Bounds on the Number of Switchings with Scale-Independent Hysteresis: Applications to Supervisory Control[1]

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Abstract

In this paper we analyze the Scale-Independent Hysteresis Switching Logic introduced in recent work. We show that, under suitable "open-loop" assumptions, one can establish an upper bound on the number of switchings produced by the logic on any given interval. This bound comes as a function of the variation of the inputs to the logic on that interval. In this paper it is also shown that, in a supervisory control context, this leads to switching that is slow-on-the-average, allowing us to study the stability of hysteresis-based adaptive control systems in the presence of measurement noise.

1 Introduction

Adaptive control algorithms that employ a logic-based supervisor to orchestrate the switching between a family of candidate controllers have been quite successful in numerous applications [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The need for switching usually arises from the fact that no single candidate controller would be capable, by itself, of guaranteeing stability and good performance when connected with a poorly modeled process. This type of supervisory control results in a switched closed-loop system of the form

$$\dot{x} = f_\sigma(x,t), \qquad x(0) = x_0, \qquad x \in \mathcal{X},\ t \geq 0, \tag{1}$$

where $\mathcal{X}$ is a finite dimensional space, $\{f_p : p \in \mathcal{P}\}$ an indexed family of locally Lipschitz functions, and σ a piecewise constant *switching signal* taking values on $\mathcal{P}$, generated by a *hybrid* switching logic combining continuous dynamics with discrete logic. Typically, the switching logic is designed to make the *monitoring signals*

$$\mu_p(t) := \Pi(p,x,t), \qquad t \geq 0,\ p \in \mathcal{P},\ x \in \mathcal{X} \tag{2}$$

have certain desired properties. Here, Π is a *monitoring function* from $\mathcal{P} \times \mathcal{X} \times [0,\infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments, for frozen values of the first. The points of discontinuity of σ are called *switching times*. For a given σ and $0 \leq \tau < t$, it is convenient to denote by $N_\sigma(t,\tau)$ the number of switching times—i.e., discontinuities of σ—on the interval (τ,t).

For some switching logics the supervisor guarantees, by construction, that there is a minimum time τ_D between switchings[1, 2, 3, 4, 5, 6]. Any switching signal generated by such logics thus satisfy $N_\sigma(\tau,t) = 0$, $\forall\, t-\tau \leq \tau_D$. The *dwell-time* τ_D is then a design parameter chosen so that (1) remains stable. Unfortunately, with nonlinear systems this may lead to finite escape of the closed-loop. Adaptive switching algorithms for nonlinear systems have therefore avoided a fixed dwell-time, and have been mostly based on hysteresis switching [13, 14], or on its more recent scale-independent version [8, 15].

To date, the analysis of algorithms based on hysteresis switching relied heavily on showing that switching stops in finite time [7, 8, 9, 10, 11, 12]. However, in the presence of noise and disturbance inputs, this is hardly the case. In fact, the only known algorithms for which switching can be proved to stop in finite time, even in the presence of noise/disturbances, are those for which an upper bound on these signals is known a priori, or effectively estimated online [1, 16]. Unfortunately, even in the noiseless case, these algorithms usually lead to bad transient responses. The objective of this paper is to analyze the behavior of the Scale-Independent Hysteresis Switching Logic introduced in [8, 15], under assumptions compatible with the existence of noise. We show that, although switching may not stop, it is possible to derive an upper bound on the number of switchings on any given interval. This bound comes as a function of the variation of the inputs to the logic on that interval.

