Steady-state probability densities for nonlinear feedback systems perturbed by white noise

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THE PROBLEM

$$dx = Axdt + Bdw + kf(c^T x)dt \qquad \text{(Itô)}$$

Equation for s.-s. prob. density:

$$L\rho = 0$$

Assumptions:

- A stable
- (A, B) controllable

In the linear case ($k = 0$) we have

$$\rho = e^{-\frac{1}{2}x^T Q^{-1} x}$$

where

$$AQ + QA^T = -BB^T$$

Nonlinear case:

Existence results are available

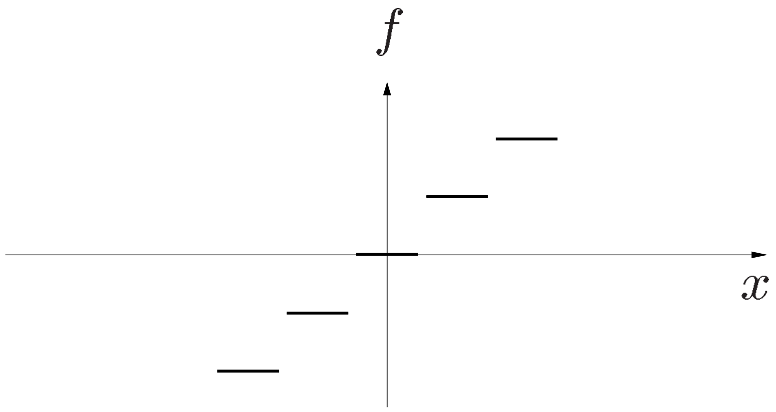
Explicit formulas – ?

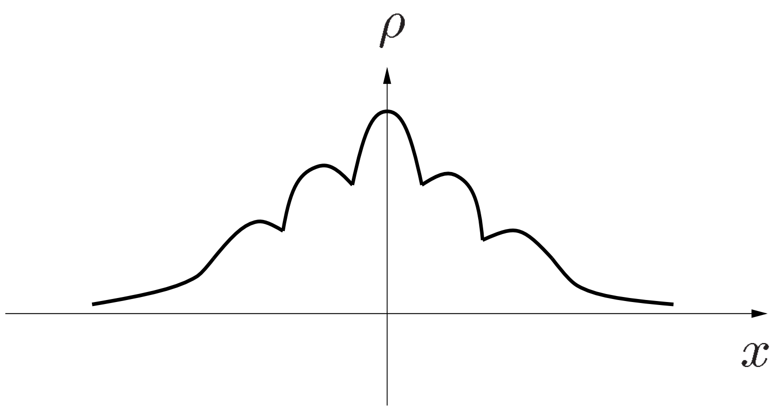
$\mathbb{R}^2$: Rueda (1993)

$\mathbb{R}^n$: Brockett (1997)

PIECEWISE CONSTANT f

$x \in \mathbb{R}^1$





$x \in \mathbb{R}^n$ – not always possible

Can get a piecewise Gaussian s.-s. prob. density if

$$\boxed{k = \lambda A Q c} \qquad \text{for some } \lambda \in \mathbb{R}$$

This is *compatibility condition*. As before

$$AQ + QA^T = -BB^T$$

GENERAL CASE

Theorem: *If the system*

$$dx = Axdt + Bdw + kf(c^T x)dt$$

is compatible, i.e.

