SWITCHED AND HYBRID SYSTEMS WITH INPUTS:
SMALL-GAIN THEOREMS, CONTROL WITH LIMITED INFORMATION,
AND TOPOLOGICAL ENTROPY

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

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Abstract

In this thesis, we study stability and stabilization of switched and hybrid systems with inputs. We consider primarily two topics in this area: small-gain theorems for interconnected switched and hybrid systems, and control of switched linear systems with limited information.

First, we study input-to-state practical stability (ISpS) of interconnections of two switched nonlinear subsystems with independent switchings and possibly non-ISpS modes. Provided that for each subsystem, the switching is slow in the sense of an average dwell-time (ADT), and the total active time of non-ISpS modes is short in proportion, Lyapunov-based small-gain theorems are established via hybrid system techniques. By augmenting each subsystem with a hybrid auxiliary timer that models the constraints on switching, we enable a construction of hybrid ISpS-Lyapunov functions, and consequently, a convenient formulation of a small-gain condition for ISpS of the interconnection. Based on our small-gain theorem, we demonstrate the stabilization of interconnected switched control-affine systems using gain-assignment techniques.

Second, we investigate input-to-state stability (ISS) of networks composed of $n \geq 2$ hybrid subsystems with possibly non-ISS dynamics. Lyapunov-based small-gain theorems are established based on the notion of candidate ISS-Lyapunov functions, which unifies and extends several previous results for interconnected hybrid and impulsive systems. In order to apply our small-gain theorem to different combinations of non-ISS dynamics, we adopt the method of modifying candidate exponential ISS-Lyapunov functions using ADT and reverse ADT timers. The effect of such modifications on the Lyapunov feedback gains between two interconnected hybrid systems is discussed in detail through a case-by-case study.

Third, we consider the problem of stabilizing a switched linear system with a completely unknown disturbance using sampled and quantized state feed-

back. The switching is assumed to be slow enough in the sense of combined dwell-time and average dwell-time, each individual mode is assumed to be stabilizable, and the data rate is assumed to be large enough but finite. By extending the approach of reachable-set approximation and propagation from an earlier result on the disturbance-free case, we develop a communication and control strategy that achieves a variant of input-to-state stability with exponential decay. An estimate of the disturbance bound is introduced to compensate for the unknown disturbance, and a novel algorithm is designed to adjust the estimate and recover the state when it escapes the range of quantization.

Last, motivated by the connection between the minimum data rate needed to stabilize a linear time-invariant system and its topological entropy, we examine a notion of topological entropy for switched systems with a known switching signal. This notion is formulated in terms of the number of initial points such that the corresponding trajectories approximate all trajectories within a certain error, and can be equivalently defined using the number of initial points that are separable up to a certain precision. We first calculate the topological entropy of a switched scalar system based on the active rates of its modes. This approach is then generalized to nonscalar switched linear systems with certain Lie structures to establish entropy bounds in terms of the active rate and eigenvalues of each mode.

To my better half, for her love and support.

Acknowledgments

First and foremost, the successful completion of this thesis is a tribute to the extraordinary guidance from my adviser, Prof. Daniel Liberzon, who has always been patient, open-minded, and supportive during my Ph.D. studies. His enthusiasm for genuine research, creativity, and attention to detail will forever inspire me to be a better researcher.

I would also like to express my gratitude to the members of my doctoral committee, Prof. Mohamed-Ali Belabbas, Prof. Sayan Mitra, and Prof. Vadim Zharnitsky, for their time and effort devoted to my thesis. My sincere thanks also goes to all my teachers at University of Illinois at Urbana-Champaign, who has always encouraged me to learn, think, and explore.

I am deeply indebted to my collaborators, Prof. Zhong-Ping Jiang, Andrii Mironchenko, and A. James Schmidt. Their invaluable inputs have helped improve my research immensely. I would also like to thank Prof. Andrew Teel and Prof. Raphaël Jungers for their insightful and constructive comments.

Many thanks go to my friends and colleagues in the Coordinated Science Laboratory, including Aneel Tanwani, Charles Yg, Shenyu Liu, Cheol Gi Bae, Amirhossein Taghvaei, Jonathan Ligo, Mayank Baranwal, Thịnh Thành Doàn, Philip Pare Jr., and the numerous others, who has made my journey as a Ph.D. student much more enjoyable.

This thesis would not have been possible without the support of my parents, Dejun Yang and Yanhua Gu, who always encourage me to pursue my dream. Finally, my deepest appreciation goes to my fiancée, Xue Hu, who endured this seemingly endless process together with me, and has always been there for me in difficult times since we first met at thirteen years old.

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Part I

Introduction

Chapter 1

Overview

1.1 Switched and hybrid systems

In systems theory, a dynamical system is typically modeled as a set of state variables, or simply states, that evolve according to certain rules in a finite-dimensional state-space. The progression of a continuous-time system is defined by a set of differential equations, while the transition of a discrete-time one is described by a sequence of isolated events. Traditionally, control theory has focused on either continuous or discrete behaviors of a dynamical system. However, in studying real-world problems, one usually finds it necessary to establish models involving interactions between continuous and discrete dynamics. Such models are named *hybrid systems*, which have attracted tremendous research interest over the past decades [1, 2]. A general modeling framework for hybrid systems was proposed in [3], which proves to be natural and effective from the viewpoint of Lyapunov stability theory [4, 5].

From the perspective of systems and control theory, a wide variety of hybrid systems can be characterized by the class of *switched systems* [6, 7]. A switched system consists of a family of continuous-time dynamics, called *modes*, and a sequence of discrete events, called *switching events*. The state evolves according to one active mode between consecutive switching events, and to two different modes before and after each switching event. In this way, we distill discrete behavior of the system into the switching mechanism, and place more emphasis on issues regarding the continuous states, such as stability analysis and control synthesis. In this thesis, we focus our attention primarily on stability and stabilization of switched systems, with the exception of Chapter 4, where we study stability properties of networks of hybrid systems.

According to their effects on the system dynamics and state, switching events in switched systems can be categorized into *state-dependent* ones, in which the switching is triggered by certain values of state, and *time-dependent* ones, in which the switching is described by a sequence of (isolated) *switching times*, or simply *switches*. Meanwhile, switching events can also be categorized into *autonomous (uncontrolled)* ones and *controlled* ones due to their sources of generation. In this thesis, we only consider switched systems with time-dependent, autonomous switching, and adopt the conventional assumption that the state trajectory is absolutely continuous (in particular, there is no instantaneous change in its value at a switch).

For switched systems with time-dependent, autonomous switching, there have been mainly two approaches for establishing asymptotic stability. The first one assumes the switching to be arbitrary, and develops sufficient conditions on dynamics of the modes. Clearly, the assumption of arbitrary switching requires all modes to be asymptotically stable. However, this condition is necessary but not sufficient (see, e.g., [6, p. 19] for a counterexample). Standard techniques in this approach include constructing a common Lyapunov function that decreases along the entire state trajectory, uniformly over all possible switchings [8]; or establishing suitable Lie structures of the set of modes such as commutativity [9] or solvability [10]. On the other hand, the second approach assumes certain stability properties of the modes, and studies constraints on the switching that ensure asymptotic stability of the switched system. In this thesis, we are interested in the second approach, and aim to formulate stability conditions for the general scenario in which some of the modes have destabilizing effects.

In the stability analysis of switched systems under constrained switching, a standard approach is to construct *multiple Lyapunov functions*, usually one for each individual mode. Considering at each time the value of the Lyapunov function corresponding to the active mode, one obtains a trajectory which is continuous and decreasing between consecutive switches, but may be discontinuous and increasing at switching times due to the change in active modes (even though the state trajectory remains absolutely continuous). This approach was introduced in [11], where it was shown that the switched system is asymptotically stable provided that at switching times, the values of Lyapunov functions corresponding to the subsequent active modes form a decreasing sequence. A generalized result was proved in [12] under a weaker condition that compares the values of each Lyapunov function at switches where the corresponding mode becomes active, separately for each mode. See [13, 14] for extensions to switched systems with unstable modes, where stability is achieved based on *candidate Lyapunov functions* whose growth between consecutive switches is bounded by a positive definite function.

Multiple Lyapunov functions prove to be particularly useful in establishing stability for switched systems with slow switching conditions. In [15], it was shown that a switched linear system with stable modes is asymptotically stable provided that the switching admits a large enough *dwell-time*, that is, a lower bound on the duration between any two consecutive switches. This result was generalized in [16] to the context of switched nonlinear systems and to the notion of *average dwell-time (ADT)*, which plays a crucial role in the stability analysis in this thesis. In [17], a similar result was developed for switched linear systems with both stable and unstable modes, by proportionally restricting the total active time of the unstable modes.

In studying dynamical systems with inputs, we adopt the *input-to-state stability (ISS)* framework proposed in [18, 19], which naturally unifies the notions of internal and external stability. Lyapunov characterizations of ISS were established in [20] for continuous-time systems, and extended in [21, 22] for discrete-time systems and in [23, 24] for hybrid systems. Towards stability of switched systems with disturbances, it was shown in [16] that ISS can be achieved under the same ADT condition as the one for stability in the disturbance-free case. This result was made explicit in [16] only for switched linear systems, and many generalizations for switched nonlinear systems have been established using similar approaches since then. Particularly relevant results include [25] for ISS with a dwell-time condition, [26] for ISS and integral input-to-state stability (iISS) with ADT conditions, and [27] for input/output-to-state stability (IOSS) with an ADT condition. In [27], IOSS was established for switched nonlinear systems with both IOSS and non-IOSS modes as well.

In this thesis, we consider primarily two topics in the area of switched and hybrid systems with inputs: small-gain theorems for interconnected switched and hybrid systems, and control of switched linear systems with limited information.

1.2 Interconnections and small-gain theorems

In studying real-world phenomena, one usually finds it helpful to transform a complicated system into an interconnection of simpler subsystems, and establish stability based on properties of the constituents. In this context, small-gain theorems prove to be useful tools for the analysis of feedback interconnections, a structure that appears frequently in the control literature. The essential idea is as follows: if each subsystem satisfies a certain stability property when its input is small enough, and such a property implies small inputs to the other subsystems, then the same stability property can be established for the interconnection provided that the composition of the feedback gains is upper bounded by the identity function.

A comprehensive overview of classical small-gain theorems using input-output gains of linear systems can be found in [28]. This technique was generalized to nonlinear feedback interconnections in [29, 30], within the input-output context. The aforementioned ISS framework allows one to establish internal and external stability properties simultaneously, making it ubiquitous in recent small-gain results. Small-gain theorems for interconnections of two ISS nonlinear systems were first established in [31], together with their extensions to the notions of *input-to-state practical stability (ISpS)* and *input-to-output practical stability (IOpS)*. Some of these results were generalized to networks of $n \geq 2$ subsystems in [32, 33], and to the discrete-time context in [21].

Small-gain theorems prove to be particularly effective in constructing ISS-Lyapunov functions for the interconnection. Lyapunov-based small-gain theorems for feedback interconnections of two subsystems were first reported in [34] for continuous-time systems and then in [35] for discrete-time systems. Further results for general networks of $n \geq 2$ subsystems can be found in [36, 37, 38], with several variations summarized in [39, 40].

In Chapter 3, we study ISpS of interconnections of two switched nonlinear subsystems.[1] We consider the general scenario in which the subsystems switch independently and both consist of ISpS and possibly non-ISpS (i.e., destabilizing) modes. Provided that for each subsystem, the switching is slow in the ADT sense, and the total active time of non-ISpS modes is short

[1] Chapter 3 is based on our work [41, 42] and the joint work with Zhong-Ping Jiang [43].

in proportion, Lyapunov-based small-gain theorems are established by introducing hybrid auxiliary timers and adopting hybrid system techniques. More specifically, we augment each switched subsystem with a hybrid auxiliary timer modeling the constraints on switching to obtain a hybrid system, and propose a construction of hybrid ISpS-Lyapunov functions. Such ISpS-Lyapunov functions not only ensure ISpS of all complete solution pairs of the corresponding hybrid systems, and consequently, ISpS of the corresponding switched subsystems, but also enable a convenient formulation of a small-gain condition for ISpS of the interconnection.[2] Based on our small-gain theorem, we stabilize interconnections of switched control-affine systems by designing a Lyapunov-based variant of the gain-assignment techniques from [44].

Due to their interactive nature, many hybrid systems can be inherently modeled as feedback interconnections [45, Section V]. During recent years, great efforts have been devoted to the development of small-gain theorems for interconnected hybrid systems. Trajectory-based small-gain theorems for interconnections of two ISS hybrid subsystems were established in [46, 47, 48], and Lyapunov-based formulations were reported in [49, 50, 45]. Some of these results were extended to networks composed of $n \geq 2$ ISS hybrid systems in [48] as well.

In this thesis, we are interested in hybrid systems in which either the continuous dynamics (called the *flow*) or the discrete dynamics (called the *jumps*) are non-ISS—a more challenging case where the results above cannot be applied directly. In the presence of non-ISS dynamics, stability properties are usually achieved by imposing restrictions on the frequency of jumps, in the sense of an aforementioned ADT (for non-ISS jumps) and/or a *reverse average dwell-time (RADT)* [51] (for a non-ISS flow).[3] The results of [45] show that one can modify the non-ISS dynamics by first augmenting the corresponding subsystems with ADT/RADT auxiliary timers, and then constructing ISS-Lyapunov functions for the augmented subsystems that de-

[2]In [27], the authors studied IOSS of switched systems with both IOSS and non-IOSS modes, and established a similar sufficient condition to the one for ISpS of the switched subsystems here. The Lyapunov-based construction in our work exhibits the following improvements: it not only yields an ISpS-Lyapunov function which is used later in the stability analysis of the interconnection, but also provides means for robustness analysis.

[3]Recall that switched systems can be viewed as a class of hybrid systems with discrete dynamics characterized by the switching. Introduced to restrict the destabilizing effects of the switching, the notion of ADT proves to be useful in studying non-ISS jumps in hybrid systems as well.

crease along entire solution trajectories, both during the flow and at jumps. One advantage of this method is that it can be applied in the presence of mixed types of non-ISS dynamics (i.e., there are non-ISS flows in some subsystems while non-ISS jumps in some other ones). However, such modifications inevitably increase the Lyapunov feedback gains, making the small-gain condition afterwards more restrictive.

A different type of Lyapunov-based small-gain theorem was proposed in [52, 53] for interconnected impulsive systems in a similar setting. The first step of this approach is to construct a candidate exponential ISS-Lyapunov function for the interconnection (i.e., one that may increase during the flow or at jumps) based on those for the subsystems and a small-gain condition. Provided that there is only one type of non-ISS dynamics in the subsystems (i.e., either the continuous or the discrete dynamics of all subsystems are ISS), the candidate exponential ISS-Lyapunov function could be used to establish ISS under suitable ADT/RADT conditions. In contrast to [45], this approach does not require modifying subsystems; thus it works under the same small-gain condition as the one for the corresponding interconnection of only ISS subsystems. However, it was developed solely for impulsive systems and requires candidate exponential ISS-Lyapunov functions for subsystems; furthermore, it is inapplicable in the presence of mixed types of non-ISS dynamics.

In Chapter 4, the two approaches above are unified and extended to the case of general networks composed of $n \geq 2$ hybrid subsystems with possibly non-ISS dynamics.[4] We start by establishing a small-gain theorem that yields a candidate ISS-Lyapunov function for the network based on those for the subsystems. The candidate ISS-Lyapunov function is then used to establish ISS properties of the interconnection for the case of only ISS subsystems, as well as the case of only one type of non-ISS dynamics in the subsystems, under a suitable ADT or RADT condition. In order to address the case of mixed types of non-ISS dynamics, we adopt the method of modifying candidate exponential ISS-Lyapunov functions for subsystems via ADT and RADT auxiliary timers from [45], and study its effects on the Lyapunov feedback gains by analyzing interconnections of two hybrid subsystems with different combinations of non-ISS dynamics.

[4]Chapter 4 is based on our joint work with Andrii Mironchenko [54, 55, 56].

1.3 Control with limited information and topological entropy

Feedback control under data-rate constraints has been an active research area for years, as surveyed in [57, 58]. In many application-related scenarios, it is important to limit the information flow in the feedback loop due to bandwidth constraints, cost concerns, physical restrictions, security considerations, etc. Besides these practical motivations, the question of how much information is needed to achieve a certain control objective is fundamental and intriguing from the theoretical viewpoint. In our work, a finite data transmission rate is achieved by generating the control input based on sampled and quantized state measurements, which is a standard modeling framework in the literature (see, e.g., [59, 60] and [6, Chapter 5]).

In this thesis, we are interested in the problem of feedback stabilization under data-rate constraints in the presence of external disturbances. In this context, [59, 60] assumed known bounds on the disturbances and addressed asymptotic stabilization with minimum data rates, while [61, 62] avoided such assumptions by alternating between "zooming-out" and "zooming-in" stages and achieved input-to-state stability. See also [63, 64] for related results in a stochastic setting.

In the context of switched systems, early works on control under data-rate constraints were devoted to quantized control of Markov jump linear systems [65, 66, 67]. However, the discrete modes in the results above were always known to the controller, which would remove a major difficulty in our problem setup, making the control problem essentially the same as in the case without switching. The problem of asymptotically stabilizing a switched linear system (without disturbance) using sampled and quantized state feedback was studied in [68], which also serves as the basis for our work. In [68], the controller was assumed to have a partial knowledge of the switching, that is, the active mode was unknown except at sampling times, and the switching was subject to a mild slow-switching condition characterized by the combination of a dwell-time and an average dwell-time. Assuming that the data rate was large enough but finite, asymptotic stability was achieved by propagating over-approximations of reachable sets of the state over sampling intervals. See [69] for a related result using output feedback.

In Chapter 5, we generalize the main result of [68] in the presence of a completely unknown disturbance.[5] By extending the approach of reachable-set approximation and propagation from [68], we develop a communication and control strategy that achieves a variant of ISS with exponential decay. Due to the unknown disturbance, the state may be forced outside the approximation of reachable set at a sampling time after it has already been inside an earlier one (i.e., the state *escapes* the range of quantization). Consequently, the closed-loop system may alternate multiple times between stabilizing and searching stages. An estimate of the disturbance bound is introduced in approximating reachable sets so that the state cannot escape unless the disturbance is larger than the estimate. A novel algorithm is designed to adjust the disturbance estimate and *recover* the state when it escapes, so that the total length of searching stages is finite and the system eventually stays in a stabilizing stage, provided that the disturbance is globally essentially bounded (by an unknown value).

For a linear time-invariant control system, the minimum data rate necessary for feedback stabilization coincides with its topological entropy in open-loop [59, 60, 73]. Entropy is a fundamental concept in systems theory, which captures essentially the growth rate of uncertainty about the state over time [74] (in Russian, translated into English in [75]). More specifically, one can think of it as the exponential growth rate of the number of system trajectories distinguishable up to a finite precision [76], or in terms of the cardinality of open covers of the state space [77]. Different entropy definitions (notably, topological and measure-theoretic ones) and relationships between them are studied in detail in the book [78] and in many other sources, and continue to be a subject of active research in the dynamical systems community. The notion of entropy also plays a central role in thermodynamics and in information theory, as discussed, e.g., in [79].

In the context of control theory, entropy provides a natural characterization of the rate at which information of the system needs to be collected to generate control for a desired behavior (such as set invariance or stabilization). Following this intuition, suitable entropy definitions for control systems have been proposed and related to minimal data rates needed for control under communication constraints. The first such result was reported in [80], where topological feedback entropy of discrete-time systems was defined in terms of

[5] Chapter 5 is based on our work [70, 71, 72].

the cardinality of open covers of the state space. An alternative definition was proposed later in [81], which instead counted the number of "spanning" control functions. The paper [82] summarized the two notions and established an equivalence between them. The formulation of [81] was extended from set invariance to exponential stabilization in [83]. Most results on entropy in systems and control theory are for time-invariant systems, as time dependence in the dynamics introduces complexities which require new methods to analyze [84, 85].

In Chapter 6, we study topological entropy of switched systems with a known switching signal.[6] We formulate a notion of topological entropy in terms of the number of initial points such that the corresponding trajectories approximate all trajectories within a certain error. This definition of topological entropy extends the one for time-invariant systems from [78, Section 3.1.b], and can be equivalently formulated using the number of initial points that are separable up to a certain precision. We first calculate the topological entropy of a switched scalar system, based on the concept of active rate of a mode, that is, the proportion of time during which the mode is active. This approach is then generalized to nonscalar switched linear systems with certain Lie structures, such as commutativity and solvability, to establish entropy bounds in terms of the active rate and eigenvalues of each mode.

[6] Chapter 6 is based on our joint work with A. James Schmidt.

Chapter 2

Mathematical preliminaries

2.1 Notations

Denote by $\mathbb{R}_+ := [0, \infty)$ the set of nonnegative real numbers, and by $\mathbb{N} := \{0, 1, 2, \ldots\}$ the set of nonnegative integers. For a complex number $a \in \mathbb{C}$, denote by $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ its real and imaginary parts, respectively.[7] For two vectors x and y, denote by $(x, y) := (x^\top, y^\top)^\top$ their concatenation. For a matrix $A \in \mathbb{R}^{n\times n}$, denote by $\lambda_1(A), \ldots, \lambda_n(A)$ its eigenvalues counting multiplicity. In addition, if A is diagonal or triangular, then $\lambda_i(A)$ denotes its i-th diagonal entry. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n, or simply by I if the dimension is clear from the context. The zero matrix in $\mathbb{R}^{n\times m}$ is denoted by $0_{n,m}$, or simply by 0_n if $m = n$. For a vector x, denote by $|x|$ its Euclidean norm, and $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$ its (Euclidean) distance to a set $\mathcal{A}$. For a set $\mathcal{A}$, denote by $\overline{\mathcal{A}}$, $\operatorname{int}\mathcal{A}$, and $\partial\mathcal{A}$ its closure, interior, and boundary, respectively.

Denote by Id the identity function. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{PD}$ if it is continuous and positive definite (i.e., $\alpha(r) = 0$ if and only if $r = 0$); it is of class $\mathcal{K}$ if $\alpha \in \mathcal{PD}$ and is strictly increasing; it is of class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$.[8] A function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is continuous and strictly decreasing, and $\lim_{t\to\infty} \eta(t) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$. Denote by $\mathcal{C}^1$ the class of continuously differentiable functions, and by $\mathcal{C}^\infty$ the class of smooth (infinitely differentiable) functions.

[7] Hence $|a| = \sqrt{\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2}$.

[8] In particular, this implies that α is globally invertible.

2.2 Switched system notions

A continuous-time dynamical system with input (disturbance) is modeled by a set of differential equations of the form

$$\dot{x} = f(x, u), \qquad x(0) = x_0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The functions $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is assumed to be locally Lipschitz (so that one can establish local existence and uniqueness of the solution [86, Theorem 3.1]) and satisfy that $f(0, 0) = 0$. Consider a family of such continuous-time dynamical systems

$$\dot{x} = f_p(x, u), \qquad p \in \mathcal{P},$$

labeled by indices p from the *index set* $\mathcal{P}$ (which can in principle be arbitrary). The corresponding switched system is modeled by

$$\dot{x} = f_\sigma(x, u), \qquad x(0) = x_0, \tag{2.2}$$

where $\sigma : \mathbb{R}_+ \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal*. Conventionally, the function f_p is called the *p-th mode*, or mode p, of the switched system (2.2), and $\sigma(t)$ is called the *active mode* at time t. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation (2.2) away from discontinuities of σ (in particular, there is no state jump). An admissible input $u(\cdot)$ is a Lebesgue measurable, locally essentially bounded function. Discontinuities of σ are called *switching times*, or simply *switches*. It is assumed that there is at most one switch at each time, and a finite number of switches on each finite time interval (i.e., the set of switches contains no accumulation point). The number of switches on a time interval $(\tau, t]$ is denoted by $N_\sigma(t, \tau)$.

Following [15], we say that the switching signal σ admits a *dwell-time* τ_d if there exists a constant $\tau_d > 0$ such that all consecutive switches t' and t'' satisfy

$$t'' - t' \geq \tau_d, \tag{2.3}$$

or equivalently,

$$N_\sigma(t, \tau) \leq 1$$

for all $\tau \geq 0$ and $t \in (\tau, \tau + \tau_d]$. This concept was generalized in [16] to the notion that σ admits an *average dwell-time (ADT)* τ_a if there exist constants $\tau_a > 0$ and $N_0 \geq 1$ such that

$$N_\sigma(t,\tau) \leq \frac{t-\tau}{\tau_a} + N_0 \qquad \forall\, t > \tau \geq 0. \tag{2.4}$$

Note that the dwell-time condition (2.3) can be written in the form of the ADT condition (2.4) with $\tau_a = \tau_d$ and $N_0 = 1$. Moreover, the ADT condition (2.4) holds with a constant $N_0 < 1$ only if there is no switch at all.

For an input $u : \mathbb{R}_+ \to \mathbb{R}^m$, denote by

$$\|u\|_t := \operatorname*{ess\,sup}_{s \in [0,t)} |u(s)|$$

its essentially supreme (Euclidean) norm over the interval $[0,t)$, and for brevity,

$$\|u\| := \operatorname*{ess\,sup}_{s \geq 0} |u(s)|$$

that over $\mathbb{R}_+$.

Following [31], the continuous-time system (2.1) is called *input-to-state practically stable (ISpS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, and a constant $\varepsilon \geq 0$ such that for all initial states $x_0 \in \mathbb{R}^n$ and inputs $u : \mathbb{R}_+ \to \mathbb{R}^m$,[9]

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_t) + \varepsilon \qquad \forall\, t \geq 0. \tag{2.5}$$

Additionally, if the ISpS estimate (2.5) holds with $\varepsilon = 0$, that is, all solutions satisfy

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_t) \qquad \forall\, t \geq 0, \tag{2.6}$$

then the system (2.1) is called *input-to-state stable (ISS)* [18]. The function γ in (2.6) is sometimes referred to as the *ISS gain function*, or simply the *ISS gain*. The same definitions of ISpS and ISS also apply to the switched system (2.2).

In studying stability properties of dynamical systems, Lyapunov analysis proves to be a particularly useful approach. A Lyapunov characterization for ISS was proposed in [20], and extended in [34] to the concept of ISpS.

[9]As mentioned in [20, p. 352], by causality, it is equivalent to use $\|u\|$ in stead of $\|u\|_t$ in the ISpS estimate (2.5) and the ISS estimate (2.6).

Definition 2.1. A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an *ISpS-Lyapunov function* for the continuous-time system (2.1) if

1. there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|) \qquad \forall\, x \in \mathbb{R}^n; \tag{2.7}$$

2. there exist functions $\phi, \alpha \in \mathcal{K}_\infty$ and a constant $\delta \geq 0$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$|x| \geq \phi(|u|) + \delta \implies \nabla V(x) \cdot f(x, u) \leq -\alpha(|x|). \tag{2.8}$$

Additionally, if (2.8) holds with $\delta = 0$, then the function V is called an ISS-Lyapunov function.

The function ϕ in (2.8) plays a similar role to that of the function γ in the definition (2.5) of ISpS; with a slight abuse of terminology, we refer to ϕ as the *gain function*, or simply the *gain*, of the ISpS-Lyapunov function V. For a multi-input system, it is usually useful to specify one gain for each different input.

Proposition 2.1 ([34, Proposition 2.1])**.** *The continuous-time system* (2.1) *is ISpS if and only if it admits an ISpS-Lyapunov function. It is ISS if and only if it admits an ISS-Lyapunov function.*

See [20] for the proof of the equivalence between the ISS property and the existence of an ISS-Lyapunov function, which replies on the notion of "weakly robust stability" and the converse Lyapunov theorem for systems with bounded inputs proved in [87]. The proof of the equivalence between the ISpS property and the existence of an ISpS-Lyapunov function relies on the notion of "ISS with respect to a compact set" [88] and is essentially along the lines of [20].

Remark 2.1. If the ISS estimate (2.6) holds with $\gamma \equiv 0$ (e.g., the case without input), that is, all solutions satisfy

$$|x(t)| \leq \beta(|x_0|, t) \qquad \forall\, t \geq 0,$$

then the system (2.1) is called *globally asymptotically stable (GAS)* [87, Proposition 2.5]. Additionally, if the GAS estimate above holds with the function $\beta(r,t) := ce^{-\lambda t}r$ for some constants $c, \lambda > 0$, that is, all solutions satisfy

$$|x(t)| \leq ce^{-\lambda t}|x_0| \qquad \forall\, t \geq 0,$$

then the system (2.1) is called *globally exponentially stable (GES)*. The same definitions of GAS and GES also apply to the switched system (2.2). See [86] for a comprehensive overview of classical Lyapunov theorems.

2.3 Hybrid system notions

Motivated by [89, 23], a hybrid system with input (disturbance) is modeled by a combination of a continuous flow and discrete jumps of the form

$$\begin{aligned} \dot{x} &\in F(x,u), && (x,u) \in \mathcal{C}, \\ x^+ &\in G(x,u), && (x,u) \in \mathcal{D}, \end{aligned} \tag{2.9}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state and $u \in \mathcal{U} \subset \mathbb{R}^m$ is the input. We call $\mathcal{C} \subset \mathcal{X} \times \mathcal{U}$ the flow set, $\mathcal{D} \subset \mathcal{X} \times \mathcal{U}$ the jump set, $F : \mathcal{C} \rightrightarrows \mathbb{R}^n$ the flow map, and $G : \mathcal{D} \rightrightarrows \mathcal{X}$ the jump map.[10] In this model, the dynamics of (2.9) is continuous if $(x,u) \in \mathcal{C}\backslash\mathcal{D}$, and discrete if $(x,u) \in \mathcal{D}\backslash\mathcal{C}$. If $(x,u) \in \mathcal{C} \cap \mathcal{D}$, then it may be either continuous or discrete. The hybrid system (2.9) is fully characterized by its *data* $\mathcal{H} := (F, G, \mathcal{C}, \mathcal{D}, \mathcal{X}, \mathcal{U})$.

Solutions of (2.9) are defined on hybrid time domains [3]. A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{k=0}^{J} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 < t_1 < \cdots < t_{J+1}$. It is a *hybrid time domain* if for each $(T, J) \in E$, the truncation $E \cap ([0,T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. Equivalently, a hybrid time domain $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last one (if existent) possibly of the form $[t_j, T) \times \{j\}$ with $T \in \mathbb{R}$ or $T = \infty$. On a hybrid time domain, there is a natural ordering of points, namely, $(s,k) \preceq (t,j)$ if $s + k \leq t + j$, and $(s,k) \prec (t,j)$ if $s + k < t + j$.

[10] Here "$\rightrightarrows$" denote set-valued mappings, that is, F maps each element of $\mathcal{C}$ to a subset of $\mathbb{R}^n$ and G maps each element of $\mathcal{D}$ to a subset of $\mathcal{X}$.

A function defined on a hybrid time domain is called a *hybrid signal*. A hybrid signal $x : \operatorname{dom} x \to \mathcal{X}$, defined on the hybrid time domain $\operatorname{dom} x$, is called a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous on $\{t : (t, j) \in \operatorname{dom} x\}$ for each fixed j. A hybrid signal $u : \operatorname{dom} u \to \mathcal{U}$ is called a *hybrid input* if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, j) \in \operatorname{dom} u\}$ for each fixed j. A hybrid arc $x : \operatorname{dom} x \to \mathcal{X}$ and a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ form a *solution pair* (x, u) of (2.9) if[11]

- $\operatorname{dom} x = \operatorname{dom} u$ and $(x(0, 0), u(0, 0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$;
- for each $j \in \mathbb{N}$, it holds that $(x(t, j), u(t, j)) \in \mathcal{C}$ for all $t \in \operatorname{int} I_j$, and $\dot{x}(t, j) \in F(x(t, j), u(t, j))$ for almost all $t \in I_j$, with the interval $I_j := \{t : (t, j) \in \operatorname{dom} x\}$;
- for each $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$, it holds that $(x(t, j), u(t, j)) \in \mathcal{D}$ and $x(t, j + 1) \in G(x(t, j), u(t, j))$.

With suitable assumptions on the data $\mathcal{H}$, one can establish local existence of solutions, which are not necessarily unique (see, e.g. [3, Proposition 2.10]). A solution pair (x, u) is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom} x$ is unbounded. In this thesis, we only consider maximal (but not necessarily complete) solution pairs. The hybrid system (2.9) is *forward complete* if for all maximal solution pairs are complete.

Following [23], for a hybrid input $u : \operatorname{dom} u \to \mathbb{R}^m$, its essential supremum (Euclidean) norm up to a hybrid time (t, j) is defined by

$$\|u\|_{(t,j)} := \max \left\{ \operatorname*{ess\,sup}_{(s,k)\in \operatorname{dom} u :\, (s,k) \preceq (t,j)} |u(s, k)|, \sup_{(s,k)\in J(u) :\, (s,k)\preceq(t,j)} |u(s, k)| \right\},$$

where $J(u) := \{(s, k) \in \operatorname{dom} u : (s, k + 1) \in \operatorname{dom} u\}$ denotes the set of hybrid jump times.[12]

Due to the nonunique nature of solutions of the hybrid system (2.9), we are interested in stability properties of certain sets of solution pairs. Let $\mathcal{A} \subset \mathcal{X}$ be a compact set. We say that a set of solution pairs $\mathcal{S}$ of the hybrid system (2.9) is *pre-input-to-state stable (pre-ISS) with respect to* $\mathcal{A}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that all solution pairs $(x, u) \in \mathcal{S}$

[11] For a hybrid signal z, denote by $z(t, j)$ its value at hybrid time (t, j), that is, at time t and after j jumps.

