Preprints, 10th IFAC Symposium on Nonlinear Control Systems
August 23-25, 2016. Monterey, California, USA

Finite data-rate stabilization of a switched linear system with unknown disturbance ⋆

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Abstract: We study the stabilization of a switched linear system with unknown disturbance using sampled and quantized state feedback. The switching is slow in the sense of combined dwell-time and average dwell-time, while the active mode is unknown except at sampling times. Each mode of the switched system is stabilizable, and the disturbance admits an unknown bound. A communication and control strategy is designed to achieve practical stability and exponential convergence w.r.t. the initial state with a nonlinear gain on the disturbance, provided the data-rate meets given lower bounds. Compared with previous results, a more involved algorithm is developed to handle effects of the unknown disturbance based on employing an iteratively updated estimate of the disturbance bound and expanding the over-approximations of reachable sets over sampling intervals from the case without disturbance.

Keywords: Quantized Control, Switched Systems, Input-to-State Stability

1. INTRODUCTION

Feedback control problems with limited information have been an active research area for years, as surveyed in Nair et al. (2007). In many application-related scenarios, the information flow in a feedback loop is an important factor due to cost concerns, physical restrictions, security considerations, etc. Besides the practical motives, the question of how much information is needed to achieve a certain control objective is quite fundamental and intriguing from the theoretical viewpoint. In this paper, a finite data transmission rate is achieved by generating the control input based on sampled and quantized measurements, which is a standard modeling framework in the literature (see, e.g., Hespanha et al. (2002); Tatikonda and Mitter (2004)).

This paper considers a finite date-rate feedback control problem in the presence of external disturbances. In this context, Hespanha et al. (2002); Tatikonda and Mitter (2004) assumed known bounds on the disturbances and addressed asymptotic stabilization with minimum data-rate, while Liberzon and Nešić (2007); Sharon and Liberzon (2012) avoided such assumptions by switching repeatedly between the "zooming-out" and "zooming-in" processes and achieved input-to-state stability (ISS) Sontag (1989).

The study of switched and hybrid systems has attracted lots of attention lately (particularly relevant results include Liberzon (2003b); Shorten et al. (2007) and many references therein). In stabilization of switched systems, a standard approach is to impose suitable slow-switching conditions, especially in the sense of dwell-time from Morse (1996) and average dwell-time (ADT) from Hespanha and Morse (1999), which also plays a crucial role in this work.

On stabilizing switched systems with disturbances, Hespanha and Morse (1999) showed that one can achieve ISS under the same ADT condition as for the case without disturbance. Their result was made explicit only for the case of switched linear systems, and many similar results for switched nonlinear systems have been established since then (see, e.g., Xie et al. (2001) for ISS with dwell-time, Vu et al. (2007) for ISS with ADT, and Müller and Liberzon (2012) for input/output-to-state stability with ADT).

Early works on control with limited information in the context of switched systems were devoted to quantized control of Markov jump linear systems, e.g., Zhang et al. (2009). However, the discrete modes in those results were always known to the controller, which would remove most of the difficulties in our formulation. The problem of asymptotically stabilizing a switched linear system using sampled and quantized state feedback was studied in Liberzon (2014), which serves as the basis of this work. In Liberzon (2014), the controller had only partial knowledge of the switching; namely, the switching signal satisfied a mild slow-switching condition described by combined dwell-time and ADT, but the active mode was unknown except at sampling times. Provided the data-rate met certain lower bounds, stabilization was achieved via propagating over-approximations of reachable sets.

This work generalizes the main result in Liberzon (2014) to the scenario where an unknown disturbance is present. The sensor and controller possess no knowledge of the disturbance except that it admits an unknown bound. A communication and control strategy is designed to achieve practical stability and exponential convergence w.r.t. the initial state, provided that the data-rate meets given lower bounds. While such bounds are derived via the concept of reachable set propagation from Liberzon (2014), a more involved algorithm is needed to handle effects of

⋆ This work was supported by the NSF grants CNS-1217811 and ECCS-1231196.

the disturbance. Due to the unknown disturbance, the possibility of the state becoming lost (i.e., outside the approximations of reachable sets) cannot be eliminated. Consequently, the closed-loop system progresses in two stages—the stabilizing stage when the state is visible, and searching stage when it is lost—alteratively. An iteratively updated estimate of the disturbance bound is employed to ensure that there is a finite number of searching stages in total, and eventually the system stays in the stabilizing stage. The preliminary case where the disturbance bound is known was studied in Yang and Liberzon (2015).

This paper is structured as follows. Section 2 introduces the problem formulation and basic assumptions. Our main result is presented in Section 3. Section 4 describes the communication and control strategy, and Section 5 constructs the approximations of reachable sets and disturbance bound. In Section 6 we provide the stability analysis with several major steps summarized as technical lemmas.

2. PROBLEM FORMULATION

2.1 System description

We study the stabilization of a switched linear system with state $x \in \mathbb{R}^{n_x}$, control $u \in \mathbb{R}^{n_u}$ and disturbance $d \in \mathbb{R}^{n_d}$:

$$\dot{x} = A_\sigma x + B_\sigma u + D_\sigma d, \qquad x(0) = x_0, \tag{1}$$

where $\{(A_p, B_p, D_p)\}_{p\in\mathcal{P}}$ is a collection of matrix triples with suitable dimensions defining the subsystems (modes), $\mathcal{P}$ is a finite *index set*, and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal* which specifies the index $\sigma(t)$ of the active mode at time t. The solution $x(\cdot)$ is absolutely continuous and satisfies (1) away from discontinuities of σ (in particular, there are no state jumps). An admissible disturbance $d(\cdot)$ is a measurable and locally essentially bounded function. The switching signal σ is fixed but unknown to the controller a priori. Discontinuities of σ are called *switching times*, or simply *switches*. The number of switches on a time interval $(\tau, t]$ is denoted by $N_\sigma(t, \tau)$.

