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A hybrid control framework for systems with quantization

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Abstract

This paper is concerned with global asymptotic stabilization of continuous-time systems subject to quantization. A hybrid control strategy originating in earlier work relies on the possibility of making discrete on-line adjustments of quantizer parameters. We explore this method here for general nonlinear systems with general types of quantizers affecting the state of the system or the control input. The analysis involves merging tools from Lyapunov stability, hybrid systems, and input-to-state stability.

1 Introduction

In the classical feedback control setting, the output of the process is assumed to be passed directly to the controller, which generates the control input and in turn passes it directly back to the process. In practice, however, this paradigm often needs to be re-examined because the interface between the controller and the process features some additional information-processing devices.

One important aspect to take into account in such situations is signal quantization. We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values. Quantization may affect the process output (this happens, for example, when the output measurements to be used for feedback are obtained by using a digital camera, stored in the memory of a digital computer, or transmitted over a digital communication channel) or the control input (examples include the standard PWM amplifier and the manual transmission on a car).

We assume that the given system evolves in continuous time. In the presence of quantization, the state space (or the input space) of the system is divided into a finite number of *quantization regions*, each corresponding to a fixed value of the quantizer. At the time of passage from one quantization region to another, the dynamics of the system change abruptly. Therefore, systems with quantization can be naturally viewed as *hybrid* systems, i.e., systems described by a coupling between continuous and discrete dynamics.

There are two well-studied phenomena which account for changes in the system's behavior caused by quantization. The first one is saturation: if the signal is outside the range of the quantizer, then the quantization error is large, and the control law designed for the ideal case of no quantization leads to instability. The second one is deterioration of performance near the equilibrium: as the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is impossible. These phenomena manifest themselves in the existence of two nested invariant regions such that all trajectories of the quantized system starting in the bigger region approach the smaller one, while no further convergence guarantees can be given.

A standard assumption made in the literature is that parameters of the quantizer are fixed in advance and cannot be changed by the control designer (see, among many sources, [3, 4, 6, 11, 12, 15]). There has been some research concerned with the question of how the choice of quantization parameters affects the behavior of the system [1, 5, 8, 10]. In this paper, building on the earlier work reported in [2, 9], we adopt the approach that it is possible to vary some parameters of the quantizer *on line*, on the basis of collected data. (In the example where a quantizer is used to represent a camera, this corresponds to zooming in or out, i.e., varying the focal length, while the number of pixels of course remains fixed.) When such manipulations are feasible, they allow one to change the range of the quantizer and the quantization error as the system evolves, thereby helping to overcome the two difficulties described above.

The quantization parameters will be updated at discrete instants of time (these *switching events* will be triggered by the values of a suitable Lyapunov function). This results in a *hybrid quantized feedback control policy.* There are several reasons for adopting a hybrid control approach rather than varying the quantization parameters continuously. First, in specific situations there may be some constraints on how many values these parameters are allowed to take and how frequently they can be adjusted. Thus a discrete adjustment policy is more natural and easier to implement than a continuous one. Secondly, the analysis of hybrid systems obtained in this way appears to be more tractable than that of systems resulting from continuous parameter tuning. In fact, we will see that a method based on computation of invariant regions defined by level sets of a Lyapunov function provides a simple and effective tool for studying the behavior of the closed-loop system. This also implies that

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precise computation of the switching times is not essential, which makes our hybrid control policies robust with respect to time delays.

The recent paper [2] by Brockett and the author thoroughly investigates the hybrid control methodology outlined above in the context of the feedback stabilization problem for linear control systems with output (or state) quantization. It is shown there that if a linear system can be stabilized by a linear feedback law, then it can also be *globally asymptotically stabilized* by a hybrid quantized feedback control policy. The control strategy is usually composed of two stages. The first, "zooming-out" stage consists in increasing the range of the quantizer until the state of the system can be adequately measured. The second, "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. The developments of [2] were restricted to quantizers giving rise to rectilinear quantization regions.

