg[\int_0^\infty \Bigg(\frac{\xi_i^{1+\delta}(0)}{a^{1+\delta}}\wedge 1\Bigg)\,\mathrm{d}a\Bigg] \\
&\leqslant \int_0^\infty \Bigg(\frac{\alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)^{1+\delta}}{a^{1+\delta}}\wedge 1\Bigg)\,\mathrm{d}a. \qquad \text{(B.6)}
\end{aligned}$$

The integral in (B.6) is finite and depends on $\|d\|_{\mathbb{R}_{\geqslant 0}}$. Indeed,

$$\begin{aligned}
&\int_0^\infty \Bigg(\frac{\alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)^{1+\delta}}{a^{1+\delta}}\wedge 1\Bigg)\,\mathrm{d}a \\
&\qquad = \int_0^{\alpha_2\left(\eta\rho\left(\|d\|_{\mathbb{R}_{\geqslant 0}}\right)\right)} 1\,\mathrm{d}a + \int_{\alpha_2\left(\eta\rho\left(\|d\|_{\mathbb{R}_{\geqslant 0}}\right)\right)}^\infty \frac{\alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)^{1+\delta}}{a^{1+\delta}}\,\mathrm{d}a \\
&\qquad = \alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)\left(1+\frac{1}{\delta}\right). \qquad \text{(B.7)}
\end{aligned}$$

Quite clearly the right-hand side of (B.7) is a $\mathcal{K}_\infty$ function of $\|d\|_{\mathbb{R}_{\geqslant 0}}$. Collecting the estimates of (B.5), (B.6), and (B.7), we get

$$\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}}\mathbf{1}_{\{\hat{t}_i<\infty\}}\Big]\leqslant M\alpha_2\big(\eta\rho(\|d\|_{\mathbb{R}_{\geqslant 0}})\big)\left(1+\frac{1}{\delta}\right)\mathsf{P}\big(\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}\big). \tag{B.8}$$

Step 5′. It is clear that for $t\in\mathbb{R}_{\geqslant 0}$ we have

$$\begin{aligned}
V_{\sigma(t)}(x(t)) = V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[0,\check{t}_1[\}} &+ \sum_{i=1}^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\check{t}_i,\hat{t}_i[\}\cap\{\check{t}_i<\infty\}} \\
&+ \sum_{i=1}^\infty V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}},
\end{aligned}$$

from which an application of the monotone convergence theorem gives

$$\begin{aligned}
\mathsf{E}\big[V_{\sigma(t)}(x(t))\big] = \mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[0,\check{t}_1[\}}\big] &+ \sum_{i=1}^\infty \mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\check{t}_i,\hat{t}_i[\}\cap\{\check{t}_i<\infty\}}\Big] \\
&+ \sum_{i=1}^\infty \mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}}\Big].
\end{aligned} \tag{B.9}$$

By Step 2 above,

$$\mathsf{E}\big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[0,\check{t}_1[\}}\big] \leqslant \beta(\|x_0\|\,, t).$$

From Step 3 above we have

$$\begin{aligned}\sum_{i=1}^{\infty}\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\check{t}_i,\hat{t}_i[\}\cap\{\check{t}_i<\infty\}}\Big] &\leqslant \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big)\sum_{i=1}^{\infty}\mathsf{P}\big(\{t\in[\check{t}_i,\hat{t}_i[\}\cap\{\check{t}_i<\infty\}\big)\\ &\leqslant \alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big).\end{aligned}$$

Without assuming the existence of the estimate in (B.5), since $\|d\|_{\mathbb{R}_{\geqslant 0}} < \infty$ we can get an ISS-M-like property from Step 4 that is valid on compact subsets of $\mathbb{R}_{\geqslant 0}$. Indeed, the family $\{f_i\}_{i\in\mathcal{P}}$ is locally Lipschitz, so there can be only finitely many excursions of $x(\cdot)$ between the sets C_1 and C_2 on the interval $[0,t]$, and we let this number be ν. Then

$$\sum_{i=1}^{\infty}\mathsf{E}\Big[V_{\sigma(t)}(x(t))\mathbf{1}_{\{t\in[\hat{t}_i,\check{t}_{i+1}[\}\cap\{\hat{t}_i<\infty\}}\Big] \leqslant \nu\left(1+\frac{1}{\delta}\right)\alpha_2\big(\eta\rho\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big)\big).$$

Substituting back in (B.9), and defining $\gamma(\cdot) := \nu\left(1+\frac{1}{\delta}\right)\alpha_2(\eta\rho(\cdot))$, we get

$$\mathsf{E}\big[V_{\sigma(s)}(x(s))\big] \leqslant \beta(\|x_0\|\,, s) + \gamma\big(\|d\|_{\mathbb{R}_{\geqslant 0}}\big) \quad \forall\, s\in[0,t].$$

If, however, the estimate in (B.5) holds, then the dependence on ν above can be removed if (B.8) is employed in (B.9).

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