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Stability Analysis and Stabilization of Randomly Switched Systems

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Abstract— **This article is concerned with stability analysis and stabilization of randomly switched systems with control inputs. The switching signal is modeled as a jump stochastic process independent of the system state; it selects, at each instant of time, the active subsystem from a family of deterministic systems. Three different types of switching signals are considered: the first is a jump stochastic process that satisfies a statistically slow switching condition; the second and the third are jump stochastic processes with independent identically distributed values at jump times together with exponential and uniform holding times, respectively. For each of the three cases we first establish sufficient conditions for stochastic stability of the switched system when the subsystems do not possess control inputs; not every subsystem is required to be stable in the latter two cases. Thereafter we design feedback controllers when the subsystems are affine in control and are not all zero-input stable, with the control space being general subsets of $\mathbb{R}^m$. Our analysis results and universal formulae for feedback stabilization of nonlinear systems for the corresponding control spaces constitute the primary tools for control design.**

I. Introduction

Randomly switched systems generally consist of a finite family of subsystems and a random switching signal that specifies at each instant of time the active subsystem. The switching signal σ is modeled as a continuous time stochastic process, which may be the state of a finite-state Markov chain, or a more general càdlàg jump stochastic process. Since the dynamics between two consecutive switching instants are governed by deterministic differential equations, these systems can be regarded as piecewise deterministic stochastic systems [1]. In this article our goal is twofold: one, to provide sufficient conditions for stochastic stability of randomly switched systems, and two, to provide a methodology for stabilizing controller synthesis when such systems possess control inputs.

A particular class of randomly switched systems has received widespread attention, namely, Markovian jump linear systems (MJLS). These systems may be realized as a family of linear subsystems, together with a switching signal generated by the state of a continuous-time Markov chain. Stability and stabilization of MJLS have been extensively investigated, specially under the assumption that the parameters of the Markov chain are completely known, see e.g. [2], [3], [4], [5] and the references therein. In particular, almost sure stabilization and mean stabilization of MJLS is discussed in [4], where the authors also establish precise equivalences between different stability notions for MJLS.

Among the several stochastic stability notions, perhaps the most interesting is almost sure global asymptotic stability (GAS a.s.). We shall concentrate on this particular notion in this article; however, it is also possible to obtain stability in the mean and stability in probability with minimal extra work, which we indicate in Remark 16. GAS a.s. of randomly switched systems was investigated in our earlier article [6]. There we assumed that each (nonlinear) subsystem was globally asymptotically stable, and σ was a general jump stochastic process having an asymptotic bound on the probability mass function of the number of switches on each time interval $[0, t[$. Unless additional structure is imposed on the switching signal, switched systems with even one unstable subsystem cannot, in general, have the GAS a.s. property; see Remark 11. In the present article, we describe two possible scenarios where sufficient structure in the probabilistic properties of the switching signal make it possible to include unstable subsystems in the family. To be precise, in the first case the set of holding times of σ is assumed to be a sequence of independent exponential variables of parameter λ, and in the second case the set of holding times is assumed to be a sequence of independent uniform random variables of parameter T. In addition, in both of the above cases we assume that values attained by σ (at each switching instant) are independent and identically distributed, and are independent of the set of holding times. It follows naturally from our results that for the switched system to be GAS a.s., the unstable subsystems must have small probability of activation; see Remarks 13 and 14.

In [6] we also established a method of designing feedback controllers to achieve GAS a.s. of closed loop switched control systems, by employing the Artstein-Sontag universal formula [7]. The control took values in $\mathbb{R}$, and every subsystem was zero-input stable. In this article the controller design scheme allows the control to take values in general subsets of $\mathbb{R}^m$, (e.g., bounded sets, Minkowski balls, etc.,) and the subsystems are not necessarily zero-input stable. Our control design methodology works whenever each subsystem is affine in control, a suitable family of control-Lyapunov functions (one for each subsystem) is available, and a universal formula for feedback stabilization is available for the set of admissible inputs.

A myriad of techniques have been employed to study stability and stabilization of piecewise deterministic stochastic systems. HJB-based optimal control schemes for piecewise deterministic stochastic systems are well-studied, see e.g., [1] for a detailed account. Linear control systems admit analytically solvable linear quadratic optimal design methods, and such techniques have been effectively combined with the stochastic nature of structural variations in [3]; stabilization schemes based on Lyapunov exponents are studied in [4].

Game-theoretic techniques [8] in the presence of disturbance inputs, and spectral theory of Markov operators [9] have also been employed to study these systems. Stabilization schemes using robust control methods are investigated in [10]; see also the references cited in it. Stochastic hybrid systems, where the switching signal and its transition probabilities are state-dependent, are studied in [11] and [12], using an extended definition of the infinitesimal generator and optimal control strategies, respectively.