The present work generalizes the contributions of [2] in three different directions. First, we consider more general types of quantizers, with quantization regions having *arbitrary shapes* (as in [11]). This extension is important for applications. For example, in the context of vision-based feedback control mentioned earlier, the image plane of the camera is divided into rectilinear regions, but the shapes of the quantization regions in the state space which result from computing inverse images of these rectangles can be rather complicated. We will demonstrate that the principal findings of [2] are still valid in this more general setting.

Another goal of this paper is to address the quantized feedback stabilization problem for *nonlinear systems*. It can be shown via a linearization argument that by using the approach of [2] one can obtain local asymptotic stability for a nonlinear system, provided that the corresponding linearized system is stabilizable (see [7]). Here we are concerned with achieving global stability results. We will show that the techniques developed in [2] can be extended in a natural way to those nonlinear systems that are *input-to-state stabilizable with respect to measurement disturbances*. We thus reveal an interesting interplay between the problem of quantized feedback stabilization, the theory of hybrid systems, and topics of current interest in nonlinear control design. A preliminary investigation of these questions has been reported in [9], but only for state quantizers with rectilinear quantization regions.

Finally, in this paper we present analogous results for systems with *input quantization*, both linear and nonlinear. In view of the examples given earlier, this expands the potential applicability of the hybrid quantized feedback control techniques. We discover that the analysis of systems with input quantization can be carried out quite similarly to the state quantization case. This analysis also yields a basis for comparing the effects of input quantization and state quantization on the performance of the system, which will be pursued elsewhere.

2 Quantizer

By a *quantizer* we mean a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. This leads to a partition of $\mathbb{R}^n$ into a finite number of *quantization regions* of the form $\{z \in \mathbb{R}^n : q(z) = l\}$, $l \in \mathcal{Q}$. The shapes of these quantization regions are arbitrary. When z does not belong to the union of quantization regions of finite size, the quantizer *saturates*. More precisely, we assume that there exist positive real numbers M and Δ such that the following two conditions hold:

1. If
$$|z| \le M \tag{1}$$
then
$$|z - q(z)| \le \Delta. \tag{2}$$
2. If $|z| > M$ then $|q(z)| > M - \Delta$.

Condition 1 gives a bound on the quantization error when the quantizer does not saturate. Condition 2 provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* of q and the *quantization error*, respectively. To preserve the equilibrium at the origin, we also assume that $q(0) = 0$. An example of a quantizer satisfying the above requirements is provided by the quantizer with rectangular quantization regions considered in earlier work [2, 9].

In the control strategies to be developed below, we will use quantized measurements of the form

$$\mu q\Big(\frac{z}{\mu}\Big)$$

where $\mu > 0$. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can think of μ as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error. We will update μ at discrete instants of time, so it will be the discrete state of the resulting hybrid closed-loop system. In the camera model mentioned in the Introduction, μ corresponds to the inverse of the focal length f.

3 State quantization

3.1 Linear systems

Consider the linear system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{3}$$

Suppose that (3) is *stabilizable*, so that for some matrix K the eigenvalues of $A + BK$ have negative real parts.

By the standard Lyapunov stability theory, there exist positive definite symmetric matrices P and Q such that

$$(A+BK)^T P + P(A+BK) = -Q. \tag{4}$$

We will let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. The inequality

$$\lambda_{\min}(P)|x|^2 \le x^T P x \le \lambda_{\max}(P)|x|^2$$

will be used repeatedly below. We will assume that $B \neq 0$ and $K \neq 0$; this is no loss of generality because the case of interest is when A is not a stability matrix.

In this section we are interested in the situation where only quantized measurements of the state are available. Since the state feedback law $u = Kx$ is not implementable, we apply the "certainty equivalence" quantized feedback control law

$$u = K\mu q\Big(\frac{x}{\mu}\Big). \tag{5}$$

Assume for the moment that μ is a fixed positive number. The closed-loop system is given by

$$\dot{x} = (A+BK)x - BK\mu\left(\frac{x}{\mu} - q\Big(\frac{x}{\mu}\Big)\right). \tag{6}$$

The behavior of trajectories of the system (6) for a fixed μ is characterized by the following result.

