QUANTIZED FEEDBACK SYSTEMS PERTURBED BY WHITE NOISE

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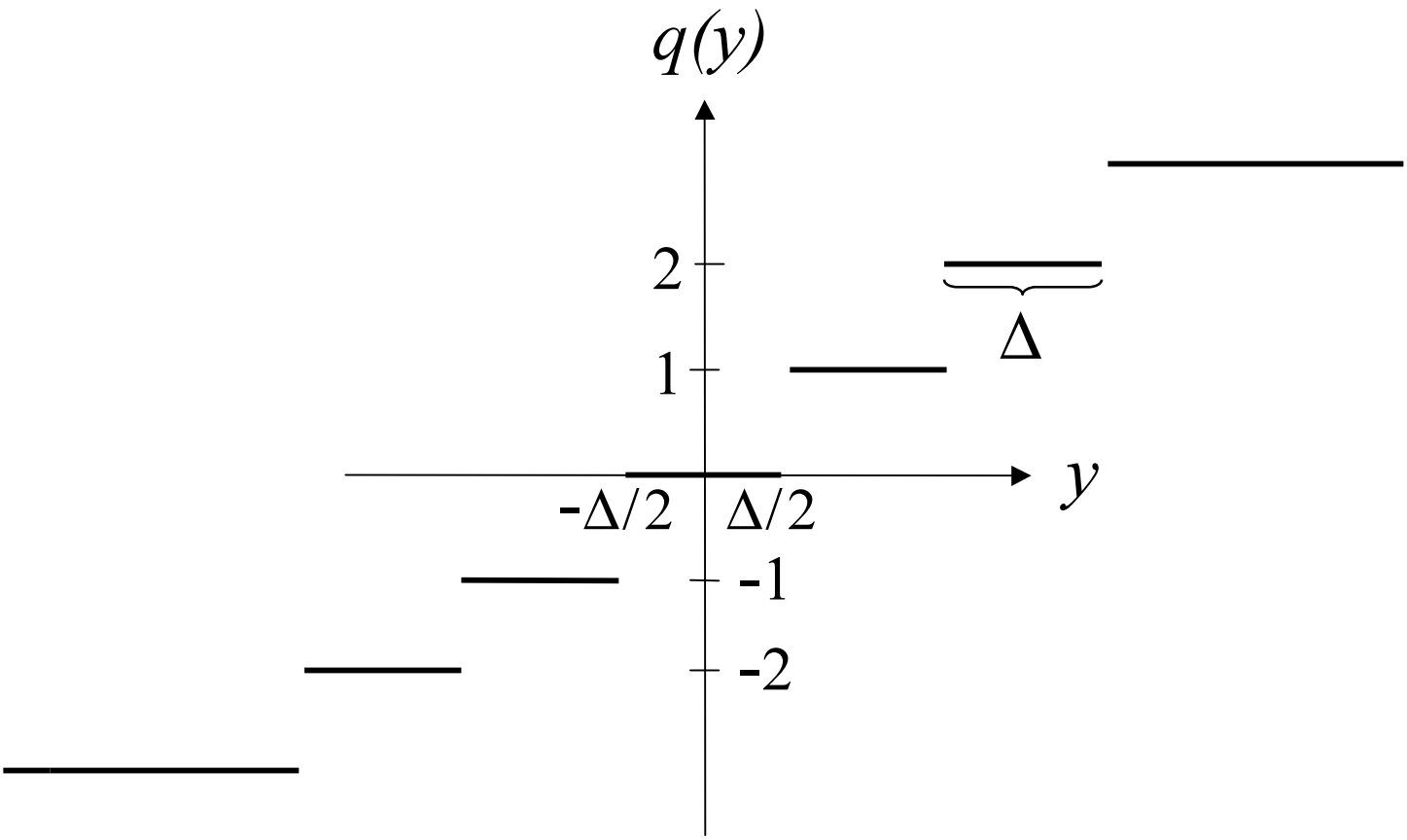
QUANTIZER

