An Asymptotic Ratio Characterization of Input-to-State Stability

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***Abstract*—For continuous-time nonlinear systems with inputs, we introduce the notion of an asymptotic ratio input-to-state stability (ISS) Lyapunov function. The derivative of such a function along solutions is upper-bounded by the difference of two terms whose ratio is asymptotically smaller than 1 for large states. This asymptotic ratio condition is sometimes more convenient to check than standard ISS Lyapunov function conditions. We show that the existence of an asymptotic ratio ISS Lyapunov function is equivalent to ISS. A related notion of ISS with nonuniform convergence rate is also explored.**

***Index Terms*—Input-to-state stability, Lyapunov function, nonlinear system.**

I. Introduction

The notion of *input-to-state stability (ISS)* characterizes the response of a nonlinear system to inputs in a way that generalizes standard notions of induced gains for linear systems while also taking into account the effect of initial conditions. Introduced by Sontag in [1], the ISS concept has since enjoyed widespread use in the nonlinear control literature. The most common way to verify ISS is by finding an *ISS Lyapunov function*, which is an extension of the classical Lyapunov function test for asymptotic stability to systems with external inputs. Such Lyapunov characterizations of ISS were presented in [2] and will be reviewed later in this note. See also [3]–[5] for some related results.

Motivated by our recent work on design of nonlinear observers robust to measurement disturbances in an ISS sense [6], in this note we propose the new notion of an *asymptotic ratio ISS Lyapunov function*. The derivative of such a function along solutions of the system has to be upper-bounded by the difference of two terms whose ratio is asymptotically smaller than 1 as the state becomes large, for each input. The form of this condition is different from the standard ISS Lyapunov function conditions mentioned above, and we will argue that it is sometimes easier to check. We show that, despite this difference, asymptotic ratio ISS Lyapunov functions provide an equivalent characterization of ISS (although they do not give direct information about the ISS gain). We will also explore a related notion of ISS with nonuniform convergence rate.

The rest of the note is organized as follows. Section II recalls the necessary notation and terminology and introduces the notion of an asymptotic ratio ISS Lyapunov function. Section III contains the main result—the equivalence between ISS and the existence of an asymptotic ratio ISS Lyapunov function—and its proof, followed by a discussion of ISS with nonuniform convergence rate and its relation to ISS and asymptotic ratio ISS Lyapunov functions. Section IV contains two illustrative examples, one of which describes a situation where our new condition is more convenient to apply than the known ones. Section V concludes the note.

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II. Preliminaries

We consider general nonlinear systems of the form

$$\dot{x} = f(x, d) \tag{1}$$

where $x(\cdot)$ is the state taking values in $\mathbb{R}^n$, $d(\cdot)$ is a measurable and locally essentially bounded input taking values in $\mathbb{R}^m$, and the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. We denote by $\|d\|_{[0,t]}$ and $\|d\|$ the essential supremum norm of d on the interval $[0, t]$ and $[0, \infty)$, respectively (with the understanding that the latter can be infinite).

A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* if $\alpha(0) = 0$ and $\alpha(r) > 0$ for all $r > 0$. A positive definite function α is of *class* $\mathcal{K}$ if it is continuous and strictly increasing. A restriction of a class $\mathcal{K}$ function to a subinterval $[0, a]$ is also said to be a class $\mathcal{K}$ function. If a class $\mathcal{K}$ function α is defined on the whole $\mathbb{R}_{\geq 0}$ and satisfies $\alpha(r) \to \infty$ as $r \to \infty$, then it is of *class* $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. We use the shorthand notation $\alpha \in \mathcal{K}$, $\beta \in \mathcal{KL}$, etc. to indicate these properties. Composition of functions will be denoted by the $\circ$ symbol.

We write $\vee$ for the maximum operator

$$a \vee b := \max\{a, b\}.$$

Following [1], we call the system (1) *input-to-state stable (ISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $x(0)$ and every input d the corresponding solution of (1) satisfies

$$|x(t)| \leq \beta\left(|x(0)|, t\right) \vee \gamma\left(\|d\|_{[0,t]}\right) \qquad \forall t \geq 0. \tag{2}$$

The function γ is sometimes called the *ISS gain function* or simply the *ISS gain*.