Lemma 1 *Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon) \tag{7}$$

where

$$\Theta_x := \frac{2\|PBK\|}{\lambda_{\min}(Q)} > 0.$$

Then the ellipsoids

$$\mathcal{R}_1 := \{x : x^T P x \le \lambda_{\min}(P)M^2\mu^2\} \tag{8}$$

and

$$\mathcal{R}_2 := \{x : x^T P x \le \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{9}$$

are invariant regions for the system (6). Moreover, all solutions of (6) that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time.

PROOF. Whenever the inequality (1), and consequently (2), hold with $z = x/\mu$, the derivative of $V(x) := x^T P x$ along solutions of (6) satisfies

$$\begin{aligned}\dot{V} &= -x^T Q x - 2x^T PBK\mu\left(\frac{x}{\mu} - q\Big(\frac{x}{\mu}\Big)\right)\\ &\le -\lambda_{\min}(Q)|x|^2 + 2|x|\|PBK\|\Delta\mu\\ &= -|x|\lambda_{\min}(Q)(|x| - \Theta_x\Delta\mu)\end{aligned}$$

This implies the following formula:

$$\Theta_x\Delta(1+\varepsilon)\mu \le |x| \le M\mu \Rightarrow \dot{V} \le -|x|\lambda_{\min}(Q)\Theta_x\Delta\varepsilon\mu. \tag{10}$$

Define the balls

$$\mathcal{B}_1 := \{x : |x| \le M\mu\}$$

and

$$\mathcal{B}_2 := \{x : |x| \le \Theta_x\Delta(1+\varepsilon)\mu\}.$$

In view of the inequality (7), we have

$$\mathcal{B}_2 \subset \mathcal{R}_2 \subset \mathcal{R}_1 \subset \mathcal{B}_1.$$

Combined with (10), this immediately implies that the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ are both invariant. The fact that the trajectories starting in $\mathcal{R}_1$ approach $\mathcal{R}_2$ in finite time follows from the bound on the derivative of V given by (10). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathcal{R}_1$ and if we let

$$T := \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2}{\Theta_x^2\Delta^2(1+\varepsilon)\lambda_{\min}(Q)\varepsilon} \tag{11}$$

then $x(t_0+T)$ is guaranteed to belong to $\mathcal{R}_2$. □

As we explained before, a hybrid quantized feedback control policy involves updating the value of μ at discrete instants of time. Using this idea and Lemma 1, it is possible to achieve global asymptotic stability.

Theorem 1 *Assume that M is large enough compared to Δ so that we have*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta\max\Big\{1, \frac{\|PBK\|}{\lambda_{\min}(Q)}\Big\}. \tag{12}$$

Then there exists a hybrid quantized feedback control policy that makes the system (6) globally asymptotically stable.

PROOF. The control strategy is divided into two stages.

The "zooming-out" stage. Set u equal to 0. Let $\mu(0) = 1$. Then increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $\|e^{At}\|$. For example, one can fix a positive number τ and let $\mu(t) = 1$ for $t \in [0,\tau)$, $\mu(t) = \tau e^{2\|A\|\tau}$ for $t \in [\tau, 2\tau)$, $\mu(t) = 2\tau e^{2\|A\|2\tau}$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \ge 0$ such that

$$\left|\frac{x(t)}{\mu(t)}\right| \le \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - 2\Delta$$

(by (12), the right-hand side of this inequality is positive). In view of condition 1 imposed in Section 2, this implies

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \le \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - \Delta. \tag{13}$$

We can thus pick a time t_0 such that (13) holds with $t = t_0$. Therefore, in view of conditions 1 and 2 of Section 2, we have

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \le \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M$$

hence $x(t_0)$ belongs to the ellipsoid $\mathcal{R}_1$ given by (8) with $\mu = \mu(t_0)$. Note that this event can be detected using only the available quantized measurements.