Our approach, in contrast to the above, parallels the one adopted in [6]. The stochastic switching signal is decoupled from the individual dynamical systems; instead of looking at the stochastic system as a whole, the properties of the random switching signal are decoupled from the deterministic properties of the switched system between consecutive switching instants. Consequently, we do not resort to infinitesimal generators for the stochastic process. The main analysis tool is the theory of multiple Lyapunov functions [13, Chapter 3], developed originally in the context of deterministic switched systems. The probabilistic properties of the switching signal, when suitably coupled with the dynamics of the Lyapunov functions, enable us to efficiently analyze the behavior of the overall switched system. Off-the-shelf universal formulae (see [7], [14], [15], [16]) and our analysis results provide the tools for our control design methodology.

The paper is arranged as follows. §II contains the definitions of randomly switched systems and the stability notion that we study. The hypotheses on the switching signal and the associated analysis results are stated in §III. Controller synthesis results are stated and proved in §IV. The proofs of all the results stated in §III are collected in §V. We conclude the paper in §VI with a brief discussion of possible directions for further investigation.

II. Preliminaries

Let the Euclidean norm be denoted by $\|\cdot\|$, the interval $[0,\infty[$ by $\mathbb{R}_{\geqslant 0}$, and the set of natural numbers $\{1,2,\ldots\}$ by $\mathbb{N}$. Recall that a continuous function $\alpha:\mathbb{R}_{\geqslant 0}\longrightarrow\mathbb{R}_{\geqslant 0}$ is of class $\mathcal{K}$ if α is strictly increasing with $\alpha(0)=0$, of class $\mathcal{K}_\infty$ if in addition $\alpha(r)\to\infty$ as $r\to\infty$; we write $\alpha\in\mathcal{K}$ and $\alpha\in\mathcal{K}_\infty$ respectively. Let $\mathrm{L}_f h$ be the directional derivative of a continuously differentiable real-valued function h defined on $\mathbb{R}^n$, along a vector field f on $\mathbb{R}^n$. For $a,b\in\mathbb{R}$, we let $a\wedge b$ and $a\vee b$ stand for $\min\{a,b\}$ and $\max\{a,b\}$, respectively.

We define the family of systems affine in control:

$$\dot{x}=f_p(x),\qquad p\in\mathcal{P},\tag{1}$$

where the state $x\in\mathbb{R}^n$, $\mathcal{P}$ is a finite index set of N elements: $\mathcal{P}=\{1,\ldots,\mathrm{N}\}$, the function $f_p:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is locally Lipschitz in x, $f_p(0)=0$, $p\in\mathcal{P}$. A *switched system* for the family (1) is generated by a *switching signal*—a piecewise constant function (continuous from the right by convention), $\sigma:\mathbb{R}_{\geqslant 0}\longrightarrow\mathcal{P}$, which specifies at every time t the index $\sigma(t)\in\mathcal{P}$ of the active subsystem:

$$\dot{x}=f_\sigma(x),\qquad x(0)=x_0,\quad t\geqslant 0.\tag{2}$$

We assume that there are no jumps in the state x at the switching instants, and let x_0 be given.

Let $\big(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geqslant 0},\mathsf{P}\big)$ be a *filtered complete probability space* [17], such that the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$ is continuous from the right and $\mathfrak{F}_0$ contains all P-null sets. Let $\mathsf{E}[\cdot]$ denote mathematical expectation. We assume that σ is a càdlàg (continuous from the right and possessing left limits) jump stochastic process adapted to $(\mathfrak{F}_t)_{t\geqslant 0}$. Let the switching instants of σ be denoted (chronologically) by τ_i, $i=1,2,\ldots$, and let $\tau_0:=0$ by convention. As a consequence of our hypotheses (see Assumptions 4, 6, and 8) there is no explosion almost surely; therefore the sequence $(\tau_i)_{i\geqslant 0}$ is divergent. Finally, we assume that for every compact subset $K\subset\mathbb{R}_{\geqslant 0}\times\mathbb{R}^n$ there exists an integrable function $m_K:\mathbb{R}_{\geqslant 0}\longrightarrow\mathbb{R}_{\geqslant 0}$ satisfying $\big\|f_{\sigma(t)}(x)\big\|\leqslant m_K(t)$ for all $(t,x)\in K$. Hence almost surely there exists a unique solution to (2) in the sense of Carathéodory [18] over a nontrivial time interval containing 0; existence and uniqueness of a global solution will follow from the hypotheses of our results. Let $x(\cdot)$ denote this solution. When it is necessary to consider the solution of (2) corresponding to a particular event $\omega\in\Omega$, we use $x(\cdot,\omega)$. For $x_0=0$, the solution to (2) is identically 0 for every σ; we shall ignore this trivial case in the sequel.