[12] In particular, the set of measure 0 of hybrid times that are ignored in computing the essential supremum norm cannot contain any jump time.

satisfy

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t+j) + \gamma(\|u\|_{(t,j)}) \qquad \forall\,(t,j) \in \operatorname{dom} x. \tag{2.10}$$

As with the continuous-time case, the function γ in (2.10) is sometimes referred to as the *ISS gain function* as well. The notion of pre-ISS above is defined for very general cases of hybrid dynamics and solution pairs, and can be reduced to more standard stability properties for slightly less generic situations, such as

- if (2.10) holds for all solution pairs of (2.9), then we say that the hybrid system (2.9) is pre-ISS with respect to $\mathcal{A}$;
- if all solution pairs in $\mathcal{S}$ are complete, then we say that the set $\mathcal{S}$ is *input-to-state stable (ISS) with respect to* $\mathcal{A}$;
- if the set $\mathcal{A} = \{0\}$, then we simply say that the set of solution pairs $\mathcal{S}$ is pre-ISS.

Remark 2.2. In [23], ISS of hybrid systems is defined using a class $\mathcal{KLL}$ function and without requiring all solution pairs to be complete, which is equivalent to our definition of pre-ISS with a class $\mathcal{KL}$ function [4, Lemma 6.1]. We choose to work with the pre-ISS notion instead of the more standard ISS notion because it corresponds more directly to the existence of ISS-Lyapunov functions, as will be clear from the results below.

Lyapunov analysis proves to be a useful tool for establishing stability properties of hybrid systems as well. In order to characterize effects of destabilizing dynamics, we adopt the following generalized notion of ISS-Lyapunov function. Due to the nonsmooth nature of dynamics of the hybrid system (2.9), the following notion of nonsmooth derivative is used in the Lyapunov analysis below. For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, its *Clarke derivative* [90] at a point x in the direction $v \in \mathbb{R}^n$ is defined by

$$V^{\circ}(x; v) := \limsup_{h \to 0^+,\, y \to x} \frac{V(y + hv) - V(y)}{h}.$$

Definition 2.2. A function $V : \mathcal{X} \to \mathbb{R}_+$ is called a *candidate ISS-Lyapunov function with respect to* $\mathcal{A}$ for the hybrid system (2.9) if it is locally Lipschitz outside $\mathcal{A}$,[13] and

[13]The Lipschitz condition here is used to ensure the existence of the Clarke derivative

1. there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|_{\mathcal{A}}) \leq V(x) \leq \psi_2(|x|_{\mathcal{A}}) \qquad \forall\, x \in \mathcal{X}; \tag{2.11}$$

2. there exist a function $\phi \in \mathcal{K}_\infty$ and a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ with $\alpha(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$,

$$\begin{aligned} &V(x) \geq \phi(|u|) \\ &\quad \Longrightarrow\; V^\circ(x; y) \leq -\alpha(V(x)) \qquad \forall\, y \in F(x, u); \end{aligned} \tag{2.12}$$

3. there exists a function $\nu \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,[14]

$$\begin{aligned} &V(x) \geq \phi(|u|) \\ &\quad \Longrightarrow\; V(y) \leq \nu(V(x)) \qquad \forall\, y \in G(x, u). \end{aligned} \tag{2.13}$$

Additionally, if α and ν satisfy that $\alpha \in \mathcal{PD}$ and $\nu < \mathrm{Id}$ on $\mathbb{R}_{>0}$, that is,

$$\alpha(r) > 0, \quad \nu(r) < r \qquad \forall\, r > 0, \tag{2.14}$$

then the function V is called an *ISS-Lyapunov function with respect to* $\mathcal{A}$.

The function ϕ in (2.12) and (2.13) plays a similar role to that of the function γ in the definition (2.10) of pre-ISS; with a slight abuse of terminology, we refer to ϕ as the *gain function*, or simply the *gain*, of the candidate ISS-Lyapunov function V as well. Again, for a multi-input system, it is usually useful to specify one gain for each different input.

The following lemma provides an alternative characterization for the candidate ISS-Lyapunov function, which will be useful in formulating the small-gain theorems in Section 4.2.

Lemma 2.1. *A function $V : \mathcal{X} \to \mathbb{R}_+$ is a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ for the hybrid system* (2.9) *if and only if it is locally Lipschitz outside $\mathcal{A}$, and*

1. there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (2.11) *holds;*

---

in (2.12), and it can be relaxed to that V is locally Lipschitz on an open set containing all $x \notin \mathcal{A}$ such that $(x, u) \in \mathcal{C}$ for some $u \in \mathcal{U}$.

[14]There is no loss of generality in requiring $\nu \in \mathcal{K}$ instead of $\nu \in \mathcal{PD}$ as in [45], since a class $\mathcal{PD}$ function is always upper bounded a class $\mathcal{K}$ one. Meanwhile, a class $\mathcal{K}$ function is needed in establishing the small-gain theorems in Section 4.2.

2. *there exist a function $\bar{\phi} \in \mathcal{K}_\infty$ and a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ with $\alpha(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$,*[15]

$$\begin{aligned} &V(x) \geq \bar{\phi}(|u|) \\ &\quad \Longrightarrow V^\circ(x; y) \leq -\alpha(V(x)) \qquad \forall\, y \in F(x, u); \end{aligned} \tag{2.15}$$

3. *there exists a function $\nu \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,*

$$V(y) \leq \max\{\nu(V(x)),\, \bar{\phi}(|u|)\} \qquad \forall\, y \in G(x, u). \tag{2.16}$$

Proof. The proof follows in principle from the proof of [53, Proposition 1] on ISS-Lyapunov functions for impulsive systems, and is omitted here. □

Proposition 2.2 ([23, Proposition 2.7])**.** *The hybrid system* (2.9) *is pre-ISS with respect to $\mathcal{A}$ if it admits an ISS-Lyapunov function with respect to $\mathcal{A}$.*

Remark 2.3. As discussed in [23, Section 3.2], the converse of Proposition 2.2 does not hold in general even if the hybrid system (2.9) is forward complete and the flow map F is real-valued and smooth; a variety of sufficient conditions for the existence of ISS-Lyapunov functions for hybrid systems were established in [23, 24].

Remark 2.4. For a set of solution pairs $\mathcal{S}$ of the hybrid system (2.9), if the definition (2.10) of pre-ISS holds with $\gamma \equiv 0$ (e.g., the case without input), that is, all solution pairs $(x, u) \in \mathcal{S}$ satisfy

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j) \qquad \forall\, (t, j) \in \operatorname{dom} x, \tag{2.17}$$

then we say that the set $\mathcal{S}$ is *globally pre-asymptotically stable (pre-GAS) with respect to $\mathcal{A}$*. Pre-GAS corresponds to the more standard notion of *global asymptotic stability (GAS)* without requiring completeness of all maximal solutions, that is, all solutions are stable and bounded, and all complete solutions converge to the set $\mathcal{A}$ (see also [3, Definition 3.6] and [45, p. 1397]). Similarly, if (2.12) and (2.13) hold with $\phi \equiv 0$, or (2.15) and (2.16) hold with $\bar{\phi} \equiv 0$, then a candidate ISS-Lyapunov functions is called a candidate Lyapunov functions, and an ISS-Lyapunov functions is called a Lyapunov functions, which guarantees pre-GAS of the hybrid system (2.9) [3, Theo-

[15] In general, the functions ϕ in Definition 2.2 and $\bar{\phi}$ in Lemma 2.1 are different.

rem 7.30]. Moreover, a converse Lyapunov theorem could only be established under additional assumptions as well; see [3, Theorem 7.31].

Suppose that the hybrid system (2.9) admits a candidate ISS-Lyapunov function such that only one of the two conditions in (2.14) holds (i.e., either the continuous or the discrete dynamics taken alone is ISS). Provided that the functions α in (2.12) and ν in (2.13) are linear, we may still be able to establish pre-ISS for certain sets of solution pairs.

Definition 2.3. For the hybrid system (2.9), a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ such that the functions α in (2.12) and ν in (2.13) satisfy

$$\alpha(r) \equiv cr, \quad \nu(r) \equiv e^{-d}r \tag{2.18}$$

with some constants $c, d \in \mathbb{R}$ is called a *candidate exponential ISS-Lyapunov function with respect to* $\mathcal{A}$ with *rate coefficients* c and d. Additionally, if $c, d > 0$, then it is called an *exponential ISS-Lyapunov function with respect to* $\mathcal{A}$.

Remark 2.5. As mentioned in [23, Section 2], following in principle the proof of [51, Theorem 2, (b) $\Rightarrow$ (c)], one can show that the hybrid system (2.9) admits an ISS-Lyapunov function if and only if it admits an exponential ISS-Lyapunov function. Also, following in principle the proof of [4, Theorem 8.1, ($\mathrm{B}_\mathcal{C}$) $\Rightarrow$ ($\mathrm{A}_\mathcal{C}$)], one can show that the hybrid system (2.9) admits a candidate exponential ISS-Lyapunov function if it is forward complete. In general, a hybrid system may admit a candidate ISS-Lyapunov function but not a candidate exponential ISS-Lyapunov function, as can be seen readily from the following simple example. The scalar system $\dot{x} = x^3$ admits the candidate ISS-Lyapunov function $V(x) := x^2$ as $\nabla V(x) \cdot x^3 = 2x^4 = 2V(x)^2$. However, it is not forward complete; thus there exists no candidate exponential ISS-Lyapunov function [91, Theorem 2].

In the following proposition, we extend [51, Theorem 1] on ISS of impulsive systems to the context of hybrid systems to establish pre-ISS for solution pairs that satisfy an additional restriction on the frequency of jumps (i.e., the number of jumps per unit interval of continuous time).

Proposition 2.3. *Suppose that the hybrid system* (2.9) *admits a candidate exponential ISS-Lyapunov function* V *with respect to* $\mathcal{A}$ *with rate coefficients*

c and d. For constants $\eta, \lambda, \mu > 0$, denote by $\mathcal{S}[\eta, \lambda, \mu]$ the set of solution pairs (x, u) such that

$$-(d-\mu)(j-k) - (c-\lambda)(t-s) \le \eta \tag{2.19}$$

for all $(s, k) \preceq (t, j)$ in the hybrid time domain $\operatorname{dom} x$. Then $\mathcal{S}[\mu, \lambda, \eta]$ is pre-ISS with respect to $\mathcal{A}$.

Proof. The proof is along the lines of the proof of [51, Theorem 1]. Consider an arbitrary solution pair $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$. Let ϕ be the gain function in (2.12) and (2.13). For all $(t_1, j_1) \preceq (t_2, j_2)$ in $\operatorname{dom} x$, if

$$V(x(s,k)) \ge \phi(\|u\|_{(s,k)}) \tag{2.20}$$

for all $(s, k) \in \operatorname{dom} x$ such that $(t_1, j_1) \preceq (s, k) \preceq (t_2, j_2)$, then from (2.12), (2.13), and (2.18), it follows that

$$V(x(t_2, j_2)) \le e^{-d(j_2-j_1)-c(t_2-t_1)} V(x(t_1, j_1)).$$

Substituting (2.19) into the previous estimate, we obtain that

$$V(x(t_2, j_2)) \le e^{-\mu(j_2-j_1)-\lambda(t_2-t_1)+\eta} V(x(t_1, j_1)). \tag{2.21}$$

Now consider an arbitrary $(t, j) \in \operatorname{dom} x$. If (2.20) holds for all $(s, k) \preceq (t, j)$ in $\operatorname{dom} x$, then (2.21), together with (2.11), implies that

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t+j) \tag{2.22}$$

with the function $\beta \in \mathcal{KL}$ defined by

$$\beta(r, \tau) := \psi_1^{-1}(e^{-\tau \min\{\lambda, \mu\}+\eta} \psi_2(r)). \tag{2.23}$$

Otherwise, let

$$(t_0, j_0) := \operatorname*{argmax}_{(s,k)\in \operatorname{dom} x :\, (s,k) \preceq (t,j)} \{s + k : V(x(s,k)) \le \phi(\|u\|_{(s,k)})\}.$$

Then (2.20) holds for all $(s, k) \in \operatorname{dom} x$ such that $(t_0, j_0) \prec (s, k) \preceq (t, j)$;

thus (2.21) implies that

$$
\begin{aligned}
V(x(t,j)) &\le e^{-\mu(j-j_0)-\lambda(t-t_0)+\eta}\max\{1,\, e^{-d}\}V(x(t_0,j_0)) \\
&\le e^{\eta}\max\{1,\, e^{-d}\}\phi(\|u\|_{(t_0,j_0)}) \\
&\le e^{\eta}\max\{1,\, e^{-d}\}\phi(\|u\|_{(t,j)}),
\end{aligned}
$$

where the term $\max\{1,\, e^{-d}\}$ is needed in case $(t_0, j_0+1) \in \operatorname{dom} x$, and $V(x(t_0,j_0)) < \phi(\|u\|_{(t_0,j_0)})$ while $V(x(t_0,j_0+1)) > \phi(\|u\|_{(t_0,j_0+1)})$. Hence

$$
|x(t,j)|_{\mathcal{A}} \le \gamma(\|u\|_{(t,j)}) \tag{2.24}
$$

with the ISS gain function $\gamma \in \mathcal{K}$ defined by

$$
\gamma(r) := \psi_1^{-1}\big(e^{\eta}\max\{1,\, e^{-d}\}\phi(r)\big).
$$

Combining (2.22) and (2.24), we obtain that the pre-ISS estimate (2.10) holds for all $(x,u) \in \mathcal{S}[\eta,\lambda,\mu]$. □

Remark 2.6. 1. If $c, d > 0$ (i.e., V is an exponential ISS-Lyapunov function), then all solution pairs of the hybrid system (2.9) satisfy the inequality (2.19) with $\lambda = c$, $\mu = d$, and an arbitrary $\eta > 0$. Hence Proposition 2.3 implies that the hybrid system (2.9) is pre-ISS with respect to $\mathcal{A}$, which is consistent with Proposition 2.2.

2. If $c > 0 \ge d$, then we can divide both sides of (2.19) by $-(d-\eta) > 0$ to transform it into an ADT condition on the frequency of jumps. Hence Proposition 2.3 can be intuitively described as follows: if the flow of the hybrid system (2.9) is stabilizing while the jumps are destabilizing, then pre-ISS can be established for solution pairs that jump slow enough.
3. If $d > 0 \ge c$, then we can divide both sides of (2.19) by $-(c-\nu) > 0$ to transform it into a *reverse average dwell-time (RADT)* condition [51] on the frequency of jumps. Hence Proposition 2.3 can be intuitively described as follows: if the jumps of (2.9) are stabilizing while the flow is destabilizing, then pre-ISS can be established for solution pairs that jump fast enough.
4. If $c, d < 0$, then the inequality (2.19) cannot hold for any complete solution pair. More specifically, for each triple of positive constants (η, λ, μ), there always exists a large enough $t \in \mathbb{R}_+$ or $j \in \mathbb{N}$ such that $\mu j + \lambda t > \eta$.

However, it may still hold for solution pairs defined on bounded hybrid time domains.

Remark 2.7. If $c > 0 > d$, then the claim of Proposition 2.3 holds for $\mu = 0$ as well. The proof remains unchanged except that (2.21) now becomes

$$\begin{aligned} V(x(t_1, j_1)) &\le e^{-d(j_1-j_0)-c(t_1-t_0)} V(x(t_0, j_0)) \\ &\le e^{-\lambda(t_1-t_0)+\eta} V(x(t_0, j_0)) \\ &\le e^{(\lambda^2/c-\lambda)(t_1-t_0)-\lambda^2(t_1-t_0)/c+\eta} V(x(t_0, j_0)) \\ &\le e^{\lambda d(j_1-j_0)/c-\lambda^2(t_1-t_0)/c+(1+\lambda/c)\eta} V(x(t_0, j_0)), \end{aligned}$$

where the second inequality follows from (2.19) with $\eta = 0$, and the last one follows from

$$e^{(\lambda^2/c-\lambda)(t_1-t_0)} = e^{\lambda(\lambda-c)(t_1-t_0)/c} \le e^{\lambda d(j_1-j_0)/c+\lambda\eta/c};$$

and the definition (2.23) becomes

$$\beta(r, \tau) := \psi_1^{-1}(e^{-\tau \min\{-\lambda d/c,\, \lambda^2/c\}+(1+\lambda/c)\mu} \psi_2(r)).$$

Analogously, if $d > 0 > c$, then the claim of Proposition 2.3 holds for $\lambda = 0$ as well.

Part II

Interconnections and small-gain theorems

Chapter 3

Lyapunov-based small-gain theorems for interconnections of switched systems with possibly non-ISpS modes

3.1 Problem formulation

Consider two switched nonlinear systems modeled by

$$\dot{x}_i = f_{i,\sigma_i}(x_i, d_i), \qquad i = 1, 2, \tag{3.1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $d_i \in \mathbb{R}^{m_i}$ is the disturbance, and $\sigma_i : \mathbb{R}_+ \to \mathcal{P}_i$ is the switching signal.[16] Suppose that they fulfill the same assumptions as those imposed on general switched systems in Section 2.2. We are interested in the case where their dynamics are coupled, in the sense that the disturbance d_i to one switched system includes the state x_j of the other one, that is,

$$d_i \equiv (x_j, w_i)$$

for $i, j \in \{1, 2\}$ with $j \neq i$. Then (3.1) becomes a feedback interconnection of two switched subsystems modeled by

$$\dot{x}_i = f_{i,\sigma_i}(x_i, x_j, w_i), \qquad i = 1, 2. \tag{3.2}$$

We refer to the dynamics of x_i as the i-th subsystem in (3.2), and denote it by Σ_i.[17] Denote by $x := (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and $w := (w_1, w_2) \in \mathbb{R}^{m_1-n_2+m_2-n_1}$ the state and the disturbance of the interconnection (3.2), respectively. Each subsystem Σ_i treats the state x_j of the other one as the internal disturbance, and w_i as the external disturbance. Note that the switchings in the subsystems are assumed to be independent.

[16] We use f_{i,σ_i} instead of f_{σ_i} to avoid confusion in case the two index sets $\mathcal{P}_1$ and $\mathcal{P}_2$ contain common elements.

[17] Throughout this chapter, we follow the convention that $i \in \{1, 2\}$ denotes the index of a subsystem, and for the i-th subsystem, $j \in \{1, 2\}$ with $j \neq i$ denotes the index of the other one.

The main objective of this chapter is to establish input-to-state practical stability (ISpS) of the interconnection (3.2) through Lyapunov-based small-gain theorems, under suitable assumptions on the dynamics and switching of each subsystem.

3.2 Lyapunov-based small-gain theorems

3.2.1 Switched systems with ISpS and non-ISpS modes

Consider the general scenario in which both switched systems in (3.1) contain ISpS and non-ISpS (i.e., destabilizing) modes. For each subsystem Σ_i, denote by $\mathcal{P}_{s,i}$ and $\mathcal{P}_{u,i}$ the index sets of ISpS and non-ISpS modes, respectively. Then $(\mathcal{P}_{s,i}, \mathcal{P}_{u,i})$ forms a partition of the index set $\mathcal{P}_i$ (i.e., $\mathcal{P}_{s,i} \cup \mathcal{P}_{u,i} = \mathcal{P}_i$ and $\mathcal{P}_{s,i} \cap \mathcal{P}_{u,i} = \emptyset$). Following [27], we denote by $T_{s,i}(t,\tau)$ the *total active time* of ISpS modes (i.e., modes from $\mathcal{P}_{s,i}$) on a time interval $(\tau, t]$, and $T_{u,i}(t,\tau)$ that of non-ISpS modes (i.e., modes from $\mathcal{P}_{u,i}$). Then clearly $T_{s,i}(t,\tau) + T_{u,i}(t,\tau) = t - \tau$.

Our first assumption is that each ISpS mode admits an ISpS-Lyapunov function, and each non-ISpS mode admits a *candidate* ISpS-Lyapunov function. Moreover, the (candidate) ISpS-Lyapunov functions are uniform in the following sense (see also Remark 3.1 below).

Assumption 3.1 (Generalized ISpS-Lyapunov)**.** For the subsystem Σ_i in (3.2), there exists a family of $\mathcal{C}^1$ functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $p_i \in \mathcal{P}_i$ such that

1. there exist functions $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ such that

$$\psi_{1,i}(|x_i|) \leq V_{i,p_i}(x) \leq \psi_{2,i}(|x_i|) \qquad \forall\, x_i \in \mathbb{R}^{n_i}, \forall\, p_i \in \mathcal{P}_i; \tag{3.3}$$

2. there exist gains $\phi_i, \phi_i^w \in \mathcal{K}_\infty$, a constant $\delta_i \geq 0$, and rate coefficients $\lambda_{s,i}, \lambda_{u,i} > 0$ such that for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$,

$$\begin{aligned} &|x_i| \geq \max\{\phi_i(|x_j|),\, \phi_i^w(|w_i|),\, \delta_i\} \\ &\quad \Longrightarrow \begin{cases} \nabla V_{i,p_s}(x_i) \cdot f_{i,p_s}(x_i, x_j, w_i) \leq -\lambda_{s,i} V_{i,p_s}(x_i); \\ \nabla V_{i,p_u}(x_i) \cdot f_{i,p_u}(x_i, x_j, w_i) \leq \lambda_{u,i} V_{i,p_u}(x_i) \end{cases} \end{aligned} \tag{3.4}$$

for all ISpS modes $p_s \in \mathcal{P}_{s,i}$ and non-ISpS modes $p_u \in \mathcal{P}_{u,i}$;

3. there exists a ratio $\mu_i \geq 1$ such that

$$V_{i,p_i}(x_i) \leq \mu_i V_{i,q_i}(x_i) \qquad \forall\, x_i \in \mathbb{R}^{n_i}, \forall\, p_i, q_i \in \mathcal{P}_i. \tag{3.5}$$

For each ISpS mode $p_s \in \mathcal{P}_{s,i}$, the existence of an ISpS-Lyapunov function V_{i,p_s} satisfying (3.3) and (3.4) follows essentially from the arguments in [92, Section 7]; for each non-ISpS mode $p_u \in \mathcal{P}_{u,i}$, the existence of a candidate ISpS-Lyapunov function V_{i,p_u} satisfying (3.3) and (3.4) follows from the forward completeness [91]. On the other hand, the condition (3.5) derives from the method of *multiple Lyapunov functions* [11, 12], and restricts the set of possible (candidate) ISpS-Lyapunov functions (see [26, Remark 1 and Section 4.1] for more discussions on this condition).

Remark 3.1. For the subsystem Σ_i, the (candidate) ISpS-Lyapunov functions $V_{i,p}$, $p \in \mathcal{P}$ are uniform in the sense that the functions $\psi_{1,i}, \psi_{2,1}, \phi_i, \phi_i^w \in \mathcal{K}_\infty$ and the constants $\delta_i, \lambda_{s,i}, \lambda_{u,i}, \mu_i$ are the same for all modes. For some particular types of index sets, parts of the uniformity can be concluded automatically. For example, if all modes are forward complete, then (3.3) holds if $\mathcal{P}_i$ is finite [26, Remark 1], while (3.4) always holds [92, 91]. Also, given a family of radially unbounded, positive definite functions V_{i,p_i}, $p_i \in \mathcal{P}_i$, the existence of a constant $\mu_i \geq 1$ satisfying (3.5) implies the existence of functions $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ satisfying (3.3).

Our second assumption is that the switching is slow in the sense of an average dwell-time; the third one is that the total active time of non-ISpS modes is short in proportion.

Assumption 3.2 (ADT)**.** For the subsystem Σ_i in (3.2), the switching signal σ_i admits an *average dwell-time (ADT)* $\tau_{a,i}$, namely, the number of switches satisfies that

$$N_{\sigma_i}(t,\tau) \leq \frac{t-\tau}{\tau_{a,i}} + N_{0,i} \qquad \forall\, t > \tau \geq 0$$

with constants $\tau_{a,i} > 0$ and $N_{0,i} \geq 1$.

Assumption 3.3 (Time-ratio)**.** For the subsystem Σ_i in (3.2), the total active time of non-ISpS modes satisfies that

$$T_{u,i}(t,\tau) \leq T_{0,i} + \rho_i(t-\tau) \qquad \forall\, t > \tau \geq 0$$

with a constant *time-ratio* $\rho_i \in [0, 1)$ and a constant $T_{0,i} \geq 0$.

The notion of ADT was introduced in [16] and has become standard in the context of switched systems, while the concept of time-ratio appears less frequently in the literature. The idea of proportionally restricting the total active time of destabilizing modes was introduced in [17].

The main result of this chapter is the following small-gain theorem.

Theorem 3.1. *Consider the interconnection* (3.2). *Suppose that for each subsystem* Σ_i*, Assumptions 3.1–3.3 hold with*

$$(1-\rho_i)\lambda_{s,i} - \rho_i\lambda_{u,i} - \frac{\ln\mu_i}{\tau_{a,i}} > 0. \tag{3.6}$$

Define the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *by*[18]

$$\chi_i(r) := \psi_{2,i}(\phi_i(\psi_{1,j}^{-1}(r)))e^{\Theta_i}, \qquad i = 1, 2 \tag{3.7}$$

with the constants

$$\Theta_i := N_{0,i}\ln\mu_i + T_{0,i}(\lambda_{s,i} + \lambda_{u,i}) > 0. \tag{3.8}$$

Provided that χ_1 *and* χ_2 *satisfy the small-gain condition*

$$\chi_1(\chi_2(r)) < r \qquad \forall\, r > 0, \tag{3.9}$$

the interconnection (3.2) *is input-to-state practically stable. In particular, the ISpS estimate* (2.5) *holds (with* w *as the input* u*) with any constant* ε *satisfying*

$$\varepsilon \geq \sqrt{2}\max\bigl\{\phi_1^{-1}(\delta_1),\, \psi_{1,1}^{-1}(\psi_{2,1}(\delta_1)e^{\Theta_1}),\, \phi_2^{-1}(\delta_2),\, \psi_{1,2}^{-1}(\psi_{2,2}(\delta_2)e^{\Theta_2})\bigr\}. \tag{3.10}$$

Remark 3.2. In the inequality (3.6), the term $(1-\rho_i)\lambda_{s,i}$ quantifies the average exponential decay rate of the ISpS-Lyapunov function for active mode due to the ISpS modes, while $\rho_i\lambda_{u,i}$ quantifies its average exponential growth rate due to the non-ISpS modes, and $\ln\mu_i/\tau_{a,i}$ quantifies that due to the switching. Thus this condition can be intuitively described as follows: for the subsystem Σ_i, the ISpS-Lyapunov function for active mode is decreasing

[18] As the functions χ_1 and χ_2 correspond to the gains between the ISpS-Lyapunov functions in the feedback interconnection, we refer to them as the *Lyapunov feedback gains*, or simply the *Lyapunov gains*.

on average.

Remark 3.3. The inequality (3.6) can be rewritten as

$$\lambda_{s,i} > \frac{1}{1-\rho_i}\left(\frac{\ln \mu_i}{\tau_{a,i}} + \lambda_{u,i}\right) - \lambda_{u,i},$$

from which it is clear that by increasing $\lambda_{s,i}$ (with all other parameters fixed), we are able to accommodate a larger time-ratio ρ_i. Meanwhile, from the small-gain condition (3.9), and the definitions (3.7) of χ_i and (3.8) of Θ_i, we see that one should work with the smallest possible $\lambda_{s,i}$ satisfying (3.6) to have the least conservative gain estimate.

Similar results to Theorem 3.1 can be established for stronger stability properties such as input-to-state stability (ISS) and global asymptotic stability (GAS), provided that the corresponding conditions hold in Assumption 3.1.

Corollary 3.2. *Consider the interconnection* (3.2)*. Suppose that for each subsystem Σ_i, Assumptions 3.1–3.3 hold with $\delta_i = 0$ in* (3.4) *and the inequality* (3.6) *holds. Provided that the Lyapunov gains χ_1 and χ_2 defined by* (3.7) *satisfy the small-gain condition* (3.9)*, the interconnection* (3.2) *is input-to-state stable. Additionally, if* (3.4) *holds with $\phi_i^w \equiv 0$ for each subsystem Σ_i (e.g., the case without external disturbance), then the interconnection* (3.2) *is globally asymptotically stable.*

3.2.2 Switched systems with only ISpS modes

A less complicated scenario arises when both switched systems in (3.1) contain only ISpS modes (i.e., $\mathcal{P}_{s,i} = \mathcal{P}_i$ and $\mathcal{P}_{u,i} = \emptyset$). In this case, ISpS of the interconnection (3.2) can be established under less restrictive assumptions.

Assumption 3.4 (ISpS-Lyapunov)**.** For the subsystem Σ_i in (3.2), there exists a family of $\mathcal{C}^1$ functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $p_i \in \mathcal{P}_i$ such that
1. there exist functions $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ such that (3.3) holds;
2. there exist gains $\phi_i, \phi_i^w \in \mathcal{K}_\infty$, a constant $\delta_i \geq 0$, and a rate coefficient

$\lambda_i > 0$ such that for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$,

$$\begin{aligned} |x_i| &\geq \max\{\phi_i(|x_j|),\, \phi_i^w(|d_i|),\, \delta_i\} \\ &\implies \nabla V_{i,p_i}(x_i) \cdot f_{i,p_i}(x_i, x_j, w_i) \leq -\lambda_i V_{i,p_i}(x_i); \end{aligned} \tag{3.11}$$

for all $p_i \in \mathcal{P}_i$;

3. there exists a ratio $\mu_i \geq 1$ such that (3.5) holds.

Corollary 3.3. *Consider the interconnection* (3.2)*. Suppose that for each subsystem Σ_i, Assumptions 3.2 and 3.4 hold with*

$$\lambda_i - \frac{\ln \mu_i}{\tau_{a,i}} > 0. \tag{3.12}$$

Define the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ by

$$\chi_i(r) := \psi_{2,i}(\phi_i(\psi_{1,j}^{-1}(r)))e^{N_{0,i} \ln \mu_i}, \qquad i = 1, 2. \tag{3.13}$$

Provided that χ_1 and χ_2 satisfy the small-gain condition (3.9)*, the interconnection* (3.2) *is input-to-state practically stable.*

Additionally, if (3.2) *holds with $\delta_i = 0$ for each subsystem Σ_i, then the interconnection* (3.2) *is input-to-state stable; if* (3.2) *holds with $\delta_i = 0$ and $\phi_i^w \equiv 0$ for each subsystem Σ_i, then the interconnection* (3.2) *is globally asymptotically stable.*

Remark 3.4. For an interconnection (3.2) in which only one of the two switched subsystems contains non-ISpS modes, ISpS can be established if Assumption 3.1 holds for this subsystem while Assumption 3.4 holds for the other one, and the small-gain condition (3.9) holds with the Lyapunov gains χ_1 and χ_2 defined according to (3.7) and (3.13), respectively.

3.3 Proof of the main theorem

In this section, we provide a thorough proof of Theorem 3.1. In Section 3.3.1, we augment each switched subsystem in (3.2) with a hybrid auxiliary timer modeling the assumptions on switching to obtain a hybrid system, and establish a correspondence between their solutions. For each hybrid system, an ISpS-Lyapunov function is constructed in Section 3.3.2, which guarantees ISpS of the corresponding switched subsystem, as explained in Section 3.3.3.[19] In Section 3.3.4, we conclude the proof by combining these hybrid ISpS-Lyapunov functions via the small-gain condition (3.9), and establishing ISpS of the interconnection (3.2) explicitly.

3.3.1 Auxiliary timers and hybrid systems

For each subsystem Σ_i in (3.2), consider the hybrid system defined by

$$\begin{aligned} \dot{z}_i &\in F_i(z_i, \tilde{d}_i), && (z_i, \tilde{d}_i) \in \mathcal{C}_i, \\ z_i^+ &\in G_i(z_i), && (z_i, \tilde{d}_i) \in \mathcal{D}_i, \end{aligned} \tag{3.14}$$

where $z_i := (\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Theta_i] =: \mathcal{Z}_i$ is the state, $\tilde{d}_i := (\tilde{u}_i, \tilde{w}_i) \in \mathbb{R}^{m_i}$ is the disturbance, and

$$\begin{aligned} F_i(z_i, \tilde{d}_i) &:= \begin{cases} \begin{bmatrix} \{f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{d}_i)\} \\ \{0\} \\ [0, \theta_i] \end{bmatrix}, & \tilde{\sigma}_i \in \mathcal{P}_{s,i}; \\ \begin{bmatrix} \{f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{d}_i)\} \\ \{0\} \\ \{\theta_i - (\lambda_{s,i} + \lambda_{u,i})\} \end{bmatrix}, & \tilde{\sigma}_i \in \mathcal{P}_{u,i}, \end{cases} \\ \mathcal{C}_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Theta_i] \times \mathbb{R}^{m_i}, \\ G_i(z_i) &:= \{\tilde{x}_i\} \times (\mathcal{P}_i \backslash \{\tilde{\sigma}_i\}) \times \{\tau_i - \ln \mu_i\}, \\ \mathcal{D}_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln \mu_i, \Theta_i] \times \mathbb{R}^{m_i} \end{aligned} \tag{3.15}$$

with the constants Θ_i defined by (3.8) and

$$\theta_i := \frac{\ln \mu_i}{\tau_{a,i}} + \rho_i(\lambda_{s,i} + \lambda_{u,i}) < \lambda_{s,i}, \tag{3.16}$$

where the inequality follows from (3.6). The following proposition characterizes the correspondence between solutions of the subsystem Σ_i in (3.2) and the hybrid system (3.14).

[19] Section 3.3.3 is not necessary for the proof of Theorem 3.1, and is considered as an independent result.