First, the switching is assumed to be slow in the sense of combined dwell-time and average dwell-time:

Assumption 1. (Switching). The switching has

1) a *dwell-time* τ_d such that $N_\sigma(t, \tau) \leq 1$ for all $\tau \geq 0$ and $t \in (\tau, \tau + \tau_d]$; and
2) an *average dwell-time (ADT)* $\tau_a > \tau_d$ such that
$$N_\sigma(t, \tau) \leq N_0 + (t - \tau)/\tau_a \qquad \forall\, t > \tau \geq 0 \tag{2}$$
with an integer $N_0 \geq 1$.

The notions of dwell-time from Morse (1996) and ADT from Hespanha and Morse (1999) are standard in the switched system context. In Assumption 1, item 1) can be rewritten in the form of (2) with $\tau_a = \tau_d$ and $N_0 = 1$; and item 2) would be implied by item 1) without $\tau_a > \tau_d$. Switching signals satisfying Assumption 1 were called "hybrid dwell-time" signals in Vu and Liberzon (2011).

Second, we assume all individual modes are stabilizable:

Assumption 2. (Stabilizability). For each $p \in \mathcal{P}$, there is a state feedback gain matrix K_p s.t. $A_p + B_p K_p$ is Hurwitz.

In this work, it is assumed that such a collection of gain matrices $\{K_p : p \in \mathcal{P}\}$ is selected and fixed. However, even

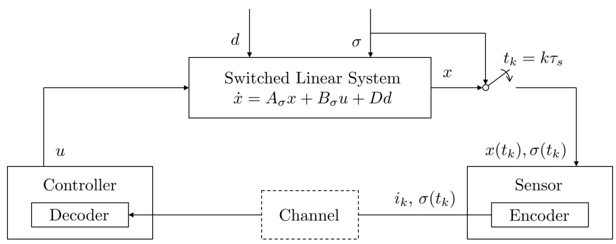

Fig. 1. Information structure

in the case without disturbance, and all individual modes are stabilized via feedback (or stable without feedback), stability of the switched system is not guaranteed in general (Liberzon, 2003b, Part II).

We use $\|\cdot\|$ to denote the ∞-norm of a vector or a matrix, that is, $\|v\| := \max_{1\leq i\leq n} |v_i|$ for $v \in \mathbb{R}^n$, and $\|M\| := \max_{1\leq i\leq n} \sum_{j=1}^{n} |M_{ij}|$ for $M \in \mathbb{R}^{n\times n}$. The right-sided limit of a piecewise absolutely continuous function z is denoted by $z(t^-) := \lim_{s\nearrow t} z(s)$. The essential supremum ∞-norm of d on an interval I is denoted by $\|d\|_I$.

Third, we assume the disturbance d is essentially bounded.

Assumption 3. (Disturbance). The disturbance d is essentially bounded, namely, there is a *disturbance bound*

$$\delta_d := \|d\|_{\mathbb{R}_{\geq 0}} < \infty.$$

The value of δ_d is *unknown* to the sensor and controller.

2.2 Information structure

The feedback loop consists of a sensor and a controller. The sensor transmits two sequences of data, indices of the active modes $\sigma(t_k)$ and quantized measurements (samples) of the state $x(t_k)$, at *sampling times* $t_k = k\tau_s, k = 0, 1, \ldots$, where $\tau_s > 0$ is the *sampling period*. Each sample is encoded by an integer i_k from 0 to N^{n_x}, where N is an odd integer. The controller generates the control input u to the switched linear system (1) based on the decoded data. As $\sigma(t_k) \in \mathcal{P}$ and $i_k \in \{0, 1, \ldots, N^{n_x}\}$, the data transmission rate between the encoder and decoder is $(\log_2 |N^{n_x} + 1| + \log_2 |\mathcal{P}|)/\tau_s$ bits per time unit, where $|\mathcal{P}|$ denotes the number of modes. Fig. 1 demonstrates the information structure. The communication and control strategy is explained in detail in Section 4.

We take the sampling period τ_s to be no larger than the dwell-time τ_d in Assumption 1, namely,

$$\tau_s \leq \tau_d, \tag{3}$$

so that there is at most one switch in any *sampling interval* $(t_k, t_{k+1}]$. Since the ADT $\tau_a > \tau_d$ in Assumption 1, switches actually occur less often than once per τ_s.

Our last basic assumption sets a lower bound on data-rate:

Assumption 4. (Data-rate). Sampling period τ_s satisfies

$$\Lambda_p := \|e^{A_p \tau_s}\| < N \qquad \forall\, p \in \mathcal{P}. \tag{4}$$

The inequality in (4) assigns a lower bound on the data-rate as it requires τ_s to be small enough w.r.t. N. This bound is the same as the one from the case without disturbance (Liberzon, 2014, Assumption 3), and similar data-rate bounds appeared in Hespanha et al. (2002); Liberzon (2003a); Tatikonda and Mitter (2004) for stabilizing non-switched linear systems; cf. (Liberzon, 2014, Section 2.2).

3. MAIN RESULT

Theorem 1. Consider the switched linear system (1). Suppose Assumptions 1–4 and (3) hold. Provided that the ADT τ_a is sufficiently large, there exists a communication and control strategy that yields:

EXPONENTIAL CONVERGENCE: There exist $\lambda, C > 0$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbance bounds $\delta_d \geq 0$ we have

$$\|x(t)\| \leq e^{-\lambda t} g(\|x_0\|) + h(\delta_d) + C \qquad \forall\, t \geq 0. \tag{5}$$

PRACTICAL STABILITY: There exists a $C' > 0$ such that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|x_0\| \leq \delta$ and $\delta_d \leq \delta$ then $\|x(t)\| \leq \varepsilon + C'$ for all $t \geq 0$.

The lower bounds on τ_a are given by (28) for exponential convergence, and by (39) for practical stability; cf. (Yang and Liberzon, 2015, Remark 2) for their relation. The *exponential decay rate* λ and the constant C are given by (37), and the gains g and h by (38). From the proof it will be clear that $g(s)$ does not go to 0 as $s \to 0$ and grows superlinearly as $s \to \infty$. Consequently, practical stability does not follow from exponential convergence, and needs to be established separately. Meanwhile, h is superlinear and positive definite. The constant C' is given by (40).

Compared to the result in Liberzon (2014) in the same setting, we see that both properties in (Liberzon, 2014, Theorem 1) are extended to the versions with disturbance. Moreover, when the disturbance $d \equiv 0$, the sensor and controller still use an arbitrarily selected $\delta_0 > 0$ as the initial estimate of the disturbance, which leads to the additional positive constants C and C'. By virtue of (Sontag and Wang, 1996, Theorem 1), the properties established in Theorem 1 is closely related to the input-to-state practical stability property in Jiang et al. (1994).

4. COMMUNICATION AND CONTROL STRATEGY

In this section we explain the communication and control strategy, assuming that suitable approximations of the disturbance bound and reachable sets of the state are available at all sampling times. (Such approximation are constructed in Section 5.) More specifically, at each sampling time t_k, the disturbance bound δ_d is approximated by a positive number δ_k, and the reachable set by a hypercube $\mathfrak{R}_k$ of radius $E_k > 0$ centered at x_k^*, namely,

$$\mathfrak{R}_k := \{v \in \mathbb{R}^{n_x} : \|v - x_k^*\| \leq E_k\}.$$

The values of x_k^*, E_k, δ_k are synchronized between the sensor and the controller at each sampling time t_k.

At $t = 0$, the initial state x_0 is unknown, and both the sensor and the controller are given $x_0^* = 0$ and arbitrarily selected initial values E_0, δ_0. Starting from $k = 0$, at each sampling time t_k, the sensor first determines if

$$\|x(t_k) - x_k^*\| \leq E_k, \tag{6}$$

that is, if the state $x(t_k)$ is inside the hypercube $\mathfrak{R}_k$. If so, we say the state is *visible*, and the sensor proceeds to the *stabilizing stage*; otherwise the state is *lost*, and the sensor proceeds to the *searching stage*. The system alternates between stabilizing and searching stages, both of which may consist of multiple sampling periods. For a j such that $x(t_{j-1}) \in \mathfrak{R}_{j-1}$ and $x(t_j) \notin \mathfrak{R}_j$, we say the state *escapes* at t_j; likewise, for an i such that $x(t_{i-1}) \notin \mathfrak{R}_{i-1}$ and $x(t_i) \in \mathfrak{R}_i$, we say the state *is recovered* at t_i.

4.1 Stabilizing stage

In a stabilizing stage, the encoder divides the hypercube $\mathfrak{R}_k$ into N^{n_x} equal hypercubic boxes, N per dimension, encodes each box by a unique index from 1 to N^{n_x}, and transmits the index i_k of the hypercubic box containing $x(t_k)$ to the decoder, along with the index $\sigma(t_k)$ of the active mode. The controller knows that (6) holds upon receiving $i_k \in \{1, \dots, N^{n_x}\}$. Following the same pre-defined indexing protocol with the encoder, the decoder is able to reconstruct the center c_k of the hypercubic box containing $x(t_k)$ from i_k. Simple calculation shows that

$$\|x(t_k) - c_k\| \leq E_k/N, \quad \|c_k - x_k^*\| \leq (1 - 1/N)E_k. \tag{7}$$

The controller sets the control $u(t) = K_{\sigma(t_k)}\hat{x}(t)$ for $t \in [t_k, t_{k+1})$, where $K_{\sigma(t_k)}$ is the gain matrix in Assumption 2, and $\hat{x}$ is the solution to the auxiliary system

$$\dot{\hat{x}} = A_{\sigma(t_k)}\hat{x} + B_{\sigma(t_k)}u = A_{\sigma(t_k)}\hat{x} + B_{\sigma(t_k)}K_{\sigma(t_k)}\hat{x} \tag{8}$$

with $\hat{x}(t_k) = c_k$ (i.e., $\hat{x}$ is readjusted to c_k at each t_k). Both the sensor and the controller maintain an identical copy of the auxiliary system (8) to calculate

$$\begin{aligned} x_{k+1}^* &:= F(\sigma(t_k), \sigma(t_{k+1}), c_k), \\ E_{k+1} &:= G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_k) \end{aligned} \tag{9}$$

for the next sampling time t_{k+1} individually. The functions F and G are designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_d), \tag{10}$$

and G is increasing in its last argument, which is δ_k in (9) and δ_d in (10). Hence the sensor learns that $\delta_k < \delta_d$ if the state escapes at t_{k+1}. The formulas for F and G are given in Subsection 5.1.

4.2 Escape

When the state escapes at t_j, the sensor learns that $\delta_{j-1} < \delta_d$, and sets $\delta_j = (1 + \varepsilon_\delta)\delta_{j-1}$ with an arbitrarily selected design parameter $\varepsilon_\delta > 0$. Estimates of the disturbance bound are unchanged in all other cases (in particular, they are increased just once per searching stage). Note that x_j^* and E_j are still calculated according to (9), and

$$E_j < \|x(t_j) - x_j^*\| \leq G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d).$$

4.3 Searching stage

In a searching stage, there exists an unknown $\hat{D}_k$ so that

$$E_k < \|x(t_k) - x_k^*\| \leq \hat{D}_k \tag{11}$$

(if the state is lost at $t_0 = 0$ then $\hat{D}_0 = \|x_0\|$; if it escapes at t_j then $\hat{D}_j = G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d)$). The encoder sends 0, the "overflow symbol", to the decoder, which consequently lets the controller set the control input $u(t) \equiv 0$ on $[t_k, t_{k+1})$. Similar to the stabilizing stage, both the sensor and the controller calculate

$$x_{k+1}^* := x_k^*, \quad E_{k+1} := \hat{G}(x_k^*, (1 + \varepsilon_E)E_k, \delta_k) \tag{12}$$

individually, where $\varepsilon_E > 0$ is a arbitrarily selected design parameter. The function $\hat{G}$ is designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq \hat{G}(x_k^*, \hat{D}_k, \delta_d). \tag{13}$$

Note the second argument of $\hat{G}$ in (12) is $(1 + \varepsilon_E)E_k$, whereas the one in (13) is $\hat{D}_k$. Introducing the coefficient $1+\varepsilon_E$ ensures that the growth rate of E_k dominates that of $\hat{D}_k$, and consequently the state is recovered in finite time, as shown in Subsection 5.2 following the formula for $\hat{G}$.

5. GENERATING APPROXIMATIONS

Now we derive the recursive formulas needed to implement our communication and control strategy. In Subsection 5.1 we consider the stabilizing stage and obtain F, G in (9). In Subsection 5.2 we consider the searching stage and obtain $\hat{G}$ in (12), together with the proof of finite time recovery.

5.1 Stabilizing stage

Sampling interval without switch When

$$\sigma(t_k) = p = \sigma(t_{k+1}) \tag{14}$$

for $p \in \mathcal{P}$, by (3) there is no switch on $(t_k, t_{k+1}]$. Combining (1) and (8) shows that the error $e := x - \hat{x}$ satisfies

$$\dot{e} = A_p e + D_p d, \qquad \|e(t_k)\| = \|x(t_k) - c_k\| \le E_k/N$$

on $[t_k, t_{k+1})$, where the boundary condition follows from $\hat{x}(t_k) = c_k$ and (7). Thus by variation of constants we get

$$\|e(t_{k+1}^-)\| \le \Lambda_p E_k/N + \Phi_p(\tau_s)\delta_d =: \hat{D}_{k+1} \tag{15}$$

with Λ_p in (4) and $\Phi_p : [0, \tau_s] \to \mathbb{R}$ defined by $\Phi_p(t) := \int_0^t \|e^{A_p s} D_p\| \mathrm{d}s$. Therefore, we let

$$E_{k+1} = G(p, p, x_k^*, E_k, \delta_k) := \Lambda_p E_k/N + \Phi_p(\tau_s)\delta_k.$$

Since x is continuous, the inequality in (10) holds with

$$x_{k+1}^* = F(p, p, c_k) := e^{(A_p + B_p K_p)\tau_s} c_k =: S_p c_k. \tag{16}$$

Sampling interval with switch When

$$\sigma(t_k) = p \ne q = \sigma(t_{k+1}) \tag{17}$$

for $p, q \in \mathcal{P}$, by (3) there is exactly one switch on $(t_k, t_{k+1}]$. Let $t_k + \bar{t}$ with $\bar{t} \in (0, \tau_s]$ denote the unknown switching time. Then $\sigma(t) = p$ for $t \in [t_k, t_k + \bar{t})$ and $\sigma(t) = q$ for $t \in [t_k + \bar{t}, t_{k+1}]$. Before the switch, we proceed as in the previous case and get that the error $e = x - \hat{x}$ satisfies

$$\|e(t_k + \bar{t})\| \le \|e^{A_p \bar{t}}\| E_k/N + \Phi_p(\bar{t})\delta_d \tag{18}$$

with Φ_p in (15). Since $\bar{t}$ is unknown, we estimate the value of $x(t_k + \bar{t})$ by comparing it with the value of $\hat{x}$ at an arbitrarily selected $t + t' \in [t_k, t_{k+1})$. As $\hat{x}(t_k + t') = e^{(A_p + B_p K_p)t'} c_k$ for all $t' \in [0, \tau_s)$, from (7), (18) and the triangle inequality we get

$$\begin{aligned}
&\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\| \\
&\le \|e(t_k + \bar{t})\| + \|\hat{x}(t_k + \bar{t}) - \hat{x}(t_k + t')\| \\
&\le \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\|(\|x_k^*\| + (1 - 1/N)E_k) \\
&\quad + \|e^{A_p \bar{t}}\| E_k/N + \Phi_p(\bar{t})\delta_d \\
&=: \hat{D}'_{k+1}(t', \bar{t}).
\end{aligned}$$

After the switch, combining (1) in mode q and (8) in mode p gives that $\dot{z} = \bar{A}_{pq} z + \bar{D}_q d$ with $z := (x^\top, \hat{x}^\top)^\top$ and

$$\bar{A}_{pq} := \begin{pmatrix} A_q & B_q K_p \\ 0_{n_x \times n_x} & A_p + B_p K_p \end{pmatrix}, \quad \bar{D}_q = \begin{pmatrix} D_q \\ 0_{n_x \times n_d} \end{pmatrix}.$$

Consider a second auxiliary system $\dot{\hat{z}} = \bar{A}_{pq}\hat{z}$ with $\hat{z}(t_k + t') = (\hat{x}(t_k + t')^\top, \hat{x}(t_k + t')^\top)^\top$, and the error $\bar{e}(t) := z(t) - \hat{z}(t - \bar{t} + t')$. Since the ∞-norm satisfies

$$\|(v^\top, w^\top)^\top\| = \max\{\|v\|, \|w\|\} \qquad \forall\, v, w \in \mathbb{R}^n, \tag{19}$$

we get $\|\bar{e}(t_k + \bar{t})\| \le \hat{D}'_{k+1}(t', \bar{t})$. Hence

$$\|\bar{e}(t_{k+1}^-)\| \le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \hat{D}'_{k+1}(t', \bar{t}) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta_d \tag{20}$$

by variation of constants with $\bar{\Phi}_{pq} : [0, \tau_s] \to \mathbb{R}$ defined by $\bar{\Phi}_{pq}(t) := \int_0^t \|e^{\bar{A}_{pq} s}\bar{D}_q\| \mathrm{d}s$. Again, we estimate the value of $z(t_{k+1}^-)$ by comparing it with the value of $\hat{z}$ at an arbitrarily selected $t_k + t'' \in [t_k, t_{k+1})$. As $\hat{z}(t_k + t'') = e^{\bar{A}_{pq}(t'' - t')}\hat{z}(t_k + t')$ for all $t'' \in [0, \tau_s)$, from (7), (19), (20) and the triangle inequality we get

$$\begin{aligned}
&\|x(t_{k+1}^-) - \hat{x}(t_k + t'')\| \le \|z(t_{k+1}^-) - \hat{z}(t_k + t'')\| \\
&\le \|\bar{e}(t_{k+1}^-)\| + \|\hat{z}(t_{k+1}^- - \bar{t} + t') - \hat{z}(t_k + t'')\| \\
&\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \hat{D}'_{k+1}(t', \bar{t}) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta_d + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} \\
&\quad - e^{\bar{A}_{pq}(t'' - t')}\|\|e^{(A_p + B_p K_p)t'}\|(\|x_k^*\| + (1 - 1/N)E_k) \\
&=: \hat{D}''_{k+1}(t', t'', \bar{t}).
\end{aligned}$$

To eliminate the dependence on the unknown $\bar{t}$, we take the maximum over $\bar{t}$ (with fixed t', t'') and get $\|x(t_{k+1}^-) - \hat{x}(t_k + t'')\| \le \max_{\bar{t} \in [0, \tau_s]} \hat{D}''_{k+1}(t', t'', \bar{t}) =: \hat{D}_{k+1}$. Therefore, we define E_{k+1} by replacing δ_d in $\hat{D}''_{k+1}(t', t'', \bar{t})$ with the estimate δ_k and then taking the maximum over $\bar{t}$ (with the same fixed t', t''). (Clearly, the design parameters t', t'' should be selected so that E_{k+1} is minimized. However, their optimal values cannot be determined without imposing further constraints on the matrices $\{A_p, B_p, D_p, K_p : p \in \mathcal{P}\}$.) Since x is continuous, (10) holds with

$$\begin{aligned}
x_{k+1}^* &= F(p, q, c_k) := H_{pq} c_k \\
&:= (I_{n_x \times n_x}\ 0_{n_x \times n_x}) e^{\bar{A}_{pq} t''} \begin{pmatrix} e^{(A_p + B_p K_p)t'} \\ e^{(A_p + B_p K_p)t'} \end{pmatrix} c_k.
\end{aligned} \tag{21}$$

A more computation-friendly upper bound is derived as

$$E_{k+1} \le \alpha_{pq}\|x_k^*\| + \beta_{pq} E_k + \gamma_{pq}\delta_k \tag{22}$$

with

$$\begin{aligned}
\alpha_{pq} &:= \|e^{(A_p + B_p K_p)t'}\| e^{\|\bar{A}_{pq}\| \max\{\tau_s, 2(t'' - t'), \tau_s + 2(t' - t'')\}} \\
&\quad \times \|\bar{A}_{pq}\| \max\{t'' - t', \tau_s + t' - t''\} + \|A_p + B_p K_p\| \\
&\quad \times e^{\|\bar{A}_{pq}\|\tau_s} e^{\|A_p + B_p K_p\| \max\{\tau_s, 2t'\}} \max\{\tau_s - t', t'\}, \\
\beta_{pq} &:= (1 - 1/N)\alpha_{pq} + e^{\|\bar{A}_{pq}\|\tau_s + \|A_p\|\tau_s}/N, \\
\gamma_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s}\Phi_p(\tau_s) + \bar{\Phi}_{pq}(\tau_s).
\end{aligned}$$

5.2 Searching stage

Recall that the control input $u \equiv 0$ in searching stages. From (11) and variation of constants we get that

$$\|x(t) - x_k^*\| \le \bar{\alpha}\|x_k^*\| + \bar{\beta}\hat{D}_k + \bar{\gamma}\delta_d =: \hat{D}_{k+1} \tag{23}$$

for all $t \in (t_k, t_{k+1}]$, where

$$\bar{\alpha} := (\hat{\Lambda} + 1)\hat{\Gamma}, \quad \bar{\beta} := \hat{\Lambda}^2 \quad \bar{\gamma} := (\hat{\Lambda} + 1)\Phi$$

with $\hat{\Gamma} := \max_p e^{\|A_p\|\tau_s}\|A_p\|\tau_s$, $\hat{\Lambda} := \max_p e^{\|A_p\|\tau_s}$, $\Phi := \max_p \Phi_p(\tau_s)$. Hence we let

$$E_{k+1} = \hat{G}(x_k^*, (1 + \varepsilon_E)E_k, \delta_k) := \bar{\alpha}\|x_k^*\| + (1 + \varepsilon_E)\bar{\beta}E_k + \bar{\gamma}\delta_k$$

with an arbitrarily selected design parameter $\varepsilon_E > 0$.

Finite time recovery Suppose the state escaped at sampling time t_j. Consider two increasing sequences $(\tilde{D}_j^l)_{l=0}^\infty$ and $(\tilde{E}_j^l)_{l=0}^\infty$ defined by

$$\begin{aligned}
\tilde{D}_j^{l+1} &:= \hat{G}(x_j^*, \tilde{D}_j^l, \delta_d), & \tilde{D}_j^0 &:= \hat{D}_j, \\
\tilde{E}_j^{l+1} &:= \hat{G}(x_j^*, (1 + \varepsilon_E)\tilde{E}_j^l, \delta_j), & \tilde{E}_j^0 &:= E_j.
\end{aligned} \tag{24}$$

Straightforward calculation gives that

$$\begin{aligned}
\tilde{D}_j^l &= \bar{\beta}^l \hat{D}_j + (\bar{\beta}^l - 1)(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_d)/(\bar{\beta} - 1), \\
\tilde{E}_j^l &= \hat{\beta}^l E_j + (\hat{\beta}^l - 1)(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)/(\hat{\beta} - 1)
\end{aligned} \tag{25}$$

with $\hat{\beta} := (1+\varepsilon_E)\bar{\beta}$. Define

$$L_j := \lceil \max\{L_j^x, L_j^d\} \rceil \tag{26}$$

with $L_j^x := \log(\hat{D}_j/E_j)/\log(1+\varepsilon_E)$, and $L_j^d := \log(((\hat{\beta}-1)\delta_d)/((\bar{\beta}-1)\delta_j))/\log(1+\varepsilon_E)$ if $\delta_d > \delta_j$; else $L_j^d := 0$. (Here $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$ is the ceiling function.) It is straightforward to verify that $\tilde{E}_j^{L_j} \geq \tilde{D}_j^{L_j}$, and hence the state will be recovered in finite time. Moreover, when the state is recovered at sampling time t_i, we get $i - j \leq L_j$.

Remark 1. By letting $j = 0$ and $\hat{D}_0 = \|x_0\|$ we see that the analysis above also applies to the case when the state is lost at $t_0 = 0$. However, the state escapes at t_j only if $\delta_{j-1} < \delta_d$, while it is lost at $t_0 = 0$ if and only if $\|x_0\| > E_0$.

6. STABILITY ANALYSIS

In this section we show that the communication and control strategy described in Section 4 fulfills Theorem 1.

6.1 Stabilizing stage

Stability analysis for stabilizing stages essentially follows from (Yang and Liberzon, 2015, Subsections VI-A–VI-D).

First consider a sampling interval $[t_k, t_{k+1}]$ without switch. For $S_p = e^{(A_p+B_pK_p)\tau_s}$ in (16), as $A_p + B_pK_p$ is Hurwitz, there exist $P_p, Q_p > 0$ such that $S_p^\top P_p S_p - P_p = -Q_p < 0$. Let $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ denote the largest and smallest eigenvalues of a matrix, and define $\chi_p := 2n_x^2\|S_p^\top P_p S_p\|^2/\underline{\lambda}(Q_p) + n_x\|S_p^\top P_p S_p\|$. The inequality in (4) implies that there exist a sufficiently small $\phi > 0$ such that $(1+\phi)\Lambda_p^2/N^2 < 1$ for all $p \in \mathcal{P}$, and a sufficiently large $\rho_p > 0$ such that

$$(1-1/N)^2\chi_p/\rho_p + (1+\phi)\Lambda_p^2/N^2 < 1.$$

Define a family of functions $\{V_r : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}\}_{r\in\mathcal{P}}$ by

$$V_r(x, E) := x^\top P_r x + \rho_r E^2. \tag{27}$$

Lemma 2. For all $k \geq 0$ such that (6) and (14) hold, the function V_p defined according to (27) satisfies

$$V_p(x_{k+1}^*, E_{k+1}) \leq \nu V_p(x_k^*, E_k) + \nu_d\delta_k^2$$

with $\nu := \max_p\{(1-1/N)^2\chi_p/\rho_p + (1+\phi)\Lambda_p^2/N^2, 1 - \underline{\lambda}(Q_p)/(2\overline{\lambda}(P_p))\}$ and $\nu_d := \max_p(1+1/\phi)\rho_p\Phi_p(\tau_s)^2$, where Λ_p and Φ_p are in (4) and (15), respectively.

Second, consider sampling interval $[t_k, t_{k+1}]$ with switch. Let h_{pq} be the largest singular value of H_{pq} in (21).

Lemma 3. For all $k \geq 0$ such that (6) and (17) hold, the functions V_p, V_q defined according to (27) satisfy

$$V_q(x_{k+1}^*, E_{k+1}) \leq \mu V_p(x_k^*, E_k) + \mu_d\delta_k^2$$

with $\mu := \max_{p,q}\{(2\overline{\lambda}(P_q)h_{pq}^2 + (2+\zeta)\alpha_{pq}^2\rho_q)/\underline{\lambda}(P_p), (2(1-1/N)^2 n_x\overline{\lambda}(P_q)h_{pq}^2 + (2+\zeta)\beta_{pq}^2\rho_q)/\rho_p\}$ and $\mu_d := \max_{p,q}(1+2/\zeta)\rho_q\gamma_{pq}^2$, where $\alpha_{pq}, \beta_{pq}, \gamma_{pq}$ are in (22), and $\zeta > 0$ is an arbitrarily selected design parameter.

By varying the design parameters t', t'', ζ we can ensure $\mu \geq 1 > \nu$ and $\mu_d \geq \nu_d$, which are assumed to hold in the following proof; cf. (Yang and Liberzon, 2015, Remark 1).

Next we derive a lower bound on ADT τ_a in Assumption 1 that ensures exponential convergence at sampling times.

Lemma 4. Suppose the state is visible at consecutive sampling times $t_i, \ldots, t_{k-1}$. If the ADT τ_a satisfies

$$\tau_a > (1 + \log(\mu)/\log(1/\nu))\tau_s, \tag{28}$$

then there exists a sufficiently small $\omega \in (0,1)$ such that $V_{\sigma(t_k)}(x_k^*, E_k) < \theta^{k-i}\Theta^{N_0}V_{\sigma(t_i)}(x_i^*, E_i) + \Theta^{N_0+1}(1 + \nu/(\omega(1-\nu)))\nu_d\delta_i^2$ with $\Theta := (\mu + \omega(1-\nu)\mu_d/\nu_d)/(\nu + \omega(1-\nu))$, $\theta := \Theta^{\tau_s/\tau_a}(\nu + \omega(1-\nu)) < 1$ and N_0 in (2).

Then the bounds for x_k^*, E_k at sampling times in stabilizing stages can be derived via the triangle inequality as

$$\begin{aligned} \|x_k^*\| &\leq \theta^{(k-i)/2}(\bar{a}_1\|x_i^*\| + \bar{b}_1E_i) + \bar{c}_1\delta_i, \\ E_k &\leq r_E(\theta^{(k-i)/2}(\bar{a}_1\|x_i^*\| + \bar{b}_1E_i) + \bar{c}_1\delta_i) \end{aligned} \tag{29}$$

for all $k > i$ with $\bar{a}_1 := \Theta^{N_0/2}\sqrt{\lambda_M/\lambda_m}$, $\bar{b}_1 := \bar{a}_1\sqrt{\rho/\lambda_M}$, $\bar{c}_1 := \Theta^{(N_0+1)/2}\sqrt{(1+\nu/(\omega(1-\nu)))\nu_d/\lambda_m}$ and $r_E := \sqrt{\lambda_m/\rho_m}$, where $\lambda_M = \max_p\overline{\lambda}(P_p)$, $\lambda_m = \min_p\underline{\lambda}(P_p)$, $\rho = \max_p\rho_p$ and $\rho_m = \min_p\rho_p$.

Finally, we derive the inter-sample bound for x in stabilizing stages. Similar analysis to Subsection 5.1.2 gives

$$\|x(t)\| \leq (\alpha^0 + 1)\|x_k^*\| + (\beta^0 + (1-1/N))E_k + \gamma\delta_d.$$

with $(\alpha^0, \beta^0, \gamma) := \max_{p,q}(\alpha_{pq}, \beta_{pq}, \gamma_{pq})$ with $t' = t'' = 0$. Combining the previous inequality with (29) gives that

$$\|x(t)\| \leq r_S(\theta^{(t/\tau_s-i-1)/2}(\bar{a}_1\|x_i^*\| + \bar{b}_1E_i) + \bar{c}_1\delta_i) + \gamma\delta_d \tag{30}$$

for all $t \in [t_i, t_j]$ with $r_S := \alpha^0 + 1 + r_E(\beta^0 + (N-1)/N)$.

6.2 Searching stage

Consider the case when the state escaped at sampling time t_j and is recovered at t_i. Let $r = \sigma(t_{j-1})$, $p = \sigma(t_j)$. Then $\hat{D}_j$ and E_j in (11) are given by $\hat{D}_j = G(r, p, x_{j-1}^*, E_{j-1}, \delta_d)$ and $E_j = G(r, p, x_{j-1}^*, E_{j-1}, \delta_{j-1})$. From the formula for G we see that $\hat{D}_j/E_j < \delta_d/\delta_{j-1} = (1+\varepsilon_\delta)\delta_d/\delta_j$. Define

$$\eta(\delta_d/\delta_j) := (\log(\delta_d/\delta_j) + \log(r_d(\delta_d/\delta_j)))/\log(1+\varepsilon_E) + 1 \tag{31}$$

with $r_d(\delta_d/\delta_j) := \max\{1+\varepsilon_\delta, (\hat{\beta}-1)/(\bar{\beta}-1)\}$ if $\delta_d > \delta_j$; else $r_d(\delta_d/\delta_j) := 1 + \varepsilon_\delta$. Then

$$i - j \leq L_j = \lceil \max\{L_j^x, L_j^d\} \rceil < \eta(\delta_d/\delta_j). \tag{32}$$

From (24) we see that $\hat{D}_k = \tilde{D}_j^{k-j}$ and $E_k = \tilde{E}_j^{k-j}$ for all $k = i, \ldots, j$. Hence $E_i = \tilde{E}_j^{i-j} \leq \tilde{E}_j^{L_j}$, and (23) implies that $\|x(t) - x_i^*\| \leq \hat{D}_i = \tilde{D}_j^{i-j} \leq \tilde{D}_j^{L_i} \leq \tilde{E}_j^{L_i}$ for all $t \in [t_j, t_i]$. Finally, (25) and (32) implies $\tilde{E}_j^{L_j} = \hat{\beta}^{L_j}E_j + (\hat{\beta}^{L_j} - 1)(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)/(\hat{\beta}-1) < \bar{b}_2^{\eta(\delta_d/\delta_j)}(\bar{a}_2\|x_j^*\| + E_j + \bar{c}_2\delta_j)$ with $\bar{a}_2 := \bar{\alpha}/(\hat{\beta}-1)$, $\bar{b}_2 := \hat{\beta}$ and $\bar{c}_2 := \bar{\gamma}/(\hat{\beta}-1)$. Hence for all $t \in [t_j, t_i]$ we have

$$\begin{aligned} \|x(t) - x_i^*\| &< \bar{b}_2^{\eta(\delta_d/\delta_j)}(\bar{a}_2\|x_j^*\| + E_j + \bar{c}_2\delta_j), \\ E_i &< \bar{b}_2^{\eta(\delta_d/\delta_j)}(\bar{a}_2\|x_j^*\| + E_j + \bar{c}_2\delta_j). \end{aligned} \tag{33}$$

The case when there is a searching stage at $t_0 = 0$ can be treated in a similar manner. Let $L_0' = \lceil \max\{L_0^x, L_0^d\} \rceil$ if $\|x_0\| > E_0$; else $L_0' = 0$, where L_0^x, L_0^d are defined in the same ways as L_j^x, L_j^d in (26) with $\hat{D}_0 = \|x_0\|$. Define

$$\eta_0^x(\|x_0\|/E_0) := L_0^x, \qquad \eta_0^d(\delta_d/\delta_0) := L_0^d + 1. \tag{34}$$

By virtue of Remark 1, we see that the previous analysis also applies to the case where $j = 0$ by letting $\hat{D}_0 = \|x_0\|$

and substituting $\eta_0^x(\|x_0\|/E_0) + \eta_0^d(\delta_d/\delta_0)$ for $\eta(\delta_d/\delta_0)$. Hence when the state is initially captured at t_{i_0} we get

$$i_0 \leq L_0' < \eta_0^x(\|x_0\|/E_0) + \eta_0^d(\delta_d/\delta_0), \tag{35}$$

and for all $t \in [0, t_{i_0}]$ we have

$$\begin{aligned} \|x(t)\| &< \bar{b}_2^{\eta_0^x(\|x_0\|/E_0)+\eta_0^d(\delta_d/\delta_0)}(E_0 + \bar{c}_2\delta_0), \\ E_{i_0} &< \bar{b}_2^{\eta_0^x(\|x_0\|/E_0)+\eta_0^d(\delta_d/\delta_0)}(E_0 + \bar{c}_2\delta_0). \end{aligned} \tag{36}$$

6.3 Exponential convergence

In this subsection we establish the first claim of Theorem 1.

The system alternates between searching and stabilizing stages. The number of searching stages are finite as there will be no more escape once $\delta_k \geq \delta_d$. Let $0 = j_0 \leq i_0 < \cdots < j_{N_s} < i_{N_s}$ be such that $[t_{i_k}, t_{j_{k+1}})$ are stabilizing stages, and $[t_{j_k}, t_{i_k})$ are searching stages. If $\delta_d > \delta_0$ then $N_s \leq \lceil(\log(\delta_d/\delta_0))/(\log(1+\varepsilon_\delta))\rceil$; else $N_s = 0$.

Combining the bounds (32), (33), (35), (36) for searching stages, and (29), (30) for stabilizing stages shows that $\|x(t)\| < \psi^{\eta_0^d(\delta_d/\delta_0)+\eta(\delta_d/\delta_{j_1})+\cdots+\eta(\delta_d/\delta_{j_k})}\Psi^k r_S\theta^{-1/2} \times (\psi^{\eta_0^x(\|x_0\|/E_0)}\theta^{t/(2\tau_s)}\bar{b}_1(E_0+\bar{c}_2\delta_0)+(1+\cdots+(r_\varepsilon/\Psi)^k)(\bar{c}_2\bar{b}_1+\bar{c}_1)\delta_0) + \gamma\delta_d$ for all $t \in [t_{j_k}, t_{j_{k+1}})$ and $k = 0, \ldots, N_s$ with $\psi := \bar{b}_2\theta^{-1/2}$, $\Psi := \bar{a}_1 + \bar{a}_2\bar{b}_1 + r_E\bar{b}_1$ and $r_\varepsilon := (1+\varepsilon_\delta)/(1+\varepsilon_E)$. Define $N_d : \mathbb{R}_{\geq 0} \to \mathbb{Z}$ by $N_d(s) := \lceil\log(s/\delta_0)/\log(1+\varepsilon_\delta)\rceil$ if $s > \delta_0$; and $N_d(s) := 0$ if $s \in [0, \delta_0]$, and $L_x, L_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $L_x(s) := \eta_0^x(s/E_0)$ and $L_d(s) := \eta_0^d(s/\delta_0) + \sum_{l=1}^{N_d(s)} \eta((1+\varepsilon_\delta)^{-l}s/\delta_0)$ with η in (31) and η_0^x, η_0^d in (34). Then $N_s \leq N_d(\delta_d)$, and $L_x(\|x_0\|) + L_d(\delta_d)$ is an upper bound on the total length of all searching stages. Hence

$$\begin{aligned} \|x(t)\| &< \psi^{L_d(\delta_d)}\Psi^{N_d(\delta_d)}r_S\theta^{-1/2}\big(\psi^{L_x(\|x_0\|)}\theta^{t/(2\tau_s)}\bar{b}_1 \\ &\times (E_0 + \bar{c}_2\delta_0) + \frac{1-(r_\varepsilon/\Psi)^{N_d(\delta_d)+1}}{1-r_\varepsilon/\Psi}(\bar{c}_2\bar{b}_1+\bar{c}_1)\delta_0\big) + \gamma\delta_d \end{aligned}$$

for all $t \in \mathbb{R}_{\geq 0}$. By applying Young's inequality with arbitrarily selected design parameters $\kappa_x, \kappa_d > 0$ such that $1/\kappa_x + 1/\kappa_d = 1$ we obtain (5) with

$$\begin{aligned} \lambda &:= -\log(\theta)/(2\tau_s) > 0, \\ C &:= \psi^{\kappa_d}r_S\theta^{-1/2}\bar{b}_1(E_0+\bar{c}_2\delta_0)/\kappa_d \\ &\quad + \psi r_S\theta^{-1/2}(\bar{c}_2\bar{b}_1+\bar{c}_1)\delta_0, \end{aligned} \tag{37}$$

and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$\begin{aligned} g(s) &:= \psi^{\kappa_x L_x(s)}r_S\theta^{-1/2}\bar{b}_1(E_0+\bar{c}_2\delta_0)/\kappa_x, \\ h(s) &:= \psi^{\kappa_d L_d(s)}\Psi^{\kappa_d N_d(s)}r_S\theta^{-1/2}\bar{b}_1(E_0+\bar{c}_2\delta_0)/\kappa_d \\ &\quad + (1-(r_\varepsilon/\Psi)^{N_d(\delta_d)+1})\psi^{L_d(s)}\Psi^{N_d(s)}r_S\theta^{-1/2} \\ &\quad \times (\bar{c}_2\bar{b}_1+\bar{c}_1)\delta_0/(1-r_\varepsilon/\Psi) + \gamma s - C. \end{aligned} \tag{38}$$

6.4 Practical stability

In this subsection we establish the second claim of Theorem 1, which essentially follows from (Yang and Liberzon, 2015, Subsection VI-E).

Lemma 5. Suppose the average dwell-time τ_a satisfies

$$\tau_a > (1 + \log(\beta)/\log(N/\Lambda))\tau_s \tag{39}$$

with $\beta := \max_{p,q}\beta_{pq}$ and $\Lambda := \max_p \Lambda_p$. Then the practical stability property in Theorem 1 holds with

$$C' := \left(r_S\bar{b}_1\Theta_B^{N_0+1}\left(1 + \frac{\Lambda/N}{\omega_B(1-\Lambda/N)}\right)\Phi + r_S\bar{c}_1\right)\delta_0. \tag{40}$$

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