We are interested in the following definition of stability of (2).

Definition 1: The system (2) is said to be ***globally asymptotically stable almost surely*** (GAS a.s.) iff the following two properties are simultaneously verified:

(AS1) $\forall\,\varepsilon>0\quad\exists\,\delta(\varepsilon)>0$ such that $\|x_0\|<\delta(\varepsilon)\implies \mathsf{P}\Big(\sup_{t\geqslant 0}\|x(t)\|<\varepsilon\Big)=1;$

(AS2) $\forall\,r,\varepsilon'>0\quad\exists\,T(r,\varepsilon')\geqslant 0$ such that $\|x_0\|<r\implies \mathsf{P}\Big(\sup_{t\geqslant T(r,\varepsilon')}\|x(t)\|<\varepsilon'\Big)=1.$ ◇

More generally a property of a random variable is said to hold almost surely if the set of events for which the property is true has probability measure 1.

III. Stability under Random Switching

In this section we establish sufficient conditions for almost sure global asymptotic stability of the switched system (2). We treat three cases of different assumptions on σ, and corresponding to each assumption we present one theorem. The applicability and the differences among the theorems are discussed in the remarks that follow; the steps of the proofs may be found in §V. We mention that Theorem 5 was stated and proved in [6]; since it takes very little extra work, we provide some of the details once again for completeness.

Hereafter we shall denote the number of switches on the time interval $[t,t'[$ by $N_\sigma(t,t')$.

We make use of multiple Lyapunov functions (see [13, Chapter 3] for an extensive treatment of multiple Lyapunov functions in the deterministic case), one for each subsystem. The following assumption collects the properties we shall require from them.

Assumption 2: There exist a family of continuously differentiable real-valued functions $(V_p)_{p\in\mathcal{P}}$ on $\mathbb{R}^n$, functions

$\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, numbers $\mu > 1$ and $\lambda_p \in \Lambda \subseteq \mathbb{R}$, $p \in \mathcal{P}$, such that

(V1) $\alpha_1(\|x\|) \leqslant V_p(x) \leqslant \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n, \ \forall\, p \in \mathcal{P}$;
(V2) $\mathrm{L}_{f_p} V_p(x) \leqslant -\lambda_p V_p(x) \quad \forall\, x \in \mathbb{R}^n, \ \forall\, p \in \mathcal{P}$;
(V3) $V_{p_1}(x) \leqslant \mu V_{p_2}(x) \quad \forall\, x \in \mathbb{R}^n, \ \forall\, p_1, p_2 \in \mathcal{P}$. ◊

Remark 3: (V1) is a fairly standard hypothesis, ensuring V_p's are each positive definite and radially unbounded. (V2) furnishes a quantitative estimate of the degree of stability or instability, depending on the sign of λ_p, of each subsystem of the family (1). The possible values that the λ_p's are allowed to take is specified by the set Λ. (To wit, if there are unstable subsystems, we allow Λ to contain negative real numbers so that the corresponding λ_p's may be negative; if there are no unstable subsystems, Λ is a subset of the positive real numbers.) The right-hand side of the inequality in (V2) being linear in V_p is no loss of generality, see e.g., [19, Theorem 2.6.10] for details. (V3) certainly restricts the class of functions that the family $(V_p)_{p\in\mathcal{P}}$ can belong to; however, this hypothesis is commonly employed in the deterministic case [13, Chapter 3]. Quadratic Lyapunov functions universally utilized in the case of linear subsystems satisfy this hypothesis. ◁

We now present the results of this section in the three different cases below.

First case. In this case σ is a general càdlàg jump stochastic process, and merely an upper bound of its asymptotic probability distribution is known. The temporal probability distribution of σ on $\mathcal{P}$ is completely unknown.

Assumption 4: The switching signal is characterized by: $\exists\, M \in \mathbb{N} \cup \{0\}$ and $\overline{\lambda}, \widetilde{\lambda} > 0$, such that $\forall\, k \geqslant M$ we have $\mathsf{P}(N_\sigma(0,t) = k) \leqslant \left(\overline{\lambda} t\right)^k \mathrm{e}^{-\widetilde{\lambda} t}/k!$. ◊

Theorem 5 ([6]): Consider the system (2). Suppose that

(G1) Assumption 2 holds with $\Lambda = \{\lambda_\circ\}$, $\lambda_\circ > 0$;
(G2) σ satisfies Assumption 4;
(G3) $\mu < \left(\lambda_\circ + \widetilde{\lambda}\right)/\overline{\lambda}$.