Lemma 3.1. *Let x_i be a solution of the subsystem Σ_i in* (3.2) *with an internal disturbance x_j, an external disturbance w_i and a switching signal σ_i. Suppose that Assumptions 3.1–3.3 and the inequality* (3.6) *hold. Then there exists a complete solution pair $(z_i, \tilde{d}_i)$ of the hybrid system* (3.14) *with $z_i = (\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ and $\tilde{d}_i = (\tilde{u}_i, \tilde{w}_i)$ such that*

$$\tilde{x}_i(t,k) = x_i(t), \quad \tilde{u}_i(t,k) = x_j(t), \quad \tilde{w}_i(t,k) = w_i(t) \tag{3.17}$$

for all $(t,k) \in \operatorname{dom} z_i$.[20]

Proof. See Appendix A.1. □

3.3.2 Hybrid ISpS-Lyapunov functions

For the hybrid system (3.14), consider the function $V_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by

$$V_i(z_i) := V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i}, \tag{3.18}$$

where V_{i,p_i}, $p_i \in \mathcal{P}_i$ are the (candidate) ISpS-Lyapunov functions in Assumption 3.1. As all V_{i,p_i} are $\mathcal{C}^1$, it follows that $V_i(z_i) = V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ is continuously differentiable in $\tilde{x}_i$ and τ_i. Moreover, it has the following properties of a hybrid ISpS-Lyapunov function.

Lemma 3.2. *Suppose that Assumptions 3.1–3.3 and the inequality* (3.6) *hold. Then the function V_i defined by* (3.18) *satisfies that*
1. for the set $\mathcal{A}_i$ defined by $\mathcal{A}_i := \{0\} \times \mathcal{P}_i \times [0, \Theta_i] \subset \mathcal{Z}_i$,

$$\tilde{\psi}_{1,i}(|z_i|_{\mathcal{A}_i}) \le V_i(z_i) \le \tilde{\psi}_{2,i}(|z_i|_{\mathcal{A}_i}) \qquad \forall\, z_i \in \mathcal{Z}_i \tag{3.19}$$

with the functions $\tilde{\psi}_{1,i}, \tilde{\psi}_{2,i} \in \mathcal{K}_\infty$ defined by

$$\tilde{\psi}_{1,i}(r) := \psi_{1,i}(r), \quad \tilde{\psi}_{2,i}(r) := \psi_{2,i}(r) e^{\Theta_i}; \tag{3.20}$$

[20] As will be clear from the proof, Lemma 3.1 still holds if the additional condition $\tilde{\sigma}_i(t,k) = \sigma_i(t)$ for all $(t,k) \in \operatorname{dom} z_i$ is added in (3.17). However, this additional condition is not required for the proof of Theorem 3.1.

2. for all $(z_i, \tilde{u}_i, \tilde{w}_i) \in \mathcal{C}_i$*,*

$$\begin{aligned} |z_i|_{\mathcal{A}_i} &\geq \max\{\phi_i(|\tilde{u}_i|),\, \phi_i^w(|\tilde{w}_i|),\, \delta_i\} \\ \implies \nabla V_i(z_i) \cdot v_i &\leq -\lambda_i V_i(z_i) \qquad \forall\, v_i \in F_i(z_i, \tilde{u}_i, \tilde{w}_i) \end{aligned} \tag{3.21}$$

with the rate coefficient

$$\lambda_i := \lambda_{s,i} - \theta_i > 0; \tag{3.22}$$

3. for all $z_i \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln \mu_i, \Theta_i]$*,*

$$V_i(z_i^+) \leq V_i(z_i) \qquad \forall\, z_i^+ \in G_i(z_i). \tag{3.23}$$

Proof. See Appendix A.2. □

From the proof of Lemma 3.2, it is clear that the hybrid auxiliary timer τ_i is designed to compensate the increases in the value $V_{i,\sigma_i}(x_i)$ of the (candidate) ISpS-Lyapunov function for the active mode. Similar techniques have been used in [6] for switched systems, in [51] for impulsive systems, and in [45] for hybrid systems. Our auxiliary timer is more general in the sense that it is able to compensate the undesirable increases in $V_{i,\sigma_i}(x_i)$ both at switches and when non-ISpS modes are active. In the latter case, our construction introduces more decay in τ_i as a counterbalance, as shown in the formula of the flow map F_i in (3.15).

3.3.3 Digression on ISpS of switched systems

The hybrid ISpS-Lyapunov function V_i constructed in Section 3.3.2 provides a convenient way to establish ISpS of the corresponding switched system in (3.1). While not directly related to the proof of Theorem 3.1, this result is presented here for its own value, and to demonstrate the advantage of Lyapunov analysis in comparison with a similar result using trajectory analysis in [27].

Based on the properties in V_i in Lemma 3.2, it is straightforward to establish the following ISpS estimate of the hybrid system (3.14), for which the proof is quite standard and is omitted here.

Lemma 3.3. *Suppose that Assumptions 3.1–3.3 and the inequality* (3.6) *hold. Then all solution pairs* $(z_i, \tilde{d}_i)$ *of the hybrid system* (3.14) *satisfy that*

$$|z_i(t,k)|_{\mathcal{A}_i} \leq \beta_i(|z_i(0,0)|_{\mathcal{A}_i}, t) + \gamma_i(\|\tilde{d}_i\|_{(t,k)}) + \varepsilon_i \qquad \forall\,(t,k) \in \operatorname{dom} z_i \tag{3.24}$$

with the function $\beta_i \in \mathcal{KL}$ *and the ISS gain function* $\gamma_i \in \mathcal{K}_\infty$ *defined by*

$$\begin{aligned} \beta_i(r,t) &:= \tilde{\psi}_{1,i}^{-1}(\tilde{\psi}_{2,i}(r)e^{-\lambda_i t}), \\ \gamma_i(r) &:= \max\big\{\tilde{\psi}_{1,i}^{-1}(\tilde{\psi}_{2,i}(\phi_i(r))),\, \tilde{\psi}_{1,i}^{-1}(\tilde{\psi}_{2,i}(\phi_i^w(r)))\big\}, \end{aligned}$$

and the constant

$$\varepsilon_i := \tilde{\psi}_{1,i}^{-1}(\tilde{\psi}_{2,i}(\delta_i)) \geq 0.$$

Combining Lemmas 3.1 and 3.3, we establish the following ISpS estimate of the corresponding switched system in (3.1).

Proposition 3.4. *For each* $i \in \{1,2\}$*, provided that Assumptions 3.1–3.3 and the inequality* (3.6) *hold, the corresponding switched system in* (3.1) *is input-to-state practically stable.*

3.3.4 ISpS of the interconnection

Following [34, Lemma A.1], if the small-gain condition (3.9) holds, then there exists a function $\chi \in \mathcal{K}_\infty$ such that χ is continuous differentiable with $\chi' > 0$ on $\mathbb{R}_{>0}$, and that

$$\chi_1^{-1}(r) > \chi(r) > \chi_2(r) \qquad \forall\, r > 0. \tag{3.25}$$

Let $z := (z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2 =: \mathcal{Z}$ and $\tilde{w} := (\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}^{m_1 - n_2 + m_2 - n_1} =: \mathcal{W}$. Consider the function $V : \mathcal{Z} \to \mathbb{R}_+$ defined by

$$V(z) := \max\{\chi(V_1(z_1)),\, V_2(z_2)\} \tag{3.26}$$

where V_1 and V_2 are defined by (3.18) for $i = 1, 2$. As each $V_i(z_i) = V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ is continuously differentiable in $\tilde{x}_i$ and τ_i, and χ is continuously differentiable on $R_{>0}$, it follows that V is locally Lipschitz, and hence absolutely continuous and differentiable almost everywhere (away from its zero set); see Rademacher's theorem [93]. Moreover, following Lemma 3.2, it has the following properties of a hybrid ISpS-Lyapunov function.

Lemma 3.4. *Suppose that Assumptions 3.1–3.3, the inequality* (3.6)*, and the small-gain condition* (3.9) *hold. Then the function V defined by* (3.26) *satisfies that*

1. for the set $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{Z}$,

$$\psi_1(|z|_{\mathcal{A}}) \leq V(z) \leq \psi_2(|z|_{\mathcal{A}}) \qquad \forall\, z \in \mathcal{Z} \tag{3.27}$$

with the functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ defined by

$$\begin{aligned} &\psi_1(r) := \min\big\{\chi(\tilde{\psi}_{1,1}(r/\sqrt{2})),\ \tilde{\psi}_{1,2}(r/\sqrt{2})\big\}, \\ &\psi_2(r) := \max\big\{\chi(\tilde{\psi}_{2,1}(r)),\ \tilde{\psi}_{2,2}(r)\big\}; \end{aligned} \tag{3.28}$$

2. for all $(z, \tilde{w}) \in \mathcal{C} := \mathcal{Z} \times \mathcal{W}$,

$$\begin{aligned} &V(z) \geq \max\{\phi^w(|\tilde{w}|),\, \delta\} \\ &\quad \Longrightarrow\ V^\circ(z; v) \leq -h(V(z)) \qquad \forall\, v \in F(z, \tilde{w}) \end{aligned} \tag{3.29}$$

with the gain $\phi^w \in \mathcal{K}_\infty$ defined by

$$\phi^w(r) := \max\big\{\chi(\tilde{\psi}_{2,1}(\phi^w_1(r))),\ \tilde{\psi}_{2,2}(\phi^w_2(r))\big\}, \tag{3.30}$$

the constant

$$\delta := \max\{\chi(\tilde{\psi}_{2,1}(\delta_1)),\ \tilde{\psi}_{2,2}(\delta_2)\} \geq 0, \tag{3.31}$$

the function $h \in \mathcal{PD}$ (i.e., positive definite and continuous) defined by

$$h(r) := \min\{\chi'(\chi^{-1}(r))\, \lambda_1 \chi^{-1}(r),\ \lambda_2 r\}, \tag{3.32}$$

and the flow function

$$F(z, \tilde{w}) := \begin{bmatrix} F_1(z_1, \tilde{x}_2, \tilde{w}_1) \\ F_2(z_2, \tilde{x}_1, \tilde{w}_2) \end{bmatrix};$$

3. for all $(z, w) \in \mathcal{D} := (\mathcal{D}^z_1 \times \mathcal{Z}_2) \cup (\mathcal{Z}_1 \times \mathcal{D}^z_2)$ with

$$\mathcal{D}^z_i := \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln \mu_i, \Theta_i], \qquad i = 1, 2,$$

it holds that

$$V(z^+) \leq V(z) \qquad \forall\, z^+ \in G(z) \tag{3.33}$$

with the jump function[21]

$$G(z) := \begin{cases} \begin{bmatrix} G_1(z_1) \\ G_2(z_2) \end{bmatrix}, & z \in \mathcal{D}_1^z \times \mathcal{D}_2^z, \\ \begin{bmatrix} G_1(z_1) \\ \{z_2\} \end{bmatrix}, & z \in \mathcal{D}_1^z \times \mathcal{Z}_2, \\ \begin{bmatrix} \{z_1\} \\ G_2(z_2) \end{bmatrix}, & z \in \mathcal{Z}_1 \times \mathcal{D}_2^z. \end{cases}$$

Proof. First, the fact that $\tilde{\psi}_{1,1}, \tilde{\psi}_{1,2}, \tilde{\psi}_{2,1}, \tilde{\psi}_{2,2} \in \mathcal{K}_\infty$ implies that $\psi_1, \psi_2 \in \mathcal{K}_\infty$, and (3.27) follows from (3.19). In particular,

$$\begin{aligned} \psi_1(|z|_{\mathcal{A}}) &= \min\big\{\chi(\tilde{\psi}_{1,1}(|z|_{\mathcal{A}}/\sqrt{2})),\, \tilde{\psi}_{1,2}(|z|_{\mathcal{A}}/\sqrt{2})\big\} \\ &\leq \min\big\{\max\big\{\chi(\tilde{\psi}_{1,1}(|z_1|_{\mathcal{A}_1})),\, \chi(\tilde{\psi}_{1,1}(|z_2|_{\mathcal{A}_2}))\big\}, \\ &\quad \max\big\{\tilde{\psi}_{1,2}(|z_1|_{\mathcal{A}_1}),\, \tilde{\psi}_{1,2}(|z_2|_{\mathcal{A}_2})\big\}\big\} \\ &\leq \max\big\{\chi(\tilde{\psi}_{1,1}(|z_1|_{\mathcal{A}_1})),\, \tilde{\psi}_{1,2}(|z_2|_{\mathcal{A}_2})\big\} \\ &\leq \max\{\chi(V_1(z_1)),\, V_2(z_2)\} \\ &= V(z). \end{aligned}$$

Second, the fact that χ_1 is continuously differentiable with $\chi' > 0$ on $\mathbb{R}_{>0}$ and $\lambda_1, \lambda_2 > 0$ in (3.22) implies that $h \in \mathcal{PD}$. Consider an arbitrary $(z, \tilde{w}) \in \mathcal{C}$ such that

$$V(z) \geq \max\{\chi^w(|\tilde{w}|),\, \delta\}. \tag{3.34}$$

Regarding the relation between $\chi(V_1(z_1))$ and $V_2(z_2)$, there are three possibilities.

1. If $\chi(V_1(z_1)) > V_2(z_2)$, then $V(z) = \chi(V_1(z_1))$. Hence

$$\begin{aligned} |z_1|_{\mathcal{A}_1} &\geq \tilde{\psi}_{2,1}^{-1}(V_1(z_1)) \\ &\geq \tilde{\psi}_{2,1}^{-1}(\chi^{-1}(V_2(z_2))) \\ &\geq \tilde{\psi}_{2,1}^{-1}(\chi_1(\tilde{\psi}_{1,2}(|z_2|_{\mathcal{A}_2}))) \\ &= \phi_1(|z_2|_{\mathcal{A}_2}), \end{aligned} \tag{3.35}$$

where the equality follows from (3.7). Substituting (3.30) and (3.31) into

[21] The three cases here are due to the assumption that the switchings in the subsystems are independent.

(3.34), we obtain that

$$
\begin{aligned}
|z_1|_{\mathcal{A}_1} &\geq \tilde{\psi}_{2,1}^{-1}(V_1(z_1)) \\
&\geq \tilde{\psi}_{2,1}^{-1}(\chi^{-1}(V(z))) \\
&\geq \max\{\phi_1^w(|\tilde{w}|),\, \delta_1)\} \\
&\geq \max\{\phi_1^w(|\tilde{w}_1|),\, \delta_1)\}. \qquad (3.36)
\end{aligned}
$$

Thus (3.21) with $i = 1$ implies that for all $v = (v_1, v_2) \in F_1(z_1, \tilde{x}_2, \tilde{w}_1) \times F_2(z_2, \tilde{x}_1, \tilde{w}_2)$,

$$
\begin{aligned}
V^\circ(z; v) &= \chi'(V_1(z_1))\nabla V_1(z_1) \cdot v_1 \\
&\leq -\chi'(V_1(z_1))\lambda_1 V_1(z_1) \\
&= -\chi'(\chi^{-1}(V(z)))\lambda_1 \chi^{-1}(V(z)) \\
&\leq -h(V(z)),
\end{aligned}
$$

where the last inequality follows from (3.32).

2. If $\chi(V_1(z_1)) < V_2(z_2)$ then $V(z) = V_2(z_2)$. Hence

$$
\begin{aligned}
|z_2|_{\mathcal{A}_2} &\geq \tilde{\psi}_{2,2}^{-1}(V_2(z_2)) \\
&\geq \tilde{\psi}_{2,2}^{-1}(\chi(V_1(z_1))) \\
&\geq \tilde{\psi}_{2,2}^{-1}(\chi_2(\tilde{\psi}_{1,1}(|z_1|_{\mathcal{A}_1}))) \\
&= \phi_2(|z_1|_{\mathcal{A}_1}), \qquad (3.37)
\end{aligned}
$$

where the equality follows from (3.7). Substituting (3.30) and (3.31) into (3.34), we obtain that

$$
\begin{aligned}
|z_2|_{\mathcal{A}_2} &\geq \tilde{\psi}_{2,2}^{-1}(V_2(z_2)) \\
&\geq \tilde{\psi}_{2,2}^{-1}(V(z)) \\
&\geq \max\{\phi_2^w(|\tilde{w}|),\, \delta_2)\} \\
&\geq \max\{\phi_2^w(|\tilde{w}_2|),\, \delta_2)\}. \qquad (3.38)
\end{aligned}
$$

Thus (3.21) with $i = 2$ implies that for all $v = (v_1, v_2) \in F_1(z_1, \tilde{x}_2, \tilde{w}_1) \times F_2(z_2, \tilde{x}_1, \tilde{w}_2)$,

$$
V^\circ(z; v) = \nabla V_2(z_2) \cdot v_2 \leq -\lambda_2 V_2(z_2) = -\lambda_2 V(z) \leq -h(V(z)),
$$

where the last inequality follows from (3.32).

3. Otherwise $V(z) = \chi(V_1(z_1)) = V_2(z_2)$. Then (3.35)–(3.38) all hold. For all $v = (v_1, v_2) \in F_1(z_1, \tilde{x}_2, \tilde{w}_1) \times F_2(z_2, \tilde{x}_1, \tilde{w}_2)$, by virtue of [45, Lemma II.1], which is a direct consequence of [90, Propositions. 2.1.2 and 2.3.12], it follows that the Clarke derivative $V^\circ(z; v)$ is well-defined and satisfies that

$$V^\circ(z; v) \le \max\{\chi'(V_1(z_1))\nabla V_1(z_1) \cdot v_1,\, \nabla V_2(z_2) \cdot v_2\} \le -h(V(z)),$$

where the last inequality follows directly from the proof of the first two cases.

Last, consider an arbitrary $(z, \tilde{w}) \in \mathcal{D}$. Then (3.33) follows from (3.23). In particular,

$$\begin{aligned} V(z^+) &\le \max\{\chi(V_1(z_1^+)),\, \chi(V_1(z_1)),\, V_2(z_2^+),\, V_2(z_2)\} \\ &\le \max\{\chi(V_1(z_1)),\, V_2(z_2)\} \\ &= V(z). \end{aligned}$$

□

Let $x = (x_1, x_2)$ be a solution of the interconnection (3.2) with a disturbance $w = (w_1, w_2)$. Then for each subsystem Σ_i, the function x_i is a solution with the internal disturbance x_j and the external disturbance w_i. From Lemma 3.1, it follows that there exists a complete solution pair $(\bar{z}_i, \bar{d}_i)$ of the corresponding hybrid system (3.14) with $\bar{z}_i = (\bar{x}_i, \bar{\sigma}_i, \bar{\tau}_i)$ and $\bar{d}_i = (\bar{u}_i, \bar{w}_i)$ such that

$$\bar{x}_i(t, k) = x_i(t), \quad \bar{u}_i(t, k) = x_j(t), \quad \bar{w}_i(t, k) = w_i(t) \qquad \forall\, (t, k) \in \operatorname{dom} \bar{z}_i.$$

In the following, we construct a hybrid arc z and a hybrid input $\tilde{w}$ that merge the solution pairs $(\bar{z}_1, \bar{d}_1)$ and $(\bar{z}_2, \bar{d}_2)$ in a suitable manner. As the switchings in the subsystems are assumed to be independent, the hybrid time domains $\operatorname{dom} \bar{z}_1$ and $\operatorname{dom} \bar{z}_2$ are different in general. First, define a hybrid time domain $E \subset \mathbb{R}_+ \times \mathbb{N}$ so that for each $(t, k) \in E$, the hybrid time $(t, k+1) \in E$ if and only if for at least one $i \in \{1, 2\}$, there exists a hybrid time $(t, l) \in \operatorname{dom} \bar{z}_i$ such that $(t, l+1) \in \operatorname{dom} \bar{z}_i$. Next, define a hybrid arc $z = (z_1, z_2) : E \to \mathcal{Z}$ as follows. For each $(t, k) \in E$, there are two possibilities.

1. If $(t, k-1)$, $(t, k+1) \notin E$, then for each $i \in \{1, 2\}$, there is a unique

$l_i \in \mathbb{Z}_+$ such that $(t, l_i) \in \operatorname{dom} \bar{z}_i$. Set $z_i(t,k) = \bar{z}_i(t, l_i)$.

2. If $(t, k+1) \in E$, consider each $i \in \{1, 2\}$ separately. If there exists a $l_i \in \mathbb{Z}_+$ such that both $(t, l_i), (t, l_i+1) \in \operatorname{dom} \bar{z}_i$, then set $z_i(t,k) = \bar{z}_i(t, l_i)$ and $z_i(t, k+1) = \bar{z}_i(t, l_i+1)$; otherwise there exists a unique $l_i \in \mathbb{Z}_+$ such that $(t, l_i) \in \operatorname{dom} \bar{z}_i$, and set $z_i(t,k) = z_i(t,k+1) = \bar{z}_i(t, l_i)$.

Last, define a hybrid input $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) : E \to \mathcal{W}$ based on $\bar{w}_1$ and $\bar{w}_2$ the same way as z is defined based on $\bar{z}_1$ and $\bar{z}_2$. Hence we have constructed a hybrid arc $z = (z_1, z_2)$ and a hybrid input $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$ such that $(z, \tilde{w})$ satisfies the inclusions in (3.14) with $\tilde{u}_i = \tilde{x}_j$, and that

$$\tilde{x}(t,k) = x(t), \quad \tilde{w}(t,k) = w(t), \qquad \forall\, (t,k) \in \operatorname{dom} z.$$

In particular,

$$|x(t)| = |z(t,k)|_{\mathcal{A}} \qquad \forall\, (t,k) \in \operatorname{dom} z. \tag{3.39}$$

Remark 3.5. In fact, $(z, \tilde{w})$ is a complete solution pair of the hybrid system defined by

$$\begin{aligned} \dot{z} &\in F(z, \tilde{w}), & \quad (z, \tilde{w}) &\in \mathcal{C}, \\ z^+ &\in G(z), & \quad (z, w) &\in \mathcal{D}. \end{aligned}$$

Lemma 3.5. *There exists a function $\beta_V \in \mathcal{KL}$ such that*

$$\beta_V(r, 0) = r \qquad \forall\, r \geq 0,$$

and that for all $(t_1, k_1) \preceq (t_2, k_2)$ in $\operatorname{dom} z$, if

$$V(z(s,l)) \geq \max\{\phi^w(\|\tilde{w}\|_{(s,l)}),\, \delta\} \tag{3.40}$$

for all $(s,l) \in \operatorname{dom} z$ satisfying $(t_1,k_1) \preceq (s,l) \preceq (t_2,k_2)$, then

$$V(x(t_2,k_2)) \leq \beta_V(V(z(t_1,k_1)), t_2 - t_1). \tag{3.41}$$

Proof. See Appendix A.3. □

Now consider an arbitrary $(t,k) \in \operatorname{dom} z$. If (3.40) holds for all $(s,l) \preceq (t,k)$ in $\operatorname{dom} z$, then (3.41) implies that

$$V(x(t,k)) \leq \beta_V(V(z(0,0)), t). \tag{3.42}$$

Otherwise, let

$$(t_0, k_0) := \operatorname*{argmax}_{(s,l)\in\operatorname{dom} z:\,(s,l)\preceq(t,k)} \{s + l : V(z(s,l)) \leq \max\{\phi^w(\|\tilde{w}\|_{(s,l)}),\, \delta\}\}.$$

Then (3.40) holds for all $(s, l) \in \operatorname{dom} z$ such that $(t_0, k_0) \preceq (s, l) \preceq (t, k)$; thus (3.41) implies that

$$\begin{aligned} V(z(t,k)) &\leq \beta_V(V(z(t_0,k_0)), t - t_0) \\ &\leq \beta_V(V(z(t_0,k_0)), 0) \\ &= V(z(t_0,k_0)) \\ &\leq \max\{\phi^w(\|\tilde{w}\|_{(t_0,k_0)},\, \delta\} \\ &\leq \max\{\phi^w(\|\tilde{w}\|_{(t,k)},\, \delta\}. \end{aligned} \tag{3.43}$$

Combining (3.42) and (3.43), we obtain that

$$V(z(t,k)) \leq \max\{\beta_V(V(z(0,0)), t),\, \phi^w(\|\tilde{w}\|_{(t,k)}),\, \delta\}.$$

Finally, from (3.27), (3.39), and the previous estimate, it follows that the ISpS estimate (2.5) holds for all solutions x of the interconnection (3.2) (with w as the input u) with the functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ defined by

$$\beta(r,t) := \psi_1^{-1}(\beta_V(\psi_2(r), t)), \quad \gamma(r) := \psi_1^{-1}(\phi^w(r)),$$

and any constant ε satisfying

$$\varepsilon \geq \psi_1^{-1}(\delta). \tag{3.44}$$

We conclude the proof of Theorem 3.1 by noting that (3.10) implies (3.44); more specifically,

$$\begin{aligned} \psi_1^{-1}(\delta) &= \sqrt{2}\max\big\{\tilde{\psi}_{1,1}^{-1}(\chi^{-1}(\delta)),\, \tilde{\psi}_{1,2}^{-1}(\delta)\big\} \\ &= \sqrt{2}\max\big\{\tilde{\psi}_{1,1}^{-1}(\chi^{-1}(\chi(\tilde{\psi}_{2,1}(\delta_1)))),\, \tilde{\psi}_{1,1}^{-1}(\chi^{-1}(\tilde{\psi}_{2,2}(\delta_2))), \\ &\quad \tilde{\psi}_{1,2}^{-1}(\chi(\tilde{\psi}_{2,1}(\delta_1))),\, \tilde{\psi}_{1,2}^{-1}(\tilde{\psi}_{2,2}(\delta_2))\big\} \\ &\leq \sqrt{2}\max\big\{\tilde{\psi}_{1,1}^{-1}(\tilde{\psi}_{2,1}(\delta_1)),\, \tilde{\psi}_{1,1}^{-1}(\chi_2^{-1}(\tilde{\psi}_{2,2}(\delta_2))), \\ &\quad \tilde{\psi}_{1,2}^{-1}(\chi_1^{-1}(\tilde{\psi}_{2,1}(\delta_1))),\, \tilde{\psi}_{1,2}^{-1}(\tilde{\psi}_{2,2}(\delta_2))\big\} \\ &= \sqrt{2}\max\big\{\psi_{1,1}^{-1}(\psi_{2,1}(\delta_1)e^{\Theta_1}),\, \phi_2^{-1}(\delta_2),\, \phi_1^{-1}(\delta_1),\, \psi_{1,2}^{-1}(\psi_{2,2}(\delta_2)e^{\Theta_2})\big\}, \end{aligned}$$

where the inequality follows from the inequalities in (3.25), and the equalities follows from the definitions (3.7), (3.20), (3.28), and (3.31).

3.4 Stabilization via small-gain approach

From the definition (3.7) of the Lyapunov gains χ_1 and χ_2, we see that the existence of switchings and non-ISpS modes results in the additional constants $e^{\Theta_1}, e^{\Theta_2} > 1$, making the small-gain condition more restrictive. In this section, we study interconnections of switched subsystems in a control-affine form, and design feedback controls that guarantee the small-gain condition (3.9), through a Lyapunov-based variant of the gain-assignment techniques introduced in [44].

Consider an interconnection of two switched control-affine systems modeled by

$$\dot{x}_i = f^0_{i,\sigma_i}(x_i, x_j, w_i) + G_{i,\sigma_i}(x_i, x_j, w_i)\, u_i, \qquad i = 1, 2. \tag{3.45}$$

Again, denote by Σ_i the i-th subsystem, for which $x_i \in \mathbb{R}^{n_i}$ is the state, $x_j \in \mathbb{R}^{n_j}$ is the internal disturbance, $w_i \in \mathbb{R}^{m_i - n_j}$ is the external disturbance, and $u_i \in \mathbb{R}^{n_i}$ is the the control. Denote by $x := (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ and $w = (w_1, w_2) \in \mathbb{R}^{m_1-n_2+m_2-n_1}$ the state and the disturbance of the interconnection (3.45), respectively. The functions $f^0_{i,p_i} : \mathbb{R}^{n_i+m_i} \to \mathbb{R}^{n_i}$, $p_i \in \mathcal{P}_i$ define the dynamics of the modes of the subsystem Σ_i in open-loop (i.e., without the control u_i), and satisfy the same assumption as those imposed on f in Section 2.2. The matrix-valued functions $G_{i,p_i} : \mathbb{R}^{n_i+m_i} \to \mathbb{R}^{n_i \times n_i}$, $p_i \in \mathcal{P}_i$ are locally Lipschitz. An admissible feedback control is of the form $u_i = \kappa_{i,\sigma_i}(x_i)$ with a family of continuous functions κ_{i,p_i}, $p_i \in \mathcal{P}_i$ such that all $\kappa_{i,p_i}(0) = 0$. In particular, we allow the feedback control to be mode-dependent. Our goal is to develop feedback controls u_1 and u_2 such that the ISpS estimate (2.5) holds for the interconnection (3.45) with an arbitrarily small constant $\varepsilon > 0$, under similar assumptions to those in Section 3.2. In particular, we consider the general scenario in which both switched subsystems in (3.45) contain ISS and non-ISS modes in open-loop.

Assumption 3.5 (Generalized ISS-Lyapunov)**.** For the subsystem Σ_i in (3.45), there exists a family of $\mathcal{C}^1$ functions $V_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $p_i \in \mathcal{P}_i$ such that the gradients $\nabla V_{i,p_i}$ are locally Lipschitz and nowhere vanishing away from the origin, and that

1. there exist functions $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ such that (3.3) holds;
2. there exist gains $\bar{\phi}_i, \phi_i^w \in \mathcal{K}_\infty$ and rate coefficients $\lambda_{s,i}, \lambda_{u,i} > 0$ such that for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$,

$$\begin{aligned} &|x_i| \geq \phi_i^w(|w|) \\ &\implies \begin{cases} \nabla V_{i,p_s}(x_i) \cdot f_{i,p_s}^0(x_i, x_j, w_i) \leq -\lambda_{s,i} V_{i,p_s}(x_i) + \bar{\phi}_i(|x_j|); \\ \nabla V_{i,p_u}(x_i) \cdot f_{i,p_u}^0(x_i, x_j, w_i) \leq \lambda_{u,i} V_{i,p_u}(x_i) + \bar{\phi}_i(|x_j|) \end{cases} \end{aligned} \tag{3.46}$$

   for all ISpS modes $p_s \in \mathcal{P}_{s,i}$ and non-ISpS modes $p_u \in \mathcal{P}_{u,i}$;
3. there exists a ratio $\mu_i \geq 1$ such that (3.5) holds.

Also, the matrix-valued functions G_{i,p_i}, $p_i \in \mathcal{P}_i$ are lower bounded in the following sense.

Assumption 3.6. For the subsystem Σ_i in (3.45), there exists a family of constants $\{\varepsilon_{i,p_i}^G > 0 : p_i \in \mathcal{P}_i\}$ such that for each $p_i \in \mathcal{P}_i$,

$$G_{i,p_i}(x_i, x_j, w_i) + G_{i,p_i}(x_i, x_j, w_i)^\top - 2\varepsilon_{i,p_i}^G I \geq 0 \tag{3.47}$$

for all $x_i \in \mathbb{R}^{n_1}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$, that is, the matrix on the left-hand side is positive semi-definite everywhere.

Assumption 3.6 ensures that it does not require an arbitrarily large control to achieve stabilization, and allows us to generalize our result from the case that all $G_{i,p_i} \equiv I$. Similar assumptions can be found in the literature such as [44, Assumptions 5 and 9].

The control objective is to achieve the following ISpS property for arbitrary open-loop gains $\bar{\phi}_1, \bar{\phi}_2 \in \mathcal{K}_\infty$.

Theorem 3.5. *Consider the interconnection* (3.45)*. Suppose that for each subsystem Σ_i, Assumptions 3.2, 3.3, 3.5, 3.6 and the inequality* (3.6) *hold. Then for each $\varepsilon > 0$, there exist feedback controls u_1 and u_2 such that, for* (3.45) *in closed-loop, the ISpS estimate* (2.5) *holds (with w as the input u) with the constant ε.*

3.4.1 Gain assignment

By extend the gain-assignment techniques proposed in [44] (see also [40, Section 2.3]), we establish a feedback control that achieves an arbitrary closed-loop gain. Apart from being developed for switched systems, the gain-assignment scheme here is different in the sense that we assume knowledge of the gradients of the ISS-Lyapunov functions instead of the $\mathcal{K}_\infty$ bounds of the dynamics as in [44].

Proposition 3.6. *Consider the subsystem Σ_i in (3.45). Suppose that Assumptions 3.5 and 3.6 hold. Given arbitrary gain function $\phi_i \in \mathcal{K}_\infty$ and constant $\delta_i > 0$, there exists a feedback control $u_i = \kappa_{i,\sigma_i}(x_i)$ (given by (3.48) below) such that Assumption 3.1 holds for the closed-loop system. In particular, for all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$, the condition (3.4) holds with*

$$f_{i,\sigma_i}(x_i, x_j, w_i) = f^0_{i,\sigma_i}(x_i, x_j, w_i) + G_{i,\sigma_i}(x_i, x_j, w_i)\,\kappa_{i,\sigma_i}(x_i)$$

for all $p_s \in \mathcal{P}_{s,i}$ and $p_u \in \mathcal{P}_{u,i}$.