The "zooming-in" stage. Pick an $\varepsilon > 0$ such that the inequality (7) is satisfied; this is possible because of (12). We know that $x(t_0)$ belongs to $\mathcal{R}_1$ with $\mu = \mu(t_0)$. We now apply the control law (5). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T)$, where T is given by the formula (11). Then $x(t_0 + T)$ belongs to the ellipsoid $\mathcal{R}_2$ given by (9) with $\mu = \mu(t_0)$. For $t \in [t_0 + T, t_0 + 2T)$, let

$$\mu(t) = \Omega\mu(t_0)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(P)}M}.$$

We have $\Omega < 1$ by (7), hence $\mu(t_0 + T) < \mu(t_0)$. The ellipsoid $\mathcal{R}_2$ with the old value $\mu = \mu(t_0)$ is the same as the ellipsoid $\mathcal{R}_1$ with the new value $\mu = \mu(t_0+T)$. This means that we can continue the analysis for $t \ge t_0+T$ as before. Namely, $x(t_0 + 2T)$ belongs to the ellipsoid $\mathcal{R}_2$ defined by (9) with $\mu = \mu(t_0+T)$. For $t \in [t_0 + 2T, t_0 + 3T)$, let $\mu(t) = \Omega\mu(t_0 + T)$. Repeating this procedure, we obtain the desired control policy. Indeed, stability of the equilibrium $x = 0$ of the continuous dynamics in the sense of Lyapunov follows directly from the adjustment policy for μ. Moreover, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. □

3.2 Nonlinear systems

Consider the system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{14}$$

It is natural to assume that there exists a state feedback law $u = k(x)$ that makes the closed-loop system globally asymptotically stable. Actually, we need to assume that k satisfies the following stronger condition: there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x, e \in \mathbb{R}^n$ we have

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{15}$$

and

$$|x| \ge \rho(|e|) \Rightarrow \nabla V(x) f(x, k(x+e)) \le -\alpha_3(|x|). \tag{16}$$

According to the results of [13, 14], this is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x+e)) \tag{17}$$

is *input-to-state stable* (ISS) with respect to the measurement disturbance input e.

Since only quantized measurements of the state are available, we again consider the "certainty equivalence" quantized feedback control law, which in this case is

$$u = k\Big(\mu q\Big(\frac{x}{\mu}\Big)\Big). \tag{18}$$

For a fixed μ, the closed-loop system is given by

$$\dot{x} = f\left(x, k\Big(\mu q\Big(\frac{x}{\mu}\Big)\Big)\right) \tag{19}$$

and this takes the form (17) with

$$e = \mu q\Big(\frac{x}{\mu}\Big) - x. \tag{20}$$

The behavior of trajectories of (19) for a fixed value of μ is characterized by the following lemma (here and below $\circ$ denotes function composition).

Lemma 2 *Assume that we have*

$$\alpha_1(M\mu) > \alpha_2 \circ \rho(\Delta\mu). \tag{21}$$

Then the sets

$$\mathcal{R}_1 := \{x : V(x) \le \alpha_1(M\mu)\} \tag{22}$$

and

$$\mathcal{R}_2 := \{x : V(x) \le \alpha_2 \circ \rho(\Delta\mu)\} \tag{23}$$

are invariant regions for the system (19). *Moreover, all solutions of* (19) *that start in the set* $\mathcal{R}_1$ *enter the smaller set* $\mathcal{R}_2$ *in finite time.*

PROOF. Whenever the inequality (1), and consequently (2), hold with $z = x/\mu$, the quantization error e given by (20) satisfies

$$|e| = \left|\mu q\Big(\frac{x}{\mu}\Big) - \mu\frac{x}{\mu}\right| \le \Delta\mu.$$

Combined with (16), this implies the following formula:

$$\rho(\Delta\mu) \le |x| \le M\mu \quad \Rightarrow \quad \dot{V} \le -\alpha_3(|x|) \tag{24}$$

where $\dot{V}$ denotes the derivative of V along solutions of (19). Define the balls

$$\mathcal{B}_1 := \{x : |x| \le M\mu\}$$

and

$$\mathcal{B}_2 := \{x : |x| \le \rho(\Delta\mu)\}.$$

As before, in view of (15) and (21) we have

$$\mathcal{B}_2 \subset \mathcal{R}_2 \subset \mathcal{R}_1 \subset \mathcal{B}_1.$$

Combined with (24), this implies that the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ are both invariant. The fact that the trajectories starting in $\mathcal{R}_1$ approach $\mathcal{R}_2$ in finite time follows from the bound on the derivative of V deduced from (24). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathcal{R}_1$ and if we let

$$T_\mu := \frac{\alpha_1(M\mu) - \alpha_2 \circ \rho(\Delta\mu)}{\alpha_3 \circ \rho(\Delta\mu)} \tag{25}$$

then $x(t_0 + T_\mu)$ is guaranteed to belong to $\mathcal{R}_2$. □

We have the following nonlinear version of Theorem 1.

Theorem 2 *Assume that the system $\dot{x} = f(x, 0)$ is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \max\{\rho(\Delta\mu), \chi(\mu) + 2\Delta\mu\} \qquad \forall \mu > 0 \tag{26}$$

for some class $\mathcal{K}_\infty$ function χ. Then there exists a hybrid quantized feedback control policy that makes the system (19) globally asymptotically stable.

Proof. *The "zooming-out" stage.* Set the control equal to 0. Let $\mu(0) = 1$. Increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x(t)|$. For example, fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \chi^{-1}(2\max_{|x(0)|, t \le \tau} |\xi(x(0), t)|)$ for $t \in [\tau, 2\tau)$, $\mu(t) = \chi^{-1}(2\max_{|x(0)|, t \le 2\tau} |\xi(x(0), t|)$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \ge 0$ such that

$$|x(t)| \le \chi(\mu(t)) < \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta\mu(t)$$

where the second inequality follows from (26). This implies

$$\left|\frac{x(t)}{\mu(t)}\right| < \frac{1}{\mu(t)} \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta$$

and by virtue of condition 1 of Section 2 we have

$$\left| q\left(\frac{x(t)}{\mu(t)}\right)\right| \le \frac{1}{\mu(t)} \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - \Delta. \tag{27}$$

Picking a time t_0 at which (27) holds and using conditions 1 and 2 of Section 2, we obtain

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \le \frac{1}{\mu(t_0)} \alpha_2^{-1} \circ \alpha_1(M\mu(t_0))$$

hence $x(t_0)$ belongs to the set $\mathcal{R}_1$ given by (22) with $\mu = \mu(t_0)$.

The "zooming-in" stage. We have established that $x(t_0)$ belongs to $\mathcal{R}_1$ with $\mu = \mu(t_0)$. We will now use the control law (18). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T_{\mu(t_0)})$, where $T_{\mu(t_0)}$ is given by the formula (25). Then $x(t_0 + T_{\mu(t_0)})$ will belong to the set $\mathcal{R}_2$ given by (23) with $\mu = \mu(t_0)$. Calculate $T_{\omega(\mu(t_0))}$ using (25) again, where the function ω is defined as

$$\omega(r) := \frac{1}{M} \alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta r), \qquad r \ge 0.$$

For $t \in [t_0 + T_{\mu(t_0)}, t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$, let

$$\mu(t) = \omega(\mu(t_0)).$$

We have $\omega(r) < r$ for all $r > 0$ by (26), thus $\mu(t_0 + T_{\mu(t_0)}) < \mu(t_0)$. The set $\mathcal{R}_2$ with $\mu = \mu(t_0)$ is the same as the set $\mathcal{R}_1$ with $\mu = \mu(t_0 + T_{\mu(t_0)})$. We can now finish the analysis as in the linear case. □

The assumption of input-to-state stabilizability with respect to measurement disturbances is quite strong. It is possible to obtain weaker stability results under weaker hypotheses (cf. [9]).