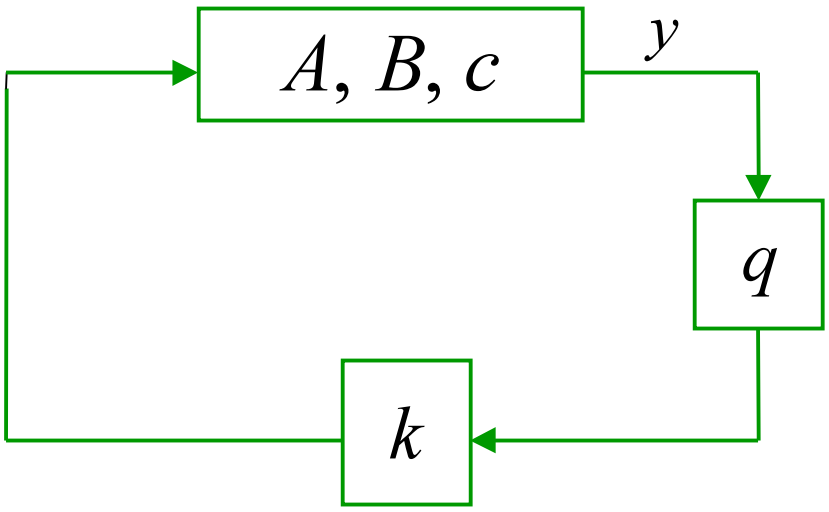


Δ – sensitivity of the quantizer

Examples: digital computer
temperature sensor
camera with zoom

THE SYSTEM

$$dx = Axdt + Bdw + kq(c^T x)dt$$



Assumptions:

- A stable
- (A,B) controllable

THE PROBLEM

Steady state prob. density:

$$L\rho = 0$$

In the linear case ($dx = Axdt + Bdw$) we have

$$\rho = e^{-\frac{1}{2}x^T Q^{-1} x}$$

where

$$AQ + QA^T = -BB^T$$

Nonlinear case:

Existence results are available

Explicit formulas - ?

$\mathbb{R}^2$: Rueda (1993)

$\mathbb{R}^n$: Brockett (1997)