Then (2) is GAS a.s.

Second case. In this case Assumption 4 is replaced by Assumption 6 below; this imposes additional structure on the stochastic properties of σ.

Assumption 6: The switching signal σ is characterized by:

(EH1) $(S_i)_{i\in\mathbb{N}}$, with $S_i := \tau_i - \tau_{i-1}$, is a sequence of independent identically distributed sequence of exponential-λ random variables;[1]
(EH2) $\exists\, q_p \in [0,1]$, $p \in \mathcal{P}$ such that $\forall\, i \in \mathbb{N}$, $\mathsf{P}\big(\sigma(\tau_i) = p \mid (\sigma(\tau_j))_{j=0}^{i-1}\big) = q_p$;
(EH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$. ◊

Theorem 7: Consider the system (2). Suppose that

(E1) Assumption 2 holds with $\Lambda = \mathbb{R}$;
(E2) σ satisfies Assumption 6;
(E3) $\lambda_p + \lambda > 0 \quad \forall\, p \in \mathcal{P}$;
(E4) $\displaystyle\sum_{p\in\mathcal{P}} \frac{\mu q_p}{(1 + \lambda_p/\lambda)} < 1$.

Then (2) is GAS a.s.

Third case. In this case Assumption 6 is replaced by Assumption 8 below; this imposes a different structure on the stochastic properties of σ compared to the second case.

Assumption 8: The switching signal σ is characterized by:

(UH1) $(S_i)_{i\in\mathbb{N}}$, with $S_i := \tau_i - \tau_{i-1}$, is a sequence of independent identically distributed sequence of uniform-T random variables;[2]
(UH2) $\exists\, q_p \in [0,1]$, $p \in \mathcal{P}$ such that $\forall\, i \in \mathbb{N}$, $\mathsf{P}\big(\sigma(\tau_i) = p \mid (\sigma(\tau_j))_{j=0}^{i-1}\big) = q_p$.
(UH3) $(S_i)_{i\in\mathbb{N}}$ is independent of $(\sigma(\tau_i))_{i\in\mathbb{N}}$. ◊

Theorem 9: Consider the system (2). Suppose that

(U1) Assumption 2 holds with $\Lambda = \mathbb{R}$;
(U2) σ satisfies Assumption 8;
(U3) $\displaystyle\sum_{p\in\mathcal{P}} \left(\frac{\mu q_p \left(1 - \mathrm{e}^{-\lambda_p T}\right)}{\lambda_p T} \right) < 1$.

Then (2) is GAS a.s.

Remarks and discussion. We now examine in detail the three cases listed above.

Remark 10: Intuitively, Assumption 4 requires that statistically the rate of switching is not too large in the long run. More specifically, the expected number of switches on the interval $[0,t[$ grows at most exponentially with t. Indeed, $\mathsf{E}[N_\sigma(0,t)] = \sum_{k=0}^\infty k\mathsf{P}(N_\sigma(0,t) = k)$, and this is upper bounded by $S + \sum_{k=M}^\infty k\mathsf{P}(N_\sigma(0,t) = k)$, where S is a constant depending on M, which finally is in turn upper bounded by $S' + (\overline{\lambda} t)\mathrm{e}^{(\overline{\lambda} - \widetilde{\lambda})t}$, where S' is a constant depending on M and greater than S. Assumption 4 may therefore be regarded as a statistically slow switching condition. ◁

Remark 11: On the one hand, note that Assumption 4 does not put any restrictions on the temporal probability distribution of σ on $\mathcal{P}$. Consequently, if one subsystem in the family $(f_p)_{p\in\mathcal{P}}$ is unstable, and the switching signal obeys Assumption 4 but activates this subsystem for most of the time, the switched system may well become unstable. It follows that this assumption alone is not strong enough for GAS a.s. of the switched system, and a further necessary (but not sufficient) condition is that each subsystem is stable. On the other hand both Assumption 6 and Assumption 8 require the existence of a stationary and memoryless transition probability distribution of σ on $\mathcal{P}$ ((EH2) and (UH2), respectively), and are therefore better equipped to take into account instabilities of some subsystems. ◁

Remark 12: Theorem 5 is intuitively quite appealing; it states that if each subsystem has sufficient *stability margin*, and σ switches sufficiently slowly on an average, then the switched system is GAS a.s.. By (G1), there is a uniform stability margin (in terms of the Lyapunov functions) among the family of subsystems. (G3) links the deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2, with the properties of the switching signal furnished by (G2). It is clear that the more stable the

[1] Recall that the probability distribution function of an exponential random variable X of parameter λ is $\mathsf{P}(X \leqslant s) = 1 - \mathrm{e}^{-\lambda s}$ if $s \geqslant 0$, and 0 otherwise; see e.g. [20] for further details.