Proof. Let arbitrary gain function $\phi_i \in \mathcal{K}_\infty$ and constant $\delta_i > 0$ be given and fixed. For each mode $p_i \in \mathcal{P}_i$, define the function $\xi_{i,p_i} : \mathbb{R}_+ \to \mathbb{R}_+$, $p_i \in \mathcal{P}_i$ by

$$\xi_{i,p_i}(r) := \begin{cases} \min\limits_{\delta_i \le |y| \le r} |\nabla V_{i,p_i}(y)|^2, & r > \delta_i; \\ \min\limits_{|y| = \delta_i} |\nabla V_{i,p_i}(y)|^2, & r \le \delta_i, \end{cases}$$

where V_{i,p_i} is the (candidate) ISS-Lyapunov function in Assumption 3.5. Then ξ_{i,p_i} is continuous, (strictly) positive, and (non-strictly) decreasing. Hence the function $\bar{\nu}_{i,p_i} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\bar{\nu}_{i,p_i}(r) := \frac{\bar{\phi}_i(\phi_i^{-1}(r))}{\xi_{i,p_i}(r)}$$

is of class $\mathcal{K}_\infty$. Following [44, Lemma 1], there exists a function $\nu_{i,p_i} \in \mathcal{K}_\infty$ such that ν_{i,p_i} is smooth (infinitely differentiable) on $\mathbb{R}_{>0}$, and that

$$\nu_{i,p_i}(r) \ge \bar{\nu}_{i,p_i}(r) \qquad \forall\, r \ge \delta_i.$$

Consider the feedback control $u_i = \kappa_{i,\sigma_i}(x_i)$ with the family of functions

$\kappa_{i,p_i} : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $p_i \in \mathcal{P}_i$ defined by

$$\kappa_{i,p_i}(x_i) := -\frac{\nu_{i,p_i}(|x_i|)}{\varepsilon^G_{i,p_i}} \nabla V_{i,p_i}(x_i), \tag{3.48}$$

where ε^G_{i,p_i} is the constant in Assumption 3.6. For all $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, and $w_i \in \mathbb{R}^{m_i - n_j}$, if $|x_i| \geq \max\{\phi_i(|x_j|), \delta_i\}$, then

$$\begin{aligned}
&\nabla V_{i,p_i}(x_i) \cdot G_{i,p_i}(x_i, x_j, w_i)\, u_i \\
&= -\frac{\nabla V_{i,p_i}(x_i)^\top G_{i,p_i}(x_i, x_j, w_i) \nabla V_{i,p_i}(x_i)\, \nu_{i,p_i}(|x_i|)}{\varepsilon^G_{i,p_i}} \\
&\leq -\frac{\nabla V_{i,p_i}(x_i)^\top G_{i,p_i}(x_i, x_j, w_i) \nabla V_{i,p_i}(x_i)\, \bar{\phi}_i(\phi_i^{-1}(|x_i|))}{\varepsilon^G_{i,p_i} \xi_{i,p_i}(|x_i|)} \\
&\leq -\frac{\nabla V_{i,p_i}(x_i)^\top G_{i,p_i}(x_i, x_j, w_i) \nabla V_{i,p_i}(x_i)}{\varepsilon^G_{i,p_i} \min_{\delta_i \leq |y| \leq |x_i|} |\nabla V_{i,p_i}(y)|^2}\, \bar{\phi}_i(\phi_i^{-1}(|x_i|)) \\
&\leq -\bar{\phi}_i(|x_j|)
\end{aligned}$$

for all $p_i \in \mathcal{P}_i$, where the last inequality follows partially from (3.47). Substituting the previous bound into (3.46) result in (3.4). □

3.4.2 Control synthesis

By combining Proposition 3.6 with Theorem 3.1, we construct feedback controls that fulfill the claim of Theorem 3.5

First, select closed-loop gains $\phi_1, \phi_2 \in \mathcal{K}_\infty$ so that the Lyapunov gains χ_1 and χ_2 defined by (3.7) satisfy the small-gain condition (3.9).

Second, given an arbitrary constant $\varepsilon > 0$, select small enough constants $\delta_1, \delta_2 > 0$ so that (3.10) holds, such as

$$\delta_i = \min\big\{\psi_{2,i}^{-1}(\psi_{1,i}(\varepsilon/\sqrt{2})/e^{\Theta_i}),\ \phi_i(\varepsilon/\sqrt{2})\big\}, \qquad i = 1, 2.$$

Finally, for each subsystem Σ_i in (3.45), invoke Proposition 3.6 to formulate the feedback control u_i so that Assumption 3.1 holds for the closed-loop system. Then Theorem 3.1 implies that, for the interconnection (3.45) in closed-loop, the ISpS estimate (2.5) holds (with w as the input u) with the constant ε.

3.5 Future work

In this chapter, for each switched subsystem, we categorized its modes by their stability properties (i.e., ISpS or non-ISpS), while assumed only a destabilizing effect from its switching (i.e., the condition (3.5) in Assumption 3.1). This lack of symmetry has drawn our attention, and the case in which switching between some modes produces a stabilizing effect could become a future research topic.

Chapter 4

Lyapunov-based small-gain theorems for networks of hybrid systems with possibly non-ISS dynamics

4.1 Problem formulation

Consider the hybrid system with input (2.9) in Section 2.3. We are interested in the case where (2.9) is transformed into a network composed of $n \geq 2$ hybrid subsystems modeled by

$$\begin{aligned} \dot{x}_i &\in F_i(x,u), \quad i = 1,\dots,n, \qquad (x,u) \in \mathcal{C}, \\ x_i^+ &\in G_i(x,u), \quad i = 1,\dots,n, \qquad (x,u) \in \mathcal{D}, \end{aligned} \tag{4.1}$$

where $x := (x_1,\dots,x_n) \in \mathcal{X} \subset \mathbb{R}^N$ with $x_i \in \mathcal{X}_i \subset \mathbb{R}^{N_i}$ is the state, and $u \in \mathcal{U} \subset \mathbb{R}^M$ is the external input (disturbance). In particular, $N = N_1 + \cdots + N_n$ and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. The data of (2.9) is decomposed accordingly, that is, $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathcal{C}_u$ with $\mathcal{C}_i \subset \mathcal{X}_i$ and $\mathcal{C}_u \subset \mathcal{U}$ is the flow set, $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \times \mathcal{D}_u$ with $\mathcal{D}_i \subset \mathcal{X}_i$ and $\mathcal{D}_u \subset \mathcal{U}$ is the jump set, $F = (F_1,\dots,F_n)$ with $F_i : \mathcal{C} \rightrightarrows \mathbb{R}^{N_i}$ is the flow map, and $G = (G_1,\dots,G_n)$ with $G_i : \mathcal{D} \rightrightarrows \mathcal{X}_i$ is the jump map. We refer to the dynamics of x_i as the i-th subsystem of (4.1), and denote it by Σ_i.[22] For each subsystem Σ_i, the states of all other subsystems are treated as internal inputs.

Remark 4.1. In (4.1), all subsystems Σ_i, as well as the entire network, have the same flow set $\mathcal{C}$ and jump set $\mathcal{D}$, which justifies viewing (4.1) as a hybrid system transformed into a network composed of n hybrid subsystems, instead of as n individual hybrid systems interconnected together; cf. Sections 3.3.1 and 3.3.4.

The main objective of this chapter is to establish pre-input-to-state stability (pre-ISS) of the network (4.1) through Lyapunov-based small-gain theo-

[22]Throughout this chapter, we follow the convention that $i \in \{1,\dots,n\}$ denotes the index of a subsystem, and for the i-th subsystem, $j \in \{1,\dots,n\}$ with $j \neq i$ denotes the index of another one.

rems.

4.2 Lyapunov-based small-gain theorems

Let $\mathcal{A}_i \subset \mathcal{X}_i$, $i = 1, \ldots, n$ be a collection of compact sets. The basic assumption is that each subsystem admits a candidate ISS-Lyapunov function.

Assumption 4.1. There exists a family of functions $V_i : \mathcal{X}_i \to \mathbb{R}_+$, $i = 1, \ldots, n$ such that for each subsystem Σ_i of (4.1), the function V_i is locally Lipschitz outside $\mathcal{A}_i$, and that

1. there exist functions $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ such that

$$\psi_{1,i}(|x_i|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{2,i}(|x_i|_{\mathcal{A}_i}) \qquad \forall\, x_i \in \mathcal{X}_i; \tag{4.2}$$

2. there exist *Lyapunov gains* $\chi_{ij} \in \mathcal{K}_\infty$ for $j \neq i$ and $\chi_{ii} \equiv 0$, a gain $\phi_i \in \mathcal{K}$, and a continuous function $\alpha_i : \mathbb{R}_+ \to \mathbb{R}$ with $\alpha_i(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x_i \notin \mathcal{A}_i$, if

$$V_i(x_i) \geq \max\Big\{ \max_{j=1,\ldots,n} \chi_{ij}(V_j(x_j)),\, \phi_i(|u|) \Big\}, \tag{4.3}$$

then

$$V_i^\circ(x_i; v_i) \leq -\alpha_i(V_i(x_i)) \qquad \forall\, v_i \in F_i(x, u); \tag{4.4}$$

3. there exists a function $\nu_i \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,

$$V_i(x_i^+) \leq \max\Big\{ \nu_i(V_i(x_i)),\, \max_{j=1,\ldots,n} \chi_{ij}(V_j(x_j)),\, \phi_i(|u|) \Big\} \quad \forall\, x_i^+ \in G_i(x, u). \tag{4.5}$$

Following Lemma 2.1, it is straightforward to verify that each V_i in Assumption 4.1 is a candidate ISS-Lyapunov function with respect to $\mathcal{A}_i$ for the subsystem Σ_i of (4.1).

Under Assumption 4.1, whether the network (4.1) is pre-ISS depends on properties of the *gain operator* $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\Gamma(r_1, \ldots, r_n) := \Big(\max_{j=1,\ldots,n} \chi_{1j}(r_j), \ldots, \max_{j=1,\ldots,n} \chi_{nj}(r_j) \Big). \tag{4.6}$$

In order to construct a candidate ISS-Lyapunov function for the network (4.1), we adopt the notion of Ω-path [36]. For two vectors $x = (x_1, \ldots, x_n)$

and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we say that $x > y$ or $x \geq y$ if the corresponding inequality holds in all scalar components, and that $x \ngeq y$ if there exists at least one scalar component for which $x_i < y_i$.

Definition 4.1. Given a gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, a function $\sigma := (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$ is called an *Ω-path* with respect to Γ if

1. all σ_i^{-1} are locally Lipschitz on $\mathbb{R}_{>0}$;
2. for each compact set $P \subset \mathbb{R}_{>0}$, there exist finite constants $K_2 > K_1 > 0$ such that for all i,
$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2$$
for all points of differentiability of σ_i^{-1} in P;
3. the function Γ is a contraction on $\sigma(\cdot)$, that is,

$$\Gamma(\sigma(r)) < \sigma(r) \qquad \forall\, r > 0. \tag{4.7}$$

Remark 4.2. In this work, we consider primarily Ω-paths with respect to the gain operator Γ defined by (4.6), due to the term $\max_{j=1,\ldots,n} \chi_{ij}(V_j(x_j))$ in (4.3) and (4.5) (which will be clear from the statement and proof of Theorem 4.1 below). However, there are other equivalent characterizations of ISS-Lyapunov functions for subsystems, which would naturally lead to gain operators of other forms (see, e.g., [32, 36]). In particular, if (4.3) and (4.5) were formulated in terms of $\sum_{j=1}^n \chi_{ij}(V_j(x_j))$ instead of $\max_{j=1,\ldots,n} \chi_{ij}(V_j(x_j))$, it would result in the gain operator $\Gamma_\Sigma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma_\Sigma(r_1, \ldots, r_n) := \left(\sum_{j=1}^n \chi_{1j}(r_j), \ldots, \sum_{j=1}^n \chi_{nj}(r_j) \right).$$

As $\Gamma_\Sigma(v) \geq \Gamma(v)$ for all $v \in \mathbb{R}^n$, every Ω-path with respect to Γ_Σ is also an Ω-path with respect to Γ. This alternative construction will be useful in establishing Corollary 4.3 below for the case with linear Lyapunov gains.

We say that a gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the *small-gain condition* if

$$\Gamma(v) \ngeq v \qquad \forall\, v \in \mathbb{R}^n_+ \backslash \{0\}, \tag{4.8}$$

or equivalently,

$$\Gamma(v) \geq v \iff v = 0.$$

As reported in [33, Proposition 2.7 and Remark 2.8] (see also [36, Theorem 5.2]), if (4.8) holds for the gain operator Γ defined by (4.6), then there exists an Ω-path σ with respect to Γ. Furthermore, σ can be made smooth (infinitely differentiable) on $\mathbb{R}_{>0}$ using standard mollification techniques [94, Appendix B.2]. In this case, a candidate ISS-Lyapunov function for the interconnection (4.1) can be constructed in terms of the candidate ISS-Lyapunov functions in Assumption 4.1 and the Ω-path.

Theorem 4.1. *Consider the network* (4.1)*. Suppose that Assumption 4.1 holds, and that the gain operator* Γ *defined by* (4.6) *satisfies the small-gain condition* (4.8)*. Then there exists an* Ω*-path* $\sigma := (\sigma_1, \dots, \sigma_n)$ *with respect to* Γ *which is smooth on* $\mathbb{R}_{>0}$*, and the function* $V : \mathcal{X} \to \mathbb{R}_+$ *defined by*

$$V(x) := \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)) \tag{4.9}$$

is a candidate ISS-Lyapunov function with respect to the set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *for* (4.1)*.*

Proof. As every $\sigma_i \in \mathcal{K}_\infty$ are smooth on $\mathbb{R}_{>0}$ and every V_i are locally Lipschitz outside $\mathcal{A}_i$, it follows that every $\sigma_i^{-1} \circ V_i$ is locally Lipschitz outside $\mathcal{A}_i$. Hence the function V defined by (4.9) is locally Lipschitz outside $\mathcal{A}$. In the following, we prove that V satisfies the properties of a candidate ISS-Lyapunov function in Lemma 2.1, by combining and extending the arguments in the proofs of [36, Theorem 5.3] and [45, Theorem III.1].

First, consider the functions $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\psi_1(r) := \min_{i=1,\dots,n} \sigma_i^{-1}(\psi_{1,i}(r/\sqrt{n})), \quad \psi_2(r) := \max_{i=1,\dots,n} \sigma_i^{-1}(\psi_{2,i}(r)).$$

The fact that all $\sigma_i, \psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ implies that $\psi_1, \psi_2 \in \mathcal{K}_\infty$, and the property (2.11) follows from (4.2). In particular,

$$\begin{aligned}
\psi_1(|x|_{\mathcal{A}}) &= \min_{i=1,\dots,n} \sigma_i^{-1}(\psi_{1,i}(|x|_{\mathcal{A}}/\sqrt{n})) \\
&\le \min_{i=1,\dots,n} \sigma_i^{-1}\Big(\psi_{1,i}\Big(\max_{j=1,\dots,n} |x_j|_{\mathcal{A}_j}\Big)\Big) \\
&\le \max_{j=1,\dots,n} \sigma_j^{-1}(\psi_{1,j}(|x_j|_{\mathcal{A}_j})) \\
&\le \max_{j=1,\dots,n} \sigma_j^{-1}(V_j(x_j)) \\
&= V(x).
\end{aligned}$$

Second, consider the gain $\bar{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\bar{\phi}(r) := \max_{i=1,\ldots,n} \sigma_i^{-1}(\phi_i(r)), \tag{4.10}$$

and the function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\alpha(r) := \min_{i=1,\ldots,n} (\sigma_i^{-1})'(\sigma_i(r))\, \alpha_i(\sigma_i(r)). \tag{4.11}$$

As all $\sigma_i \in \mathcal{K}_\infty$ are smooth on $\mathbb{R}_{>0}$, all $\phi_i \in \mathcal{K}$, and all α_i are continuous with $\alpha_i(0) = 0$, it follows that $\bar{\phi} \in \mathcal{K}$, and that α is continuous with $\alpha(0) = 0$. Consider the family of sets $\{\mathcal{M}_i : i = 1, \ldots, n\}$ defined by

$$\mathcal{M}_i := \Big\{ x \in \mathcal{X} : \sigma_i^{-1}(V_i(x_i)) > \max_{j=1,\ldots,n:\, j\neq i} \sigma_j^{-1}(V_j(x_j)) \Big\}.$$

The fact that all V_i and σ_i^{-1} are continuous implies that all $\mathcal{M}_i$ are open in $\mathcal{X}$, that $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $j \neq i$, and that the union of their closure covers $\mathcal{X}$, that is,

$$\mathcal{X} = \bigcup_{i=1}^{n} \overline{\mathcal{M}}_i.$$

Consider an arbitrary $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$ such that $V(x) \geq \bar{\phi}(|u|)$. Regarding the relation between x and the family of sets $\{\mathcal{M}_i : i = 1, \ldots, n\}$, there are two possibilities.

1. There exists a unique index $i \in \{1, \ldots, n\}$ such that $x \in \mathcal{M}_i$. Then

$$V(x) = \sigma_i^{-1}(V_i(x_i)) > \max_{j=1,\ldots,n:\, j\neq i} \sigma_j^{-1}(V_j(x_j)). \tag{4.12}$$

   In particular, from $V_i(x_i) > 0$, it follows that $x_i \notin \mathcal{A}_i$. Combining (4.6), (4.7), and (4.12), we obtain that

$$V_i(x_i) = \sigma_i(V(x)) > \max_{j=1,\ldots,n} \chi_{ij}(\sigma_j(V(x))) \geq \max_{j=1,\ldots,n} \chi_{ij}(V_j(x_j)).$$

Also, following (4.10) and (4.12), the condition $V(x) \geq \bar{\phi}(|u|)$ implies that

$$\begin{aligned} V_i(x_i) &= \sigma_i(V(x)) \\ &\geq \sigma_i(\bar{\phi}(|u|)) \\ &= \sigma_i\Big(\max_{j=1,\dots,n} \sigma_j^{-1}(\phi_j(|u|))\Big) \\ &\geq \sigma_i(\sigma_i^{-1}(\phi_i(|u|))) \\ &= \phi_i(|u|). \end{aligned}$$

Hence (4.3), and therefore (4.4), is satisfied. From (4.4), (4.11), and (4.12), it follows that for all $v \in (v_1, \dots, v_n) \in F(x,u)$,

$$\begin{aligned} V^\circ(x;v) &= (\sigma_i^{-1})'(V_i(x_i))\, V_i^\circ(x_i;v_i) \\ &\leq -(\sigma_i^{-1})'(\sigma_i(V(x)))\, \alpha_i(\sigma_i(V(x))) \\ &\leq -\alpha(V(x)). \end{aligned}$$

2. There exists a subset $I(x) \subseteq \{1,\dots,n\}$ of indices with the cardinality $|I(x)| \geq 2$ such that

$$x \in \bigcap_{i\in I(x)} \partial\mathcal{M}_i,$$

where $\partial\mathcal{M}_i$ denotes the boundary of M_i in $\mathcal{X}$ and satisfies that $\partial\mathcal{M}_i = \overline{\mathcal{M}}_i \backslash \mathcal{M}_i$ as $\mathcal{M}_i$ is open in $\mathcal{X}$. Then for each $i \in I(x)$,

$$V(x) = \sigma_i^{-1}(V_i(x_i)) > \max_{j=1,\dots,n:\, j\notin I(x)} \sigma_j^{-1}(V_j(x_j)). \tag{4.13}$$

Following essentially the calculations from the previous case while using (4.13) in stead of (4.12), we see that $x_i \notin \mathcal{A}_i$ and (4.4) holds for all $i \in I(x)$. By virtue of [45, Lemma II.1], which is a direct consequence of [90, Propositions. 2.1.2 and 2.3.12], it follows that for all $v \in (v_1,\dots,v_n) \in F(x,u)$, the Clarke derivative $V^\circ(x;v)$ is well-defined and satisfies that

$$\begin{aligned} V^\circ(x;v) &\leq \max_{i\in I(x)} (\sigma_i^{-1})'(V_i(x_i))\, V_i^\circ(x_i;v_i) \\ &\leq -\min_{i\in I(x)} (\sigma_i^{-1})'(\sigma_i(V(x)))\, \alpha_i(\sigma_i(V(x))) \\ &\leq -\alpha(V(x)), \end{aligned}$$

where the last inequalities follows directly from the proof of the first case.

Hence (2.12) holds for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$.

Last, consider the function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\nu(r) := \max_{i=1,\ldots,n} \Big\{ \sigma_i^{-1}(\nu_i(\sigma_i(r))), \max_{j=1,\ldots,n} \sigma_i^{-1}(\chi_{ij}(\sigma_j(r))) \Big\}. \tag{4.14}$$

As all $\sigma_i \in \mathcal{K}_\infty$, all $\chi_{ij} \in \mathcal{K}_\infty$ for $j \neq i$, all $\chi_{ii} \equiv 0$, and all $\nu_i \in \mathcal{K}$, it follows that $\nu \in \mathcal{K}$. Consider an arbitrary $(x, u) \in \mathcal{D}$. Combining (4.9) and (4.14), we obtain that

$$\nu(V(x)) \geq \max_{i=1,\ldots,n} \Big\{ \sigma_i^{-1}(\nu_i(V_i(x_i))), \max_{j=1,\ldots,n} \sigma_i^{-1}(\chi_{ij}(V_j(x_j))) \Big\}.$$

Meanwhile, (4.10) implies that

$$\bar{\phi}(|u|) \geq \max_{i=1,\ldots,n} \sigma_i^{-1}(\phi_i(|u|)).$$

Combining the previous two inequalities with (4.5) and (4.9), we obtain that

$$V(x^+) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i^+)) \leq \max\{\nu(V(x)), \bar{\phi}(|u|)\}$$

for all $x^+ = (x_1^+, \ldots, x_n^+) \in G(x, u)$. Hence (2.13) holds for all $(x, u) \in \mathcal{D}$.

Therefore, Lemma 2.1 implies that V defined by (4.9) is a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ for the network (4.1). $\square$

If each subsystem of (4.1) admits an ISS-Lyapunov function, then Theorem 4.1 implies the following result, which generalizes [48, Theorem 3.6] and [45, Theorem III.1].

Corollary 4.2. *Consider the network* (4.1). *Suppose that Assumption 4.1 holds with $\alpha_i \in \mathcal{PD}$ in* (4.4) *and $\nu_i < \mathrm{Id}$ on $\mathbb{R}_{>0}$ in* (4.5) *for each subsystem Σ_i, and that the gain operator Γ defined by* (4.6) *satisfies the small-gain condition* (4.8). *Then there exists an Ω-path $\sigma := (\sigma_1, \ldots, \sigma_n)$ with respect to Γ which is smooth on $\mathbb{R}_{>0}$, and the function V defined by* (4.9) *is an ISS-Lyapunov function with respect to $\mathcal{A}$, and the network* (4.1) *is pre-ISS with respect to $\mathcal{A}$.*

Proof. Following Theorem 4.1, the function V defined by (4.9) is a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ for (4.1). We will show that V is an ISS-Lyuapunov function with respect to $\mathcal{A}$. First, as all $\sigma_i \in \mathcal{K}_\infty$ is smooth on $\mathbb{R}_{>0}$, and all $\alpha_i \in \mathcal{PD}$, it follows that the function α defined by (4.11) is of class $\mathcal{PD}$. Second, (4.7) implies that $\sigma_i^{-1} \circ \chi_{ij} \circ \sigma_j < \mathrm{Id}$ on $\mathbb{R}_{>0}$, and from the fact that all $\sigma_i \in \mathcal{K}_\infty$ and all $\nu_i < \mathrm{Id}$ on $\mathbb{R}_{>0}$, it follows that all $\sigma_i^{-1} \circ \nu_i \circ \sigma_i < \mathrm{Id}$ on $\mathbb{R}_{>0}$; thus the function ν defined by (4.14) satisfies that $\nu < \mathrm{Id}$ on $\mathbb{R}_{>0}$. Therefore, V is an ISS-Lyapunov function with respect to $\mathcal{A}$, and (4.1) is pre-ISS with respect to $\mathcal{A}$ due to Proposition 2.2. □

Now we consider the case in which for some subsystems Σ_i, either $\phi_i \notin \mathcal{PD}$ in (4.4) or $\alpha_i(r) \geq r$ for some $r > 0$ in (4.5). In this case, we cannot prove pre-ISS of the network (4.1) by applying directly Corollary 4.2, and our goal is to establish pre-ISS for solution pairs satisfying suitable conditions on the frequency of jumps based on Proposition 2.3. In general, Theorem 4.1 cannot provide the candidate exponential ISS-Lyapunov function needed in Proposition 2.3. Next, we construct such a function, assuming that each subsystem admits a candidate exponential ISS-Lyapunov function with linear Lyapunov gains, as described by the following assumption.

Assumption 4.2. In addition to Assumption 4.1, for each subsystem Σ_i of (4.1), the functions χ_{ij} in (4.3) and (4.5) satisfy that

$$\chi_{ij}(r) \equiv \xi_{ij} r \tag{4.15}$$

with some constants $\xi_{ij} > 0$ for $j \neq i$ and $\xi_{ii} = 0$, and the functions α_i in (4.4) and ν_i in (4.5) satisfy that

$$\alpha_i(r) \equiv c_i r, \quad \nu_i(r) \equiv e^{-d_i} r \tag{4.16}$$

with some constants $c_i, d_i \in \mathbb{R}$.

Following Lemma 2.1 and Definition 2.3, it is straightforward to verify that each V_i in Assumption 4.2 is a candidate exponential ISS-Lyapunov function with respect to $\mathcal{A}_i$ with rate coefficients c_i and d_i for the subsystem Σ_i of (4.1).

Under Assumption 4.2, we consider the *gain matrix*

$$\Xi := (\xi_{ij}) \in \mathbb{R}^{n \times n}. \tag{4.17}$$

As reported in [95, p. 78], if the small-gain condition (4.8) holds for the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(v) := \Xi v, \tag{4.18}$$

then there exists a linear Ω-path σ with respect to Γ of the form

$$\sigma(r) := (s_1 r, \dots, s_n r) \tag{4.19}$$

with some constants $s_1, \dots, s_n > 0$.[23]

Corollary 4.3. *Consider the network* (4.1)*. Suppose that Assumption 4.2 holds, and that the gain operator* Γ *defined by* (4.18) *satisfies the small-gain condition* (4.8)*. Then there exists a linear* Ω*-path* σ *with respect to* Γ *of the form* (4.19)*, and the function* $V : \mathcal{X} \to \mathbb{R}_+$ *defined by*

$$V(x) := \max_{i=1,\dots,n} \frac{1}{s_i} V_i(x_i) \tag{4.20}$$

is a candidate exponential ISS-Lyapunov function with respect to $\mathcal{A}$ *with rate coefficients*

$$c := \min_{i=1,\dots,n} c_i, \quad d := \min_{i,j=1,\dots,n:\, j\neq i} \left\{ d_i,\, -\ln\left(\frac{s_j}{s_i}\xi_{ij}\right)\right\}. \tag{4.21}$$

Proof. In view of Remark 4.2, σ is also an Ω-path with respect to the gain operator Γ defined by (4.6). Following Theorem 4.1, the function V defined by (4.20) is a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ for (4.1). Substituting (4.16) into (4.11) and (4.14), we obtain that

$$\alpha(r) = \min_{i=1,\dots,n} c_i, \quad \nu(r) = \max_{i=1,\dots,n} \left\{ e^{-d_i r},\ \max_{j=1,\dots,n} \frac{s_j}{s_i}\xi_{ij} r \right\} \qquad \forall\, r \geq 0,$$

that is, V is a candidate exponential ISS-Lyapunov function with respect to $\mathcal{A}$ with rate coefficients c and d defined by (4.21). $\square$

Remark 4.3. For the more general case in which the Lyapunov gains χ_{ij} in Assumption 4.1 are power functions instead of linear ones, a candidate exponential ISS-Lyapunov function can be constructed through a similar approach; cf. [53, Theorem 9].

[23] This case corresponds to the alternative gain operator Γ_Σ in Remark 4.2; see [96] for more results regarding existence and properties of Ω-paths.

Remark 4.4. For the gain matrix Ξ defined by (4.17), if its spectral radius satisfies that

$$\rho(\Xi) < 1,$$

then the small-gain condition (4.8) holds for the gain operator Γ defined by (4.18) [32, p. 110]. Additionally, if Ξ is irreducible, then $\rho(\Xi)$ is the Perron–Frobenius eigenvalue of Ξ, and the corresponding eigenvector $\bar{s} = (s_1, \dots, s_n)$ satisfies that $\bar{s} > 0$ (Perron–Frobenius theorem [97, Theorem 2.1.3]). As $\Xi\bar{s} = \rho(\Xi)\,\bar{s} < \bar{s}$, it follows that the function $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ defined by $\sigma(r) = \bar{s}r$ is a linear Ω-path with respect to the gain operator Γ defined by (4.18). For this Ω-path, as

$$\max_{j=1,\dots,n} \frac{s_j}{s_i}\xi_{ij} \leq \frac{1}{s_i}\sum_{j=1}^{n} s_j\xi_{ij} = \rho(\Xi)$$

for each subsystem Σ_i, the rate coefficient d defined in (4.21) satisfies that

$$d \geq \min\Big\{ \min_{i=1,\dots,n} d_i,\ -\ln(\rho(\Xi))\Big\}.$$

Having applied Corollary 4.3, we are able to establish pre-ISS for solution pairs satisfying suitable conditions on the frequency of jumps based on Proposition 2.3. However, if there exist subsystems Σ_i and Σ_j such that the rate coefficients c_i and d_j are negative, then the rate coefficients c and d defined by (4.21) are negative as well, and Proposition 2.3 cannot be applied for complete solution pairs (see Remark 2.6). Such cases can be addressed via the method of modifying candidate ISS-Lyapunov functions for subsystems based on auxiliary timers from [45]. In the following section, we provide a case-by-case study of the effects of such modifications on the Lyapunov feedback gains. Striving for as simple a setting as possible, we distill the main features of the formulations from [45, 52] and highlight their relative advantages.

4.3 Modifying candidate ISS-Lyapunov functions: a case-by-case study

In order to better focus on investigating effects of modifications and implications on the Lyapunov gains, we consider the special case of an interconnection of two hybrid subsystems with single-valued maps, and omit the external input. Consider a hybrid system transformed into an interconnection of two subsystems modeled by

$$\begin{aligned} \dot{x}_i &= f_i(x), \quad i = 1, 2, \qquad && x \in \mathcal{C}, \\ x_i^+ &= g_i(x), \quad i = 1, 2, \qquad && x \in \mathcal{D}, \end{aligned} \tag{4.22}$$

where $x := (x_1, x_2) \in \mathcal{X} \subset \mathbb{R}^N$ is the state. Recall that Σ_i denotes the i-th subsystem, for which $x_i \in \mathcal{X}_i \subset \mathbb{R}^{N_i}$ is the state and $x_j \in \mathcal{X}_j \subset \mathbb{R}^{N_j}$ is the internal input.

For brevity, we call a function V characterized by Lemma 2.1 a candidate ISS-Lyapunov function with respect to $\mathcal{A}$ with bounds ψ_1, ψ_2, gain $\bar{\phi}$, and rates α, ν. Additionally, if there exist constants $c, d \in \mathbb{R}$ such that (2.18) holds, then we call V a candidate exponentially ISS-Lyapunov function with respect to $\mathcal{A}$ with bounds ψ_1, ψ_2, gain $\bar{\phi}$, and rate coefficients c, d. Also, if $\mathcal{A} = \{0\}$, then the term "with respect to $\mathcal{A}$" is omitted.

Assumption 4.3. Each subsystem Σ_i of (4.22) admits a candidate exponential ISS-Lyapunov function V_i with bounds $\psi_{1,i}, \psi_{2,i}$, gain ϕ_i, and rate coefficients c_i, d_i.

Under Assumption 4.3, for each subsystem Σ_i, the corresponding Lyapunov gain $\chi_i \in \mathcal{K}_\infty$ is defined by

$$\chi_i(r) := \phi_i(\psi_{1,j}^{-1}(r)). \tag{4.23}$$

In the current setting, the *small-gain condition* (4.8) becomes

$$\chi_1(\chi_2(r)) < r \qquad \forall\, r > 0, \tag{4.24}$$

or equivalently,

$$\phi_1(\psi_{1,2}^{-1}(\phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.25}$$

Lemma 4.1 ([34, Lemma A.1])**.** *If two functions $\chi_1, \chi_2 \in \mathcal{K}_\infty$ satisfy the small-gain condition* (4.24)*, then there exists a function $\chi \in \mathcal{K}_\infty$ such that χ is continuously differentiable with $\chi' > 0$ on $\mathbb{R}_{>0}$, and that*

$$\chi_1^{-1}(r) > \chi(r) > \chi_2(r) \qquad \forall\, r > 0. \tag{4.26}$$

Based on Lemma 4.1, we are able to construct a candidate Lyapunov function for the interconnection (4.22) using the candidate exponential ISS-Lyapunov functions in Assumption 4.3. The following result follows directly from Theorem 4.1 and Corollary 4.2, and is consistent with [45, Theorem III.1 and Corollary III.2].

Corollary 4.4. *Consider the interconnection* (4.22)*. Suppose that Assumption 4.3 holds, and that the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by* (4.23) *satisfy the small-gain condition* (4.24)*. Let $\chi \in \mathcal{K}_\infty$ be the corresponding function in Lemma 4.1. Then the function $V : \mathcal{X} \to \mathbb{R}_+$ defined by*

$$V(x) := \max\{\chi(V_1(x_1)),\, V_2(x_2)\} \tag{4.27}$$

is a candidate Lyapunov function for (4.22) *with the rates α, ν defined by*

$$\begin{aligned}
\alpha(r) &:= \min\{\chi'(\chi^{-1}(r))\, c_1\chi^{-1}(r),\, c_2 r\},\\
\nu(r) &:= \max\{\chi(e^{-d_1}\chi^{-1}(r)),\, e^{-d_2}r,\, \chi(\chi_1(r)),\, \chi_2(\chi^{-1}(r))\}.
\end{aligned}$$

Additionally, if $c_1, c_2, d_1, d_2 > 0$ in Assumption 4.3, then (2.14) *holds; thus V is a Lyapunov function, and* (4.22) *is pre-GAS.*

If $c_i \le 0$ or $d_i \le 0$ in Assumption 4.3, then (2.14) does not hold for the corresponding rate α or ν; thus the function V defined by (4.27) is not a Lyapunov function, and we cannot conclude pre-GAS. In the following subsections, we investigate such cases and establish pre-GAS for solutions satisfying suitable conditions on the frequency of jumps.

4.3.1 Destabilizing flows: RADT modification

Consider the case that Assumption 4.3 holds with the rate coefficients satisfying $c_1, c_2 \le 0 < d_1, d_2$, that is, flows of the subsystems have destabilizing effects.[24] We will establish pre-GAS for solutions that jump fast enough, in the sense of a reverse average dwell-time [51]. We say that a solution $x : \operatorname{dom} x \to \mathcal{X}$ admits a *reverse average dwell-time (RADT)* τ_a^* if there exist constants $\tau_a^* > 0$ and $N_0^* \geq 1$ such that

$$k - l \geq \frac{t-s}{\tau_a^*} - N_0^* \tag{4.28}$$

for all $(s, l) \preceq (t, k)$ in $\operatorname{dom} x$. (If (4.28) holds with $N_0^* = 1$, then any two consecutive jumps are separated by at most τ_a^*; in this case, we say that the solution admits a *reverse dwell-time* τ_a^*.) Following [5, Appendix] and [45, Section IV.B], a solution $x : \operatorname{dom} x \to \mathcal{X}$ satisfies the RADT condition (4.28) if and only if $\operatorname{dom} x$ is the domain of an *RADT timer* τ modeled by[25]

$$\begin{aligned} &\dot{\tau} = 1/\tau_a^*, && \tau \in [0, N_0^*], \\ &\tau^+ = \max\{0,\, \tau - 1\}, && \tau \in [0, N_0^*]. \end{aligned} \tag{4.29}$$

For an RADT $\tau_a^* > 0$ and a constant $N_0^* \geq 1$, consider the augmented interconnection with the state $(x, \tau) \in \mathcal{X} \times [0, N_0^*]$ modeled by

$$\begin{aligned} &\dot{x}_i = f_i(x), \quad i = 1, 2, \quad \dot{\tau} = 1/\tau_a^*, && (x, \tau) \in \bar{\mathcal{C}}^*, \\ &x_i^+ = g_i(x), \quad i = 1, 2, \quad \tau^+ = \max\{0,\, \tau - 1\}, && (x, \tau) \in \bar{\mathcal{D}}^*, \end{aligned} \tag{4.30}$$

where $\bar{\mathcal{C}}^* = \mathcal{C} \times [0, N_0^*]$ is the flow set and $\bar{\mathcal{D}}^* = \mathcal{D} \times [0, N_0^*]$ is the jump set. Following [45, Proposition IV.4], for each (x_i, τ)-subsystem of (4.30) with x_j as the input, the function $W_i : \mathcal{X}_i \times [0, N_0^*] \to \mathbb{R}_+$ defined by

$$W_i(x_i, \tau) := e^{-L_i \tau} V_i(x_i)$$

with some constant $L_i > 0$ is a candidate exponential ISS-Lyapunov function with respect to $\mathcal{A}_i := \{0_{N_i}\} \times [0, N_0^*]$ with bounds $e^{-L_i N_0^*} \psi_{1,i}, \psi_{2,i}$, gain ϕ_i, and the rate coefficients $\bar{c}_i^*, \bar{d}_i^*$ defined by

$$\bar{c}_i^* := c_i + L_i/\tau_a^*, \quad \bar{d}_i^* := d_i - L_i. \tag{4.31}$$

[24] The cases where only one of c_1 and c_2 is nonpositive can be addressed via a similar approach; see the discussion after Corollary 4.6.

[25] There is a scaling difference between the RADT timers here and the ones in [5, 45].

Therefore, if the RADT τ_a^* satisfies that

$$-c_i\tau_a^* < d_i, \tag{4.32}$$

then we are able to find a constant L_i such that

$$-c_i\tau_a^* < L_i < d_i, \tag{4.33}$$

which renders $\bar{c}_i^*, \bar{d}_i^* > 0$; thus W_i becomes an exponential ISS-Lyapunov function with respect to $\mathcal{A}_i$ for the (x_i, τ)-subsystem of (4.30) with x_j as the input.

Next, we construct a Lyapunov function for the augmented interconnection (4.30) based on Corollary 4.4. In the current setting, this requires that the small-gain condition (4.24) holds for the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by

$$\chi_i(r) := \phi_i(\psi_{1,j}^{-1}(e^{L_j N_0^*} r)), \tag{4.34}$$

or equivalently,

$$e^{L_1 N_0^*}\phi_1(\psi_{1,2}^{-1}(e^{L_2 N_0^*}\phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.35}$$

The next lemma provides a small-gain condition in terms of the bounds $\psi_{1,1}$ and $\psi_{1,2}$, and the gains ϕ_1 and ϕ_2.

Lemma 4.2. *1. Suppose that there exists a constant $\varepsilon > 0$ such that*[26]

$$(1+\varepsilon)\phi_1(\psi_{1,2}^{-1}((1+\varepsilon)\phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.36}$$

For an RADT $\tau_a^* > 0$ and a constants $N_0^* \geq 1$ satisfying (4.32) and

$$-c_i N_0^* \tau_a^* < \ln(1+\varepsilon) \tag{4.37}$$

for each $i \in \{1, 2\}$, there exist constants $L_1, L_2 > 0$ such that (4.33) holds for each $i \in \{1, 2\}$ and (4.35) holds.

2. If there exist an RADT $\tau_a^* > 0$ and constants $N_0^* \geq 1$ and $L_1, L_2 > 0$ such that (4.33) holds for each $i \in \{1, 2\}$ and (4.35) holds, then there exists a constant $\varepsilon > 0$ such that (4.36) holds.

[26] Here it is equivalent to require "$\leq$" instead of "$<$".

Proof. 1. From (4.36), it follows that (4.35) holds with all constants $L_1, L_2 > 0$ satisfying $L_1, L_2 \leq \ln(1+\varepsilon)/N_0^*$. Then for each $i \in \{1,2\}$, the inequality (4.37) implies that there exists a constant $L_i \leq \ln(1+\varepsilon)/N_0^*$ such that (4.33) holds.

2. From (4.35), it follows that (4.36) holds with the constant

$$\varepsilon := \min\bigl\{e^{L_1 N_0^*} - 1,\, e^{L_2 N_0^*} - 1\bigr\}. \qquad \square$$

Combining Corollary 4.4 with the results above, we obtain the following small-gain theorems.

Proposition 4.5. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_1, c_2 \leq 0 < d_1, d_2$, *and that there exists a constant* $\varepsilon > 0$ *such that* (4.36) *holds. For an RADT* $\tau_a^* > 0$ *and a constant* $N_0^* \geq 1$ *satisfying* (4.32) *and* (4.37) *for each* $i \in \{1,2\}$, *there exist constants* $L_1, L_2 > 0$ *such that* (4.33) *holds for each* $i \in \{1,2\}$, *and the Lyapunov gains* $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *defined by* (4.34) *satisfy the small-gain condition* (4.24). *Let* $\chi \in \mathcal{K}_\infty$ *be the corresponding function in Lemma 4.1. Then the function* $W : \mathcal{X} \times [0, N_0^*] \to \mathbb{R}_+$ *defined by*

$$W(x,\tau) := \max\bigl\{\chi(e^{-L_1\tau}V_1(x_1)),\, e^{-L_2\tau}V_2(x_2)\bigr\}$$

is a Lyapunov function with respect to $\mathcal{A} := \{0_N\} \times [0, N_0^*]$ *for the augmented interconnection* (4.30).

Corollary 4.6. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_1, c_2 \leq 0 < d_1, d_2$, *and that there exists a constant* $\varepsilon > 0$ *such that* (4.36) *holds. Then the pre-GAS estimate* (2.17) *holds for all solutions such that the RADT condition* (4.28) *holds with an RADT* $\tau_a^* > 0$ *and a constant* $N_0^* \geq 1$ *satisfying* (4.32) *and* (4.37) *for each* $i \in \{1,2\}$.

For arbitrary rate coefficients satisfying $c_1, c_2 \leq 0 < d_1, d_2$ and constant $\varepsilon > 0$ satisfying (4.36), there always exists a small enough RADT $\tau_a^* > 0$ such that (4.32) and (4.37) hold for each $i \in \{1,2\}$. Meanwhile, if a rate coefficient $c_i \geq 0$, then (4.32) and (4.37) hold automatically. Moreover, if $c_1, c_2 \geq 0$, then Proposition 4.5 and Corollary 4.6 hold with arbitrary RADT $\tau_a^* > 0$ and constant $N_0^* \geq 1$.

The term $1+\varepsilon$ in (4.36) can be made arbitrarily close to 1 by selecting a small enough $\varepsilon > 0$. Consider the following small-gain conditions.

(SG1) The condition (4.25) holds.

(SG2) There exists a constant $\varepsilon > 0$ such that (4.36) holds.

We say that (SG2) is generic in (SG1), in the sense that every pair of gains $\phi_1, \phi_2 \in \mathcal{K}_\infty$ satisfying (SG1) can be approximated by a pair satisfying (SG2). Therefore, the approach of RADT modification will not result in fixed minimum increases in the Lyapunov feedback gains.

4.3.2 Destabilizing jumps: ADT modification

Consider the case that Assumption 4.3 holds with the rate coefficients satisfying $c_1, c_2 > 0 \geq d_1, d_2$, that is, jumps of the subsystems have destabilizing effects.[27] We will establish pre-GAS for solutions that jump slowly enough, in the sense of an average dwell-time [16]. We say that a solution $x : \operatorname{dom} x \to \mathcal{X}$ admits an *average dwell-time (ADT)* τ_a if there exist constants $\tau_a > 0$ and $N_0 \geq 1$ such that

$$k - l \leq \frac{t-s}{\tau_a} + N_0 \tag{4.38}$$

for all $(s,l) \preceq (t,k)$ in $\operatorname{dom} x$. (If (4.38) holds with $N_0 = 1$, then any two consecutive jumps are separated by at least τ_a; in this case, we say that the solution admits a *dwell-time* τ_a [15].) Following [45, Section IV.A], a solution $x : \operatorname{dom} x \to \mathcal{X}$ satisfies the ADT condition (4.38) if and only if $\operatorname{dom} x$ is the domain of an *ADT timer* τ modeled by

$$\begin{aligned} &\dot{\tau} \in [0, 1/\tau_a], && \tau \in [0, N_0], \\ &\tau^+ = \tau - 1, && \tau \in [1, N_0]. \end{aligned} \tag{4.39}$$

Remark 4.5. The notion of ADT timer for hybrid systems first appeared in [5, Appendix] (see also [98] for a related earlier construction), where the timer was modeled by

$$\begin{cases} \dot{\tau} \in \eta_\delta(\tau) & \tau \in [0, N_0] \\ \tau^+ = \tau - 1 & \tau \in [1, N_0] \end{cases} \tag{4.40}$$

[27] The cases where only one of d_1 and d_2 is nonpositive can be addressed via a similar approach; see the discussion after Corollary 4.8.

with

$$\eta_\delta(\tau) := \begin{cases} 1/\tau_a & \tau \in [0, N_0) \\ [0, 1/\tau_a] & \tau = N_0. \end{cases}$$

The models in (4.39) and (4.40) are equivalent in the following sense. First, as $1/\tau_a \in [0, 1/\tau_a]$, an ADT timer modeled by (4.40) always satisfies (4.39). Second, given an ADT timer modeled by (4.39) that increases on $[0, N_0)$ with a speed less than $1/\tau_a$, there always exists an ADT timer modeled by (4.40) that increases on $[0, N_0)$ with the speed $1/\tau_a$ but stays longer at N_0 so that their hybrid time domains are the same.

For an ADT $\tau_a > 0$ and a constant $N_0 \geq 1$, consider the augmented interconnection with the state $(x, \tau) \in \mathcal{X} \times [0, N_0]$ modeled by

$$\begin{aligned} \dot{x}_i &= f_i(x), \quad i = 1, 2, \quad \dot{\tau} \in [0, 1/\tau_a], \qquad (x, \tau) \in \bar{\mathcal{C}}, \\ x_i^+ &= g_i(x), \quad i = 1, 2, \quad \tau^+ = \tau - 1, \qquad (x, \tau) \in \bar{\mathcal{D}}, \end{aligned} \tag{4.41}$$

where $\bar{\mathcal{C}} = \mathcal{C} \times [0, N_0]$ is the flow set and $\bar{\mathcal{D}} = \mathcal{D} \times [1, N_0]$ is the jump set. Following [45, Proposition IV.1], for each (x_i, τ)-subsystem of (4.41) with x_j as the input, the function $W_i : \mathcal{X}_i \times [0, N_0] \to \mathbb{R}_+$ defined by

$$W_i(x_i, \tau) := e^{L_i \tau} V_i(x_i)$$

with some constant $L_i > 0$ is a candidate exponential ISS-Lyapunov function with respect to $\mathcal{A}_i := \{0_{N_i}\} \times [0, N_0]$ with bounds $\psi_{1,i}, e^{L_i N_0} \psi_{2,i}$, gain $e^{L_i N_0} \phi_i$, and the rates coefficients $\bar{c}_i, \bar{d}_i$ defined by

$$\bar{c}_i := c_i - L_i/\tau_a, \quad \bar{d}_i := d_i + L_i. \tag{4.42}$$

Therefore, if the ADT τ_a satisfies that

$$c_i \tau_a > -d_i, \tag{4.43}$$

then we are able to find a constant L_i such that

$$c_i \tau_a > L_i > -d_i, \tag{4.44}$$

which renders $\bar{c}_i, \bar{d}_i > 0$; thus W_i becomes an exponential ISS-Lyapunov function with respect to $\mathcal{A}_i$ for the (x_i, τ)-subsystem of (4.41) with x_j as the input.

Next, we construct a Lyapunov function for the augmented interconnection (4.41) based on Corollary 4.4. In the current setting, this requires that the small-gain condition (4.24) holds for the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by

$$\chi_i(r) := e^{L_i N_0} \phi_i(\psi_{1,j}^{-1}(r)), \tag{4.45}$$

or equivalently,

$$e^{L_1 N_0} \phi_1(\psi_{1,2}^{-1}(e^{L_2 N_0} \phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.46}$$

The next lemma provides a small-gain condition in terms of the bounds $\psi_{1,1}$ and $\psi_{1,2}$, and the gains ϕ_1 and ϕ_2.

Lemma 4.3. *1. Suppose that there exists a constant $\varepsilon > 0$ such that*

$$(1+\varepsilon) e^{-d_1} \phi_1(\psi_{1,2}^{-1}((1+\varepsilon) e^{-d_2} \phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.47}$$

For an ADT $\tau_a > 0$ satisfying (4.43) for each $i \in \{1,2\}$ and a constant $N_0 \geq 1$ satisfying

$$-d_i(N_0 - 1) < \ln(1+\varepsilon) \tag{4.48}$$

for each $i \in \{1,2\}$, there exist constants $L_1, L_2 > 0$ such that (4.44) holds for each $i \in \{1,2\}$ and (4.46) holds.

2. If there exist an ADT $\tau_a > 0$ and constants $N_0 \geq 1$ and $L_1, L_2 > 0$ such that (4.44) holds for each $i \in \{1,2\}$ and (4.46) holds, then there exists a constant $\varepsilon > 0$ such that (4.47) holds.

Proof. 1. From (4.47), it follows that (4.46) holds with all constant $L_1, L_2 > 0$ satisfying $L_1 \leq (\ln(1+\varepsilon) - d_1)/N_0$ and $L_2 \leq (\ln(1+\varepsilon) - d_2)/N_0$. Then for each $i \in \{1,2\}$, the inequality (4.48) implies that there exists a constant $L_i \leq (\ln(1+\varepsilon) - d_i)/N_0$ such that (4.44) holds.

2. From (4.46), it follows that (4.47) holds with the constant

$$\varepsilon := \min\{e^{L_1 N_0 + d_1} - 1,\, e^{L_2 N_0 + d_2} - 1\}. \qquad \square$$

Combining Corollary 4.4 with the results above, we obtain the following small-gain theorems.

Proposition 4.7. *Consider the interconnection* (4.22)*. Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_1, c_2 > 0 \geq d_1, d_2$*, and that there exists a constant* $\varepsilon > 0$ *such that* (4.47) *holds. For an ADT* $\tau_a > 0$ *satisfying* (4.43) *for each* $i \in \{1, 2\}$*, and a constant* $N_0 \geq 1$ *satisfying* (4.48) *for each* $i \in \{1, 2\}$*, there exists constant* $L_1, L_2 > 0$ *such that* (4.44) *holds for each* $i \in \{1, 2\}$*, and the Lyapunov gains* $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *defined by* (4.45) *satisfy the small-gain condition* (4.24)*. Let* $\chi \in \mathcal{K}_\infty$ *be the corresponding function in Lemma 4.1. Then the function* $W : \mathcal{X} \times [0, N_0] \to \mathbb{R}_+$ *defined by*

$$W(x, \tau) := \max\bigl\{\chi(e^{L_1 \tau} V_1(x_1)),\, e^{L_2 \tau} V_2(x_2)\bigr\}$$

is a Lyapunov function with respect to $\mathcal{A} := \{0_N\} \times [0, N_0]$ *for the augmented interconnection* (4.41)*.*

Corollary 4.8. *Consider the interconnection* (4.22)*. Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_1, c_2 > 0 \geq d_1, d_2$*, and that there exists a constant* $\varepsilon > 0$ *such that* (4.47) *holds. Then the pre-GAS estimate* (2.17) *holds for all solutions such that the ADT condition* (4.38) *holds with an ADT* $\tau_a > 0$ *satisfying* (4.43) *for each* $i \in \{1, 2\}$*, and a constant* $N_0 \geq 1$ *satisfying* (4.48) *for each* $i \in \{1, 2\}$*.*

For arbitrary constant $\varepsilon > 0$ and rate coefficients c_1, c_2, d_1 and d_2 satisfying $c_1, c_2 > 0 \geq d_1, d_2$ and (4.47), there always exists a large enough ADT $\tau_a > 0$ such that (4.43) holds for each $i \in \{1, 2\}$.[28] Meanwhile, if a rate coefficient $d_i \geq 0$, then (4.43) and (4.48) hold automatically. Moreover, if $d_1, d_2 \geq 0$, then Proposition 4.7 and Corollary 4.8 hold with arbitrary ADT $\tau_a > 0$ and constant $N_0 \geq 1$.

Compared with $1 + \varepsilon$ in (4.36), the terms $(1 + \varepsilon)e^{-d_1}, (1 + \varepsilon)e^{-d_2}$ in (4.47) are lower bounded by $e^{-d_1}, e^{-d_2} > 1$, respectively. Consider the following small-gain condition.

(SG3) There exists a constant $\varepsilon > 0$ such that (4.47) holds.

Unlike (SG2) in Section 4.3.1, (SG3) is clearly not generic in (SG1) in the same sense. Therefore, the approach of ADT modification will result in fixed minimum increases in the Lyapunov feedback gains.

[28] Unlike the case with (4.37) and the RADT τ_a^*, the inequality (4.48) introduce no constraint on the ADT τ_a.

4.3.3 Destabilizing jumps: an alternative construction

Consider again the case that Assumption 4.3 holds with the rate coefficients satisfying $c_1, c_2 > 0 \geq d_1, d_2$. If the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by (4.23) are linear, that is,

$$\chi_1(r) \equiv \xi_1 r, \quad \chi_2(r) \equiv \xi_2 r \tag{4.49}$$

with some constants $\xi_1, \xi_2 > 0$, then we are able to establish pre-GAS under the less restrictive small-gain condition (SG1) instead of (SG3), by applying the ADT modification to the interconnection (4.22) instead of its subsystems.[29] In this case, (SG1) becomes $\xi_1\xi_2 < 1$; thus we can select a function $\chi \in \mathcal{K}_\infty$ defined by $\chi(r) := \mu r$ with some constant $\mu \in (\xi_2, 1/\xi_1)$ so that it fulfills the claim of Lemma 4.1. Following Corollary 4.4, for the interconnection (4.22), the function V defined by (4.27) is a candidate exponential Lyapunov function with the rate coefficients c, d defined by

$$c := \min\{c_1, c_2\}, \quad d := \min\{d_1, d_2\}. \tag{4.50}$$

Then the ADT modification yields the following small-gain theorems.

Proposition 4.9. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying $c_1, c_2 > 0 \geq d_1, d_2$, and that the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by* (4.23) *satisfy* (4.49) *and the small-gain condition* (4.24)*, that is, $\xi_1\xi_2 < 1$. For an ADT satisfying $\tau_a > -d/c$ with the constants c and d defined by* (4.50)*, and a constant $N_0 \geq 1$, the function $W : \mathcal{X} \times [0, N_0] \to \mathbb{R}_+$ defined by*

$$W(x, \tau) := e^{L\tau} \max\{\mu V_1(x_1),\, V_2(x_2)\}$$

with any constants $L \in (-d, c\tau_a)$ and $\mu \in (\xi_2, 1/\xi_1)$ is an exponential Lyapunov function with respect to $\mathcal{A} := \{0_N\} \times [0, N_0]$ for the augmented interconnection (4.41).

Corollary 4.10. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying $c_1, c_2 > 0 \geq d_1, d_2$, and that*

[29] Also, in the case with linear Lyapunov gains, the small-gain conditions (SG1) and (SG2) are equivalent, that is, the RADT modification in Section 4.3.1 does not alter the small-gain condition at all.

the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by (4.23) *satisfy* (4.49) *and the small-gain condition* (4.24)*, that is, $\xi_1 \xi_2 < 1$. Then the pre-GAS estimate* (2.17) *holds for all solutions such that the ADT condition* (4.38) *holds with an ADT satisfying $\tau_a > -d/c$ with the constants c and d defined by* (4.50)*, and a constant $N_0 \geq 1$.*

Remark 4.6. The lower bound on the ADT τ_a in Proposition 4.9 (and Corollary 4.10) is greater than or equal to the one in Proposition 4.7 (and Corollary 4.8), that is,

$$\frac{\max\{-d_1,\, -d_2\}}{\min\{c_1,\, c_2\}} \geq \max\left\{\frac{-d_1}{c_1},\, \frac{-d_2}{c_2}\right\}.$$

On the other hand, if the assumptions in Proposition 4.9 and Corollary 4.10 hold with

$$d_1 + d_2 \leq \ln(\xi_1 \xi_2),$$

then (4.47) does not hold for any $\varepsilon > 0$; thus Proposition 4.7 and Corollary 4.8 cannot be applied.

4.3.4 Destabilizing flow and jumps

Consider the case that the flow of one subsystem and the jumps of the other one have destabilizing effects. Without loss of generality, suppose that Assumption 4.3 holds with the rate coefficients satisfying $c_2, d_1 > 0 \geq c_1, d_2$. We will establish pre-GAS for solutions that jump neither too fast nor too slowly, in the sense of combined RADT and ADT.

For an RADT $\tau_a^* > 0$, an ADT $\tau_a > 0$, and constants $N_0^*, N_0 \geq 1$, consider the augmented interconnection with the state $(x, \tau_1, \tau_2) \in \mathcal{X} \times [0, N_0^*] \times [0, N_0]$ modeled by

$$\begin{aligned} &\left.\begin{aligned} &\dot{x}_i = f_i(x), \quad i = 1, 2, \\ &\dot{\tau}_1 = 1/\tau_a^*, \quad \dot{\tau}_2 \in [0, 1/\tau_a], \end{aligned}\right\} \quad (x, \tau_1, \tau_2) \in \tilde{\mathcal{C}}, \\ &\left.\begin{aligned} &x_i^+ = g_i(x), \quad i = 1, 2 \\ &\tau_1^+ = \max\{0,\, \tau_1 - 1\}, \quad \tau_2^+ = \tau_2 - 1, \end{aligned}\right\} \quad (x, \tau_1, \tau_2) \in \tilde{\mathcal{D}}, \end{aligned} \tag{4.51}$$

where $\tilde{\mathcal{C}} = \mathcal{C} \times [0, N_0^*] \times [0, N_0]$ is the flow set and $\tilde{\mathcal{D}} = \mathcal{D} \times [0, N_0^*] \times [1, N_0]$ is the jump set. Following [45, Proposition IV.4], for the (x_1, τ_1)-subsystem of

(4.51) with x_2 as the input, the function $W_1 : \mathcal{X}_1 \times [0, N_0^*] \to \mathbb{R}_+$ defined by

$$W_1(x_1, \tau_1) := e^{-L_1\tau_1} V_1(x_1) \tag{4.52}$$

with some constant $L_1 > 0$ is a candidate exponential ISS-Lyapunov function with respect to $A_1 := \{0_{N_1}\} \times [0, N_0^*]$ with bounds $e^{-L_1 N_0^*}\psi_{1,1}, \psi_{2,1}$, gain χ_1, and the rate coefficients $\bar{c}_1^*, \bar{d}_1^*$ defined by (4.31) with $i = 1$; it becomes an exponential ISS-Lyapunov function if (4.33) holds with $i = 1$. Meanwhile, following [45, Proposition IV.1], for the (x_2, τ_2)-subsystem of (4.51) with x_1 as the input, the function $W_2 : \mathcal{X}_2 \times [0, N_0]$ defined by

$$W_2(x_2, \tau_2) := e^{L_2\tau_2} V_2(x_2)$$

with some constant $L_2 > 0$ is a candidate exponential ISS-Lyapunov function with respect to $A_2 := \{0_{N_2}\} \times [0, N_0]$ with bounds $\psi_{1,2}, e^{L_2 N_0}\psi_{2,2}$, gain $e^{L_2 N_0}\chi_2$, and the rate coefficients $\bar{c}_2, \bar{d}_2$ defined by (4.42) with $i = 2$; it becomes an exponential ISS-Lyapunov function if (4.44) holds with $i = 2$.

Next, we construct a Lyapunov function for the augmented interconnection (4.51) based on Corollary 4.4. In the current setting, this requires that the small-gain condition (4.24) holds for the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by

$$\chi_1(r) := \phi_1(\psi_{1,2}^{-1}(r)), \quad \chi_2(r) := e^{L_2 N_0}\phi_2(\psi_{1,1}^{-1}(e^{L_1 N_0^*} r)), \tag{4.53}$$

or equivalently,

$$e^{L_1 N_0^*}\phi_1(\psi_{1,2}^{-1}(e^{L_2 N_0}\phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.54}$$

Following essentially the proofs of Lemmas 4.2 and 4.3, we formulate a small-gain condition in terms of the bounds $\psi_{1,1}$ and $\psi_{1,2}$, and the gains ϕ_1 and ϕ_2.

Lemma 4.4. *1. Suppose that there exists a constant $\varepsilon > 0$ such that*

$$(1+\varepsilon)\phi_1(\psi_{1,2}^{-1}((1+\varepsilon)e^{-d_2}\phi_2(\psi_{1,1}^{-1}(r)))) < r \qquad \forall\, r > 0. \tag{4.55}$$

For an RADT $\tau_a^* > 0$ and a constant $N_0^* \geq 1$ satisfying (4.32) and (4.37) for $i = 1$, an ADT $\tau_a > 0$ satisfying (4.43) for $i = 2$, and a constant $N_0 \geq 1$ satisfying (4.48) for $i = 2$, there exist constants $L_1, L_2 > 0$ such *that* (4.33) *holds for* $i = 1$, (4.44) *holds for* $i = 2$, *and* (4.54) *holds.*

2. *If there exist an RADT* $\tau_a^* > 0$, *an ADT* $\tau_a > 0$ *and constants* $N_0^*, N_0 \geq 1$ *and* $L_1, L_2 > 0$ *such that* (4.33) *holds for* $i = 1$, (4.44) *holds for* $i = 2$, *and* (4.54) *holds, then there exists a constant* $\varepsilon > 0$ *such that* (4.55) *holds.*

Combining Corollary 4.4 with the results above, we obtain the following small-gain theorems.

Proposition 4.11. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_2, d_1 > 0 \geq c_1, d_2$, *and that there exists a constant* $\varepsilon > 0$ *such that* (4.55) *holds. For an RADT* $\tau_a^* > 0$ *and a constant* $N_0^* \geq 1$ *satisfying* (4.32) *and* (4.37) *for* $i = 1$, *an ADT* $\tau_a > 0$ *satisfying* (4.43) *for* $i = 2$, *and a constants* $N_0 \geq 1$ *satisfying* (4.48) *for* $i = 2$, *there exist constants* $L_1, L_2 > 0$ *such that* (4.33) *holds for* $i = 1$, (4.44) *holds for* $i = 2$, *and the Lyapunov gains* $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *defined by* (4.53) *satisfy the small-gain condition* (4.24). *Let* $\chi \in \mathcal{K}_\infty$ *be the corresponding function in Lemma 4.1. Then the function* $W : \mathcal{X} \times [0, N_0^*] \times [0, N_0] \to \mathbb{R}_+$ *defined by*

$$W(x, \tau_1, \tau_2) := \max\{\chi(e^{-L_1\tau_1}V_1(x_1)),\, e^{L_2\tau_2}V_2(x_2)\}$$

is a Lyapunov function with respect to $\mathcal{A} := \{0_N\} \times [0, N_0^*] \times [0, N_0]$ *for the augmented interconnection* (4.51).

Corollary 4.12. *Consider the interconnection* (4.22). *Suppose that Assumption 4.3 holds with the rate coefficients satisfying* $c_2, d_1 > 0 \geq c_1, d_2$, *and that there exists a constant* $\varepsilon > 0$ *such that* (4.55) *holds. Then the pre-GAS estimate* (2.17) *holds for all solutions such that the RADT condition* (4.28) *holds with an RADT* $\tau_a^* > 0$ *and a constant* $N_0^* \geq 1$ *satisfying* (4.32) *and* (4.37) *for* $i = 1$, *and the ADT condition* (4.38) *holds with an ADT* $\tau_a > 0$ *satisfying* (4.43) *for* $i = 2$, *and a constants* $N_0 \geq 1$ *satisfying* (4.48) *for* $i = 2$.

Remark 4.7. If there is a solution $x : \operatorname{dom} x \to \mathcal{X}$ satisfying both the RADT condition (4.28) and the ADT condition (4.38), then

$$(\tau_a - \tau_a^*)(t - s) \leq (N_0 + N_0^*)\tau_a\tau_a^*,$$
$$(\tau_a - \tau_a^*)(k - l) \leq N_0\tau_a + N_0^*\tau_a^*$$

for all $(s, l) \preceq (t, k)$ in $\operatorname{dom} x$. Hence the solution x could be complete only if

$$\tau_a \leq \tau_a^*. \tag{4.56}$$

Furthermore, if the RADT τ_a^* and the constant N_0^* satisfy (4.32) and (4.37) for $i = 1$, the ADT τ_a satisfies (4.43) for $i = 2$, and the constant N_0 satisfies (4.48) for $i = 2$, then it is necessary that

$$c_1 d_2 < c_2 \ln(1 + \varepsilon), \tag{4.57}$$

$$c_1 d_2 < c_2 d_1. \tag{4.58}$$

Consider the following small-gain condition.

(SG4) There exists a constant $\varepsilon > 0$ such that (4.55) holds.

Similar to (SG3) in Section 4.3.2, (SG4) is clearly not generic in (SG1) due to the fix minimum increase in the Lyapunov feedback gain of the (x_2, τ_2)-subsystem resulted from the ADT modification.

4.3.5 Destabilizing flow and jumps: an alternative construction

Consider again the case that Assumption 4.3 holds with the rate coefficients satisfying $c_2, d_1 > 0 \geq c_1, d_2$. If the Lyapunov gains $\chi_1, \chi_2 \in \mathcal{K}_\infty$ defined by (4.23) are linear, that is, (4.49) holds with some constants $\xi_1, \xi_2 > 0$, then we are able to establish pre-GAS under the less restrictive small-gain condition (SG1) instead of (SG4), by first constructing the (x_1, τ_1)-subsystem of (4.51) through the RADT modification, and then applying the ADT modification to the (x_1, x_2, τ_1)-interconnection. In this case, (SG1) becomes $\xi_1 \xi_2 < 1$; thus we can select a function $\chi \in \mathcal{K}_\infty$ defined by $\chi(r) := \mu r$ with some constant $\mu \in (\xi_2, 1/\xi_1)$ so that it fulfills the claim of Lemma 4.1. Following Corollary 4.4 and the results in Section 4.3.1, for the (x_1, x_2, τ_1)-interconnection, the function $\bar{W} : \mathcal{X} \times [0, N_0^*] \to \mathbb{R}_+$ defined by

$$\bar{W}(x, \tau_1) := \max\left\{\mu e^{-L_1 \tau_1} V_1(x_1),\, V_2(x_2)\right\}$$

with some constant $L_1 > 0$ satisfying

$$e^{L_1 N_0^*} \xi_1 \xi_2 < 1 \tag{4.59}$$

is a candidate exponential Lyapunov function with respect to $\bar{A} := \{0_N\} \times [0, N_0^*]$ with the bounds $\bar{\psi}_1, \bar{\psi}_2$ defined by

$$\bar{\psi}_1(r) := \min\bigl\{\mu e^{-L_1 N_0^*}\psi_{11}(r/\sqrt{2}),\, \psi_{12}(r/\sqrt{2})\bigr\},$$
$$\bar{\psi}_2(r) := \max\bigl\{\mu\psi_{21}(r),\, \psi_{22}(r)\bigr\},$$

and the rate coefficients $\bar{c}, \bar{d}$ define by

$$\bar{c} := \min\{c_1 + L_1/\tau_a^*,\, c_2\}, \quad \bar{d} := \min\{d_1 - L_1,\, d_2\}.$$

Note that $\bar{c} > 0$ when

$$L_1 > -c_1\tau_a^*, \tag{4.60}$$

while $\bar{d} \leq d_2 \leq 0$; thus, unlike the case in Section 4.3.1, it is unnecessary to require that $L_1 < d_1$. Following the results in Section 4.3.2, in order to obtain an exponential Lyapunov function through the ADT modification, it is necessary that (4.44) holds for $i = 2$ with c_2 replaced by $\bar{c}$ and d_2 replaced by $\bar{d}$, that is,

$$\tau_a \min\{c_1 + L_1/\tau_a^*,\, c_2\} > L_2 > \max\{L_1 - d_1,\, -d_2\} \tag{4.61}$$

Note that there exist constant $L_1, L_2 > 0$ such that (4.59)–(4.61) hold if and only if the RADT $\tau_a^* > 0$, the ADT $\tau_a > 0$, and the constant $N_0^* \geq 1$ satisfy

$$\tau_a \min\{c_1 - \ln(\xi_1\xi_2)/(N_0^*\tau_a^*),\, c_2\} > \max\{-c_1\tau_a^* - d_1,\, -d_2\}. \tag{4.62}$$

Combining the results above, we obtain the following small-gain theorems.

Proposition 4.13. *Consider the interconnection* (4.22)*. Suppose that Assumption 4.3 holds with rate coefficients satisfying* $c_2, d_1 > 0 \geq c_1, d_2$*, and that the Lyapunov gains* $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *defined by* (4.23) *satisfy* (4.49) *and the small-gain condition* (4.25)*, that is,* $\xi_1\xi_2 < 1$*. For an RADT* $\tau_a^* > 0$*, an ADT* $\tau_a > 0$*, and a constant* $N_0^* \geq 1$ *satisfying* (4.62)*, and a constant* $N_0 \geq 1$*, there exists constants* $L_1, L_2 > 0$ *such that* (4.59)*–*(4.61) *hold. Then the function* $W : \mathcal{X} \times [0, N_0^*] \times [0, N_0] \to \mathbb{R}_+$ *defined by*

$$W(x, \tau_1, \tau_2) := e^{L_2\tau_2} \max\bigl\{\mu e^{-L_1\tau_1} V_1(x_1),\, V_2(x_2)\bigr\}$$

with any constant $\mu \in (\xi_2, 1/\xi_1)$ *is an exponential Lyapunov function with* *respect to* $\mathcal{A} := \{0_N\} \times [0, N_0^*] \times [0, N_0]$ *for the augmented interconnection* (4.51).

Corollary 4.14. *Consider the interconnection* (4.22)*. Suppose that Assumption 4.3 holds with rate coefficients satisfying* $c_2, d_1 > 0 \geq c_1, d_2$*, and that the Lyapunov gains* $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *defined by* (4.23) *satisfy* (4.49) *and the small-gain condition* (4.25)*, that is,* $\xi_1\xi_2 < 1$*. Then the pre-GAS estimate* (2.17) *holds for all solutions such that the RADT condition* (4.28) *and the ADT condition* (4.38) *hold with an RADT* $\tau_a^* > 0$*, an ADT* $\tau_a > 0$*, and a constant* $N_0^* \geq 1$ *satisfying* (4.62)*, and a constant* $N_0 \geq 1$*.*

Remark 4.8. As in Remark 4.6, if the assumptions in Proposition 4.13 (and Corollary 4.14) hold with

$$d_2 \leq \ln(\xi_1\xi_2),$$

then (4.55) does not hold for any $\varepsilon > 0$; thus Proposition 4.11 (and Corollary 4.12) cannot be applied.

Remark 4.9. As in Remark 4.7, there is a complete solution x such that the RADT condition (4.28) and the ADT condition (4.38) both hold only if the RADT τ_a^* and the ADT τ_a satisfy (4.56), which, combined with (4.62), implies that

$$\tau_a^* \min\{c_1 - \ln(\xi_1\xi_2)/(N_0^*\tau_a^*),\ c_2\} > \max\{-c_1\tau_a^* - d_1,\ -d_2\}, \tag{4.63}$$

or equivalently,

$$\begin{aligned} -2c_1\tau_a^* &< d_1 - \ln(\xi_1\xi_2)/N_0^*, \\ -(c_1 + c_2)\tau_a^* &< d_1, \\ -c_1\tau_a^* &< d_2 - \ln(\xi_1\xi_2)/N_0^*, \\ c_2\tau_a^* &> -d_2. \end{aligned}$$

There exists an RADT $\tau_a^* > 0$ and a constant $N_0^* \geq 1$ such that (4.63) holds if and only if

$$\xi_1\xi_2 < e^{d_2(1-c_1/c_2)}, \tag{4.64}$$

$$\xi_1\xi_2 < e^{d_2(d_1/d_2-2c_1/c_2)}, \tag{4.65}$$

and

$$c_2 d_1 > (c_1 + c_2)d_2. \tag{4.66}$$

If $d_2 < 0$, then (4.64) is more restrictive than the small-gain condition $\xi_1\xi_2 < 1$. Meanwhile, (4.58) implies (4.66), and combining (4.57) with (4.55) yields (4.64) and (4.65). Therefore, if

$$e^{d_2(1-2c_1/c_2)} \leq \xi_1\xi_2 < e^{d_2(1-c_1/c_2)},$$

then pre-GAS can be established using Proposition 4.13 and Corollary 4.14, but Proposition 4.11 and Corollary 4.12 cannot be applied.

4.4 Future work

For the case with mixed types of non-ISS dynamics, in light of Sections 4.3.4 and 4.3.5, one should apply first the RADT modification to subsystems with non-ISS flows, and then the ADT modification to the interconnection. However, this scheme requires linear Lyapunov feedback gains, which is rather restrictive in real-world problems. To relax this requirement, Proposition 2.2 needs to be extended to the case of candidate ISS-Lyapunov functions with nonlinear rates. Similar results have been established in [53, Theorems 1 and 3] for impulsive systems, and we conjecture that they can be generalized to hybrid systems as well.

Part III

Control with limited information and topological entropy

Chapter 5

Feedback stabilization of switched linear systems with unknown disturbances under data-rate constraints

5.1 Problem formulation

5.1.1 System definition

We are interested in stabilizing a switched linear control system modeled by

$$\dot{x} = A_\sigma x + B_\sigma u + D_\sigma d, \qquad x(0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control, and $d \in \mathbb{R}^{n_d}$ is the external disturbance. The set $\{(A_p, B_p, D_p) : p \in \mathcal{P}\}$ denotes a family of matrix triples defining the modes, where $\mathcal{P}$ is a finite index set. Suppose that (5.1) fulfills the same assumptions as those imposed on general switched systems in Section 2.2. The switching signal σ is fixed but unknown to the sensor and the controller a priori.

Our first basic assumption is that the switching is slow in the sense of combined dwell-time and average dwell-time.

Assumption 5.1 (Switching)**.** The switching signal σ admits

1. a dwell-time $\tau_d > 0$ such that (2.3) holds for all consecutive switches t' and t'', and
2. an average dwell-time (ADT) $\tau_a > \tau_d$ such that (2.4) holds with a constant $N_0 \geq 1$.

The notions of dwell-time [15] and ADT [16] have become standard in the literature on switched systems. In Assumption 5.1, the ADT condition (item 2) would be implied by the dwell-time condition (item 1) if the constraint $\tau_a > \tau_d$ is violated. Switching signals satisfying Assumption 5.1 were referred to as "hybrid dwell-time" signals in [99].

Our second basic assumption is that every individual mode is stabilizable.

Assumption 5.2 (Stabilizability)**.** For each $p \in \mathcal{P}$, the pair (A_p, B_p) is stabilizable, that is, there exists a state feedback gain matrix K_p such that $A_p + B_p K_p$ is Hurwitz.

In the following analysis, it is assumed that such a family of stabilizing gain matrices K_p, $p \in \mathcal{P}$ has been selected and fixed. However, even in the disturbance-free case, and when all individual modes are stabilized through state feedback (or stable without feedback), stability of the switched system is not necessarily guaranteed (see, e.g., [6, p. 19]).

Throughout this chapter, $\|\cdot\|$ denotes the ∞-norm of a vector, or the (induced) ∞-norm of a matrix, that is,

$$\|v\| := \|v\|_\infty := \max_{i=1,\dots,n} |v_i|$$

for a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, and

$$\|M\| := \|M\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |M_{ij}|$$

for a matrix $M = (M_{ij}) \in \mathbb{R}^{n \times n}$. The left-sided limit of a piecewise absolutely continuous function z approaching t is denoted by $z(t^-) := \lim_{s \nearrow t} z(s)$.

We let δ_d denote the essential supremum ∞-norm of the disturbance d, that is,

$$\delta_d := \|d\|_\infty := \operatorname*{ess\,sup}_{s \ge 0} \|d(s)\| \le \infty, \tag{5.2}$$

and refer to it as the *disturbance bound*. In the following analysis, it is assumed that δ_d is finite (as the state bound (5.6) in our main result below holds trivially when $\delta_d = \infty$). However, its value is *unknown* to the sensor and the controller.

5.1.2 Information structure

The feedback loop consists of a *sensor* and a *controller*. The sensor measures two sequences of data—quantized measurements (samples) of the state $x(t_k)$, and indices of the active modes $\sigma(t_k)$—and transmits them to the controller at a sequence of *sampling times* $t_k = k\tau_s$, where $\tau_s > 0$ is the *sampling period* and $k \in \mathbb{N}$. Each sample is encoded by an integer i_k from 0 to N^{n_x}, where

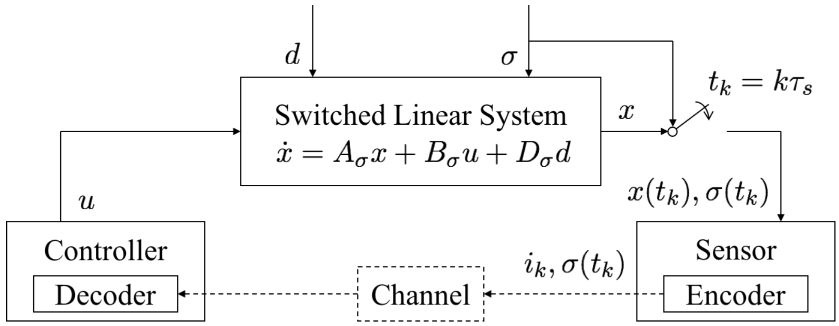


Figure 5.1: Information structure.

N is an odd integer (so that the equilibrium at the origin is preserved). The controller generates the control input $u(\cdot)$ to the switched linear system (5.1) based on the decoded data. As $\sigma(t_k) \in \mathcal{P}$ and $i_k \in \{0, 1, \dots, N^{n_x}\}$, the data transmission rate between the encoder and the decoder is given by

$$R = \frac{\log_2 |N^{n_x} + 1| + \log_2 |\mathcal{P}|}{\tau_s} \tag{5.3}$$

bits per unit of time, where $|\mathcal{P}|$ is the cardinality of the index set $\mathcal{P}$ (i.e., the number of modes). As illustrated in Figure 5.1, this information structure allows us to separate the sensing and the control tasks in the following sense: the sensor does not have access to the exact control objective, and the controller does not have access to the exact state. The communication and control strategy is explained in detail in Section 5.3.

The sampling period τ_s is assumed to be no larger than the dwell-time τ_d in Assumption 5.1, that is,

$$\tau_s \leq \tau_d, \tag{5.4}$$

so that there is at most one switch on each *sampling interval* $(t_k, t_{k+1}]$. Due to the ADT $\tau_a > \tau_d$ in Assumption 5.1, switches actually occur less often than once per sampling period.

Our last basic assumption imposes a lower bound on the data rate R:

Assumption 5.3 (Data rate)**.** The sampling period τ_s satisfies

$$\Lambda_p := \|e^{A_p \tau_s}\| < N \qquad \forall\, p \in \mathcal{P}. \tag{5.5}$$

The inequality in (5.5) can be interpreted as a lower bound on the data rate R since it requires the sampling period τ_s to be small enough with respect to the integer N, which defines the number of bits in each transmission. This bound is the same as the one for the disturbance-case [68, Assumption 3], and similar data-rate bounds appeared in [59, 100, 60] for stabilizing non-switched linear systems; see [62, Section V] and [68, Section 2.2] for more discussions on their relation.

5.2 Main result

The control objective is to stabilize the switched linear control system defined in Section 5.1.1 under the data-rate constraint described in Section 5.1.2 in a robust sense. More precisely, we intend to establish the following ISS-like property.

Theorem 5.1 (Exponential decay)**.** *Consider the switched linear control system* (5.1)*. Suppose that Assumptions 5.1–5.3 and the inequality* (5.4) *hold. Then there is a communication and control strategy that yields the following property: Provided that the average dwell-time τ_a is large enough, there exist a constant $\lambda > 0$ and gain functions $g, h : \mathbb{R}_+ \to \mathbb{R}_{>0}$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_+ \to \mathbb{R}^{n_d}$,*

$$\|x(t)\| \le e^{-\lambda t} g(\|x_0\|) + h(\|d\|_\infty) \qquad \forall\, t \ge 0. \tag{5.6}$$

The communication and control strategy is described in Section 5.3. The lower bound on τ_a is given by (5.49) in Section 5.5.1. The *exponential decay rate* λ is given by (5.64), and the nonlinear gain functions g and h are given by (5.65), both in Section 5.5.3. From the proof, it will be clear that both g and h can be made continuous and strictly increasing. However, $g(0) > 0$ due to the sampling and quantization, $h(0) > 0$ due to the unknown disturbance, and both $g(s)$ and $h(s)$ have superlinear growth rates as $s \to \infty$, which is consistent with [101, Corollary 2.3]. Consequently, the state bound (5.6) does not give the standard notion of input-to-state stability (ISS) [18], but rather the input-to-state practical stability (ISpS) [31] with exponential decay, that is,

$$\|x(t)\| \le e^{-\lambda t}\gamma_x(\|x_0\|) + \gamma_d(\|d\|_\infty) + C \qquad \forall\, t \ge 0$$

with the gain functions $\gamma_x, \gamma_d \in \mathcal{K}_\infty$ defined by

$$\gamma_x(s) := g(s) - g(0), \quad \gamma_d(s) := h(s) - h(0) \tag{5.7}$$

and the constant

$$C := g(0) + h(0) > 0. \tag{5.8}$$

Remark 5.1. Following essentially the analysis from [88, Section VI], the state bound (5.6) can be restated as ISS with respect to a set. More specifically, (5.6) implies that the *uniform asymptotic gain* (UAG) property [88] holds for the set $\mathcal{A} := \{v \in \mathbb{R}^{n_x} : \|v\| \leq h(0)\}$, that is, for each pair $\varepsilon, \delta > 0$, there exists a time $T_{\varepsilon,\delta} := \max\{\ln(g(h(0) + \delta)/\varepsilon)/\lambda,\, 0\}$ such that

$$\begin{aligned} &\|x_0\|_{\mathcal{A}} \leq \delta \\ &\implies \|x(t)\|_{\mathcal{A}} \leq \gamma_d(\|d\|_\infty) + \varepsilon \qquad \forall\, t \geq T_{\varepsilon,\delta} \end{aligned}$$

with the gain function $\gamma_d \in \mathcal{K}_\infty$ defined in (5.7), where $\|v\|_{\mathcal{A}} := \inf_{v' \in \mathcal{A}} \|v - v'\|$ is the (Chebyshev) distance from a point v to the set $\mathcal{A}$. In the context of non-switched systems, it has been shown that if UAG holds for $\mathcal{A}$, then the system is ISS with respect to the closure of the reachable set from $\mathcal{A}$ with $d \equiv 0$ [88, Lemma VI.2].

The state bound (5.6) also implies the following stability property.

Corollary 5.2 (Practical stability)**.** *Consider the switched linear control system* (5.1)*. Suppose that Assumptions 5.1–5.3 and the inequality* (5.4) *hold. Then there is a communication and control strategy that yields the following property: Provided that the average dwell-time τ_a is large enough, for each $\varepsilon > 0$, there exists a small enough $\delta > 0$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_+ \to \mathbb{R}^{n_d}$,*

$$\begin{aligned} &\|x_0\|, \|d\|_\infty \leq \delta \\ &\implies \|x(t)\| \leq \varepsilon + C \qquad \forall\, t \geq 0 \end{aligned} \tag{5.9}$$

with the constant C defined by (5.8)*.*

Corollary 5.2 means that, if the initial state and the disturbance are both small, then the solution is confined within a neighborhood of the hypercube of radius C centered at the origin. In Section 5.5.4, we establish practical stability with a smaller constant C through a more direct approach.

5.3 Communication and control strategy

In this section we describe the communication and control strategy in detail, assuming that suitable approximations of reachable sets of the state are available at all sampling times. (Such approximations are derived in the next section.)

The initial state x_0 is unknown. At $t_0 = 0$, the sensor and the controller are both provided with $x_0^* = 0$ and arbitrarily selected initial estimates $E_0 > 0$ and $\delta_0 > 0$ (for $\|x_0\|$ and the disturbance bound δ_d defined in (5.2), respectively). Starting from $t_0 = 0$, at each sampling time t_k, the sensor determines if the state $x(t_k)$ is inside the hypercube of radius E_k centered at x_k^* denoted by

$$\mathcal{S}_k := \{v \in \mathbb{R}^{n_x} : \|v - x_k^*\| \leq E_k\},$$

or equivalently, if

$$\|x(t_k) - x_k^*\| \leq E_k. \tag{5.10}$$

The hypercube $\mathcal{S}_k$ is the approximation of the reachable set at t_k, which is also used as the range of quantization. If (5.10) holds (i.e., if $x(t_k) \in \mathcal{S}_k$), we say the state is *visible*, and the system is in a *stabilizing stage* described in Section 5.3.1. Otherwise the state is *lost*, and the system is in a *searching stage* described in Section 5.3.2.

Due to the unknown disturbance, we introduce an estimate δ_k of the disturbance bound δ_d in calculating E_{k+1}. Note that if $\delta_k < \delta_d$, then it is possible that $x(t_k) \in \mathcal{S}_k$ but $x(t_{k+1}) \notin \mathcal{S}_{k+1}$ (unlike in the disturbance-free case, where $x(t_k) \in \mathcal{S}_k$ implies that $x(t_l) \in \mathcal{S}_l$ for all $l \geq k$).

If the state is visible at t_k, then the system is in a stabilizing stage until the first sampling time $t_j > t_k$ such that $x(t_j) \notin \mathcal{S}_j$; in this case, we say that the state *escapes* at t_j. Likewise, if the state is lost at t_k, then the system is in a searching stage until the first sampling time $t_i > t_k$ such that $x(t_i) \in \mathcal{S}_i$; in this case, we say that the state *is recovered* at t_i. Due to the unknown disturbance, the system may alternate multiple times between stabilizing and searching stages. The rule for adjusting the estimate δ_k so that there are only a finite number of escapes are described in Section 5.3.3.

The communication and control strategy is summarized in Algorithm 1 at the end of this section.

5.3.1 Stabilizing stage

At each sampling time t_k in a stabilizing stage, the encoder divides the hypercube $\mathcal{S}_k$ into N^{n_x} equal hypercubic boxes, N per dimension, encodes each box by a unique integer index from 1 to N^{n_x}, and transmits the index i_k of the box containing $x(t_k)$ to the decoder, along with the active mode $\sigma(t_k)$. The controller learns that (5.10) holds upon receiving $i_k \in \{1, \ldots, N^{n_x}\}$. The decoder follows the same pre-defined indexing protocol as the encoder, so that it is able to reconstruct the center c_k of the hypercubic box containing $x(t_k)$ from i_k. Simple calculation shows that

$$\|x(t_k) - c_k\| \leq \frac{1}{N} E_k, \quad \|c_k - x_k^*\| \leq \frac{N-1}{N} E_k. \tag{5.11}$$

The controller then generates the control input $u(t) = K_{\sigma(t_k)} \hat{x}(t)$ for $t \in [t_k, t_{k+1})$, where $K_{\sigma(t_k)}$ is the state feedback gain matrix in Assumption 5.2, and $\hat{x}$ is the state of the auxiliary system

$$\dot{\hat{x}} = A_{\sigma(t_k)}\, \hat{x} + B_{\sigma(t_k)}\, u = (A_{\sigma(t_k)} + B_{\sigma(t_k)} K_{\sigma(t_k)})\, \hat{x} \tag{5.12}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{5.13}$$

In particular, the auxiliary state $\hat{x}$ is reset to c_k at each sampling time t_k in a stabilizing stage. Both the sensor and the controller then use two functions F and G to calculate

$$\begin{aligned} x_{k+1}^* &:= F(\sigma(t_k), \sigma(t_{k+1}), c_k), \\ E_{k+1} &:= G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_k) \end{aligned} \tag{5.14}$$

for the next sampling time t_{k+1} without further communication. The functions F and G are designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_d), \tag{5.15}$$

and G is strictly increasing in the last argument, which is δ_k in (5.14) and δ_d in (5.15). Hence the state may escape at t_{k+1} only if $\delta_k < \delta_d$. (However, $x(t_{k+1}) \in S_{k+1}$ does not imply that $\delta_k \geq \delta_d$.) The formulas for F and G are derived in Section 5.4.1.

5.3.2 Searching stage

At each sampling time t_k in a searching stage, there is an unknown $\hat{D}_k$ such that

$$E_k < \|x(t_k) - x_k^*\| \leq \hat{D}_k. \tag{5.16}$$

For example, if the state escapes at t_j, then (5.15) implies that

$$\hat{D}_j = G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d);$$

while if it is lost at $t_0 = 0$, then $\hat{D}_0 = \|x_0\|$. The encoder sends $i_k = 0$, the "overflow symbol", to the decoder. Upon receiving $i_k = 0$, the controller learns the state is lost, and sets the control input to be $u \equiv 0$ on $[t_k, t_{k+1})$. Both the sensor and the controller then use a function $\hat{G}$ to calculate

$$\begin{aligned} x_{k+1}^* &:= x_k^*, \\ E_{k+1} &:= \hat{G}(x_k^*, (1+\varepsilon_E)E_k, \delta_k) \end{aligned} \tag{5.17}$$

for the next sampling time t_{k+1} without further communication, where $\varepsilon_E > 0$ is an arbitrary design parameter. The function $\hat{G}$ is designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq \hat{G}(x_k^*, \hat{D}_k, \delta_d), \tag{5.18}$$

and it is strictly increasing in the last two arguments. Note that the second argument of $\hat{G}$ in (5.18) is $\hat{D}_k$, whereas the one in (5.17) is $(1+\varepsilon_E)E_k$. With the additional coefficient $1+\varepsilon_E$, it is ensured that the growth rate of E_k dominates that of $\hat{D}_k$; thus the state will be recovered in a finite time, as shown in Section 5.4.2 following the derivation of $\hat{G}$.

5.3.3 Adjusting the estimate of the disturbance bound

When the state escapes at a sampling time t_j, the sensor and the controller learn that $\delta_{j-1} < \delta_d$, and adjust the estimate by enlarging it to $\delta_j = (1+\varepsilon_\delta)\delta_{j-1}$, where $\varepsilon_\delta > 0$ is an arbitrary design parameter. The estimate remains unchanged in all other cases; in particular, it is adjusted only once per searching stage. Thus it is ensured that there is a finite number of searching stages in total, as the estimate becomes greater than or equal to the disturbance bound δ_d after finitely many adjustments, and the state cannot escape after that.

5.3.4 Algorithm

The communication and control strategy is summarized in Algorithm 1.

Algorithm 1 Communication and control strategy

Input: $x^*_{k-1}, E_{k-1}, \delta_{k-1}, i_{k-1}, \sigma(t_{k-1}), \sigma(t_k), x(t_k)$
Output: $x^*_k, E_k, \delta_k, i_k$

```
c_{k-1} ← decode(i_{k-1}, x*_{k-1}, E_{k-1})     // decode
if i_{k-1} ≠ 0 then     // stabilizing on [t_{k-1}, t_k)
    x*_k ← F(σ(t_{k-1}), σ(t_k), c_{k-1})
    E_k ← G(σ(t_{k-1}), σ(t_k), x*_{k-1}, E_{k-1}, δ_{k-1})
    if ‖x(t_k) − x*_k‖ ≤ E_k then     // stabilizing on [t_k, t_{k+1})
        δ_k ← δ_{k-1}
        i_k ← encode(x(t_k), x*_k, E_k)     // encode
    else     // escape at t_k
        δ_k ← (1 + ε_δ)δ_{k-1}
        i_k ← 0
    end if
else     // searching on [t_{k-1}, t_k)
    x*_k ← x*_{k-1}
    E_k ← Ĝ(x*_{k-1}, (1 + ε_E)E_{k-1}, δ_{k-1})
    δ_k ← δ_{k-1}
    if ‖x(t_k) − x*_k‖ ≤ E_k then     // recover at t_k
        i_k ← encode(x(t_k), x*_k, E_k)     // encode
    else     // searching on [t_k, t_{k+1})
        i_k ← 0
    end if
end if
```

5.4 Approximation of reachable sets

In this section we derive the recursive formulas needed to implement the communication and control strategy. In Section 5.4.1, we consider a stabilizing stage, and formulate the functions F and G in (5.14) so that (5.15) holds. In Section 5.4.2, we consider a searching stage, formulate the function $\hat{G}$ in (5.17) so that (5.18) holds, and prove that the state is ensured to be recovered in a finite time.

5.4.1 Stabilizing stage

Suppose that the state is visible at a sampling time t_k, that is, (5.10) holds.

Sampling interval with no switch

When

$$\sigma(t_k) = p = \sigma(t_{k+1}) \tag{5.19}$$

with some $p \in \mathcal{P}$, there is no switch on $(t_k, t_{k+1}]$ due to (5.4). Combining the switched linear system (5.1) and the auxiliary system (5.12), we obtain that

$$\begin{aligned} \dot{x} &= A_p x + B_p u + D_p d, \\ \dot{\hat{x}} &= A_p \hat{x} + B_p u. \end{aligned}$$

The error $e := x - \hat{x}$ satisfies taht

$$\dot{e} = A_p e + D_p d, \qquad \|e(t_k)\| = \|x(t_k) - c_k\| \leq \frac{1}{N} E_k$$

on $[t_k, t_{k+1})$, where the boundary condition follows from (5.11) and (5.13). Hence

$$\begin{aligned} \|e(t_{k+1}^-)\| &= \left\| e^{A_p \tau_s} e(t_k) + \int_{t_k}^{t_{k+1}} e^{A_p(t_{k+1}-\tau)} D_p d(\tau) \mathrm{d}\tau \right\| \\ &\leq \|e^{A_p \tau_s}\| \|e(t_k)\| + \left(\int_0^{\tau_s} \|e^{A_p s} D_p\| \mathrm{d}s \right) \delta_d \\ &\leq \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s) \delta_d =: \hat{D}_{k+1} \end{aligned}$$

with the constant Λ_p in (5.5) and the increasing function $\Phi_p : [0, \tau_s] \to \mathbb{R}$ defined by

$$\Phi_p(t) := \int_0^t \|e^{A_p s} D_p\| \mathrm{d}s. \tag{5.20}$$

Therefore, we set

$$E_{k+1} = G(p, p, x_k^*, E_k, \delta_k) := \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s)\delta_k. \tag{5.21}$$

As x is continuous, (5.15) holds with x_{k+1}^* set as the auxiliary state $\hat{x}$ approaching t_{k+1}, that is,

$$x_{k+1}^* = F(p, p, c_k) := \hat{x}(t_{k+1}^-) = S_p c_k \tag{5.22}$$

with the matrix $S_p := e^{(A_p + B_p K_p)\tau_s}$.

Sampling interval with a switch

When

$$\sigma(t_k) = p \neq q = \sigma(t_{k+1}) \tag{5.23}$$

with some $p, q \in \mathcal{P}$, there is exactly one switch on $(t_k, t_{k+1}]$ due to (5.4). Let $t_k + \bar{t}$ with $\bar{t} \in (0, \tau_s]$ denote the unknown switching time. Then

$$\sigma(t) = \begin{cases} p, & t \in [t_k, t_k + \bar{t}), \\ q, & t \in [t_k + \bar{t}, t_{k+1}]. \end{cases}$$

Before the switch, mode p is active on $[t_k, t_k + \bar{t})$. Following essentially the calculations from the case with no switch, the error $e = x - \hat{x}$ satisfies that

$$\|e(t_k + \bar{t})\| \leq \frac{\|e^{A_p \bar{t}}\|}{N} E_k + \Phi_p(\bar{t})\delta_d$$

with the function Φ_p defined by (5.20). As the switching time $t_k + \bar{t}$ is unknown, we replace $x(t_k + \bar{t})$ with $\hat{x}(t_k + t') = e^{(A_p + B_p K_p)t'} c_k$ of the auxiliary system (5.12) at an arbitrarily selected time $t_k + t' \in [t_k, t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned}
&\|\hat{x}(t_k+\bar{t}) - \hat{x}(t_k+t')\| \\
&\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\|c_k\| \\
&\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right),
\end{aligned}$$

where the last inequality follows partially from (5.11). Then

$$\begin{aligned}
&\|x(t_k+\bar{t}) - \hat{x}(t_k+t')\| \\
&\leq \|\hat{x}(t_k+\bar{t}) - \hat{x}(t_k+t')\| + \|e(t_k+\bar{t})\| \\
&\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \frac{\|e^{A_p\bar{t}}\|}{N}E_k + \Phi_p(\bar{t})\delta_d \\
&=: \hat{D}'_{k+1}(t', \bar{t}, \delta_d).
\end{aligned} \tag{5.24}$$

After the switch, mode q is active on $[t_k+\bar{t}, t_{k+1}]$. Combining the switched linear system (5.1) and the auxiliary system (5.12) with $u = K_p\hat{x}$, we obtain that

$$\dot{z} = \bar{A}_{pq}z + \bar{D}_q d$$

for $z := (x, \hat{x}) \in \mathbb{R}^{2n_x}$ with the matrices

$$\bar{A}_{pq} := \begin{bmatrix} A_q & B_qK_p \\ 0_{n_x} & A_p + B_pK_p \end{bmatrix}, \quad \bar{D}_q = \begin{bmatrix} D_q \\ 0_{n_x,n_d} \end{bmatrix}.$$

Combining it with a second auxiliary system

$$\dot{\hat{z}} = \bar{A}_{pq}\hat{z}, \quad \hat{z}(t_k+t') = (\hat{x}(t_k+t'), \hat{x}(t_k+t')), \tag{5.25}$$

we obtain that

$$\begin{aligned}
\dot{z} &= \bar{A}_{pq}z + \bar{D}_q d, \\
\dot{\hat{z}} &= \bar{A}_{pq}\hat{z}
\end{aligned}$$

with the boundary condition

$$\begin{aligned}
&\|z(t_k+\bar{t}) - \hat{z}(t_k+t')\| \\
&= \max\{\|x(t_k+\bar{t}) - \hat{x}(t_k+t')\|, \|\hat{x}(t_k+\bar{t}) - \hat{x}(t_k+t')\|\} \\
&\leq \hat{D}'_{k+1}(t', \bar{t}, \delta_d),
\end{aligned}$$

where the first inequality follows from the property that the ∞-norms of two vectors v, w and their concatenation (v, w) satisfy

$$\|(v,w)\| = \max\{\|v\|,\ \|w\|\}. \tag{5.26}$$

Hence

$$\begin{aligned}
&\|z(t_{k+1}^-) - \hat{z}(t_{k+1} - \bar{t} + t')\| \\
&= \left\| e^{\bar{A}_{pq}(\tau_s - \bar{t})} z(t_k + \bar{t}) + \int_{t_k+\bar{t}}^{t_{k+1}} e^{\bar{A}_{pq}(t_{k+1}-\tau)} \bar{D}_q d(\tau) \mathrm{d}\tau - e^{\bar{A}_{pq}(\tau_s - \bar{t})} \hat{z}(t_k + t') \right\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \|z(t_k + \bar{t}) - \hat{z}(t_k + t')\| + \left(\int_0^{\tau_s - \bar{t}} \|e^{\bar{A}_{pq}s} \bar{D}_q\| \mathrm{d}s \right) \delta_d \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \hat{D}'_{k+1}(t', \bar{t}, \delta_d) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta_d
\end{aligned}$$

with the increasing function $\bar{\Phi}_{pq} : [0, \tau_s] \to \mathbb{R}$ defined by

$$\bar{\Phi}_{pq}(t) := \int_0^t \|e^{\bar{A}_{pq}s} \bar{D}_q\| \mathrm{d}s.$$

Again, we replace $z(t_{k+1}^-)$ with $\hat{z}(t_k + t'') = e^{\bar{A}_{pq}(t''-t')} \hat{z}(t_k + t')$ of the second auxiliary system (5.25) at an arbitrarily selected time $t_k + t'' \in [t_k, t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned}
&\|\hat{z}(t_{k+1} - \bar{t} + t') - \hat{z}(t_k + t'')\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\| \|\hat{z}(t_k + t')\| \\
&= \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\| \|\hat{x}(t_k + t')\| \\
&\leq \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t''-t')}\| \|e^{(A_p + B_p K_p)t'}\| \left(\|x_k^*\| + \frac{N-1}{N} E_k \right),
\end{aligned}$$

where the equality follows from (5.26), and the last inequality follows partially from (5.11). Then

$$
\begin{aligned}
&\|z(t_{k+1}^-) - \hat{z}(t_k + t'')\| \\
&\le \|z(t_{k+1}^-) - \hat{z}(t_{k+1} - \bar{t} + t')\| + \|\hat{z}(t_{k+1} - \bar{t} + t') - \hat{z}(t_k + t'')\| \\
&\le \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\|\hat{D}'_{k+1}(t', \bar{t}, \delta_d) + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t'' - t')}\| \\
&\quad \times \|e^{(A_p + B_p K_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \bar{\Phi}_{pq}(\tau_s - \bar{t})\delta_d \\
&=: \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d).
\end{aligned}
\tag{5.27}
$$

To remove the dependence on the unknown $\bar{t}$, we take the supremum over $\bar{t}$ (with fixed t' and t'') and obtain that

$$
\|z(t_{k+1}^-) - \hat{z}(t_k + t'')\| \le \sup_{\bar{t} \in (0, \tau_s]} \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d) =: \hat{D}_{k+1}.
$$

Therefore, we set E_{k+1} by first replacing the disturbance bound δ_d in the formula of $\hat{D}''_{k+1}(t', t'', \bar{t}, \delta_d)$ with the estimate δ_k, and then taking the maximum over $\bar{t}$ (with the same fixed t' and t''), that is,

$$
\begin{aligned}
E_{k+1} &= G(p, q, x_k^*, E_k, \delta_k) \\
&:= \sup_{\bar{t} \in (0, \tau_s]} \hat{D}''_{k+1}(t', t'', \bar{t}, \delta_k) \\
&= \sup_{\bar{t} \in (0, \tau_s]} \Bigg\{ \Big(\|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\| \\
&\quad + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t'' - t')}\| \|e^{(A_p + B_p K_p)t'}\| \Big) \|x_k^*\| \\
&\quad + \Bigg(\frac{N-1}{N} \Big(\|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \|e^{(A_p + B_p K_p)\bar{t}} - e^{(A_p + B_p K_p)t'}\| \\
&\quad + \|e^{\bar{A}_{pq}(\tau_s - \bar{t})} - e^{\bar{A}_{pq}(t'' - t')}\| \|e^{(A_p + B_p K_p)t'}\| \Big) \\
&\quad + \frac{1}{N} \|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \|e^{A_p \bar{t}}\| \Bigg) E_k \\
&\quad + \Big(\|e^{\bar{A}_{pq}(\tau_s - \bar{t})}\| \Phi_p(\bar{t}) + \bar{\Phi}_{pq}(\tau_s - \bar{t}) \Big) \delta_k \Bigg\}.
\end{aligned}
\tag{5.28}
$$

(Clearly, the design parameters t' and t'' should be selected so that E_{k+1} is minimized. However, their optimal values cannot be determined without imposing further constraints on the matrices $\{A_p, B_p, D_p, K_p : p \in \mathcal{P}\}$.) As x is continuous, (5.15) holds with x_{k+1}^* set as the projection of the second auxiliary state $\hat{z}$ approaching $t_k + t''$ onto the x-component, that is,

$$x_{k+1}^* = F(p, q, c_k) := (I_{n_x} \ 0_{n_x})\, \hat{z}(t_k + t'') = H_{pq} c_k \tag{5.29}$$

with the matrix

$$H_{pq} := (I_{n_x} \ 0_{n_x}) e^{\bar{A}_{pq}(t''-t')} \begin{pmatrix} I_{n_x} \\ I_{n_x} \end{pmatrix} e^{(A_p + B_p K_p)t'}.$$

In the remainder of this subsection, we derive a simpler but more conservative bound of E_{k+1}, which is more useful for computations. First, the norm of the difference of two matrix exponentials can be simplified via the following result.[30]

Lemma 5.1. *For all square matrices X and Y,*

$$\|e^{X+Y} - e^X\| \leq e^{\|X\|+\|Y\|}\|Y\|.$$

Proof. See Appendix A.4. □

Based on Lemma 5.1 and the property that

$$\|e^{Ms}\| \leq e^{\|M\||s|} \qquad \forall\, M \in \mathbb{R}^{n \times n}, \forall\, s \in \mathbb{R},$$

from (5.28) it follows that

$$E_{k+1} \leq \alpha_{pq}\|x_k^*\| + \beta_{pq} E_k + \gamma_{pq}\delta_k \tag{5.30}$$

with the constants

$$\begin{aligned} \alpha_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s} e^{\|A_p + B_p K_p\| \max\{\tau_s, 2t'\}} \|A_p + B_p K_p\| \max\{\tau_s - t', t'\} \\ &\quad + e^{\|\bar{A}_{pq}\| \max\{\tau_s, 2(t''-t'), \tau_s + 2(t'-t'')\}} \|\bar{A}_{pq}\| \\ &\quad \times \max\{t'' - t', \tau_s + t' - t''\} \|e^{(A_p + B_p K_p)t'}\|, \\ \beta_{pq} &:= \frac{N-1}{N}\alpha_{pq} + \frac{1}{N} e^{(\|\bar{A}_{pq}\| + \|A_p\|)\tau_s}, \\ \gamma_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s} \Phi_p(\tau_s) + \bar{\Phi}_{pq}(\tau_s). \end{aligned} \tag{5.31}$$

[30]Using Lemma 5.1 instead of the inequality that $\|M - I\| \leq \|M\| + 1$ for all square matrices M as in [68, eq. (20)] ensures that $\alpha_{pq} \to 0$ as $\tau_s \to 0$, a property we will use in the comparison to [16] in Remark 5.4. However, for a large enough τ_s, it is possible that the bound in Lemma 5.1 is worse.

Remark 5.2. If we set $t'' = t' = 0$, then (5.30) becomes

$$E_{k+1} \leq \alpha^0_{pq}\|x^*_k\| + \beta^0_{pq}E_k + \gamma_{pq}\delta_k$$

with the constants

$$\begin{aligned}
\alpha^0_{pq} &:= e^{\|\bar{A}_{pq}\|\tau_s} e^{\|A_p+B_pK_p\|\tau_s}\|A_p + B_pK_p\|\tau_s + e^{\|\bar{A}_{pq}\|\tau_s}\|\bar{A}_{pq}\|\tau_s, \\
\beta^0_{pq} &:= \frac{N-1}{N}\alpha^0_{pq} + \frac{1}{N}e^{(\|\bar{A}_{pq}\|+\|A_p\|)\tau_s}.
\end{aligned} \tag{5.32}$$

Although this choice of t' and t'' considerably simplifies the formula of the bound, it does not necessarily minimize E_{k+1}.

5.4.2 Searching stage

Suppose that the state is lost at a sampling time t_k, that is, (5.16) holds.

Reachable-set approximation

Let $p = \sigma(t_k)$, and consider an arbitrary $t \in (t_k, t_{k+1}]$. If $\sigma(t) = p$, then there is no switch on $(t_k, t]$ due to (5.4); thus

$$\begin{aligned}
&\|x(t) - x^*_k\| \\
&= \left\| e^{A_p(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A_p(t-\tau)}D_p d(\tau)\mathrm{d}\tau - x^*_k \right\| \\
&\leq \|e^{A_p(t-t_k)} - I\|\|x^*_k\| + \|e^{A_p(t-t_k)}\|\|x(t_k) - x^*_k\| + \left(\int_0^{t-t_k} \|e^{A_ps}D_p\|\mathrm{d}s\right)\delta_d \\
&\leq \bar{\Gamma}\|x^*_k\| + \bar{\Lambda}\hat{D}_k + \bar{\Phi}\delta_d
\end{aligned}$$

with the constants

$$\begin{aligned}
\bar{\Gamma} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt} - I\|, \\
\bar{\Lambda} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt}\| \geq 1, \\
\bar{\Phi} &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \Phi_p(t) = \max_{p\in\mathcal{P}} \Phi_p(\tau_s).
\end{aligned} \tag{5.33}$$

If $\sigma(t) = q \neq p$, then there is exactly one switch on $(t_k, t]$ due to (5.4); thus

$$\begin{aligned}
&\|x(t) - x_k^*\| \\
&= \left\| e^{A_q(t-t_k-\bar{t})} x(t_k + \bar{t}) + \int_{t_k+\bar{t}}^{t} e^{A_q(t-\tau)} D_q d(\tau) \mathrm{d}\tau - x_k^* \right\| \\
&\leq \|e^{A_q(t-t_k-\bar{t})} - I\| \|x_k^*\| + \|e^{A_q(t-t_k-\bar{t})}\| \|x(t_k+\bar{t}) - x_k^*\| \\
&\quad + \left(\int_0^{t-t_k-\bar{t}} \|e^{A_q s} D_q\| \mathrm{d}s \right) \delta_d \\
&\leq \bar{\Gamma} \|x_k^*\| + \bar{\Lambda} \|x(t_k+\bar{t}) - x_k^*\| + \bar{\Phi}\delta_d \\
&\leq \bar{\Gamma} \|x_k^*\| + \bar{\Lambda}(\bar{\Gamma}\|x_k^*\| + \bar{\Lambda}\hat{D}_k + \bar{\Phi}\delta_d) + \bar{\Phi}\delta_d \\
&\leq (\bar{\Lambda}+1)\bar{\Gamma}\|x_k^*\| + \bar{\Lambda}^2 \hat{D}_k + (\bar{\Lambda}+1)\bar{\Phi}\delta_d,
\end{aligned}$$

where $t_k + \bar{t}$ denotes the unknown switching time. As $\bar{\Lambda} \geq 1$, the bound for the second case holds for both cases, that is,

$$\|x(t) - x_k^*\| \leq \bar{\alpha}\|x_k^*\| + \bar{\beta}\hat{D}_k + \bar{\gamma}\delta_d =: \hat{D}_{k+1} \qquad \forall\, t \in (t_k, t_{k+1}] \tag{5.34}$$

with the constants

$$\bar{\alpha} := (\bar{\Lambda}+1)\bar{\Gamma}, \quad \bar{\beta} := \bar{\Lambda}^2, \quad \bar{\gamma} := (\bar{\Lambda}+1)\bar{\Phi}.$$

From $\bar{\beta} = \bar{\Lambda}^2 \geq 1$, it follows that $\hat{D}_{k+1} \geq \hat{D}_k$. In order to dominate the growth rate of $\hat{D}_{k+1}$, we set

$$E_{k+1} = \hat{G}(x_k^*, (1+\varepsilon_E)E_k, \delta_k) := \bar{\alpha}\|x_k^*\| + (1+\varepsilon_E)\bar{\beta}E_k + \bar{\gamma}\delta_k \tag{5.35}$$

with the arbitrary design parameter $\varepsilon_E > 0$.

Recovery in a finite time

Suppose that the state escapes at a sampling time t_j (or it is lost at $t_j = t_0 = 0$), and remains lost at $t_{j+1}, \dots, t_{k-1}$. Then the disturbance estimate satisfies that $\delta_{k-1} = \cdots = \delta_{j+1} = \delta_j = (1+\varepsilon_\delta)\delta_{j-1}$. From the recursive formulas (5.34) and (5.35), it follows that

$$\begin{aligned}\hat{D}_k &= \bar{\beta}^{k-j}\hat{D}_j + \frac{\bar{\beta}^{k-j}-1}{\bar{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_d),\\ E_k &= \hat{\beta}^{k-j}E_j + \frac{\hat{\beta}^{k-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)\end{aligned} \tag{5.36}$$

with the constant[31]

$$\hat{\beta} := (1+\varepsilon_E)\bar{\beta} > \bar{\beta}.$$

Let $c_\beta := (\hat{\beta}-1)/(\bar{\beta}-1)$, and consider the integer-valued functions $\eta_E, \eta_\delta : \mathbb{R}_+ \to \mathbb{N}$ defined by

$$\begin{aligned}\eta_E(s) &:= \begin{cases}\lceil \log_{1+\varepsilon_E} s\rceil, & s > 1;\\ 0, & 0 \le s \le 1,\end{cases}\\ \eta_\delta(s) &:= \begin{cases}\lceil \log_{1+\varepsilon_E}(c_\beta s)\rceil, & s > 1;\\ 0, & 0 \le s \le 1,\end{cases}\end{aligned} \tag{5.37}$$

where $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function, that is, $\lceil s\rceil := \min\{m \in \mathbb{Z} : m \ge s\}$. Consider the integer

$$k' := j + \max\{\eta_E(\hat{D}_j/E_j),\, \eta_\delta(\delta_d/\delta_j)\}.$$

First, it holds that

$$\hat{\beta}^{k'-j}E_j \ge \bar{\beta}^{k'-j}(1+\varepsilon_E)^{\eta_E(\hat{D}_j/E_j)}E_j \ge \bar{\beta}^{k'-j}\hat{D}_j.$$

Second, if $\delta_d \le \delta_j$, then

$$\frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j \ge \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d$$

[31]From (5.33), it follows that $\bar{\beta} = \bar{\Lambda}^2 \ge 1$, and $\bar{\Lambda} = 1$ only if all eigenvalues of all A_p have nonpositive real parts. In the following analysis, we assume that $\bar{\beta} > 1$ (so that the first formula in (5.36) is well-defined), which can be achieved by letting $\bar{\beta} = \max\{\bar{\Lambda}^2,\, 1+\varepsilon\}$ for an arbitrary $\varepsilon > 0$ if necessary. The special case where $\bar{\beta} = 1$ can be treated using similar arguments, and is omitted here.

due to $\hat{\beta} > \bar{\beta}$ and $k' \geq j$; otherwise

$$\begin{aligned}
\frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j &= \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}\frac{(1+\varepsilon_E)^{k'-j}\bar{\beta}^{k'-j}-1}{\bar{\beta}^{k'-j}-1}\delta_j \\
&> \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}(1+\varepsilon_E)^{k'-j}\delta_j \\
&\geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}(1+\varepsilon_E)^{\eta_\delta(\delta_d/\delta_j)}\delta_j \\
&\geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d.
\end{aligned}$$

Hence $E_{k'} \geq \hat{D}_{k'}$, that is, the state is recovered no later than $t_{k'}$. Denote by t_i the sampling time of recovery. Then[32]

$$i - j \leq \max\{\eta_E(\hat{D}_j/E_j),\, \eta_\delta(\delta_d/\delta_j)\}. \tag{5.38}$$

However, δ_d being unknown implies that neither the sensor nor the controller is able to predict how long it will take to recover the state.

5.5 Stability analysis

In this section, we show that the communication and control strategy described in Section 5.3 fulfills the claim of Theorem 5.1. In Section 5.5.1, we formulate a Lyapunov-based bound with exponential decay in stabilizing stages. Then we derive its exponential growth in searching stages in Section 5.5.2. In Section 5.5.3, we calculate the maximum number of searching stages, and prove the variant of ISS with exponential decay in Theorem 5.1. A stronger version of Corollary 5.2 is established in Section 5.5.4.

[32] The function η_δ is piecewise-defined since if $\delta_d \leq \delta_j$ in (5.38)—which is possible as the escape only implies $\delta_d > \delta_{j-1} = \delta_j/(1+\varepsilon_\delta)$—then the second term on the right-hand side of the second formula in (5.36) is larger than or equal to that of the first formula for all $k \geq j$. Similarly, the function η_E is piecewise-defined since if $\hat{D}_0 = \|x_0\| \leq E_0$ in (5.57) below, then there is no searching stage at the beginning.

5.5.1 Stabilizing stage

Sampling interval with no switch

Consider a sampling interval $[t_k, t_{k+1}]$ such that (5.19) holds, as in Section 5.4.1. As $A_p + B_pK_p$ is Hurwitz, there exist positive definite matrices $P_p, Q_p \in \mathbb{R}^{n_x \times n_x}$ such that

$$S_p^\top P_p S_p - P_p = -Q_p < 0 \tag{5.39}$$

with the matrix S_p in (5.22). Let $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$ denote the largest and smallest eigenvalues of a matrix M, respectively, and define the constant

$$\chi_p := \frac{2n_x^2 \|S_p^\top P_p S_p\|^2}{\underline{\lambda}(Q_p)} + n_x \|S_p^\top P_p S_p\|. \tag{5.40}$$

Due to the inequality in (5.5), there exists a small enough constant $\phi_1 > 0$ such that $(1+\phi_1)\Lambda_{p'}^2 < N^2$ for all $p' \in \mathcal{P}$. Then for each p', there exists a large enough constant $\rho_{p'} > 0$ such that

$$\frac{(N-1)^2}{N^2}\frac{\chi_{p'}}{\rho_{p'}} + \frac{(1+\phi_1)\Lambda_{p'}^2}{N^2} < 1. \tag{5.41}$$

Consider a family of positive definite functions $V_{p'} : \mathbb{R}^{n_x} \times \mathbb{R}_+ \to \mathbb{R}_+$, $p' \in \mathcal{P}$ defined by

$$V_{p'}(x, E) := x^\top P_{p'} x + \rho_{p'} E^2. \tag{5.42}$$

For the sampling interval $[t_k, t_{k+1}]$ with no switch, the following lemma provides an upper bound of $V_{\sigma(t_{k+1})}(x_{k+1}^*, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x_k^*, E_k)$ and the disturbance estimate δ_k.

Lemma 5.2. *Consider a sampling interval* $[t_k, t_{k+1}]$ *such that* (5.10) *and* (5.19) *hold. Then*

$$V_p(x_{k+1}^*, E_{k+1}) \le \nu V_p(x_k^*, E_k) + \nu_d \delta_k^2 \tag{5.43}$$

with the constants[33]

$$
\begin{aligned}
\nu &:= \max_{p\in\mathcal{P}} \nu_p,\\
\nu_p &:= \max\left\{\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p} + \frac{(1+\phi_1)\Lambda_p^2}{N^2},\ 1-\frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right\}, \qquad (5.44)\\
\nu_d &:= \max_{p\in\mathcal{P}} \left(1+\frac{1}{\phi_1}\right)\rho_p\Phi_p(\tau_s)^2.
\end{aligned}
$$

Proof. See Appendix A.5. □

Sampling interval with a switch

Consider a sampling interval $[t_k, t_{k+1}]$ such that (5.23) holds, as in Section 5.4.1. Let h_{pq} be the largest singular value of the matrix H_{pq} in (5.29), that is,

$$h_{pq} := \sqrt{\overline{\lambda}(H_{pq}^\top H_{pq})}.$$

Consider the functions V_p and V_q defined by (5.42). For the sampling interval $[t_k, t_{k+1}]$ with a switch, the following lemma provides an upper bound of $V_{\sigma(t_{k+1})}(x^*_{k+1}, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x^*_k, E_k)$ and the disturbance estimate δ_k.

Lemma 5.3. *Consider a sampling interval* $[t_k, t_{k+1}]$ *such that* (5.10) *and* (5.23) *hold. Then*

$$V_q(x^*_{k+1}, E_{k+1}) \le \mu V_p(x^*_k, E_k) + \mu_d \delta_k^2 \qquad (5.45)$$

[33]The denominator in the second term of the maximum in the definition of ν_p in (5.44) is reduced to $1/n_x$ of the corresponding term in [68, eq. (34)]. This improvement is due to the more suitable inequalities (A.4) and (A.5) from linear algebra. The first numerator in the first term of the maximum in the definition of μ_{pq} in (5.46) is reduced to $1/n_x$ of the corresponding term in [68, eq. (37)] for the same reason.

with the constants

$$
\begin{aligned}
\mu &:= \max_{p,q\in\mathcal{P}} \mu_{pq}, \\
\mu_{pq} &:= \max\Bigg\{ \frac{2\overline{\lambda}(P_q)h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2\rho_q}{\underline{\lambda}(P_p)}, \\
&\qquad \frac{(N-1)^2}{N^2}\frac{2n_x\overline{\lambda}(P_q)h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2\rho_q}{\rho_p}\Bigg\}, \\
\mu_d &:= \max_{p,q\in\mathcal{P}} \left(1+\frac{2}{\phi_2}\right)\rho_q\gamma_{pq}^2,
\end{aligned}
\tag{5.46}
$$

where $\phi_2 > 0$ is an arbitrary design parameter.

Proof. See Appendix A.6. □

Remark 5.3. From the definition of ν in (5.44) and the inequality (5.41), it follows that

$$\nu \le \max_{p\in\mathcal{P}} \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right) < 1.$$

Meanwhile, if we set $t' = t'' = 0$ in (5.29), then $h_{pq} = 1$ for all $p, q \in \mathcal{P}$; thus from the definition of μ in (5.46), it follows that

$$\mu = \max_{p,q\in\mathcal{P}} \mu_{pq} > \max_{p,q\in\mathcal{P}} \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} \ge 1 > \nu.$$

While this may not hold for general $t', t'' \in [0, \tau_s]$, we are able to ensure

$$\mu > \nu \tag{5.47}$$

by letting $\mu = 1$ if all $\mu_{pq} < 1$. Meanwhile, the relation between μ_d and ν_d depends on the values of ϕ_1 and ϕ_2. As (5.45) holds for all $\phi_2 > 0$, given an arbitrary ϕ_1, a small enough ϕ_2 (e.g., $\phi_2 = \phi_1$) can be selected so that

$$\mu_d \ge \nu_d. \tag{5.48}$$

(Alternatively, we can simply replace μ_d with $\max\{\mu_d,\, \nu_d\}$ if necessary.) In the following analysis, we assume that the inequalities (5.47) and (5.48) hold. Consequently, the bound in (5.45) holds for all sampling intervals in stabilizing stages, regardless of whether there is a switch.

Combined bound at sampling times

Combining the bounds (5.43) and (5.45), we derive a lower bound on the average dwell-time τ_a in Assumption 5.1 that ensures a bound with exponential decay of $V_{\sigma(t_k)}(x_k^*, E_k)$ at sampling times t_k in a stabilizing stage.

Lemma 5.4. *Consider a sequence of consecutive sampling times $t_i, \ldots, t_{k-1}$ in a stabilizing stage. Provided that the average dwell-time τ_a satisfies*

$$\tau_a > \left(1 + \frac{\ln \mu}{\ln(1/\nu)}\right)\tau_s, \tag{5.49}$$

there exists a small enough constant $\phi_3 \in (0,1)$ such that

$$V_{\sigma(t_k)}(x_k^*, E_k) < \Theta^{N_0}\left(\theta^{k-i} V_{\sigma(t_i)}(x_i^*, E_i) + \Theta_d \delta_i^2\right) \tag{5.50}$$

with the constants N_0 in Assumption 5.1 and

$$\begin{aligned} \theta &:= \frac{(\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{\tau_s/\tau_a}}{(\nu + \phi_3(1-\nu))^{\tau_s/\tau_a - 1}} < 1, \\ \Theta &:= \frac{\mu + \phi_3(1-\nu)\mu_d/\nu_d}{\nu + \phi_3(1-\nu)} > 1, \\ \Theta_d &:= \frac{\mu}{\phi_3(1-\nu)}\nu_d + \mu_d. \end{aligned} \tag{5.51}$$

Proof. See Appendix A.7. □

Remark 5.4. In [16], the authors considered switched linear systems with inputs (disturbances) and derived a lower bound on the average dwell-time that ensured a variant of ISS with exponential decay.[34] The lower bound (5.49) on the average dwell-time τ_a in Lemma 5.4, in the absence of sampling and quantization, is consistent with the one in [16, Theorem 2]. More specifically, the case without sampling and quantization can be approximated by letting $\tau_s \to 0$ and $N \to \infty$. Consequently, $S_p \to I + (A_p + B_p K_p)\tau_s$ in (5.22), $H_{pq} \to I$ in (5.29), and $\alpha_{pq}, \beta_{pq} \to 0$ in (5.31); thus

$$\nu \to 1 - \min_{p \in \mathcal{P}} \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}, \qquad \mu \to \max_{p,q \in \mathcal{P}} \frac{2\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)}$$

[34] More precisely, the result in [16] is stated in terms of "input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_\infty$-induced norm", which ensures an exponential decay rate.

in (5.44) and (5.46) with large enough ρ_p, $p \in \mathcal{P}$. Moreover, the first order approximation in τ_s of the Lyapunov equation (5.39) is given by

$$\big((A_p + B_pK_p)^\top P_p + P_p(A_p + B_pK_p)\big)\tau_s = -Q_p.$$

As the index set $\mathcal{P}$ is finite, Assumption 5.2 implies that there exists a constant $\lambda_0 > 0$ such that all $A_p + B_pK_p + \lambda_0 I$ are Hurwitz. Hence the (approximated) Lyapunov equation above holds with P_p satisfying

$$(A_p + B_pK_p + \lambda_0 I)^\top P_p + P_p(A_p + B_pK_p + \lambda_0 I) = -I,$$

and $Q_p = (2\lambda_0 P_p + I)\tau_s$. Then (5.49) can be approximated by

$$\tau_a > \frac{\ln(2\mu^*)}{\min\limits_{p\in\mathcal{P}} \left(\dfrac{\underline{\lambda}(P_p)}{2\overline{\lambda}(P_p)}2\lambda_0 + \dfrac{1}{2\overline{\lambda}(P_p)}\right)}$$

with

$$\mu^* := \max_{p,q\in\mathcal{P},\ p\neq q} \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)},$$

which is in a similar form as the lower bound

$$\tau_a \geq \tau_a^* > \frac{\ln \mu^*}{2\lambda_0}$$

in [16] (see [6, eq. (3.10)] for an explicit bound on τ_a^*). The additional terms are due to the more complex Lyapunov functions (5.42) we used due to the sampling and quantization. In particular, the additional coefficients in the numerator and the first term of the denominator are generated when completing the squares. Meanwhile, we can make $\overline{\lambda}(P_p)$ arbitrarily large (i.e., the second term of the denominator arbitrarily small) by selecting a small enough λ_0.

5.5.2 Searching stage

Recovery

Suppose that the state escapes at a sampling time t_j and is recovered at a sampling time t_i, as in Section 5.4.2. At t_j, (5.15) and (5.16) imply that

$$E_j < \|x(t_j) - x_j^*\| \leq \hat{D}_j$$

with

$$\begin{aligned}\hat{D}_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d),\\ E_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_{j-1}).\end{aligned}$$

From the formulas (5.21) and (5.28) of G, it follows that

$$\hat{D}_j/E_j < \delta_d/\delta_{j-1} = (1+\varepsilon_\delta)\delta_d/\delta_j.$$

Let $c_\varepsilon := \max\{1+\varepsilon_\delta, (\hat{\beta}-1)/(\bar{\beta}-1)\}$, and consider the integer-valued function $\eta : \mathbb{R}_+ \to \mathbb{N}$ defined by

$$\begin{aligned}\eta(s) &:= \max\{\eta_E((1+\varepsilon_\delta)s),\ \eta_\delta(s)\}\\ &= \begin{cases} \lceil \log_{1+\varepsilon_E}(c_\varepsilon s)\rceil, & s > 1;\\ \lceil \log_{1+\varepsilon_E}((1+\varepsilon_\delta)s)\rceil, & (1+\varepsilon_\delta)^{-1} < s \leq 1;\\ 0, & 0 \leq s \leq (1+\varepsilon_\delta)^{-1}.\end{cases}\end{aligned} \tag{5.52}$$

Then (5.38) becomes

$$i - j \leq \eta(\delta_d/\delta_j), \tag{5.53}$$

which, combined with (5.36), implies that

$$\begin{aligned}E_i &= \hat{\beta}^{i-j}E_j + \frac{\hat{\beta}^{i-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)\\ &< \hat{\beta}^{i-j}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right)\\ &< \hat{\beta}^{\eta(\delta_d/\delta_j)}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right).\end{aligned} \tag{5.54}$$

For the searching stage $[t_j, t_i)$, the following lemma provides a bound of $V_{\sigma(t_i)}(x_i^*, E_i)$ at the recovery in terms of $V_{\sigma(t_j)}(x_j^*, E_j)$ and the disturbance estimate δ_j at the escape, and the disturbance bound δ_d.

Lemma 5.5. *Suppose that the state escapes at a sampling time t_j and is recovered at a sampling time t_i. Then*

$$V_{\sigma(t_i)}(x_i^*, E_i) \leq \hat{\beta}^{2\eta(\delta_d/\delta_j)}\big(\omega V_{\sigma(t_j)}(x_j^*, E_j) + \omega_d \delta_j^2\big) \tag{5.55}$$

with the constants

$$\begin{aligned} \omega &:= \max_{p,q\in\mathcal{P}} \omega_{pq}, \\ \omega_{pq} &:= \max\left\{\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2\rho_q}{(\hat{\beta}-1)^2\underline{\lambda}(P_p)}, \frac{(2+\phi_4)\rho_q}{\rho_p}\right\}, \\ \omega_d &:= \max_{q\in\mathcal{P}}\left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2\rho_q}{(\hat{\beta}-1)^2}, \end{aligned} \tag{5.56}$$

where $\phi_4 > 0$ is an arbitrary design parameter.

Proof. See Appendix A.8. □

Initial capture

The case where the state is lost at $t_0 = 0$ and is recovered at t_{i_0} for the first time can be analyzed in a similar manner. From (5.38) with $j = 0$ and $\hat{D}_0 = \|x_0\|$, it follows that

$$i_0 \leq \eta_E(\|x_0\|/E_0) + \eta_\delta(\delta_d/\delta_0), \tag{5.57}$$

which, combined with (5.36) and $x_0^* = 0$, implies that

$$E_{i_0} < \hat{\beta}^{\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0)}\left(E_0 + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_0\right).$$

For the searching stage $[0, t_{i_0})$, the following lemma provides a bound of $V_{\sigma(t_{i_0})}(0, E_{i_0})$ at the first recovery in terms of $V_{\sigma(0)}(0, E_0) = \rho_{\sigma(0)}E_0^2$ and the initial estimates δ_0 and E_0 at $t = 0$, the initial value $\|x_0\|$, and the disturbance bound δ_d.

Lemma 5.6. *Suppose that the state is lost at $t_0 = 0$ and is recovered at a* *sampling time t_{i_0}. Then*

$$V_{\sigma(t_{i_0})}(0, E_{i_0}) \leq \hat{\beta}^{2(\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0))}\big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d \delta_0^2\big) \tag{5.58}$$

with the constant

$$\omega_0 := \max_{q\in\mathcal{P}} \left(1 + \frac{\phi_4}{2}\right) \frac{\rho_q}{\rho_{\sigma(0)}} \leq \frac{1}{2}\omega,$$

where ϕ_4 and w are the design parameter and the constant in (5.56)*, respectively.*

Proof. The proof is essentially the same as the one of Lemma 5.5, and is omitted here. □

5.5.3 Exponential decay

Number of searching stages

As explained in Section 5.3.3, the closed-loop system alternates between a finite number of searching and stabilizing stages, and eventually stays in a stabilizing stage. Let $0 = j_0 \leq i_0 < j_1 < i_1 < \cdots < j_{N_s} < i_{N_s}$ be such that $[t_{j_m}, t_{i_m})$ is a searching stage and $[t_{i_m}, t_{j_{m+1}})$ is a stabilizing stage for each $m \in \{0, \ldots, N_s\}$.[35] As the disturbance estimate is enlarged by a factor of $1 + \varepsilon_\delta$ every time the state escapes, it satisfies that

$$\delta_k = (1 + \varepsilon_\delta)^m \delta_0 \qquad \forall\, k \in \{j_m, \ldots, j_{m+1} - 1\}.$$

Hence

$$N_s \leq N_d(\delta_d)$$

with the integer-valued function $N_d : \mathbb{R}_+ \to \mathbb{N}$ defined by

$$N_d(s) := \begin{cases} \lceil \log_{1+\varepsilon_\delta}(s/\delta_0) \rceil, & s > \delta_0; \\ 0, & 0 \leq s \leq \delta_0. \end{cases} \tag{5.59}$$

[35]There is a searching stage at the beginning (i.e., $i_0 > 0$) if and only if $\|x_0\| > E_0$; in order to represent the final stabilizing stage, we let $j_{N_s+1} := \infty$ and $t_{j_{N_s+1}} := \infty$.

Global bound at sampling times

Combining the bound in Lemma 5.4 for stabilizing stages and the ones in Lemmas 5.5, 5.6 for searching stages, we establish a global bound of $V_{\sigma(t_k)}(x_k^*, E_k)$ in stabilizing stages in terms of the coefficient $\rho_{\sigma(0)}$ and the initial estimates δ_0 and E_0 at $t = 0$, the initial value $\|x_0\|$, and the disturbance bound δ_d.

Lemma 5.7. *Consider a sampling time t_k such that* (5.10) *holds. Then*

$$V_{\sigma(t_k)}(x_k^*, E_k) \le \Theta^{N_0} \Psi^{N_d(\delta_d)} \psi^{2L_d(\delta_d)} \big(\theta^k \psi^{2L_x(\|x_0\|)} \times (\omega_0 \rho_{\sigma(0)} E_0^2 + \omega_d \delta_0^2) + C_d(\delta_d)\delta_0^2\big)$$

with the functions $L_x, L_d : \mathbb{R}_+ \to \mathbb{N}$ and $C_d : \mathbb{R}_+ \to \mathbb{R}_{>0}$ defined by[36]

$$\begin{aligned} L_x(s) &:= \eta_E(s/E_0), \\ L_d(s) &:= \eta_\delta(s/\delta_0) + \sum\nolimits_{l=1}^{N_d(s)} \eta((1+\varepsilon_\delta)^{-l} s/\delta_0), \\ C_d(s) &:= \Theta_d + (\Theta_d + \omega_d) \sum\nolimits_{l=1}^{N_d(s)} \psi_d^l, \end{aligned}$$

and the constants

$$\psi := \hat{\beta}\theta^{-1/2}, \quad \Psi := \omega\Theta^{N_0}, \quad \psi_d := (1+\varepsilon_\delta)^2/\Psi.$$

Proof. See Appendix A.9. □

Remark 5.5. The gain functions N_d, L_x, L_d, and C_d in Lemma 5.7 are piecewise constant, and satisfy that $L_x(s) = 0$ for all $0 \le s \le E_0$, and that $N_d(s) = L_d(s) = 0$ for all $0 \le s \le \delta_0$. A more conservative bound that depends continuously on $\|x_0\|$ and δ_d can be established by replacing them with continuous, strictly increasing gain functions as follows. First, $N_d(s) \le \bar{N}_d(s)$ for all $s \ge 0$ with the function $\bar{N}_d \in \mathcal{K}_\infty$ defined by

$$\bar{N}_d(s) := \begin{cases} 1 + \log_{1+\varepsilon_\delta}(s/\delta_0), & s > \delta_0; \\ s/\delta_0, & 0 \le s \le \delta_0. \end{cases}$$

[36] The sum $L_x(\|x_0\|) + L_d(\delta_d)$ gives a bound of the total length of all searching stages (in terms of sampling intervals).

