Quantized feedback systems perturbed by white noise

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Abstract

This paper is concerned with a class of nonlinear feedback systems perturbed by white noise, for which the nonlinearity takes the form of a piecewise constant function of a certain type. We obtain explicit formulae for steady-state probability densities associated with such systems. This result is used to treat a stochastic optimal control problem that can be interpreted as minimization of the cost of implementing a feedback control law.

1 Introduction

Given a positive integer M and a nonnegative real number Δ, we define the *quantizer* $q : \mathbb{R} \to \mathbb{Z}$ with *sensitivity* Δ and *saturation value* M by the formula

$$q(x) = \begin{cases} M, & x \geq (M+1/2)\Delta \\ -M, & x < -(M+1/2)\Delta \\ \left\lfloor \dfrac{x}{\Delta} + \dfrac{1}{2} \right\rfloor, & -(M+1/2)\Delta \leq x < (M+1/2)\Delta \end{cases}$$

Thus on each interval $J_k := [(k-1/2)\Delta, (k+1/2)\Delta)$, where $k \in \mathbb{Z}$ and $-M \leq k \leq M$, the function q takes on the value k.

We consider the quantized feedback system

$$dx = Ax dt + G dw + bq(c^T x) dt \qquad (1)$$

where $x, b, c \in \mathbb{R}^n$, w is an m-dimensional Wiener process, and A and G are matrices of appropriate dimensions. All stochastic differential equations in this paper are to be interpreted in the Itô sense. Let us make the following two assumptions regarding the system (1).

a) All eigenvalues of A have negative real parts.

b) (A,G) is a controllable pair.

The linear case ($b = 0$) is well understood: under the above assumptions, the steady-state probability density (i.e., the solution of the equation $L\rho(x) = 0$, where L is the Fokker-Planck operator) is a Gaussian with mean 0 and variance Q satisfying the equation

$$AQ + QA^T + GG^T = 0. \qquad (2)$$

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The situation in the nonlinear case is much more difficult. Lyapunov-like methods have been used to prove the existence of steady-state probability distributions for quite general classes of nonlinear systems (this work can be traced back to at least [5]). However, these results say nothing about a particular form of a steady-state density. We will formulate a condition on the parameters of the system (1) which enables us to obtain an explicit formula for a steady-state probability density, and use this result to investigate a stochastic optimal control problem associated with the steady-state performance of (1).

2 Compatible systems

Let us denote by Q the positive definite solution of (2). It is not hard to see that the function

$$\rho(x) = N e^{-\frac{1}{2}(x+A^{-1}bk)^T Q^{-1}(x+A^{-1}bk)+d_k} \quad \text{if} \quad c^T x \in J_k$$

with arbitrary constants N and d_k satisfies the equation for a steady-state probability density associated with (1) almost everywhere. This function is piecewise Gaussian. Clearly, if $x \in \mathbb{R}$, we can always determine particular values of d_k so as to make ρ continuous. However, this is not necessarily true in the multidimensional case. We will say that the system (1) is *compatible* if the following *compatibility condition* is satisfied:

$$b = \lambda AQc \quad \text{for some } \lambda \in \mathbb{R}. \qquad (3)$$

This condition is precisely what makes it possible to obtain a continuous steady-state probability density.

Theorem 1 *If the compatibility condition (3) is satisfied, then the process described by the system (1) admits a steady-state probability distribution with a continuous piecewise Gaussian density.*

The proof is straightforward and will not be given. The above result can in fact be directly generalized to a larger class of nonlinear feedback systems [2]. In [2] it is also shown that, at least for some initial probability distributions, convergence to steady state can be established, and that for an important class of systems the steady-state probability distribution is unique.

3 Stochastic optimal control

Consider the quantized output feedback system (1). Let us assume that it is compatible, and that q has saturation value $M = 1$:

$$q(x) = \begin{cases} 1, & x \geq \Delta/2 \\ 0, & -\Delta/2 \leq x < \Delta/2 \\ -1, & x < -\Delta/2 \end{cases}$$

In [3], several optimal control problems of the quadratic-gaussian type were considered. The performance criteria were formulated in terms of the steady-state probability density. In view of the remarks at the end of Section 2, solutions to such problems provide information about the behaviour of (1) for large t, when the difference between $\rho(t,x)$ and the steady-state can be viewed as negligible.