As shown in [2], [7], the system (1) is ISS if and only if there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the bounds

$$\alpha_1\left(|x|\right) \leq V(x) \leq \alpha_2\left(|x|\right) \qquad \forall x \tag{3}$$

for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, whose derivative

$$\dot{V} = \dot{V}(x, d) := \frac{\partial V}{\partial x} f(x, d)$$

along solutions of (1) satisfies the inequality

$$\dot{V} \leq -\alpha_3(|x|) + \chi(|d|) \qquad \forall x, d \tag{4}$$

for some $\alpha_3, \chi \in \mathcal{K}_\infty$; an equivalent property results if (4) is replaced by

$$|x| \geq \rho\left(|d|\right) \quad \Rightarrow \quad \dot{V} \leq -\alpha_4\left(|x|\right) \tag{5}$$

for some $\rho, \alpha_4 \in \mathcal{K}_\infty$, or by

$$|x| \geq \rho\left(|d|\right) \quad \Rightarrow \quad \dot{V} \leq -\alpha\left(V(x)\right) \tag{6}$$

with ρ still of class $\mathcal{K}_\infty$ but α merely continuous positive definite. (It is easy to see that (3) and (5) imply (6) with $\alpha := \alpha_4 \circ \alpha_2^{-1} \in \mathcal{K}_\infty$.) Such functions V are called *ISS Lyapunov functions*. From an ISS Lyapunov function V satisfying (3) and (6), one obtains the ISS bound (2) with

$$\beta(r,t) = \alpha_1^{-1} \circ \eta_\alpha^{-1}(\eta_\alpha \circ \alpha_2(r) + t), \qquad \gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \rho(s) \tag{7}$$

where η_α is defined by[1]

$$\eta_\alpha(r) := -\int_1^r \frac{dv}{\alpha(v)}. \tag{8}$$

Here, instead of (4), (5) or (6) we want to consider a condition in a different form, introduced in the following definition.

Definition 1: A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is an *asymptotic ratio ISS Lyapunov function* for the system (1) if it satisfies (3) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and its derivative along solutions satisfies

$$\dot{V} \le -\alpha_3(|x|) + g(|x|, |d|) \tag{9}$$

where $\alpha_3 \in \mathcal{K}$ and $g : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a continuous nonnegative function with the following properties: for each r, the function $g(r, \cdot)$ is nondecreasing with $g(r, 0) = 0$, and:

$$\limsup_{r\to\infty} \frac{g(r,s)}{\alpha_3(r)} < 1 \qquad \forall s \ge 0. \tag{10}$$

The term "asymptotic ratio" is motivated by the form of the condition (10).

Remark 1: In the special case when g decomposes as $g(r,s) = g_1(r)g_2(s)$ with g_2 unbounded, (10) is equivalent to

$$\limsup_{r\to\infty} \frac{g_1(r)}{\alpha_3(r)} = 0.$$

This type of condition was used in [9] in the context of ISS controller design and in [10] in the context of ISS observer design.

Remark 2: If V satisfies (4) with α_3, χ of class $\mathcal{K}$ and bounded (not class $\mathcal{K}_\infty$), then (9) holds with $g(r,s) := \chi(s)$ and (10) reduces to $\lim_{r\to\infty} \alpha_3(r) > \lim_{s\to\infty} \chi(s)$.

III. Main Result

Our main result (Theorem 1 below) says that the existence of an asymptotic ratio ISS Lyapunov function is equivalent to ISS. The primary utility of this result lies in the fact that the asymptotic ratio condition may sometimes be more convenient to check than either one of the standard ISS Lyapunov function conditions (4)–(6); an example of such a situation will be given in the next section. (Unlike the standard ISS Lyapunov functions, however, asymptotic ratio ISS Lyapunov functions do not provide direct information about the ISS gain; in other words, they are useful for quickly showing ISS but not for estimating the ISS gain.) Following the proof of Theorem 1, we discuss another implication of the asymptotic ratio condition.