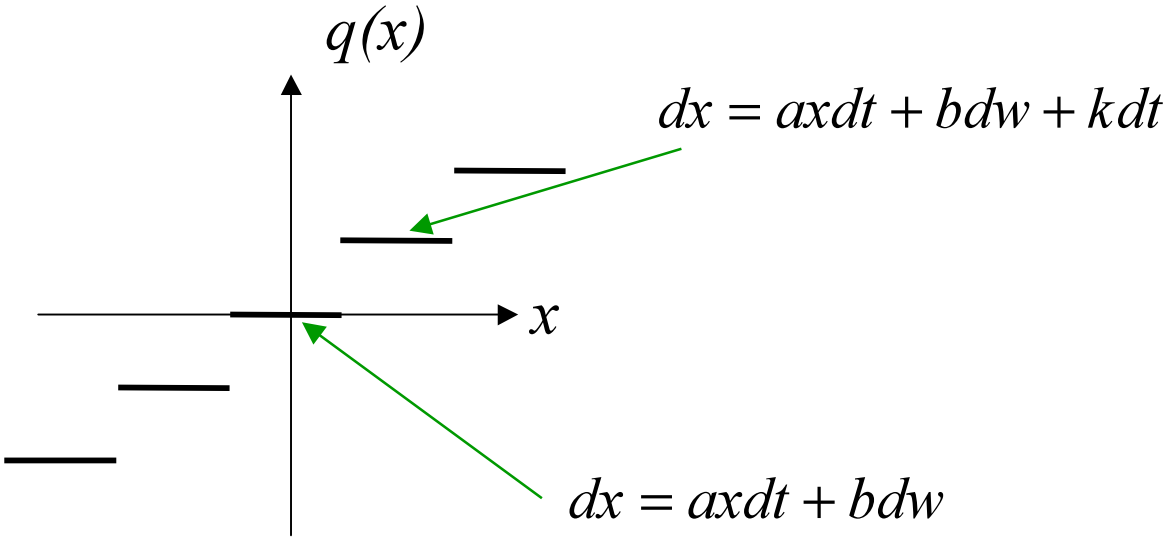
PLAN

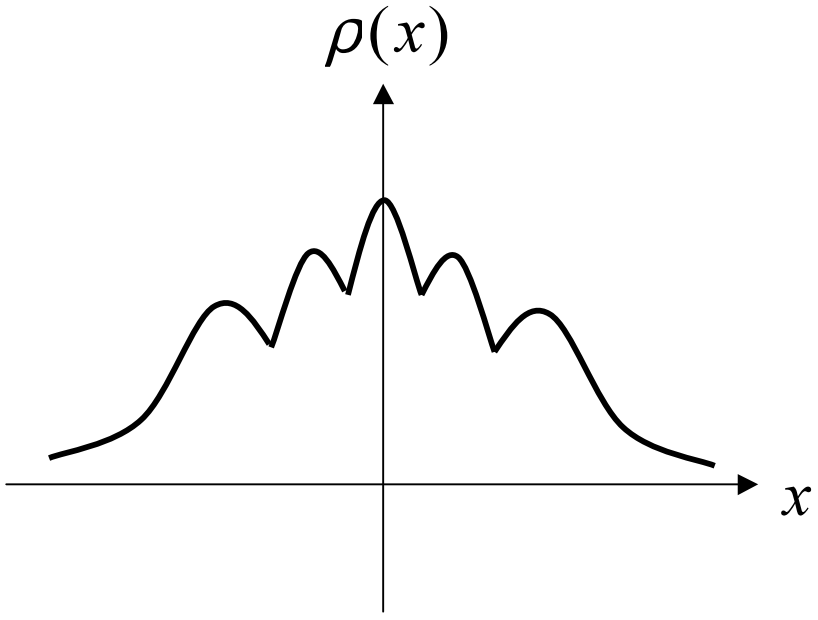
$$dx = Axdt + Bdw + kq(c^T x)dt$$

- Under what conditions can we compute the steady state density explicitly ?
- How does this steady state density depend on k and Δ ?

SCALAR CASE

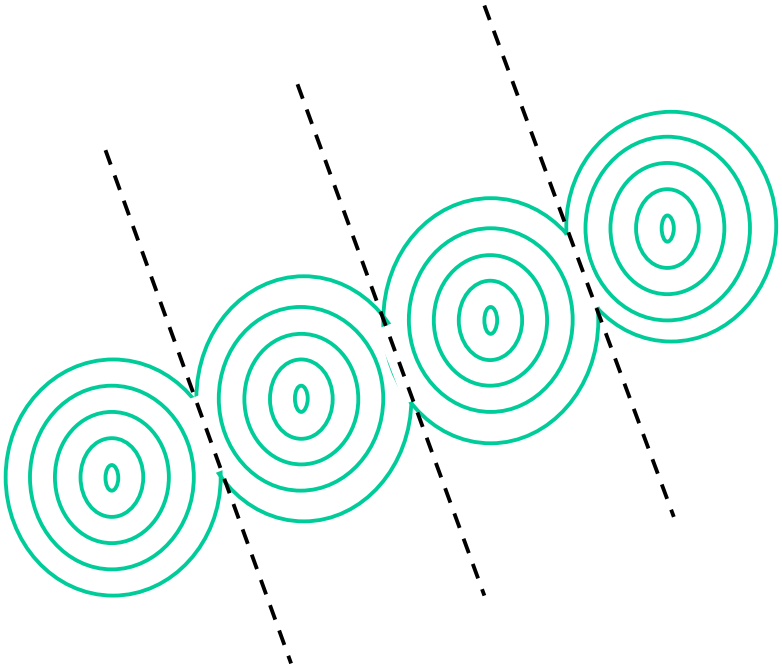
$$dx = axdt + bdw + kq(x)dt, \qquad x \in \mathbb{R}$$





COMPATIBILITY CONDITION

$$dx = Axdt + Bdw + kq(c^T x)dt, \qquad x \in \mathbb{R}^n$$



$\exists$ continuous piecewise Gaussian s.s.p.d. if

$$\boxed{k = \lambda AQc} \quad \text{for some } \lambda \in \mathbb{R}$$

As before, $AQ + QA^T = -BB^T$

MORE GENERAL CASE

Theorem. If the system

$$dx = Axdt + Bdw + kf(c^T x)dt$$

is compatible, i.e.