The algorithm used to generate σ considered in this paper is called a *Scale-Independent Hysteresis Switching Logic* and can be regarded as a hybrid dynamical sys-

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tem $\mathbb{S}_{\mathbb{H}}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_{\mathbb{H}}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant.* $\mathbb{S}_{\mathbb{H}}$'s internal logic is then defined by the computer diagram shown in Figure 1, where the μ_p are defined by (2) and, at each time t, $q := \arg\min_{p\in\mathcal{P}} \Pi(p, x, t)$. In interpret-

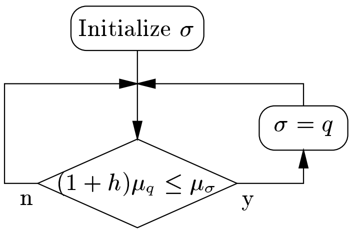

Figure 1: Computer Diagram of $\mathbb{S}_{\mathbb{H}}$.

ing this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_{\mathbb{H}}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_{\mathbb{H}}$ has just changed the value of σ to p. σ is then held fixed at this value unless and until there is a time $t_1 > t_0$ at which $(1+h)\mu_q \leq \mu_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on.

The main result of this paper is the Scale-Independent Hysteresis Switching Theorem. This theorem states that under appropriate "open-loop" assumptions, the number of switchings $N_\sigma(t, \tau)$ on any given interval (τ, t) can be bounded in terms of the variation of the monitoring signals on that interval. Using the Scale-Independent Hysteresis Switching Theorem, we will show that, although scale-independent hysteresis does not guarantee the existence of a fixed dwell-time between switchings, it can produce switching that is slow-on-the-average [17]. This will allow us to analyze hysteresis-based supervisory control algorithms in the presence of noise. Scale Independent Hysteresis and Dwell-time Switching are apparently the only two Certainty Equivalence based switching logics proposed thus far for which an analysis has proved possible under circumstances for which switching may not stop.

The switching logic described above is "scale-independent" in that its output σ remains unchanged if its monitoring function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta(t)\Pi(p, x, t), \qquad \forall p \in \mathcal{P},\ t \geq 0$$

where ϑ is a positive time function. This is because, for any fixed time t, (i) the value of p that minimizes $\Pi(p, x, t)$ also minimizes $\bar{\Pi}(p, \bar{x}, t)$ and (ii) $(1+h)\Pi(q, x, t) \leq \Pi(p, x, t)$ is exactly equivalent to $(1+h)\bar{\Pi}(q, \bar{x}, t) \leq \bar{\Pi}(p, \bar{x}, t)$ for every $p, q \in \mathcal{P}$. Scale-independence simplifies considerably the analysis of estimator-based supervisory control algorithms [2, 3, 7, 9, 10, 11, 12].

This paper is organized as follows. Section 2 contains the Scale-Independent Hysteresis Switching Theorem. The reader is referred to [18] for the proof of this theorem. In Section 3 it is illustrated how the scale-independence property can be used to effectively relax the "open-loop" assumptions required by this theorem. In Section 4 we use the previous results to analyze a supervisory control algorithm based on the Scale-Independent Hysteresis Switching Logic. Finally, Section 5 contains some concluding remarks and directions for future research.

2 Scale-Independent Hysteresis Switching Theorem

Let $\mathcal{X}_0$ denote a given subset of $\mathcal{X}$, and $\mathcal{S}$ the class of all piecewise-constant functions $s : [0, \infty) \to \mathcal{P}$. In what follows, for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, $T_{\{x_0,s\}}$ denotes the length of the maximal interval of existence of solution to the equations

$$\dot{x} = f_{s(t)}(x, t), \qquad x(0) = x_0, \qquad t \geq 0,$$

and $x_{\{x_0,s\}}$ the corresponding solution. The following "open-loop" assumptions are made:

Assumption 1 (Open-Loop) *For each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, the monitoring signals $\mu_p(t) := \Pi(p, x_{\{x_0,s\}}, t)$, $t \in [0, T_{\{x_0,s\}})$, $p \in \mathcal{P}$, are such that:*

1. *There exists a positive constant ϵ such that $\mu_p(0) \geq \epsilon$, for each $p \in \mathcal{P}$.*
2. *Each monitoring signal μ_p, $p \in \mathcal{P}$, is monotone nondecreasing on $[0, T_{\{x_0,s\}})$.*

Using standard arguments (see [8, 15]), one can show that there must be an interval $[0, T)$ of maximal length on which there is a unique pair $\{x, \sigma\}$ with x continuous and σ piecewise constant, which satisfies (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$. Moreover, on each strictly proper subinterval $[0, \tau) \subset [0, T)$, σ can switch at most a finite number of times.