[2] Recall that the probability distribution function of a uniform random variable X of parameter λ is $\mathsf{P}(X \leqslant s) = s/T$ if $s \in [0,T]$, and 0 otherwise; see e.g. [20] for further details.

subsystems (larger the $\lambda_\circ$), the faster can be the switching signal (larger the $\overline{\lambda}$) that still renders (2) GAS a.s.. This result is reminiscent of the well-known result [13, Theorem 3.2] on global asymptotic stability of deterministic switched systems under average dwell-time switching; see [6] for a detailed comparison. Moreover, this theorem applies to the case of σ being the state of a continuous-time Markov chain with a given generator matrix; for further details on this important case please see [6]. $\lhd$

Remark 13: Let us examine the statement of Theorem 7 in some detail. Firstly, note that by (E1) not all subsystems are required to be stable, i.e., for some $p \in \mathcal{P}$, λ_p can be negative; then (V2) provides a measure of the rate of instability of the corresponding subsystems. Secondly, note that condition (E3) is always satisfied if each $\lambda_p > 0$. However, if $\lambda_p < 0$ for some $p \in \mathcal{P}$, then (E3) furnishes a maximum instability margin of the corresponding subsystems that can still lead to GAS a.s. of (2). Intuitively, in the latter case, the process $N_\sigma(0,t)$ must switch fast enough ($\lambda > 0$ large enough) so that the unstable subsystems are not active for too long. Potentially this fast switching may have a destabilizing effect. Indeed, it may so happen that for a given μ, a fixed set $(q_p)_{p\in\mathcal{P}}$, and a choice of functions $(V_p)_{p\in\mathcal{P}}$, (E3) and (E4) may be impossible to satisfy simultaneously, due to a very high degree of instability of even one subsystem for which the corresponding q_p is also large. Then we need to search for a different family of functions $(V_p)_{p\in\mathcal{P}}$ for which the hypotheses hold. Thirdly, (E4) links the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2, with the properties of the switching signal. From (E4) it is clear that larger degrees of instability of a subsystem (larger λ_p) can be compensated by a smaller probability (smaller q_p) of the switching signal activating the corresponding subsystem. $\lhd$

Remark 14: Let us observe some features of Theorem 9. Just like Theorem 7, note that by (U1) not all subsystems are required to be stable, i.e., for some $p \in \mathcal{P}$, λ_p can be negative. (U3) connects the properties of deterministic subsystem dynamics, furnished by the family of Lyapunov functions satisfying Assumption 2, with the properties of the switching signal. Also from (U3) it is clear that larger degrees of instability (larger λ_p) of a subsystem can be compensated by a smaller probability (smaller q_p) of the switching signal activating the corresponding subsystem. $\lhd$

Remark 15: It may appear that Theorem 7 requires a larger set of hypotheses compared to Theorem 9; however, this is only natural. Indeed, the switching signal in the latter case is constrained to switch at least once in T units of time, whereas no such constraint is present on the switching signal in the former case. We observed in Remark 13 that it is necessary for the switching signal to switch fast enough if there are unstable subsystems in the family (1), which accounted for (E3). This fast enough switching is automatic if σ satisfies Assumption 8, provided T is related to the instability margin of the subsystems in a particular way. (U3) captures this relationship, for, observe that if λ_p is negative and large in magnitude for some $p \in \mathcal{P}$, the ratio $\left(1 - \mathrm{e}^{-\lambda_p T}\right)/(\lambda_p T)$ is small provided T is small, and a smaller ratio is better for GAS a.s. of (2); also for a given T, large and positive λ_p's (i.e., subsystems with high margins of stability) make the aforesaid ratio small. $\lhd$

Remark 16: We provided sufficient conditions for *global asymptotic stability in the mean* (GAS-M) of (2) in [6], under the hypotheses of Theorem 5. It is not difficult to establish that if the hypotheses of Theorem 7 hold, and if α_1 in (V1) is convex, then (2) is GAS-M; the proof utilizes Jensen's inequality in (V1), coupled with a little careful analysis involving the final step of Lemma 24. An identical conclusion holds with Theorem 9 in place of Theorem 7 in the previous statement. Even without convexity assumption on α_1, it is possible to prove GAS-M following the approach of [6, Corollary 3.19]. These details are documented in [21]. Once GAS-M of (2) is established, the *global asymptotic stability in probability* of (2) follows via a standard application of Chebyshev's inequality; see e.g., [22] for details. Also note that the case of σ being the state of a continuous-time Markov chain is not covered by Assumptions 6 or 8; the extension of our results involving unstable subsystems to this important case is a subject of future work. $\lhd$

IV. Stabilization under Random Switching

In this section we provide a methodology for designing controllers that ensure almost sure global asymptotic stability of control-affine randomly switched systems in closed loop.