$$k = \lambda AQc, \qquad \lambda \in \mathbb{R}$$

then the function

$$\rho(x) = Ne^{-\frac{1}{2}x^T Q^{-1}x - \lambda F(c^T x)}$$

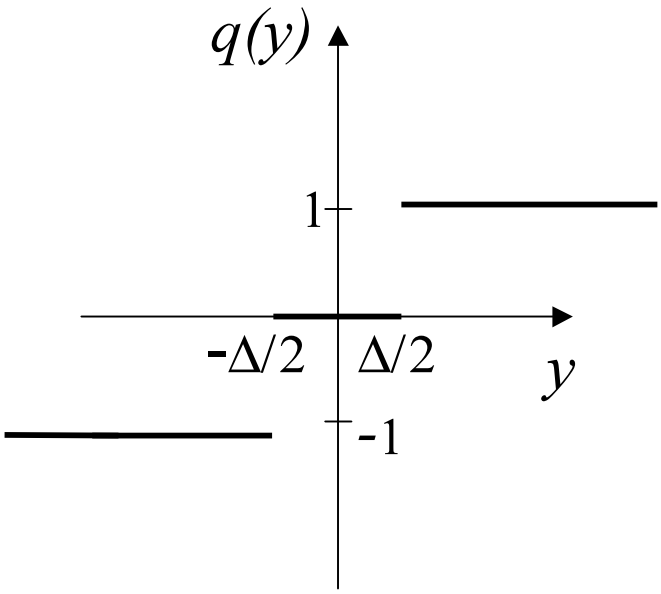
is a steady state density when $\rho \in L^1$

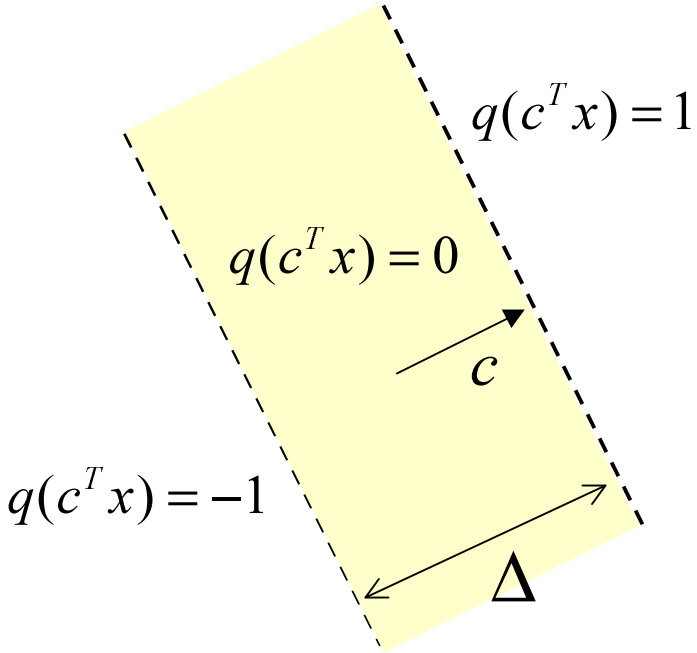
(true if $\lambda > 0$ and $x\,f(x) \geq 0$ for all x)

More on this in Brockett-Liberzon (ACC'98)