To establish existence of solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$, only the first Open-Loop Assumption is needed. The other assumption enable us to draw conclusions regarding the behavior of σ and the μ_p on $[0, T)$. The following is the main result of this paper.

Theorem 1 (Scale-Independent Hysteresis Switching) *Let $\mathcal{P}$ be a finite set with m elements and assume that the Open-Loop Assumptions hold. For a fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (1) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ and let*

$[0,T)$ *denote the largest interval on which this solution is defined. For any* $\ell \in \mathcal{P}$,

$$N_\sigma(t,t_0) \le 1 + m + \frac{m \log\left(\frac{\mu_\ell(t)}{\min_{p\in\mathcal{P}} \mu_p(t_0)}\right)}{\log(1+h)}, \qquad (3)$$

$0 \le t_0 < t < T$, *and, when the monitoring signals are differentiable, we also have that*

$$\int_{t_0}^{t} \dot{\mu}_\sigma(\tau) d\tau \le m\Big((1+h)\mu_\ell(t) - \min_{p\in\mathcal{P}} \mu_p(t_0)\Big), \qquad (4)$$

$0 \le t_0 < t < T$, *where* $\dot{\mu}_\sigma(\tau)$ *is defined to be equal to* $\frac{\mathrm{d}\mu_p}{\mathrm{d}\tau}(\tau)$ *on intervals where* σ *is constant and equal to* $p \in \mathcal{P}$, *and zero at points of discontinuity of* σ.

When there is some μ_ℓ, $\ell \in \mathcal{P}$, bounded on $[0,T)$, σ can only have a finite number of discontinuities on $[0,T)$ because of (3). This means that there must be a time $T^* < T$ beyond which σ is constant. Moreover, since $\sigma = \sigma(T^*)$ on $[T^*,T)$,

$$\begin{aligned}\mu_{\sigma(T^*)}(t) &= \mu_{\sigma(T^*)}(T^*) + \mu_{\sigma(T^*)}(t) - \mu_{\sigma(T^*)}(T^*)\\ &\le \mu_{\sigma(T^*)}(T^*) + \int_{T^*}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\mu_{\sigma(\tau)}(\tau)\Big) d\tau,\end{aligned}$$

$t \in [T^*,T)$. Thus $\mu_{\sigma(T^*)}$ must be bounded on $[0,T)$ because of (4). Theorem 1 thus generalizes previous results in [8, 15]. The proof of this theorem can be found in [18].

Remark 1 *Equation* (4) *can be generalized for the case when the monitoring signals are not piecewise differentiable. In this case the left-hand-side of* (4) *must be reinterpreted as a summation of the variation of the corresponding monitoring signals over the intervals on which* σ *is constant. Equation* (4) *must then be replaced by*

$$\begin{aligned}&\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\mu_{\sigma(t_k)}(t_{k+1}) - \mu_{\sigma(t_k)}(t_k)\Big)\\ &\qquad\qquad \le m\Big((1+h)\mu_\ell(t) - \min_{p\in\mathcal{P}} \mu_p(t_0)\Big), \quad (5)\end{aligned}$$

$0 \le t_0 < t < T$, *where* $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ *denote the discontinuities of* σ *on* (t_0,t) *and* $t_{N_\sigma(t,t_0)+1} := t$. *Since* (5) *is a generalization of* (4), *we shall prove* (5) *instead of* (4).

It is useful to consider monitoring functions of the form

$$\Pi(p,x,t) := \min_{q\in\mathcal{Q}_p} \tilde{\Pi}(q,x,t), \qquad (6)$$

where $\{\mathcal{Q}_p : p \in \mathcal{P}\}$ is a parameterized family of compact sets and $\tilde{\Pi}$ a continuous function from $\mathcal{Q} \times \mathcal{X} \times [0,\infty)$ to $\mathbb{R}$, $\mathcal{Q} := \cup_{p\in\mathcal{P}} \mathcal{Q}_p$. In this case the monitoring signals can be written as $\mu_p(t) := \min_{q\in\mathcal{Q}_p} \tilde{\mu}_q(t)$, $p \in \mathcal{P}$, where $\tilde{\mu}_q(t) := \tilde{\Pi}(q,x,t)$, $q \in \mathcal{Q}$. In supervisory control, typically, each $p \in \mathcal{P}$ corresponds to a particular candidate controller and each $q \in \mathcal{Q}_p$ to a particular process model that can be stabilized by that controller. Because of this the $\tilde{\mu}_q$ are called *process monitoring signals* and the μ_p are called *control monitoring signals.* A similar convention is used for the monitoring functions.

We say that a piecewise constant signal ρ taking values in $\mathcal{Q}$ is $\{\mathcal{Q}_p : p \in \mathcal{P}\}$*-consistent* with a switching signal σ on an interval (τ,t) when

1. $\rho(s) \in \mathcal{Q}_{\sigma(s)}$ for all $s \in (\tau,t)$,
2. the set of discontinuities of ρ on (τ,t) is a subset of the set of discontinuities of σ.

Although, in general, the control monitoring signals are not differentiable, the process monitoring signals are. The following result is a direct corollary of Theorem 1.

Corollary 1 *Let* $\mathcal{P}$ *be a finite set with* m *elements and assume that the Open-Loop Assumptions hold for the process monitoring function. For a fixed initial state* $\{x_0,\sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, *let* $\{x,\sigma\}$ *denote the unique solution to* (1) *with* σ *generated by* $\mathbb{S}_{\mathbb{H}}$ *and let* $[0,T)$ *denote the largest interval on which this solution is defined. For any* $\ell \in \mathcal{Q}$,

$$N_\sigma(t,t_0) \le 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{\tilde{\mu}_\ell(t)}{\min_{q\in\mathcal{Q}} \tilde{\mu}_q(t_0)}\Big), \qquad (7)$$

$0 \le t_0 < t < T$, *and, when the process monitoring signals are differentiable, we also have that there exists a signal* ρ, *which is* $\{\mathcal{Q}_p : p \in \mathcal{P}\}$*-consistent with* σ *on* (t_0,t), *such that*

$$\int_{t_0}^{t} \dot{\tilde{\mu}}_\rho(\tau) d\tau \le m\Big((1+h)\tilde{\mu}_\ell(t) - \min_{q\in\mathcal{Q}} \tilde{\mu}_q(t_0)\Big), \qquad (8)$$

$0 \le t_0 < t < T$, *where* $\dot{\tilde{\mu}}_\rho(\tau)$ *is defined to be equal to* $\frac{\mathrm{d}\tilde{\mu}_q}{\mathrm{d}\tau}(\tau)$ *on intervals where* ρ *is constant and equal to* $q \in \mathcal{Q}$, *and zero at points of discontinuity of* ρ.

For an application of these ideas, see [19].

3 Relaxing the Open-Loop Assumptions

The scale-independence property mentioned in Section 1 can be used to somewhat relax the Open-Loop Assumptions 1. Suppose that the following assumptions hold.

Assumption 2 (Relaxed Open-Loop) *There exists a positive time-function* ϑ *such that for each pair* $\{x_0,s\} \in \mathcal{X}_0 \times \mathcal{S}$, *the* scaled monitoring signals $\bar{\mu}_p(t) := \vartheta(t)\Pi(p,x_{\{x_0,s\}},t)$, $t \in [0,T_{\{x_0,s\}})$, $p \in \mathcal{P}$, *are such that:*

1. *There exists a positive constant* ϵ *such that* $\bar{\mu}_p(0) \ge \epsilon$, *for each* $p \in \mathcal{P}$.