Theorem 1 (Asymptotic Ratio Characterization of ISS): The system (1) is ISS if and only if it admits an asymptotic ratio ISS Lyapunov function in the sense of Definition 1.

Proof: The "only if" part is easy: (1) being ISS, we know that there exists an ISS Lyapunov function satisfying (3) and (4) with $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$, and then all the properties required in Definition 1 hold with $g(r,s) := \chi(s)$.

The "if" part: Let V be an asymptotic ratio ISS Lyapunov function. For $s \ge 0$, let

$$\theta(s) := \limsup_{r\to\infty} \frac{g(r,s)}{\alpha_3(r)}. \tag{11}$$

We know from Definition 1 that $\theta(0) = 0$, the function $\theta(\cdot)$ is nondecreasing (but may not be continuous[2]) because $g(r, \cdot)$ is nondecreasing for each r, and $0 \le \theta(s) < 1$ for all s in view of (10). Define

$$\bar{\theta}_k := \frac{1}{2} + \frac{1}{2}\theta(k), \qquad k \in \mathbb{N}.$$

Then $\{\bar{\theta}_k\}$ is a nondecreasing sequence such that

$$\theta(s) < \bar{\theta}_k < 1, \qquad k-1 < s \le k, \quad k \in \mathbb{N} \tag{12}$$

because, for $s \in (k-1, k]$, we have $\theta(s) < (1/2) + (1/2)\theta(s) \le (1/2) + (1/2)\theta(k) = \bar{\theta}_k < 1$ using (10), (11), and the monotonicity of θ. Let a sequence of positive numbers $\{m_k\}$ be such that

$$|x| \ge m_k \quad \Rightarrow \quad \frac{g(|x|, k)}{\alpha_3(|x|)} \le \bar{\theta}_k, \qquad k \in \mathbb{N}$$

whose existence follows from (11) and (12). By the fact that $g(|x|, \cdot)$ is nondecreasing, we have

$$|x| \ge m_k \quad \Rightarrow \quad \frac{g(|x|, |d|)}{\alpha_3(|x|)} \le \bar{\theta}_k, \quad k-1 < |d| \le k, \quad k \in \mathbb{N}. \tag{13}$$

By (9), this in turn implies that

$$|x| \ge m_k \quad \Rightarrow \quad \dot{V} \le -(1 - \bar{\theta}_k)\alpha_3(|x|), \qquad k-1 < |d| \le k, \quad k \in \mathbb{N}. \tag{14}$$

On the other hand, by continuity of g and α_3 and the fact that $g(\cdot, 0) \equiv 0$, there exists a class $\mathcal{K}$ function $\ell : [0, m_1] \to \mathbb{R}$ such that

$$g(|x|, |d|) \le \bar{\theta}_1 \alpha_3(|x|) \quad \text{for all } |d| \le \ell(|x|),\ 0 \le |x| \le m_1$$

and we suppose that $\ell(m_1) \le 1$ without loss of generality. (To construct such an ℓ, first define $\bar{\ell}(r) := \min\{1, \inf\{s : g(r,s) = \bar{\theta}_1 \alpha_3(r)\}\}$, which is positive definite and lower semi-continuous.[3] As such, $\bar{\ell}$ is bounded away from 0 except at the origin, and so we can lower-bound it by a function $\ell \in \mathcal{K}$.) This implies that

$$|x| \ge \ell^{-1}(|d|) \quad \Rightarrow \quad \dot{V} \le -(1 - \bar{\theta}_1)\alpha_3(|x|), \qquad 0 \le |d| \le \ell(m_1) \le 1, \quad 0 \le |x| \le m_1. \tag{15}$$

Now, pick a function $\rho \in \mathcal{K}_\infty$ such that

$$\rho(s) \ge \begin{cases} \ell^{-1}(s), & 0 \le s \le \ell(m_1) \\ m_1, & \ell(m_1) < s \le 1 \\ m_k, & k-1 < s \le k,\ k \ge 2 \end{cases} \tag{16}$$

[1]Decreasing α near zero if necessary, we can assume with no loss of generality that $\lim_{r\to 0+} \eta_\alpha(r) = \infty$, hence we use the conventions $\eta_\alpha(0) = \infty$ and $\eta_\alpha^{-1}(\infty) = 0$ (cf. the proof of Lemma 4.4 in [7]).

[2]An example of this is when $g(r,s) = (1/2)\mathrm{sat}(rs)r^2$ and $\alpha_3(r) = r^2$, where $\mathrm{sat}(r) := \mathrm{sign}(r)\min\{|r|, 1\}$, because

$$\theta(s) = \begin{cases} 0, & s = 0 \\ \frac{1}{2}, & s > 0. \end{cases}$$

This case arises when we analyze the system $\dot{x} = -x + (1/2)\ \mathrm{sat}(|x||d|)x$ with $V(x) = (1/2)x^2$.

[3]Lower semi-continuity means that $\bar{\ell}(r) \le \liminf_{v\to r} \bar{\ell}(v)$ for all r. This property easily follows from the definition of $\bar{\ell}$ and from continuity of g and α_3. Indeed, $\bar{\ell}(r)$ cannot exceed the limit of the values $\bar{\ell}(r_i)$ for any sequence $\{r_i\} \to r$ because the limit of any (sub)sequence of points s_i at which the infimum in the definition of $\bar{\ell}(r_i)$ is achieved is included in the set over which the infimum in the definition of $\bar{\ell}(r)$ is being taken.

and pick a continuous nonincreasing function ϕ such that

$$0 < \phi(s) \leq 1 - \bar{\theta}_k, \qquad k-1 < s \leq k, \quad k \in \mathbb{N} \tag{17}$$

and $\phi(0) = \lim_{s \to 0^+} \phi(s)$. Then, from (14)–(17), we have

$$|x| \geq \rho(|d|) \quad \Rightarrow \quad \dot{V} \leq -\phi(|d|)\,\alpha_3(|x|) \tag{18}$$

for all $|d|$. Since ϕ is nonincreasing, this implies

$$|x| \geq \rho(|d|) \quad \Rightarrow \quad \dot{V} \leq -\phi\left(\rho^{-1}(|x|)\right)\alpha_3(|x|). \tag{19}$$

In view of the bounds (3) and the fact that $\alpha_3 \in \mathcal{K}$ (and using again the fact that ϕ is nonincreasing), we obtain

$$|x| \geq \rho(|d|) \quad \Rightarrow \quad \dot{V} \leq -\alpha(V(x))$$

where

$$\alpha(r) := \phi \circ \rho^{-1} \circ \alpha_1^{-1}(r) \cdot \alpha_3 \circ \alpha_2^{-1}(r)$$

is continuous positive definite. We have shown that V satisfies the ISS Lyapunov function condition (6), and hence the ISS bound (2) holds with β and γ as defined in (7) and (8). □

Remark 3: It is easy to check that, with minor modifications, the above proof still works and establishes the same result if the definition of an asymptotic ratio ISS Lyapunov function is changed by replacing (9) and (10) with

$$\dot{V} \leq -\bar{\alpha}_3(|x|, |d|) + \bar{g}(|x|, |d|) \tag{20}$$

and

$$\limsup_{r \to \infty} \frac{\bar{g}(r,s)}{\bar{\alpha}_3(r,s)} < 1 \qquad \forall s \geq 0 \tag{21}$$

respectively, where $\bar{\alpha}_3 \in \mathcal{KL}$ and $\bar{g}$ has the same properties as g in Definition 1. It can also be shown that (20), (21) in fact imply (9), (10) with $\alpha_3(r) := \bar{\alpha}_3(r,0)$ and $g(r,s) := \bar{\alpha}_3(r,0) - \bar{\alpha}_3(r,s) + \bar{g}(r,s)$. Example 1 in Section IV will provide an illustration of this construction. Passing from (9), (10) to (20), (21) appears to be more difficult: one can obtain (20) by defining $\bar{\alpha}_3(r,s) := \alpha_3(r)\mu(s)$ where μ is some function that takes values in (0, 1) and decreases to 0 as $s \to \infty$, but then (21) does not hold unless the limsup in (10) equals 0 for all s.

A. ISS With Nonuniform Convergence Rate

The existence of an asymptotic ratio ISS Lyapunov function also implies that the solutions of (1) satisfy the following bound:

$$|x(t)| \leq \bar{\beta}\left(|x(0)|, \phi\left(\|d\|_{[0,t]}\right)t\right) \vee \gamma\left(\|d\|_{[0,t]}\right) \qquad \forall t \geq 0 \tag{22}$$

where $\bar{\beta} \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, and ϕ is a continuous, positive, and nonincreasing function (which was constructed in the proof of Theorem 1). This property resembles ISS, but the convergence rate (characterized by the dependence of $\bar{\beta}$ on t) is affected by the size of the input. To establish (22), we return to (18) in the above proof and then proceed differently as follows (loosely along the lines of [11, Section 4.1]). For every essentially bounded d, since $\rho \in \mathcal{K}_\infty$ and ϕ is nonincreasing, we have from (18) that

$$\begin{aligned} &|x| \geq \rho(\|d\|) \quad \Rightarrow \\ &\quad \dot{V} \leq -\phi(\|d\|)\,\alpha_3(|x|) \leq -\phi(\|d\|)\,\alpha_3\left(\alpha_2^{-1}(V(x))\right) \end{aligned}$$

or, equivalently

$$|x| \geq \rho(\|d\|) \quad \Rightarrow \quad \frac{dV}{d\left(\phi(\|d\|)\,t\right)} \leq -\bar{\alpha}(V(x))$$

where

$$\bar{\alpha}(r) := \alpha_3 \circ \alpha_2^{-1}(r).$$

From this, by the same standard arguments that we used to finish the proof of Theorem 1 (see also [11, Lemma A.4]), we obtain

$$|x(t)| \leq \bar{\beta}\left(|x(0)|, \phi(\|d\|)\,t\right) \vee \gamma(\|d\|) \qquad \forall t \geq 0$$

where $\bar{\beta} \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ are defined in (7)-(8) as before but with $\bar{\alpha}$ in place of α, so that $\bar{\beta}$ is different from β while γ is the same. Finally, by causality we can replace $\|d\|$ with $\|d\|_{[0,t]}$ and arrive at (22).

The property (22) is seemingly weaker than ISS (unless $\phi \equiv 1$), but in fact the two are equivalent. One way to prove that (22) implies ISS is to note that (22) implies

$$\limsup_{t \to \infty} |x(t)| \leq \gamma(\|d\|)$$

and

$$|x(t)| \leq \bar{\beta}(|x(0)|, 0) \vee \gamma(\|d\|).$$

These two bounds guarantee that the system (1) has the "asymptotic gain" (AG) property and the "global stability" (GS) property as defined in [8], from which ISS follows by [8, Theorem 1]. Another, more direct proof that (22) implies ISS is as follows (cf. again [11, Section 4.1]). Observe that $\bar{\beta}(r, \phi(s)t)$ is increasing in r, nondecreasing in s, and decreasing in t. Considering the two cases $|x(0)| \geq \|d\|_{[0,t]}$ and $|x(0)| \leq \|d\|_{[0,t]}$, we can write

$$\begin{aligned} &\bar{\beta}\left(|x(0)|, \phi\left(\|d\|_{[0,t]}\right)t\right) \\ &\quad \leq \bar{\beta}\left(|x(0)|, \phi(|x(0)|)\,t\right) \vee \bar{\beta}\left(\|d\|_{[0,t]}, \phi\left(\|d\|_{[0,t]}\right)t\right) \\ &\quad \leq \bar{\beta}\left(|x(0)|, \phi(|x(0)|)\,t\right) \vee \bar{\beta}\left(\|d\|_{[0,t]}, 0\right) \\ &\quad = \hat{\beta}\left(|x(0)|, t\right) \vee \bar{\beta}\left(\|d\|_{[0,t]}, 0\right) \end{aligned}$$