OPTIMAL CONTROL

$$dx = Axdt + Bdw + kq(c^T x)dt$$





E – expectation w.r.t. steady state

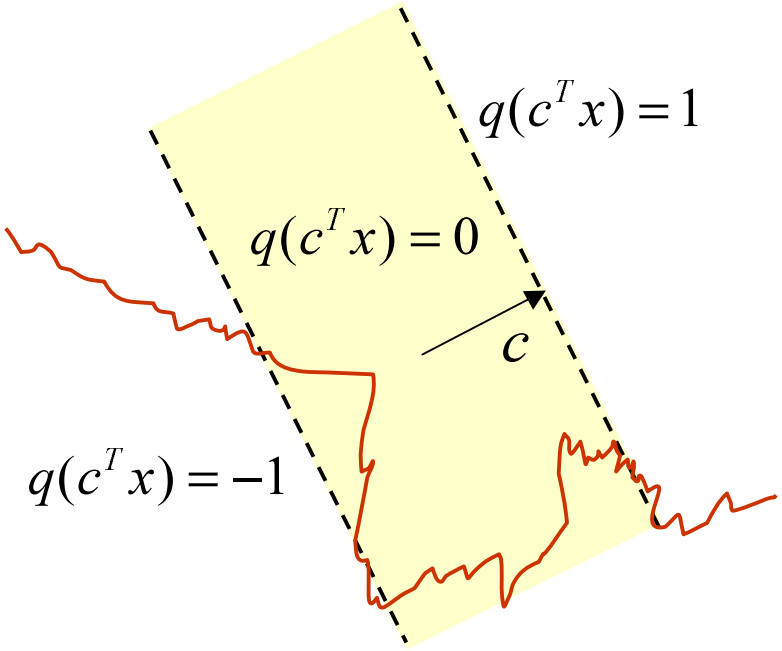
Variance cost: $V := E\left(c^T x\right)^2$

Control cost: $C := P\left(|\, c^T x \,| > \Delta/2\right)$

$V(\Delta)$ is increasing

$C(\Delta)$ is decreasing

ATTENTION COST



C := expected # of crossings per unit time

Rice's formula:

$$C = \frac{2}{\pi}\sqrt{\frac{-r''(0)}{r(0)}} \cdot e^{-\Delta^2/8r(0)}$$

where r is the autocorrelation function:

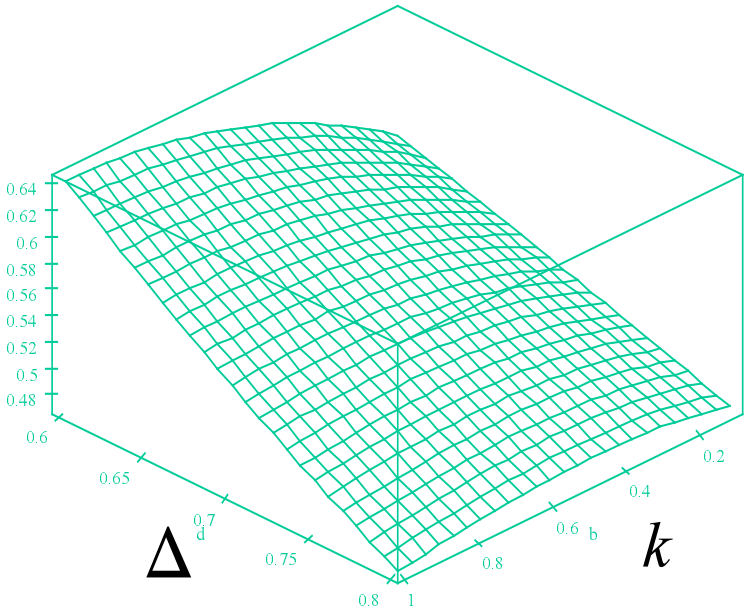
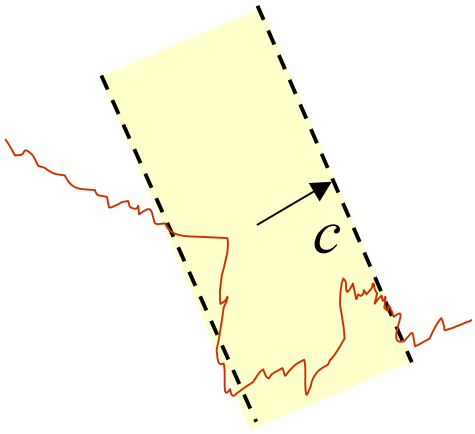
$$r(\tau) = \lim_{t \to \infty} E\, c^T x(t) c^T x(t+\tau)$$

EXAMPLE

1. $dx = -xdt + dw - kq(x)dt\,, \qquad x \in \mathbb{R},\ k > 0$

Attention cost is infinite

2. $dx = -x\,dt + dw - k\,q(y)\,dt$

$dy = b\,x\,dt - b\,y\,dt\,, \qquad b > 0$

Attention cost is finite: