A NEW DEFINITION OF THE MINIMUM-PHASE PROPERTY FOR NONLINEAR SYSTEMS

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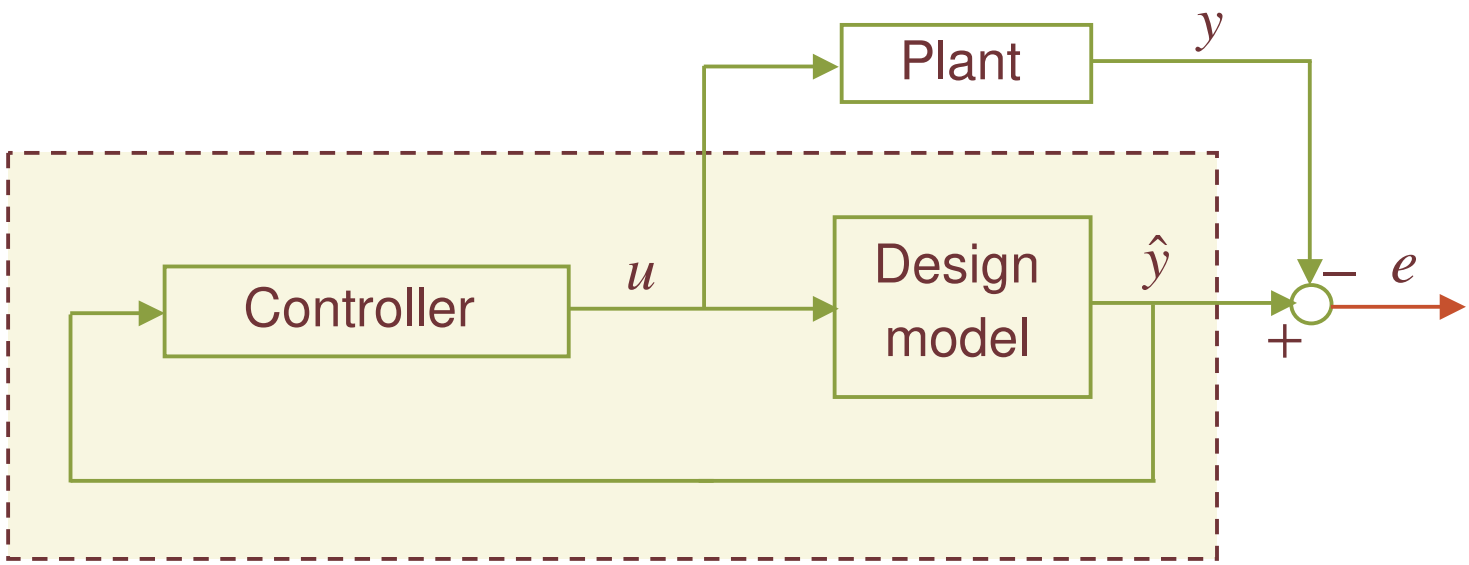
MOTIVATION

	$\dot{x} = Ax + Bu$ $y = Cx$	$\dot{x} = f(x,u)$ $y = h(x)$
stable	$\mathrm{Spec}(A) \subset \{\lambda : \mathrm{Re}\,\lambda < 0\}$ finite induced norms $PA + A^T P < 0$	ISS: $\lvert x(t)\rvert \le \beta(\lvert x(0)\rvert, t) + \gamma(\sup_{0\le s\le t} \lvert u(s)\rvert)$ $\dot{V} \le -\alpha(\lvert x\rvert) + \chi(\lvert u\rvert)$
detectable	stable unobs. modes $PA + A^T P < C^T C$	$\lvert x(t)\rvert \le \beta(\lvert x(0)\rvert, t) + \gamma(\sup_{0\le s\le t} \lvert y(s)\rvert)$ $\dot{V} \le -\alpha(\lvert x\rvert) + \chi(\lvert y\rvert)$
minimum phase	zeros of $g(s)$ are stable stable inverse: $y^{(r)} \to u$	? ? ?

MOTIVATION: Adaptive Control



If:
- the system in the box is output-stabilized
- the plant is minimum-phase

Then the closed-loop system is detectable through e ("tunable" – Morse '92)

DEFINITION

Call the system

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

minimum-phase if $\exists$ integer N and functions $\beta \in KL,\ \gamma \in K_\infty$ s.t. $\forall x(0), u \in C^{N-1}$

$$\left\| \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\| \leq \beta(|x(0)|, t) + \gamma(\sup_{0 \leq s \leq t} |y^N(s)|)$$

where $y^N := (y, \dot{y}, \ldots, y^{(N)})$

Example: $\dot{y} = u$

Reduces to usual def for linear systems (MIMO too)

DISCUSSION

Split into two:

$$\left|\left(x(t)\right)\right| \leq \beta(|x(0)|,t) + \gamma(\sup_{0\leq s\leq t}|y^N(s)|)$$

This is detectability (w.r.t. extended output)

Have a Lyapunov sufficient condition

$$\left|\left(u(t)\right)\right| \leq \beta(|x(0)|,t) + \gamma(\sup_{0\leq s\leq t}|y^N(s)|)$$

This is related to relative degree

RELATIVE DEGREE

Call r a uniform relative degree if $y, \dot{y}, \ldots, y^{(r-1)}$ don't depend on u and

$$|u| \leq \rho_1(|x|) + \rho_2(|y^{(r)}|)$$

for some $\rho_1, \rho_2 \in K_\infty$

For affine systems: $y^{(r)} = a(x) + b(x)u$
this reduces to the usual definition $(b(x) \neq 0)$

However, the system $\dot{y} = \arctan u$
doesn't have a relative degree

SOME RESULTS

- Detectability w.r.t. extended output: $\left|\left(x(t)\right)\right| \leq \beta(|x(0)|,t)+\gamma(\sup_{0\leq s\leq t}|y^N(s)|)$

  plus relative degree: $|u|\leq \rho_1(|x|)+\rho_2(|y^{(r)}|)$

  imply minimum-phase: $\left|\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}\right| \leq \beta(|x(0)|,t)+\gamma(\sup_{0\leq s\leq t}|y^N(s)|)$

- Minimum-phase implies detectability

- Main result: for SISO systems analytic in controls, minimum-phase implies relative degree

  Example: affine systems $y^{(r)} = a(x)+b(x)u$

  Not true for MIMO systems !

FEEDBACK DESIGN

$$\dot{\xi}_1 = \xi_2 \qquad\qquad y = \xi_1$$
$$\vdots$$
$$\dot{\xi}_r = a(\xi,\eta) + b(\xi,\eta)u \qquad\qquad r \text{ — relative degree}$$

Apply u to have $\dot{\xi} = A\xi$ where A is stable

If the system is minimum-phase then

$$\left|\left(x(t)\right)\right| \le \beta(|x(0)|,t) + \gamma(\sup_{0\le s\le t}|y^N(s)|)$$

implies $x \to 0$

No global normal form is needed !

With global normal form:

$$\dot{\eta} = q(\eta,\xi)$$

minimum-phase corresponds to ISS internal dynamics

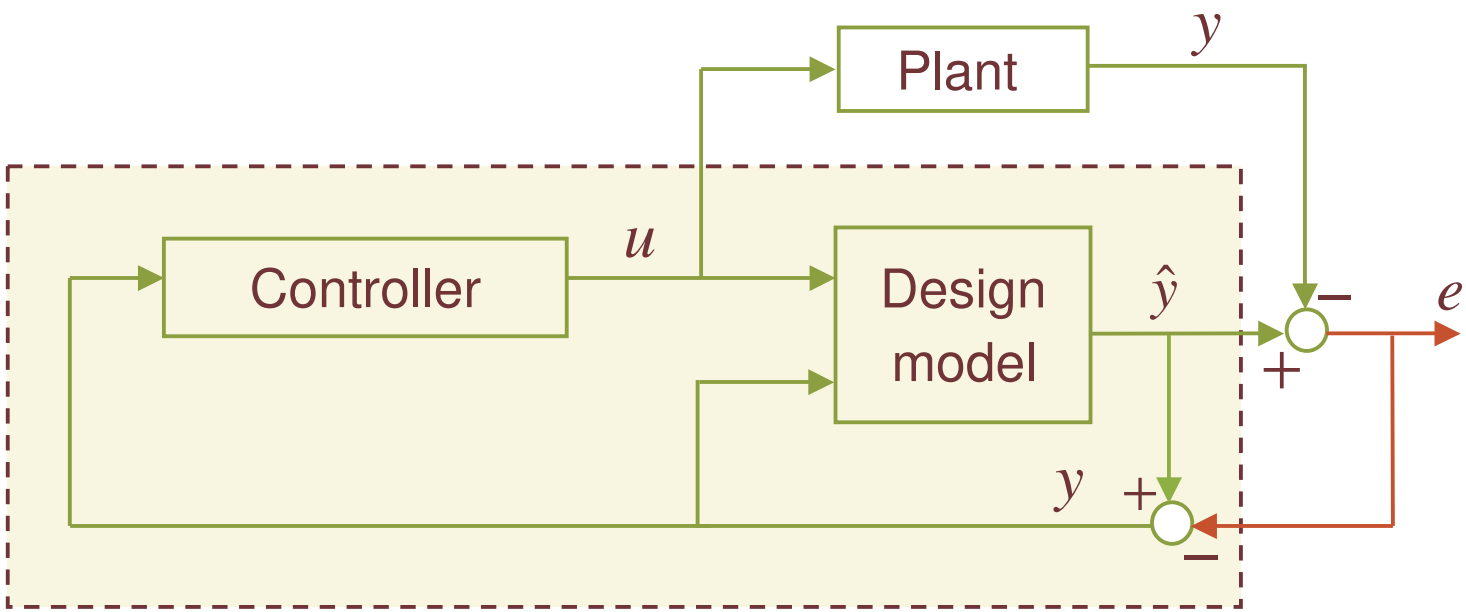
CASCADE SYSTEMS

If:
- Σ_1 is detectable (IOSS)
- Σ_2 is minimum-phase ($N=r$)

Then the cascade system is detectable (IOSS) w.r.t. u and extended output y^r

For linear systems recover usual detectability (observability decomposition)

ADAPTIVE CONTROL



If:
- the plant is minimum-phase ($N=r$)
- the system in the box is input-to-output stable (IOS) from e^{r-1} to $\hat{y}^r$

Then the closed-loop system is detectable through e^r ("weakly tunable")