Let $\mathcal{E}$ denote the expectation with respect to the steady-state probability density. Every time the solution trajectory crosses one of the *switching hyperplanes* $c^T x = \pm\Delta/2$, we need to communicate to the controller a request to change the control value. This reflects the amount of "attention" needed for implementing the given control law (a similar idea is exploited in [1] in the context of deterministic systems with smooth control functions). One might thus be interested in minimizing the number of such crossings per unit time. Making use of the fact that the steady-state probability density is an even function of x, let us define the *attention cost* to be $C = 2\mathcal{E}C_{\Delta/2}(c^T x(t))$, where $\mathcal{E}C_u(\xi(t))$ stands for the mean number of crossings of a level u per unit time by a scalar stochastic process $\xi(t)$.

Since the expectation is computed with respect to the steady-state probability distribution, we may treat the process $c^T x(t)$ as stationary, assuming that it "has reached steady state". Therefore, we may use the Rice's formula for the mean number of crossings [4]

$$\mathcal{E}C_u(\xi(t)) = \frac{1}{\pi}\sqrt{\frac{-r''(0)}{r(0)}} \cdot e^{-u^2/2r(0)}$$

where $r(\tau) = \mathcal{E}\xi(t)\xi(t+\tau)$ is the autocorrelation function associated with a stationary stochastic process $\xi(t)$. In our case $r(\tau) = \lim_{t\to+\infty} c^T\mathcal{E}x(t)x^T(t+\tau)c$. Let us first study the following question: when is $\mathcal{E}C_{\Delta/2}(c^T x(t))$ finite? We will need the following easy

Lemma 2 *Assume that $r''(\tau)$ exists in some neighborhood of zero, possibly excluding zero itself. Then $\mathcal{E}C_{\Delta/2}(c^T x(t)) < \infty$ if and only if $r'(0) = 0$.*

Example 1. For the equation

$$dx = -xdt + dw - bq(x)dt, \quad x \in \mathbb{R},\ b > 0 \qquad (4)$$

we have $r'(0) = -\mathcal{E}x(t)(-x(t) - bq(x(t))) < 0$, hence the condition of Lemma 2 is not satisfied. Thus we see that it is in fact a nontrivial task to construct a control system with a finite attention cost.

Consider a general linear stochastic system

$$\begin{aligned} dx &= Axdt + Gdw \\ y &= c^T x \end{aligned}$$

We have $\lim_{t\to\infty}\mathcal{E}y(t)y(t+\tau) = \lim_{t\to\infty} c^T\mathcal{E}x(t)x(t+\tau)c = c^T Q e^{A^T\tau}c$, where Q is the steady-state variance matrix satisfying the equation (2). Therefore $r'(0) = c^T QA^T c = \langle c, AQc\rangle$. Premultiplying (2) by c^T and postmultiplying by c, we obtain $2\langle c, AQc\rangle = -\langle G^T c, G^T c\rangle$. Thus the condition of Lemma 2 is satisfied if and only if $G^T c = 0$. Thus we will have a finite attention cost if all the directions in which the noise can propagate are parallel to the switching hyperplanes. On the other hand, if G is a nonsingular $n \times n$ matrix, the attention cost will always be infinite.

The above discussion suggests replacing (4) by

$$\begin{aligned} dx &= -xdt + dw - bq(y)dt, \\ dy &= \beta xdt - \beta ydt \end{aligned} \qquad (5)$$

with $\beta > 0$. This system is compatible, and the corresponding attention cost is finite. It is not very difficult to compute it directly using our knowledge of the steady-state density $\rho(x,y)$ associated with (5). We have

$$\mathcal{E}y^2 = b^2\mathcal{P}(|y| > \Delta/2) + \frac{\beta}{\beta+1}\left(\frac{1}{2} - \mu b e^{-(\beta+1)\Delta^2/4\beta}\sqrt{\frac{\pi}{\beta+1}}\right)$$

where $\mu = \rho(0,0)$. The attention cost is given by

$$C = \frac{\sqrt{\beta}}{\pi}\sqrt{\frac{\frac{\beta}{2(\beta+1)}}{\mathcal{E}y^2}} \cdot 2\,e^{-\Delta^2/8\mathcal{E}y^2}.$$

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