2. *Each scaled monitoring signal $\bar{\mu}_p$, $p \in \mathcal{P}$, is monotone nondecreasing on $[0, T_{\{x_0, s\}})$.*

In light of the scale independence property, $\mathbb{S}_{\mathbb{H}}$'s output σ remains unchanged if its monitoring function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, x\}$ satisfying

$$\bar{\Pi}(p, x, t) = \vartheta(t)\Pi(p, x, t), \qquad \forall p \in \mathcal{P},\ t \geq 0.$$

Since the Relaxed Open-Loop Assumptions guarantee that the original Open-Loop Assumptions hold for the pair $\{\bar{\Pi}, x\}$, we obtain the following corollary of Theorem 1.

Corollary 2 *Let $\mathcal{P}$ be a finite set with m elements and assume that the Relaxed Open-Loop Assumptions hold. For a fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$ and let $[0, T)$ denote the largest interval on which this solution is defined. For any $\ell \in \mathcal{P}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m \log\left(\frac{\bar{\mu}_\ell(t)}{\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)}\right)}{\log(1 + h)},$$

$0 \leq t_0 < t < T$, and, when the scaled monitoring signals are differentiable, we also have that

$$\int_{t_0}^{t} \dot{\bar{\mu}}_\sigma(\tau) d\tau \leq m\Big((1 + h)\bar{\mu}_\ell(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)\Big), \qquad (9)$$

$0 \leq t_0 < t < T$, where $\dot{\mu}_\sigma(\tau)$ is defined to be equal to $\frac{\mathrm{d}\mu_p}{\mathrm{d}\tau}(\tau)$ on intervals where σ is constant and equal to $p \in \mathcal{P}$, and zero at points of discontinuity of σ.

It turns out that in supervisory control the original Open-Loop Assumptions may often be violated, whereas the relaxed ones can be shown to hold (cf. Section 4 and [2, 3, 7, 9, 10, 11, 12]).

4 Supervisory Control

In this section we show how the previous results can be used in the context of supervisory control. We follow closely the formulation in [2, 3].

The problem addressed here is the set-point control of an imprecisely modeled process $\mathbb{P}$. In particular, we want to generate the control input u to the process so as to drive its output y to a constant reference r. The process has two other exogenous inputs that cannot be measured: a bounded measurement noise signal $\mathbf{n}$ and a bounded disturbance $\mathbf{d}$. For simplicity the signals u, y, $\mathbf{n}$, and $\mathbf{d}$ are scalar. $\mathbb{P}$ is assumed linear, time-invariant, with a stabilizable (through u) and detectable realization

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}x + B_{\mathbb{P}}u + D_{\mathbb{P}}\mathbf{d}, \qquad y = C_{\mathbb{P}}x + \mathbf{n}, \qquad (10)$$

but precise values for $A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}, D_{\mathbb{P}}$ are not known. It is known, however, that $\mathbb{P}$'s transfer function τ, from u to y, belongs to a family of transfer functions of the form $\mathcal{N} := \bigcup_{p \in \mathcal{P}} \mathcal{N}_p$, where p is an unknown parameter taking values in some parameter set $\mathcal{P}$ and each $\mathcal{N}_p$ denotes a family of transfer functions centered around a known, *nominal transfer function* ν_p, e.g., $\mathcal{N}_p := \{(1 + \delta)\nu_p : \|\delta\|_\infty \leq \epsilon\}$. Here, ϵ denotes some small positive constant and δ a stable transfer function with $\mathcal{H}_\infty$-norm smaller than ϵ. For simplicity, in the sequel we assume that $\epsilon = 0$ and the set $\mathcal{P}$ is finite and equal to $\{1, 2, \ldots, m\}$. In a future paper we shall consider the general case of $\epsilon > 0$ and infinite $\mathcal{P}$.