where $\hat{\beta}(r,t) := \bar{\beta}(r, \phi(r)t) \in \mathcal{KL}$. Plugging this bound into (22), we recover the ISS property with the ISS gain function $\hat{\gamma}(s) := \bar{\beta}(s,0) \vee \gamma(s)$. Obviously, this function $\hat{\gamma}$ is never smaller than γ, which is the ISS gain function constructed in the course of the proof of Theorem 1. The function $\hat{\beta}$, on the other hand, looks quite different from the function β that we reached at the end of the proof of Theorem 1, and it is not clear how to compare the two. Finally, we point out that the alternative proof of ISS given in this subsection remains valid if ρ is of class $\mathcal{K}$ but not $\mathcal{K}_\infty$ (i.e., ρ is bounded), while our proof of Theorem 1 required the function ρ to be of class $\mathcal{K}_\infty$ because its inverse was used in (19). Of course a class $\mathcal{K}$ function ρ can always be increased to obtain a class $\mathcal{K}_\infty$ function, but this increases the function γ in (7) which determines the ISS gain. So, in order not to introduce extra conservatism into the ISS gain estimate this step needs to be done carefully; for example, if starting with $\rho \in \mathcal{K}$ we consider the function $\bar{\rho}(s) := \max\{\rho(s), \varepsilon s\}$ for arbitrarily small $\varepsilon > 0$ and use it in place of ρ in the proof of Theorem 1, then the resulting ISS gain is not affected. As for the ISS gain function $\hat{\gamma}$ constructed in this subsection, it is always of class $\mathcal{K}_\infty$ (even when γ is bounded) because $\bar{\beta}(s,0) \geq s$ for all $s \geq 0$.

IV. EXAMPLES

Our first example illustrates that the asymptotic ratio condition may be easier to check than the more standard ISS Lyapunov function conditions.

Example 1: Consider the scalar system

$$\dot{x} = -\frac{1}{1+d^2}x + bd, \qquad b \in \mathbb{R}. \tag{23}$$

We claim that $V(x) := (1/2)x^2$ is an asymptotic ratio ISS Lyapunov function. Indeed, its derivative along (23) is given by

$$\dot{V} = -\frac{x^2}{1+d^2} + bxd = -x^2 + x^2\frac{d^2}{1+d^2} + bxd$$

and so, with $\alpha_3(r) := r^2$ and $g(r,s) := r^2s^2/(1+s^2) + |b|rs$, all properties in Definition 1 are fulfilled because

$$\frac{g(r,s)}{\alpha_3(r)} = \frac{s^2}{1+s^2} + \frac{|b|s}{r} \underset{r\to\infty}{\longrightarrow} \frac{s^2}{1+s^2} < 1 \qquad \forall s.$$

Thus (23) is ISS by Theorem 1. (Note that the observations of Remark 3 also apply here.) We challenge the reader to check that this V satisfies one of the ISS Lyapunov function conditions (4)–(6). While our proof of Theorem 1 guarantees that this must be the case, verifying this fact directly requires more effort.

It is instructive to look separately at the special case of (23) when $b = 0$, so that

$$\dot{x} = -\frac{1}{1+d^2}x. \tag{24}$$

Although we know from the above argument that this system is ISS, the proof of Theorem 1 does not provide explicit expressions for the ISS gain and decay rate (because they depend on the function ρ which is obtained in a nonconstructive way). On the other hand, it is easy to see from (24) that

$$|x(t)| \le e^{-\frac{1}{1+\|d\|^2_{[0,t]}}t}\,|x(0)|\,.$$

This shows that the bound (22) holds with $\bar{\beta}(r,t) = e^{-t}r$, $\phi(s) = 1/(1+s^2)$, and $\gamma \equiv 0$; hence, in particular, $x(t) \to 0$ for all essentially bounded inputs (although the convergence becomes slower for larger inputs). We could use the calculation given at the end of Section III-A to derive the functions $\hat{\beta}$ and $\hat{\gamma}$ which give the decay rate and the ISS gain, respectively, but $\hat{\gamma}$ would be nonzero and so the ISS bound obtained in this way would not let us recover the claim that $x(t) \to 0$ for all essentially bounded d. However, let us return to the asymptotic ratio ISS Lyapunov function $V(x) = (1/2)x^2$ and its derivative