4 Input quantization

In this section[1] we present analogous results for systems whose input, rather than state, is quantized.

4.1 Linear systems

Consider the linear system (3). Suppose again that there exists a matrix K such that the eigenvalues of $A + BK$ have negative real parts, so that for some positive definite symmetric matrices P and Q the equation (4) holds.

The "certainty equivalence" quantized feedback control law

$$u = \mu q\left(\frac{Kx}{\mu}\right)$$

yields the closed-loop system

$$\dot{x} = (A + BK)x - B\mu\left(\frac{Kx}{\mu} - q\left(\frac{Kx}{\mu}\right)\right). \tag{28}$$

Its behavior for a fixed μ is characterized as follows.

Lemma 3 *Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_u\|K\|\Delta(1 + \varepsilon)$$

where

$$\Theta_u := \frac{2\|PB\|}{\lambda_{\min}(Q)}.$$

Then the ellipsoids

$$\mathcal{R}_1 := \{x : x^T P x \le \lambda_{\min}(P) M^2 \mu^2 / \|K\|^2\}$$

and

$$\mathcal{R}_2 := \{x : x^T P x \le \lambda_{\max}(P) \Theta_u^2 \Delta^2 (1 + \varepsilon)^2 \mu^2\}$$

are invariant regions for the system (28). Moreover, all solutions of (28) that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time.

Theorem 3 *Assume that*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} M > 2\Delta \frac{\|PB\|\|K\|}{\lambda_{\min}(Q)}.$$

Then there exists a hybrid quantized feedback control policy that makes the system (28) globally asymptotically stable.

[1] This material is inspired by a conversation with Mark Spong.

4.2 Nonlinear systems

Consider the nonlinear system (14). Assume that there exists a feedback law $u = k(x)$ that makes the closed-loop system globally asymptotically stable and, moreover, ensures that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the inequalities (15) and

$$|x| \geq \rho(|e|) \Rightarrow \nabla V(x) f(x, k(x) + e) \leq -\alpha_3(|x|) \quad (29)$$

for all $x, e \in \mathbb{R}^n$. According to the results of [13, 14], this is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x) + e) \quad (30)$$

is ISS with respect to the actuator disturbance input e.

The closed-loop system with the "certainty equivalence" quantized feedback control law

$$u = \mu q\left(\frac{k(x)}{\mu}\right)$$

becomes

$$\dot{x} = f\left(x, \mu q\left(\frac{k(x)}{\mu}\right)\right) \quad (31)$$

and this takes the form (30) with $e = \mu q\Big(\frac{k(x)}{\mu}\Big) - x$. The behavior of trajectories of (31) for a fixed μ is characterized by the following result.

Lemma 4 *Assume that we have*

$$\alpha_1 \circ \kappa^{-1}(M\mu) > \alpha_2 \circ \rho(\Delta\mu).$$

Then the sets

$$\mathcal{R}_1 := \{x : V(x) \leq \alpha_1 \circ \kappa^{-1}(M\mu)\}$$

and

$$\mathcal{R}_2 := \{x : V(x) \leq \alpha_2 \circ \rho(\Delta\mu)\}$$

are invariant regions for the system (31). *Moreover, all solutions of* (31) *that start in the set* $\mathcal{R}_1$ *enter the smaller set* $\mathcal{R}_2$ *in finite time.*

Theorem 4 *Assume that the system* $\dot{x} = f(x, 0)$ *is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1 \circ \kappa^{-1}(M\mu) > \rho(\Delta\mu) \qquad \forall \mu > 0.$$

Then there exists a hybrid quantized feedback control policy that makes the system (31) *globally asymptotically stable.*

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