The solution proposed in [2, 3] to solve this problem is based on Certainty Equivalence and starts with the selection of a family of linear, time-invariant *candidate controllers* $\mathcal{C} := \{\kappa_p : p \in \mathcal{P}\}$. Each κ_p would make the feedback closed-loop system in Figure 2 asymptotically stable if the process transfer function τ was known to belong to $\mathcal{N}_p$. To avoid pole-zero cancellations it is assumed that $\mathbb{P}$ does not have transmission zeros at the origin.

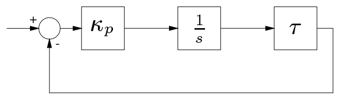

Figure 2: Feedback configuration.

In case we knew to which set $\mathcal{N}_p$ the actual process transfer function τ belonged, stability of the closed loop could be achieved with a nonadaptive, linear, time-invariant controller with transfer function equal to $\frac{1}{s}\kappa_p$. Since the process transfer function is not known in advance we build a "multi-controller" $\mathbb{C}$ that effectively allows switching between all the controller transfer functions in $\mathcal{C}$. If $\{(\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p) : p \in \mathcal{P}\}$ is a family of n-dimensional, stabilizable and detectable realizations for the transfer functions in $\mathcal{C}$, the multi-controller $\mathbb{C}$ can be defined by

$$\dot{x}_{\mathbb{C}} = \bar{A}_\sigma x_{\mathbb{C}} + \bar{B}_\sigma \mathbf{e_T}, \quad v = \bar{C}_\sigma x_{\mathbb{C}} + \bar{D}_\sigma \mathbf{e_T}, \quad \dot{u} = v, \qquad (11)$$

where $\mathbf{e_T} := r - y$ and $\sigma : [0, \infty) \to \infty$ denotes a "switching signal" that, at each instant of time, determines which candidate controller is put into the feedback loop. The system that generates the switching signal σ is called a supervisor. Here we are interested in estimator-based supervisors like the one in Figure 3. An estimator-based supervisor consists of three blocks: a multi-estimator, a monitoring signal generator, and a switching logic.

The *multi-estimator* $\mathbb{E}$ is a linear, time-invariant system whose inputs are the outputs of the process and multi-controller and whose outputs are the *output estimation errors* e_p, $p \in \mathcal{P}$. Each e_p is a signal that would converge

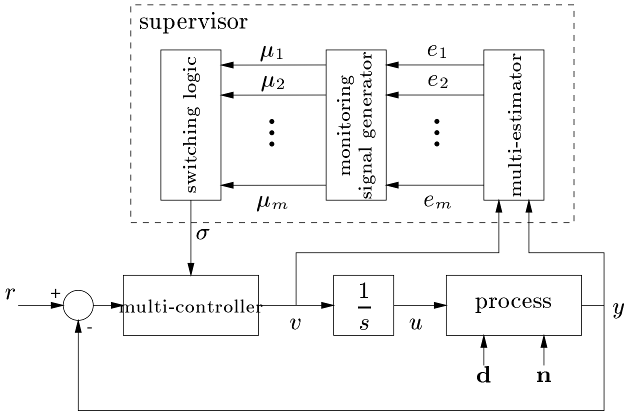

Figure 3: Supervisory control architecture.