$$k = \lambda AQc, \quad \lambda \in \mathbb{R}$$

then the function

$$\rho(x) = Ne^{-\frac{1}{2}x^T Q^{-1} x - \lambda F(c^T x)} \tag{1}$$

is s.-s. prob. density whenever $\rho \in L^1$

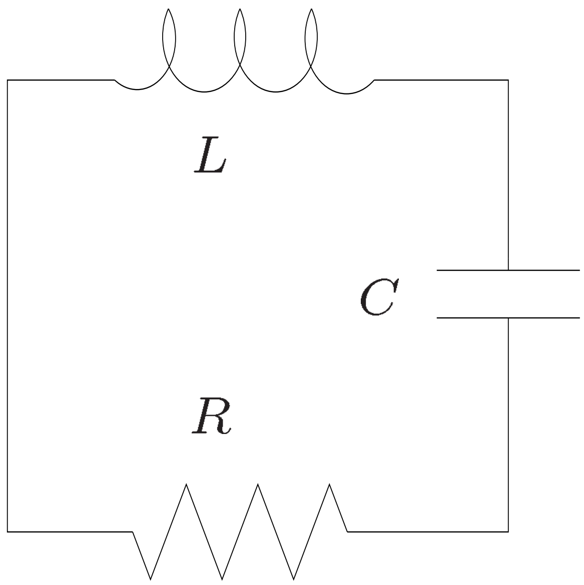
(true if $\lambda > 0$ and $xf(x) \geq 0$ for all x)

Proof – by direct verification

Convergence – via spectral analysis of L

In some cases can apply known results

MONOTEMPERATURIC SYSTEMS



Nonlinear capacitor: $V_C = f(q_C)$

Nyquist-Johnson model for resistor noise:

$$V = RI + \sqrt{2RT}\,\dot{w}$$

Here $\dot{w} = \dfrac{dw}{dt}$ is white noise, T is *temperature*

Brockett, Willems (1978):

Monotemperaturic systems. $Q = TI$

Close the loop $\longrightarrow$ compatible system

EXAMPLE: SPRING

Damped nonlinear spring:

$$\ddot{x} + a\dot{x} + f(x) = 0, \quad a > 0, \quad xf(x) > 0 \ \forall x \neq 0$$

$$\dot{x}\ddot{x} + a\dot{x}^2 + \dot{x}f(x) = 0$$

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{x}^2 + F(x)\right)}_{\text{energy, Lyapunov function}} = -a\dot{x}^2$$

Add random forces:

$$\ddot{x} + a\dot{x} + f(x) = \dot{w}$$

$$\rho(x, \dot{x}) = Ne^{-2a(\frac{1}{2}\dot{x}^2 + F(x))}$$

Can be generalized to higher dimensions

Theorem: *Consider the system*

$$p(D)x + f(c(D)x) = \dot{w}$$

where $D = \dfrac{d}{dt}$ *and* $p(D)$, $c(D)$ *are polynomials*

Suppose

$$Dc(D) = \frac{1}{a}\mathrm{Odd}\, p(D), \qquad a > 0 \tag{2}$$

Then the system is compatible.

In the previous example:

$$\ddot{x} + a\dot{x} + f(x) = \dot{w}$$

or

$$(D^2 + aD)x + f(x) = \dot{w}$$

$$\begin{matrix} p(D) = D^2 + aD \\ c(D) = 1 \end{matrix} \quad \Rightarrow \quad Dc(D) = D = \frac{1}{a}\mathrm{Odd}\, p(D)$$

SKETCH OF PROOF

$$p(D)x + f(c(D)x) = \dot{w}$$

$$Dc(D) = \frac{1}{a}\text{Odd}\, p(D)$$

Define $m(D) \triangleq Dc(D)$. Then $\dfrac{m(D)}{p(D)}$ is positive real

Spectral factorization: $\text{Ev}\, p(D)m(-D) = r^{+}(D)r^{-}(D)$

$$V(x) \triangleq \int_0^x [p(D)x\, m(D)x - (r^{-}(D)x)^2]dt + F(c(D)x)$$

Along solutions of

$$p(D)x + f(c(D)x) = 0$$

$\dot{V} = -(r^{-}(D)x)^2 \Rightarrow V$ is a Lyapunov function

$\boxed{\rho(x) = Ne^{-2aV(x)}}$ is s.-s. prob. density $\square$

Zakai (1969): this s.-s. density is unique, and

$$\mathcal{P}\left\{\frac{1}{T}\int_0^T g(x(t))dt \to \int_{\mathbb{R}^n} g(x)\rho\, dx\right\} = 1 \quad \forall g$$

FUTURE WORK

$$dx = Axdt + Bdw + kf(c^T x)dt$$

- Convergence
- Uniqueness
- "Compatible stabilization"
- Transient solutions
- Perturbation results