Consider the affine in control switched system:

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{\sigma,i}(x)u_i, \qquad x(0) = x_0, \quad t \geqslant 0, \quad (3)$$

where $x \in \mathbb{R}^n$ is the state, $u_i,\ i = 1,\ldots,m$ are the control inputs, f_p and $g_{p,i}$ are smooth vector fields on $\mathbb{R}^n$, with $f_p(0) = 0, g_{p,i}(0) = 0$, for each $p \in \mathcal{P}, i \in \{1,\ldots,m\}$. Let $\mathcal{U}$ be the set where the control $u := [u_1,\ldots,u_m]^{\mathrm{T}}$ takes its values. For the moment, we let $\mathcal{U}$ be a subset of $\mathbb{R}^m$; later we shall consider the case when $\mathcal{U}$ is a more general set, e.g. a Minkowski ball. With a feedback control function $\overline{u}_\sigma(x) = [u_{\sigma,1}(x),\ldots,u_{\sigma,m}(x)]^{\mathrm{T}}$, the closed loop system stands as:

$$\dot{x} = f_\sigma(x) + \sum_{i=1}^{m} g_{\sigma,i}(x)\overline{u}_{\sigma,i}(x), \quad x(0) = x_0, \ \ t \geqslant 0. \quad (4)$$

Our objective is to choose the control function $\overline{u}_\sigma$ so that (4) is GAS a.s. Let the switching signal σ be a stochastic process as defined in §II, and let $x_0 \neq 0$.

We now describe the controller design methodology promised in §I.

A universal formula for stabilization of control-affine nonlinear systems was first constructed in [7], for the control taking values in $\mathcal{U} = \mathbb{R}^m$. The articles [14], [15], [16] provided universal formulae for bounded controls, positive controls, and controls restricted to Minkowski balls, respectively. In view of the analysis results of §III and the universal formulae provided in the aforementioned articles, it is possible to synthesize controllers $\overline{u}_\sigma$ for (3), such that the closed loop system (4) is GAS a.s. Recall that three different types of switching signals were considered in §III; the corresponding hypotheses on them appear in Assumptions 4, 6, and 8. In general, we obtain one synthesis scheme for each type of $\mathcal{U}$ and σ; the following theorem provides a typical illustration of such a result. A complete recipe to obtain such results is provided in Remark 18.

Theorem 17: Consider the system (3), with $\mathcal{U} = \mathbb{R}^m$. Suppose that σ satisfies Assumption 6, and there exists a family of continuously differentiable functions $\big(V_p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}\big)_{p\in\mathcal{P}}$, such that

(C1) (V1) of Assumption 2 holds;

(C2) (V3) of Assumption 2 holds;

(C3) $\exists\, \lambda_p \in \Lambda = \mathbb{R},\ p \in \mathcal{P}$, such that $\forall\, x \in \mathbb{R}^n \setminus \{0\}$ and $\forall\, p \in \mathcal{P}$

$$\inf_{u\in\mathcal{U}} \left\{ \mathrm{L}_{f_p} V_p(x) + \lambda_p V_p(x) + \sum_{i=1}^{m} u_i \mathrm{L}_{g_{p,i}} V_p(x) \right\} < 0;$$

(C4) ((E3), (E4)) holds.

Then the feedback control function

$$\overline{u}_\sigma(x) = [k_{\sigma,1}(x), \ldots, k_{\sigma,m}(x)]^{\mathrm{T}},$$

where

$$k_{p,i}(x) := \begin{cases} -\mathrm{L}_{g_{p,i}} V_p(x) \cdot \varphi\Big(\overline{W}_p(x), \widetilde{W}_p(x)\Big) & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5a}$$

$$\overline{W}_p(x) := \mathrm{L}_{f_p} V_p(x) + \lambda_p V_p(x), \tag{5b}$$

$$\widetilde{W}_p(x) := \sum_{i=1}^{m} \big(\mathrm{L}_{g_{p,i}} V_p(x)\big)^2, \tag{5c}$$

and

$$\varphi(a, b) := \begin{cases} \dfrac{a + \sqrt{a^2 + b^2}}{b} & \text{if } b \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5d}$$

renders (4) GAS a.s.