to zero if the process transfer function τ was equal to the nominal transfer function ν_p. The reader is referred to [2, 3] for the precise structure of $\mathbb{E}$. Denoting by x the combined state of the multi-estimator and multi-controller (excluding the integrator), the evolution of x is determined by

$$\dot{x} = A_\sigma x + d_\sigma e_\sigma, \qquad (12)$$

$$\mathbf{e_T} = c_{p^*} x + e_{p^*}, \qquad (13)$$

where $A_p, d_p, c_p,\ p \in \mathcal{P}$ are appropriately defined matrices, and p^* is the element of $\mathcal{P}$ for which $\tau \in \mathcal{N}_{p^*}$. Equation (12) is obtained from equation (23) in [2] with $l = \sigma$ and (13) is obtained from equation (26) in [2] with $l = p^*$. It is also known that there exists a positive constant λ_0 for which each $\lambda_0 I + A_p$ is asymptotically stable (cf. Remark 4 in [2]). Moreover, from equation (28) in [2], one concludes that e_{p^*} is bounded and

$$\int_0^t e^{2\lambda\tau} e_{p^*}(\tau)^2 d\tau \le c_n e^{2\lambda t} + c_0, \qquad t \ge 0, \qquad (14)$$

$$\|e_{p^*}(t)\| \le d_n + d_0 e^{-\lambda t}, \qquad t \ge 0, \qquad (15)$$

where λ is any constant in $(0, \lambda_0)$, c_0, d_0 are positive constants that depends only on initial conditions, and c_n, d_n are positive constants that depends only on upper bounds on the norms of $\mathbf{n}$ and $\mathbf{d}$.

The *monitoring signal generator* $\mathbb{G}$ takes as inputs the output estimation errors $e_p,\ p \in \mathcal{P}$, and produces the *monitoring signals* $\mu_p,\ p \in \mathcal{P}$ defined by

$$\dot{\tilde{\mu}}_p = -2\lambda\tilde{\mu}_p + e_p^2, \qquad \mu_p = \tilde{\mu}_p + \epsilon_\mu, \qquad p \in \mathcal{P}, \qquad (16)$$

with $\lambda \in (0, \lambda_0)$ and $\epsilon_\mu > 0$ constant. $\mathbb{G}$ is initialized so that $\tilde{\mu}_p(0) \ge 0,\ p \in \mathcal{P}$.

The *switching logic* $\mathbb{S}$ generates the switching signal σ based on the values of the monitoring signals $\mu_p,\ p \in \mathcal{P}$. The logic used here is the Scale-Independent Hysteresis Switching Logic defined in Section 1.

Suppose now that we define *scaled monitoring signals* $\bar{\mu}_p := \vartheta\mu_p$, with $\vartheta(t) := e^{2\lambda t},\ t \ge 0$. From (16) one concludes that, for each $t \ge t_0 \ge 0$,

$$\bar{\mu}_p(t) = \tilde{\mu}_p(t_0) + e^{2\lambda t}\epsilon_\mu + \int_{t_0}^t e^{2\lambda\tau} e_p(\tau)^2 d\tau, \qquad (17)$$

and therefore each $\bar{\mu}_p$ is always monotone increasing and never smaller than ϵ_μ. By the Scale-Independent Hysteresis Switching Theorem (or more precisely Corollary 2), we can then conclude that, for any $\ell \in \mathcal{P}$ and $0 \le t_0 \le t < T$,

$$N_\sigma(t, t_0) \le 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{\bar{\mu}_\ell(t)}{\inf_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\Big) \qquad (18)$$

and

$$\int_{t_0}^t \left(2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau} e_\sigma^2\right) d\tau \le m\Big((1+h)\bar{\mu}_\ell(t) - \inf_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big). \qquad (19)$$

In (19) we used the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar{\mu}_{\sigma(\tau)}(\tau)\right) = 2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau} e_\sigma^2$, wherever the derivative exists. From (14) and (17) we obtain $\bar{\mu}_{p^*}(t) \le e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0,\ t \ge 0$, where $\bar{c}_0 := c_0 + \tilde{\mu}_{p^*}(0)$. Now (18)–(19) with $\ell := p^*$ and the fact that $\bar{\mu}_p(t_0) \ge e^{2\lambda t_0}\epsilon_\mu,\ p \in \mathcal{P}$ give