$$\dot{V} = -\frac{1}{1+d^2}x^2$$

which gives us (18) with $\phi(s) = 1/(1+s^2)$, $\alpha_3(r) = r^2$, and $\rho \equiv 0$. This means that (18), and consequently (19), hold with *every* $\rho \in \mathcal{K}_\infty$. For example, we can let $\rho(s) := \varepsilon s$ where $\varepsilon > 0$ is arbitrary, and arrive at (19), namely

$$|x| \ge \varepsilon|d| \quad \Rightarrow \quad \dot{V} \le -\frac{1}{1+\frac{x^2}{\varepsilon^2}}x^2.$$

Now we can conclude from (7), as at the end of the proof of Theorem 1, that the ISS gain is $\gamma(s) = \varepsilon s$ (taking $\alpha_1(r) = \alpha_2(r) = (1/2)r^2$) and, since ε can be arbitrarily small, this confirms that $x(t) \to 0$ for all essentially bounded d (albeit with a nonuniform convergence rate).

The next example demonstrates that the monotonicity assumption imposed on the function $g(r,\cdot)$ in Definition 1 is essential.

Example 2: Consider the scalar system (cf. the earlier example in footnote 2)

$$\dot{x} = -x + \frac{1}{2}\mathrm{sat}\left(|x||d|\right)\left(e^{-(|x||d|-1)^2} + 1\right)x.$$

This system is not ISS because for a constant nonzero input d it has equilibria at $x = \pm 1/|d|$ which go off to ∞ as $|d| \to 0$. With $V(x) := (1/2)x^2$, $\alpha_3(r) := r^2$, and

$$g(r,s) := \frac{1}{2}\mathrm{sat}(rs)\left(e^{-(rs-1)^2} + 1\right)r^2$$

the conditions (9) and (10) are satisfied, g is continuous and nonnegative, and $g(\cdot,0) \equiv 0$. However, $g(r,\cdot)$ is not a nondecreasing function, which is why Theorem 1 does not apply (the construction of the sequence $\{m_k\}$ satisfying the condition (13) in the proof of Theorem 1 breaks down).

V. Conclusion

We introduced the notion of an asymptotic ratio ISS Lyapunov function, and showed that the existence of such a function is equivalent to ISS. Besides the simple examples provided in this note, a similar condition (in a somewhat more complicated form) has found application in our recent work on design of robust observers [6].

References

[1] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, Apr. 1989.

[2] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. Control Lett.*, vol. 24, pp. 351–359, 1995.

[3] D. Angeli, "Input-to-state stability of PD-controlled robotic systems," *Automatica*, vol. 35:, pp. 1285–1290, 1999.

[4] Z. P. Jiang and Y. Wang, "A converse Lyapunov theorem for discrete-time systems with disturbances," *Syst. Control Lett.*, vol. 45, pp. 49–58, 2002.

[5] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. Control Lett.*, vol. 58, pp. 47–53, 2009.

[6] H. Shim and D. Liberzon, "Nonlinear observers robust to measurement disturbances in an ISS sense," *IEEE Trans. Autom. Control*, to appear.

[7] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control Optim.*, vol. 34, pp. 124–160, 1996.

[8] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, Sep. 1996.

[9] E. D. Sontag and Y. Wang, "On characterizations of input-to-state stability with respect to compact sets," in *Proc. 3rd IFAC Symp. Nonlinear Control Systems (NOLCOS'95)*, 1995, pp. 226–231.

[10] H. Shim, D. Liberzon, and J.-S. Kim, "Quasi-ISS reduced-order observers and quantized output feedback," in *Proc. 48th IEEE Conf. Decision and Control*, 2009, pp. 6680–6685.

[11] E. D. Sontag and Y. Wang, "Lyapunov characterizations of input to output stability," *SIAM J. Control Optim.*, vol. 39, pp. 226–249, 2000.