Proof: The proof relies heavily on the construction of the universal formula in [7]. Fix $t \in \mathbb{R}_{\geqslant 0}$. If $x \neq 0$, applying the definition of φ, we get

$$\begin{aligned} &\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \sum_{i=1}^{m} k_{\sigma(t),i}(x) \mathrm{L}_{g_{\sigma(t),i}} V_{\sigma(t)}(x) \\ &= \mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) - \widetilde{W}_{\sigma(t)}(x) \cdot \varphi\Big(\overline{W}_{\sigma(t)}(x), \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^2\Big) \\ &= -\lambda_{\sigma(t)} V_{\sigma(t)}(x) - \sqrt{\big(\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x)\big)^2 + \Big(\widetilde{W}_{\sigma(t)}(x)\Big)^2} \\ &< -\lambda_{\sigma(t)} V_{\sigma(t)}(x). \end{aligned}$$

Since t is arbitrary, we conclude that the above inequality holds for all $t \in \mathbb{R}_{\geqslant 0}$. Note that by (C3), if for any $p \in \mathcal{P}$, $x \in \bigcap_{i=1}^{m} \ker\big(\mathrm{L}_{g_{p,i}} V_p\big)$, we automatically have $\mathrm{L}_{f_{\sigma(t)}} V_{\sigma(t)}(x) + \lambda_{\sigma(t)} V_{\sigma(t)}(x) < 0$.

The above arguments, in conjunction with (C1) and (C2) enable us to conclude that the family $(V_p)_{p\in\mathcal{P}}$ satisfies Assumption 2 for the closed loop system (4) and $\Lambda = \mathbb{R}$. (C4) ensures that (E3) and (E4) hold, respectively, for (4). Since σ satisfies Assumption 4, (E2) holds as well. Hence, it follows from Theorem 7 that (4) is GAS a.s. ∎

Remark 18: Theorem 17 can be modified to suit a different $\mathcal{U}$ and a different type of σ using the following simple recipe. First, recall from the discussion preceding Theorem 17 that $\mathcal{U}$ may be any one among $\mathbb{R}^m$, the nonnegative orthant of $\mathbb{R}^m$, a bounded subset of $\mathbb{R}^m$, and a Minkowski ball in $\mathbb{R}^m$; σ may satisfy any one of Assumptions 4, 6, and 8. Now suppose that a $\mathcal{U}$ and a σ among the above possibilities is given to us. Then:

- (C1) and (C2) remain unchanged;
- the given $\mathcal{U}$ replaces the $\mathcal{U} = \mathbb{R}^m$ in Theorem 17;
- if the given σ satisfies Assumption 4, then this assumption replaces Assumption 6, the pair ((E3), (E4)) appearing in hypothesis (C4) is replaced by (G3), and Λ appearing in (C3) is replaced by the set $\{\lambda_\circ\}$;
- if the given σ satisfies Assumption 8, then this assumption replaces Assumption 6, the pair ((E3), (E4)) appearing in hypothesis (C4) is replaced by (U3), and Λ appearing in (C3) is replaced by the set $\mathbb{R}$;
- the universal formula corresponding to the given $\mathcal{U}$ replaces the one given in (5). ◁

V. Proofs

This section contains the key steps leading to the proofs of our results in §III. Due to constraints of space most of the proofs are omitted; see [21] for a version containing complete proofs of all the statements.

Recall that the random variable $N_\sigma(t, t')$ gives the number of switches of σ on the interval $[t, t'[$, and $(\tau_i)_{i\in\mathbb{N}}$ is the set of switching instants. We also define $N_\sigma(0, 0) := 0$. The extended real-valued random variable $\zeta := \sup_{\nu\in\mathbb{N}} \tau_\nu$ is the *explosion time* [17, Chapter 1] of the process $(N_\sigma(0, t))_{t\in\mathbb{R}_{\geqslant 0}}$. If $\zeta = \infty$, then the process $(N_\sigma(0, t))_{t\in\mathbb{R}_{\geqslant 0}}$ is said to have *no explosions*; we shall also say that under this condition σ has no explosions.

The proofs of the theorems in §III are provided after the following technical lemmas.

Lemma 19: Suppose σ satisfies Assumption 4. Then $N_\sigma(0, t) \longrightarrow \infty$ a.s. only if $t \longrightarrow \infty$; i.e., almost surely σ has no explosion.

Lemma 20: Suppose σ satisfies Assumption 6. Then $N_\sigma(0, t) \longrightarrow \infty$ a.s. if and only if $t \longrightarrow \infty$.

Lemma 21: Suppose σ satisfies Assumption 8. Then $N_\sigma(0, t) \longrightarrow \infty$ a.s. if and only if $t \longrightarrow \infty$.

Lemma 22: Consider the system (2). Suppose that Assumption 2 holds, and for every nonnegative monotonically increasing divergent sequence $(s_i)_{i\in\mathbb{N}}$, we have $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$. Then $V_{\sigma(t)}(x(t)) \longrightarrow 0$ as $t \longrightarrow \infty$ almost surely.