$$N_\sigma(t, t_0) \le 1 + m + \frac{m \log\left(e^{2\lambda(t-t_0)}\left(1 + \frac{c_n}{\epsilon_\mu}\right) + \frac{\bar{c}_0}{\epsilon_\mu}\right)}{\log(1+h)},$$

$$\int_{t_0}^t e^{2\lambda\tau} e_\sigma^2 d\tau \le m\Big((1+h)\left(e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0\right) - e^{2\lambda t_0}\epsilon_\mu\Big) - \left(e^{2\lambda t} - e^{2\lambda t_0}\right)\epsilon_\mu,$$

for every $t \ge t_0 \ge 0$. Since for $a, b > 0$, $\log(a+b) \le \log(2a) + \log(2b)$, we also conclude that

$$N_\sigma(t, t_0) \le N_0 + \frac{t - t_0}{\bar{\tau}_D}, \qquad (20)$$

$$\int_{t_0}^t e^{2\lambda\tau} e_\sigma^2 d\tau \le \bar{m} e^{2\lambda t} c_n + \bar{m}\bar{c}_0 - (m-1) e^{2\lambda t_0}\epsilon_\mu + (\bar{m} - 1) e^{2\lambda t}\epsilon_\mu, \qquad (21)$$

with $\tau_D := \frac{\log(1+h)}{2\lambda m}$, $\bar{m} = m(1+h)$, and

$$N_0 := 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{4\bar{c}_0}{\epsilon_\mu}\Big(1 + \frac{c_n}{\epsilon_\mu}\Big)\Big).$$

Now, because of Lemma 1 and Theorem 2 in [17], there is a finite constant τ_D^* such that (12) has input-to-state $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ norm uniformly bounded over the set $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$ of switching signals for which

$$N_\sigma(t, \tau) \le N_0 + \frac{t - \tau}{\tau_D^*}.$$

The signals in $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$ are said to have *average dwell-time* no larger than τ_D^* [17]. If we then choose λ and h so that $\frac{\lambda}{\log(1+h)} \leq \gamma := \frac{1}{2m\tau_D^*}$, we get $\tau_D \geq \tau_D^*$ and the output σ of the switching logic is guaranteed to be in $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. From this and (21) one concludes that x is bounded and, because of (11) and (13), $\mathbf{e_T}$ and v are also bounded. The boundedness of u and the internal state of the process follows from the detectability of the cascade formed by the integrator in (11) and the process (10). The following can then be stated.

Theorem 2 *There exists a positive constant γ such that, whenever $\frac{\lambda}{\log(1+h)} \leq \gamma$, all signals remain bounded, for any bounded $\mathbf{n}$ and $\mathbf{d}$, and any initialization of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{C}$, $\mathbb{G}$, $\mathbb{S}$, with $\tilde{\mu}_p(0) \geq 0$, $p \in \mathcal{P}$.*

The previous analysis assumed that the family of $\mathcal{N}\{\nu_p : p \in \mathcal{P}\}$ of admissible transfer functions for the process was finite. This type of analysis can be extended to the case when $\mathcal{N}$ has infinitely many elements [19].

5 Conclusions

We showed that, under suitable "open-loop" assumptions, one can establish an upper bound on the number of switchings produced by the Scale-Independent Hysteresis Switching Logic on a given interval. This bound comes as a function of the variation of the inputs to the logic on that interval. Computing upper bounds on the number of switchings produced by hysteresis-based logics has been the main difficulty in applying them to adaptive control when noise and exogenous disturbances are present. With the properties derived here we were able to show that, although this logic does not guarantee the existence of a fixed dwell-time between switchings, it can produce switching that is slow-on-the-average. This allowed us to analyze hysteresis-based supervisory control algorithms in the presence of noise. To the best of our knowledge, this is the first time that an algorithm of this type is analyzed without relying on switching stopping. The results presented here are confined to the case of a finite parameter set $\mathcal{P}$ and the absence of unmodeled dynamics. See [19] for more general results which do not require these assumptions.

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