Lemma 23: Consider the system (2). Suppose that the hypotheses of Theorem 5 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

Lemma 24: Consider the system (2). Suppose that the hypotheses of Theorem 7 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

Lemma 25: Consider the system (2). Suppose that the hypotheses of Theorem 9 hold. Then for every nonnegative, monotonically increasing, divergent sequence $(s_i)_{i\in\mathbb{N}}$ we have $\limsup_{i\to\infty} \mathsf{E}\big[V_{\sigma(s_i)}(x(s_i))\big] = 0$.

Lemma 26: The system (2) has the following property: for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that $\|x(t)\| \leqslant \|x_0\| \,\mathrm{e}^{L_\varepsilon t}\ \ \forall\, t \geqslant 0$ as long as $\|x(t)\| < \varepsilon$.

Proof: [Proof of Theorem 5] We need to establish the properties (AS1)-(AS2) of (2).

First we prove (AS2). Fix $r, \varepsilon' > 0$. Lemma 23 shows that the assertion of Lemma 22 holds. In view of (V1) and Lemma 22, we can now write $\lim_{t\to\infty} \alpha_1(\|x(t)\|) = 0$ a.s.; hence there exists $T(r,\varepsilon') \geqslant 0$ such that $\|x_0\| < r \implies \mathsf{P}\Big(\sup_{t\geqslant T(r,\varepsilon')} \alpha_1(\|x(t)\|) < \alpha_1(\varepsilon')\Big) = 1$. Since r, ε' are arbitrary, we conclude that $\forall\, r, \varepsilon' > 0$ there exists $T(r,\varepsilon') \geqslant 0$ such that $\|x_0\| < r \implies \mathsf{P}\Big(\sup_{t\geqslant T(r,\varepsilon')} \|x(t)\| < \varepsilon'\Big) = 1$. The (AS2) property of (2) follows.

It remains to prove (AS1). Fix $\varepsilon > 0$. We know from the (AS2) property proved above that there exists a nonnegative real number $T(1,\varepsilon)$, so that $\|x_0\| < 1$ implies $\mathsf{P}\Big(\sup_{t\geqslant T(1,\varepsilon)} \|x(t)\| < \varepsilon\Big) = 1$. Select $\delta(\varepsilon) = \varepsilon \mathrm{e}^{-L_\varepsilon T(1,\varepsilon)} \wedge 1$. By Lemma 26, $\|x_0\| < \delta(\varepsilon)$ implies

$$\|x(t)\| \leqslant \|x_0\| \,\mathrm{e}^{L_\varepsilon t} < \delta(\varepsilon)\mathrm{e}^{L_\varepsilon T(1,\varepsilon)} < \varepsilon \;\forall\, t \in [0, T(1,\varepsilon)].$$

Further, the (AS2) property guarantees that with the above choice of δ and x_0, we have $\mathsf{P}\Big(\sup_{t\geqslant T(1,\varepsilon)} \|x(t)\| < \varepsilon\Big) = 1$. Thus, $\|x_0\| < \delta(\varepsilon)$ implies $\mathsf{P}\big(\sup_{t\geqslant 0} \|x(t)\| < \varepsilon\big) = 1$. Since ε is arbitrary, the (AS1) property of (2) follows.

We conclude that (2) is GAS a.s. ■

Proof: [Proof of Theorem 7] The proof repeats verbatim that of Theorem 5, with just Lemma 24 substituted in place of Lemma 23. ■

Proof: [Proof of Theorem 9] The proof repeats verbatim that of Theorem 5, with just Lemma 25 substituted in place of Lemma 23. ■

VI. Conclusion and Further Work

As mentioned in §I, a necessary condition for applying our control synthesis methodology is that the controller for every subsystem can be so placed that the switching signal activates each closed loop subsystem. In other words, the controller must have perfect information of σ at each instant of time. This leads us to ask whether it is possible to design one stabilizing controller for the switched control system, which gets imperfect or no information about σ.

In the deterministic context, the problem of simultaneous stabilization of multiple systems can be thought of as a possible approach to the case when the controller gets no information about σ. Indeed, if a single controller stabilizes each subsystem, then under a sufficiently slow switching hypotheses (e.g. Assumption 4 with small enough $\overline{\lambda}$), the closed loop switched system will be GAS a.s. But in general the problem of simultaneous stabilization is restrictive and difficult. However, if there exists a controller that stabilizes a subfamily of $(f_p)_{p\in\mathcal{P}}$ and at the same time does not destabilize the others subsystems too much, the theorems of §III can be applied to the closed loop switched system. Such results will be reported elsewhere.

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