Second, $L_x(s) \le \bar{L}_x(s)$ for all $s \ge 0$ with the function $\bar{L}_x \in \mathcal{K}_\infty$ defined by

$$\bar{L}_x(s) := \begin{cases} 1 + \log_{1+\varepsilon_E}(s/E_0), & s > E_0; \\ s/E_0, & 0 \le s \le E_0. \end{cases}$$

Third, $L_d(s) \le \bar{L}_d(s)$ for all $s \ge 0$ with the function $\bar{L}_d \in \mathcal{K}_\infty$ defined by

$$\begin{aligned} \bar{L}_d(s) := {} & \log_{1+\varepsilon_E}(c_\beta s/\delta_0) + (\bar{N}_d(s) - 1)\log_{1+\varepsilon_E}(c_\varepsilon s/\delta_0) + \log_{1+\varepsilon_E}(s/\delta_0) \\ & + \bar{N}_d(s) + 1 - \big(\bar{N}_d(s)(\bar{N}_d(s)+1)/2 - 1\big)\log_{1+\varepsilon_E}(1+\varepsilon_\delta) \end{aligned}$$

for $\delta_d > \delta_0$; and

$$\bar{L}_d(s) := (2 + \log_{1+\varepsilon_E} c_\beta)s/\delta_0$$

for $0 \le \delta_d \le \delta_0$. Finally, $C_d(s) \le \bar{C}_d(s)$ for all $s \ge 0$ with the continuous, strictly increasing function $\bar{C}_d : \mathbb{R}_+ \to \mathbb{R}_{>0}$ defined by[37]

$$\bar{C}_d(s) := \Theta_d + \frac{1 - \psi_d^{\bar{N}_d(s)}}{1 - \psi_d}\psi_d(\Theta_d + \omega_d).$$

Intersample bound

First, consider an arbitrary time t in a stabilizing stage, that is, a time $t \in [t_k, t_{k+1}]$ with t_k satisfying (5.10). Following essentially the calculations from Section 5.4.1 with $t' = t'' = 0$, we replace $x(t)$ with c_k in (5.11), the center of the hypercubic box containing $x(t_k)$, via the triangle inequality. If there is no switch on $(t_k, t]$, then (5.24) holds with $t - t_k$ in place of $\bar{t}$; thus

$$\|x(t) - c_k\| = \|x(t) - \hat{x}(t_k)\| \le \hat{D}'_{k+1}(0, t - t_k, \delta_d).$$

Otherwise, there is exactly one switch on $(t_k, t]$ due to (5.4), and (5.27) holds with t in place of t_{k+1}^- (and $t_k + \bar{t} \in (t_k, t]$ denoting the unknown switching time); thus

$$\|x(t) - c_k\| \le \|z(t) - \hat{z}(t_k)\| \le \hat{D}''_{k+1}(0, 0, t - t_k, \delta_d),$$

[37]For $\bar{C}_d$ to be well-defined the design parameter ε_δ should be selected so that $\psi_d \ne 1$. The special case where $\psi_d = 1$ can be treated via similar arguments and is omitted here for brevity (cf. footnote 31).

where the first inequality follows from (5.26). Comparing the corresponding coefficients in (5.24), (5.27), (5.31) and (5.32), we see straightforwardly that in both cases

$$\|x(t) - c_k\| \le \alpha^0 \|x_k^*\| + \beta^0 E_k + \gamma \delta_d$$

with

$$\alpha^0 := \max_{p,q \in \mathcal{P}} \alpha^0_{pq}, \quad \beta^0 := \max_{p,q \in \mathcal{P}} \beta^0_{pq}, \quad \gamma := \max_{p,q \in \mathcal{P}} \gamma_{pq}. \tag{5.60}$$

Applying the triangle inequality, we obtain that

$$\|x(t)\| \le \|c_k\| + \|x(t) - c_k\| \le (\alpha^0 + 1)\|x_k^*\| + \left(\beta^0 + \frac{N-1}{N}\right) E_k + \gamma \delta_d,$$

where the second inequality follows from (5.11). Let

$$\begin{aligned} \lambda^P_{\min} &:= \min_{p \in \mathcal{P}} \underline{\lambda}(P_p), & \lambda^P &:= \max_{p \in \mathcal{P}} \overline{\lambda}(P_p), \\ \rho_{\min} &:= \min_{p \in \mathcal{P}} \rho_p, & \rho &:= \max_{p \in \mathcal{P}} \rho_p, \end{aligned} \tag{5.61}$$

and define

$$\begin{aligned} \xi &:= \frac{\lambda^P_{\min}}{\rho_{\min}} \frac{(\beta^0 + 1 - 1/N)^2}{(\alpha^0 + 1)^2}, \\ \Xi &:= \sqrt{\frac{(\alpha^0 + 1)^2}{\lambda^P_{\min}} + \frac{(\beta^0 + 1 - 1/N)^2}{\rho_{\min}}}. \end{aligned}$$

Then from Young's inequality with ξ, it follows that[38]

$$\begin{aligned} &\left((\alpha^0 + 1)\|x_k^*\| + \left(\beta^0 + \frac{N-1}{N}\right) E_k \right)^2 \\ &\le (1 + \xi)(\alpha^0 + 1)^2 \|x_k^*\|^2 + \left(1 + \frac{1}{\xi}\right)\left(\beta^0 + \frac{N-1}{N}\right)^2 E_k^2 \\ &= \Xi^2 (\lambda^P_{\min} \|x_k^*\|^2 + \rho_{\min} E_k^2) \\ &\le \Xi^2 V_{\sigma(t_k)}(x_k^*, E_k). \end{aligned}$$

Hence

$$\|x(t)\| \le \Xi \sqrt{V_{\sigma(t_k)}(x_k^*, E_k)} + \gamma \delta_d$$

[38]For an $\varepsilon > 0$, Young's inequality with ε states that $ab \le \varepsilon a^2/2 + b^2/(2\varepsilon)$ for all $a, b \in \mathbb{R}$. When $\varepsilon = 1$, the term "with ε" is omitted for brevity.

which, combined with Lemma 5.7 and Remark 5.5, implies that

$$\begin{aligned}\|x(t)\| \le\ & \Xi\Theta^{N_0/2}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(\theta^{k/2}\psi^{\bar{L}_x(\|x_0\|)} \\ & \times (\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)\delta_0}\Big) + \gamma\delta_d,\end{aligned}$$

where the last inequality follows partially from the property

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b} \qquad \forall\, a, b \ge 0. \tag{5.62}$$

Moreover, from $t \in [t_k, t_{k+1}]$ and $\theta < 1$, it follows that

$$\theta^{k/2} \le \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

Second, consider an arbitrary time t in a searching stage, that is, a time $t \in [t_{j_m}, t_{i_m})$. Then (5.34) implies that

$$\|x(t) - x^*_{j_m}\| \le \hat{D}_{i_m} \le E_{i_m}.$$

Following essentially the calculations from the first case, we obtain that

$$\begin{aligned}\|x(t)\| &\le \|x^*_{j_m}\| + E_{i_m} \\ &\le (\alpha^0 + 1)\|x^*_{i_m}\| + \left(\beta^0 + \frac{N-1}{N}\right)E_{i_m} \\ &\le \Xi\sqrt{V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m})} \\ &\le \Xi\Theta^{N_0/2}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(\theta^{i_m/2}\psi^{\bar{L}_x(\|x_0\|)} \\ &\quad \times (\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)\delta_0}\Big) + \gamma\delta_d,\end{aligned}$$

in which

$$\theta^{i_m/2} \le \theta^{t/(2\tau_s)} < \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

Finally, consider an arbitrary time $t \ge 0$. Combining the results above, we obtain that

$$\begin{aligned}\|x(t)\| \le\ & \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(\theta^{t/(2\tau_s)}\psi^{\bar{L}_x(\|x_0\|)} \\ & \times (\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)\delta_0}\Big) + \gamma\delta_d.\end{aligned} \tag{5.63}$$

From Young's inequality with an arbitrary design parameter $\phi > 0$, it follows that

$$\begin{aligned}
\|x(t)\| &\leq \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\left(\frac{1}{2\phi}\psi^{2\bar{L}_x(\|x_0\|)} + \frac{\phi}{2}\Psi^{\bar{N}_d(\delta_d)}\psi^{2\bar{L}_d(\delta_d)}\right)\theta^{t/(2\tau_s)}(\sqrt{\omega_0\rho}E_0 \\
&\quad + \sqrt{\omega_d}\delta_0) + \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\sqrt{\bar{C}_d(\delta_d)}\delta_0 + \gamma\delta_d \\
&= \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar{L}_x(\|x_0\|)}\theta^{t/(2\tau_s)} \\
&\quad + \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar{N}_d(\delta_d)}\psi^{2\bar{L}_d(\delta_d)} \\
&\quad + \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\sqrt{\bar{C}_d(\delta_d)}\delta_0\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)} + \gamma\delta_d.
\end{aligned}$$

Hence the state bound (5.6) holds with the exponential decay rate

$$\lambda := -\frac{\ln\theta}{2\tau_s} > 0, \tag{5.64}$$

and the gain functions $g, h : \mathbb{R}_+ \to \mathbb{R}_{>0}$ defined by

$$\begin{aligned}
g(s) &:= \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar{L}_x(s)}, \\
h(s) &:= \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar{N}_d(s)}\psi^{2\bar{L}_d(s)} \\
&\qquad + \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\sqrt{\bar{C}_d(s)}\delta_0\Psi^{\bar{N}_d(s)/2}\psi^{\bar{L}_d(s)} + \gamma s.
\end{aligned} \tag{5.65}$$

Remark 5.6. Young's inequality is applied here to transform the state bound (5.63) into the one (5.6) in the standard ISS form; cf. the definition (2.6) of ISS in Section 2.2. However, this not only increases the value of the state bound, but also has the following consequence. In the case where there is no disturbance and the sensor and the controller both know that, the initial disturbance estimate will be set to $\delta_0 = \delta_d = 0$. Then (5.63) becomes $\|x(t)\| \leq \Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0\theta^{t/(2\tau_s)}\psi^{\bar{L}_x(\|x_0\|)}$, that is, it reduces to a state bound of the same form as the one in [68, eq. (5)] for the disturbance-free case. On the contrary, (5.6) cannot be reduced to the same form since $h(0) = \phi\Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0/2 > 0$ even if $\delta_0 = \delta_d = 0$.

5.5.4 Practical stability

Based on the calculations in [68, Section 5.5] and Sections 5.4.1, 5.5.1, and 5.5.3, we establish the following stability result, which is a stronger version of Corollary 5.2 due to the smaller constant C.

Proposition 5.3 (Practical stability). *Consider the switched linear control system* (5.1). *Suppose that Assumptions 5.1–5.3 and the inequality* (5.4) *hold. Then there is a communication and control strategy that yields the following property: Provided that the average dwell-time τ_a is large enough, for each $\varepsilon > 0$, there exists a small enough $\delta > 0$ such that for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_+ \to \mathbb{R}^{n_d}$,* (5.9) *holds with the constant*

$$C := \Xi \Theta^{N_0/2} \sqrt{\Theta_d} \delta_0. \tag{5.66}$$

Proof. First, suppose that $\|x_0\| \le \delta \le E_0$ and $\delta_d \le \delta \le \delta_0$. Then the system is always in the stabilizing stage, and the disturbance estimate is always δ_0. Suppose that there is an integer $k_1 \ge 1$ such that $c_k = 0$ (i.e., the state $x(t_k)$ is inside the central hypercubic box) for all $k \le k_1 - 1$. Following essentially the calculations in Sections 5.4.1 and 5.4.1, we obtain that $u \equiv 0$ on $[0, t_{k_0})$ and $x_k^* = 0$ for all $k \in \{0, \dots, k_1\}$; thus

$$E_{k+1} \ge \frac{\Lambda_{\min}}{N} E_k \qquad \forall\, k \in \{0, \dots, k_1 - 1\} \tag{5.67}$$

with the constant

$$\Lambda_{\min} := \min_{p \in \mathcal{P}} \Lambda_p$$

due to (5.21) and (5.28).

Second, following essentially the calculations in Section 5.4.2, we otain that for each $k \le k_1$,

$$\|x(t)\| \le \bar{\beta}^k \|x_0\| + \frac{\bar{\beta}^k - 1}{\bar{\beta} - 1} \bar{\gamma} \delta_d \qquad \forall\, t \le t_k. \tag{5.68}$$

Third, following essentially the calculations in Section 5.5.3, we obtain that for each $k \ge k_1$,

$$\|x(t)\| \le \Xi \sqrt{V_{\sigma(t_k)}(x_k^*, E_k)} + \gamma \delta_d \qquad \forall\, t \in [t_k, t_{k+1}],$$

which, combined with (5.50) fo $i = 0$ and (5.62), implies that

$$\|x(t)\| \leq \Xi\Theta^{N_0/2}(\theta^{k_1}\sqrt{\rho}E_0 + \sqrt{\Theta_d}\delta_0) + \gamma\delta_d \qquad \forall\, t \geq t_{k_1} \tag{5.69}$$

with the coefficient ρ in (5.61).

Finally, the proof of Lemma 5.3 is completed through the following three steps. First, given an arbitrary $\varepsilon > 0$, from (5.66) and (5.69), it follows that if

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 + \gamma\delta_d \leq \varepsilon,$$

then $\|x(t)\| \leq \varepsilon + C$ for all $t \geq t_{k_1}$. Second, taking E_0 as fixed, calculate a large enough k_1 so that

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 \leq \varepsilon/2.$$

Third, calculate a small enough δ so that $\gamma\delta \leq \varepsilon/2$ and

$$\left(\bar{\beta}^{k_1} + \frac{\bar{\beta}^{k_1} - 1}{\bar{\beta} - 1}\bar{\gamma}\right)\delta \leq \varepsilon,$$

which, combined with (5.68), implies that $\|x(t)\| \leq \varepsilon$ for all $t \leq t_{k_1}$; and that $\delta \leq \delta_0$ and

$$\left(\bar{\beta}^{k_1-1} + \frac{\bar{\beta}^{k_1-1} - 1}{\bar{\beta} - 1}\bar{\gamma}\right)\delta \leq \left(\frac{\Lambda_{\min}}{N}\right)^{k_1-1}\frac{E_0}{N},$$

which, combined with (5.67) and (5.68), implies that $c_k = 0$ for all $k \leq k_1 - 1$ (in particular, the systems is always in the stabilizing stage), making the analysis above valid. $\square$

As

$$\begin{aligned} g(0) + h(0) &= \Xi\Theta^{N_0/2}\left(\left(\frac{1}{2\phi} + \frac{\phi}{2}\right)\frac{\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0}{\sqrt{\theta}} + \sqrt{\Theta_d}\delta_0\right) \\ &> \Xi\Theta^{N_0/2}\sqrt{\Theta_d}\delta_0, \end{aligned}$$

it follows that the constant C in Proposition 5.3 is smaller than the one in Corollary 5.2.

Proposition 5.3 also improves the practical stability result in [71, Theorem 1]. Moreover, from the proof, it is clear that the additional lower bound [71, eq. (39)] on the average dwell-time τ_a is not necessary for establishing

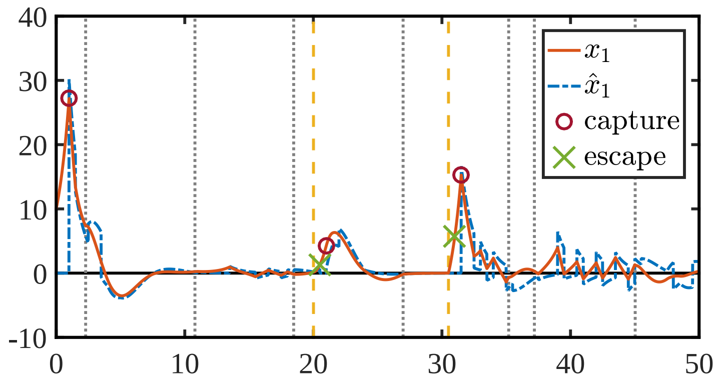


Figure 5.2: Simulation result.

practical stability.

5.6 Simulation study

Our communication and control strategy is simulated with the following data, The index set is $\mathcal{P} = \{1, 2\}$, the matrices are

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -2 & 0 \end{bmatrix};$$
$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Also, the constants are $\tau_s = 0.5$, $N = 5$, $\tau_d = 1.05$, $\tau_a = 7.55$, and $N_0 = 5$ so that the basic Assumptions 5.1–5.3 hold. We set $t' = t'' = 0$ in (5.28), $\varepsilon_E = 0.8$ in (5.17), and $\varepsilon_\delta = 1$ in (5.59). The disturbance $d(\cdot)$ is kept 0 most of the time, and is turned on for 2 sampling intervals with the constant value 10 when the state stays small (more specifically, when $\|x\| < 2$ for 10 consecutive sampling intervals). The initial disturbance estimate is $\delta_0 = 2$. Figure 5.2 plots a typical behavior of the first component x_1 of the continuous state (in orange solid line) and the corresponding component $\hat{x}_1$ of the auxiliary state (in blue dash-dot line). Switching times are denoted by vertical gray dotted lines, and sampling times at which the disturbance is turned on are denoted by vertical yellow dashed lines; captures are marked by red circles, and escapes are marked by green crosses. Observe the search-

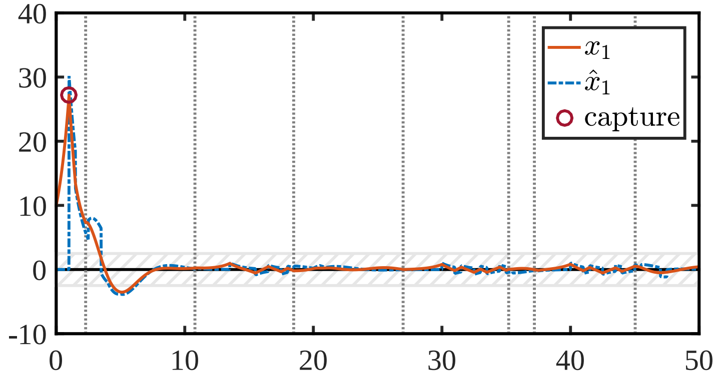


Figure 5.3: Simulation result with constant disturbance estimates $\delta_k \equiv \delta_0$: the state $x \nrightarrow 0$ even if $d \equiv 0$.

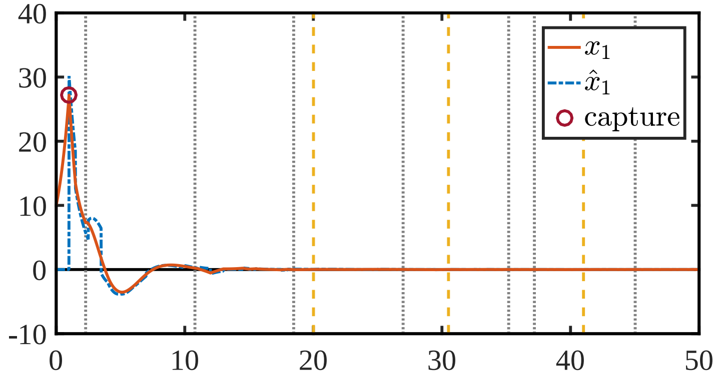


Figure 5.4: Simulation result with converging disturbance estimates $\delta_k \to 0$: the state $x \to 0$ when $d \to 0$.

ing stages at $t = 0$ (the state is lost due to $\|x_0\| > E_0$) and $t = 20.5, 31$ (the state escapes due to the disturbance), and the nonsmooth behavior of x when $\hat{x}$ experiences a jump. The value of τ_a is empirically selected to be large enough to provide consistent convergence in simulations. For this example, the theoretical lower bound (5.49) on the average dwell-time τ_a is approximately 28.13, which is rather conservative. However, our simulation result is significantly less conservative than the one in [68, Section 6] for the disturbance-free case, which generated a theoretical bound of $\tau_a \geq 85.5$ while consistent convergence was observed with $\tau_a = 7.55$. The improvement is due to the more careful calculations in the approximation and stability analysis, such as the ones explained by footnotes 30 and 33.

Figure 5.3 exhibits the cases where the unknown disturbance $d(\cdot)$ is transient or $d \equiv 0$, so that once the state is captured it will never escape. Due to the nonzero initial disturbance estimate δ_0, the state x will converge to the set $\mathcal{A} = \{v \in \mathbb{R}^{n_x} : \|v\| \leq h(0)\}$ (visualized by the shaded area) instead of the origin. Following essentially the idea of "zooming-in" from [61], we are able to make the state converge to the origin by halving the estimate δ_k every 10 sampling intervals, as shown in Figure 5.4. We conjecture that for general disturbances, a similar modification to our communication and control strategy can be made to establish ISS with respect to the origin.

5.7 Future work

As discussed in Section 5.6, we intend to advance our result via the "zooming-in" technique from [61] to establish ISS with respect to the origin. However, reducing the disturbance estimate in stabilizing stages may lead to an unbounded number of searching stages, and further work is needed to establish convergence for the improved communication and control strategy.

For a linear time-invariant control system, the minimum data rate necessary for feedback stabilization coincides with its topological entropy in open-loop [59, 60, 73]. In the context of switched systems, neither the concept of topological entropy nor the minimum data rate necessary for feedback stabilization was well-established, and there is currently a gap between the sufficient data rate in this chapter and the known entropy bounds for switched systems. These two notions and their relation are intriguing topics for future research.

Chapter 6

Topological entropy of switched linear systems with Lie structures

6.1 Entropy notions

Consider a continuous-time switched system modeled by

$$\dot{x} = f_\sigma(x), \qquad x(0) \in K, \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state, $K \subset \mathbb{R}^n$ is a compact *initial set*, and $\sigma : \mathbb{R}_+ \to \mathcal{P}$ is the switching signal with a finite index set $\mathcal{P}$. Denote by $\xi_\sigma(x, t)$ the solution of (6.1) at time t with initial state x and switching signal σ. Suppose that (6.1) fulfills the same assumptions as those imposed on general switched systems in Section 2.2, except that there is no input, and that the origin is not necessarily an equilibrium. In particular, for fixed x and σ, the function $\xi_\sigma(x, \cdot)$ is absolutely continuous and satisfies the differential equation (6.1) away from discontinuities of σ. We observe that, for a fixed switching signal, the switched system (6.1) becomes a time-varying system, and the state trajectory is uniquely determined by the initial state.

Based on the definition of topological entropy for time-invariant systems in [78, Section 3.1.b], we formulate a notion of topological entropy for the switched system (6.1) with a known switching signal as follows. Given a time horizon $T \geq 0$ and a scalar $\varepsilon > 0$, define the open ball at point $x \in K$ with radius ε over interval $[0, T]$ by

$$B_{f_\sigma}(x, \varepsilon, T) := \Big\{ y \in K : \max_{t \in [0,T]} \|\xi_\sigma(y, t) - \xi_\sigma(x, t)\| < \varepsilon \Big\}, \tag{6.2}$$

where $\|\cdot\|$ is some chosen norm on $\mathbb{R}^n$. We say that a finite set of points

$E = \{x_1, \dots, x_N\} \subset K$ is (T, ε)-*spanning* if

$$K = \bigcup_{x_k \in E} B_{f_\sigma}(x_k, \varepsilon, T), \tag{6.3}$$

or equivalently, for each initial state $x \in K$, there exists some $x_k \in E$ such that

$$\|\xi_\sigma(x, t) - \xi_\sigma(x_k, t)\| < \varepsilon \qquad \forall\, t \in [0, T]. \tag{6.4}$$

Denote by $S(f_\sigma, \varepsilon, T)$ the minimal cardinality of such a (T, ε)-spanning set, or equivalently, the cardinality of a *minimal* (T, ε)*-spanning set*. The *topological entropy* of the switched system (6.1) is defined by

$$h(f_\sigma) := \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log S(f_\sigma, \varepsilon, T). \tag{6.5}$$

Remark 6.1. Following [78, Proposition 3.1.2], the value of $h(f_\sigma)$ is the same for all metrics defining the same topology. Hence the norm $\|\cdot\|$ can be arbitrary. For simplicity and concreteness, we take $\|\cdot\|$ to be the ∞-norm in the following analysis; see Section 5.1.1 for the precise definition. Also, the value of $h(f_\sigma)$ is independent of the size or shape of the compact initial set K. Unless otherwise stated, we think of K as the closed unit ball (cube) centered at the origin, that is, $K = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

Next, we introduce an equivalent definition for the topological entropy of the switched system (6.1) with a known switching signal. With T and ε given as before, we say that a finite set of points $E = \{x_1, \dots, x_N\} \subset K$ is (T, ε)-*separated* if

$$x_{k'} \notin B_{f_\sigma}(x_k, \varepsilon, T) \qquad \forall\, x_k, x_{k'} \in E, \tag{6.6}$$

or equivalently, for each pair of points $x_k, x_{k'} \in E$, there exists some time $t \in [0, T]$ such that

$$\|\xi_\sigma(x_k, t) - \xi_\sigma(x_{k'}, t)\| \ge \varepsilon. \tag{6.7}$$

Let $N(f_\sigma, \varepsilon, T)$ denote the maximal cardinality of such a (T, ε)-separated set, or equivalently, the cardinality of a *maximal* (T, ε)*-separated set.*

Proposition 6.1. *The topological entropy of the switched system* (6.1) *sat-*

isfies that

$$h(f_\sigma) \equiv \lim_{\varepsilon \searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log N(f_\sigma, \varepsilon, T). \tag{6.8}$$

Proof. The proof is along the lines of [78, p. 110], and is omitted here. □

Remark 6.2. For the case with a time-invariant system $\dot{x} = f(x)$, as shown in [78, Section 3.1.b], the value of $h(f)$ remains the same if $\limsup$ is replaced with $\liminf$ in the definition (6.5). However, this is not necessarily the case for a time-varying system. More specifically, in [78, Section 3.1.b], the equivalence was established by comparing $S(f, \varepsilon, T)$ with $D(f, \varepsilon, T)$, the minimal number of sets of diameters at most ε over interval $[0, T]$ such that their union covers the initial set K. From [78, p. 109], it follows that

$$\begin{aligned}
\liminf_{T\to\infty} \frac{1}{T} \log D(f, 2\varepsilon, T) &\le \liminf_{T\to\infty} \frac{1}{T} \log S(f, \varepsilon, T) \\
&\le \limsup_{T\to\infty} \frac{1}{T} \log S(f, \varepsilon, T) \\
&\le \limsup_{T\to\infty} \frac{1}{T} \log D(f, \varepsilon, T).
\end{aligned}$$

Moreover, in [78, Lemma 3.1.5], it was shown that the limit

$$\lim_{T\to\infty} \frac{1}{T} \log D(f, \varepsilon, T) \tag{6.9}$$

always exists. Hence

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \liminf_{T\to\infty} \frac{1}{T} \log S(f, \varepsilon, T) &= \lim_{\varepsilon \searrow 0} \lim_{T\to\infty} \frac{1}{T} \log D(f, \varepsilon, T) \\
&= \lim_{\varepsilon \searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log S(f, \varepsilon, T) \\
&= h(f).
\end{aligned}$$

For a time-varying system, the subadditivity required in the proof of [78, Lemma 3.1.5] does not hold in general, meaning that the limit (6.9) doesn't necessarily exist.

The main objective of this chapter is to examine the topological entropy of switched linear systems modeled by

$$\dot{x} = A_\sigma x, \qquad x(0) \in K \tag{6.10}$$

with a family of matrices $\{A_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$.

6.2 Linear time-invariant systems

Consider a linear time-invariant (LTI) system modeled by

$$\dot{x} = Ax, \qquad x(0) \in K \tag{6.11}$$

with a matrix $A \in \mathbb{R}^{n\times n}$. Its topological entropy is given by the following proposition; see, e.g., [102, Theorem 2.4.2] for a corresponding result for the discrete-time case.

Proposition 6.2 ([103, Theorem 4.1])**.** *Consider the LTI system* (6.11)*. Its topological entropy is given by the sum of the nonnegative real parts of its eigenvalues, that is,*

$$h(A) = \sum_{i=1}^{n} \max\{\mathrm{Re}(\lambda_i(A)), 0\}. \tag{6.12}$$

Next, we compare the property that $h(A) = 0$ with the following stability notions for the LTI system (6.11).

Corollary 6.3 ([86, Theorem 4.5])**.** *Consider the LTI system* (6.11)*.*

- *It is stable if and only if* $\mathrm{Re}(\lambda_i(A)) \leq 0$ *for all* $i \in \{1, \dots, n\}$*, and the algebraic and geometric multiplicities are equal for each purely imaginary eigenvalue.*
- *It is globally exponentially stable (GES) if and only if* $\mathrm{Re}(\lambda_i(A)) < 0$ *for all* $i \in \{1, \dots, n\}$*.*

Clearly, both stability and GES imply that the entropy $h(A) = 0$. Their relations are summarized in Table 6.1.

Table 6.1: Characterizations for GES, stability, and $h(A) = 0$

Property	Eigenvalue	State-transition matrix
GES	$\mathrm{Re}(\lambda_i(A)) < 0$ for all i	$\lim_{t\to\infty} e^{At} = 0$
Stability	$\mathrm{Re}(\lambda_i(A)) \leq 0$ for all i, and the algebraic and geometric multiplicities are equal for each purely imaginary eigenvalue	$\lim_{t\to\infty} \lVert e^{At} \rVert$ is bounded
$h(A) = 0$	$\mathrm{Re}(\lambda_i(A)) \leq 0$ for all i	$\lim_{t\to\infty} \lVert e^{At} \rVert < e^{at}$ for all $a > 0$

6.3 Switching characterization

Given a switching signal $\sigma : \mathbb{R}_+ \to \mathcal{P}$, for a mode $p \in \mathcal{P}$, define the *active time* of mode p up to time t by

$$\tau_p(t) := \int_0^t \mathbb{1}_p(\sigma(s))\,\mathrm{d}s$$

with the indicator function

$$\mathbb{1}_p(\sigma(s)) := \begin{cases} 1, & \sigma(s) = p, \\ 0, & \sigma(s) \neq p. \end{cases}$$

Then the *active rate* of mode p up to time t is defined by

$$\rho_p(t) := \tau_p(t)/t,$$

and the *asymptotic active rate* of mode p is defined by

$$\hat{\rho}_p := \limsup_{t\to\infty} \rho_p(t). \tag{6.13}$$

Clearly, the active times and active rates satisfy that

$$\sum_{p\in\mathcal{P}} \tau_p(t) \equiv t,$$

or equivalently,

$$\sum_{p\in\mathcal{P}} \rho_p(t) \equiv 1.$$

However, due to the limit supremum in (6.13), it is possible that the asymptotic active rates $\hat{\rho}_p = 1$ for all $p \in \mathcal{P}$, as demonstrated in Example 6.1 below.

Given a family of scalars $\{a_p \in \mathbb{R} : p \in \mathcal{P}\}$, define their *asymptotic average* by

$$\hat{a} := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} a_p \rho_p(t) = \limsup_{t\to\infty} \frac{1}{t} \sum_{p\in\mathcal{P}} a_p \tau_p(t). \tag{6.14}$$

Also, define the following quantity

$$\bar{a}(T) := \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p \tau_p(t) \tag{6.15}$$

for a time horizon $T \geq 0$, which proves to be useful in calculating topological entropy of switched linear systems. Clearly, it satisfies that

$$\bar{a}(T) \geq 0, \quad \bar{a}(T) \geq \sum_{p\in\mathcal{P}} a_p \rho_p(T) \qquad \forall\, T \geq 0.$$

Hence

$$\limsup_{T\to\infty} \bar{a}(T) \geq \max\{\hat{a},\, 0\}.$$

In the following result, we establish that the opposite holds as well.

Lemma 6.1. *The limit supremum of $\bar{a}$ satisfies that*

$$\limsup_{T\to\infty} \bar{a}(T) = \max\{\hat{a},\, 0\}. \tag{6.16}$$

Proof. See Appendix A.10. □

6.4 Switched scalar systems

Consider the switched linear system (6.10). If $n = 1$, then every matrix A_p becomes a scalar $a_p \in \mathbb{R}$. Hence (6.10) becomes a switched scalar system modeled by

$$\dot{x} = a_\sigma x, \qquad x(0) \in K \subset \mathbb{R}. \tag{6.17}$$

Its topological entropy is characterized by the following theorem.

Theorem 6.4. *Consider the switched scalar system* (6.17)*. Its topological entropy is given by*

$$h(a_\sigma) = \max\{\hat{a},\, 0\}$$

with the asymptotic average $\hat{a}$ defined by (6.14)*.*

Theorem 6.4 follows from Lemma 6.1 and the following result.

Lemma 6.2. *The topological entropy of* (6.17) *satisfies that*

$$h(a_\sigma) = \limsup_{T\to\infty}\, \bar{a}(T)$$

with $\bar{a}$ defined by (6.15)*.*

Proof. For initial points $x, y \in K$, the corresponding solutions at time t satisfy that

$$\xi_\sigma(y,t) - \xi_\sigma(x,t) = e^{\sum_{p\in\mathcal{P}} a_p\tau_p(t)}(y-x).$$

For brevity, define

$$\bar{\eta}(T) := \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p\tau_p(t) = \bar{a}(T)\,T$$

for a time horizon $T \geq 0$. Then

$$\max_{t\in[0,T]} |\xi_\sigma(y,t) - \xi_\sigma(x,t)| = e^{\bar{\eta}(T)}|y-x|.$$

First, we consider the formula of topological entropy (6.5) in terms of spanning sets, and prove the upper bound. Following (6.4), a finite set of points $E = \{x_1, \ldots, x_N\} \subset K$ is (T,ε)-spanning if and only if for each $x \in K$, there is some $x_k \in E$ such that

$$|x - x_k| < e^{-\bar{\eta}(T)}\varepsilon.$$

Recall that the initial set K is taken to be the closed unit ball (interval) centered at the origin. Consider the set defined by

$$E_1 := \big\{-1,\, -1 + e^{-\bar{\eta}(T)}\varepsilon,\, \ldots,\, -1 + \big\lfloor 2e^{\bar{\eta}(T)}/\varepsilon\big\rfloor e^{-\bar{\eta}(T)}\varepsilon,\, 1\big\}.$$

It is then straightforward to verify that E_1 is a (T,ε)-spanning set. Hence the minimal cardinality of a (T,ε)-spanning set satisfies that

$$S(a_\sigma,\varepsilon,T) \le \left\lfloor 2e^{\bar{\eta}(T)}/\varepsilon \right\rfloor + 2 \le (2/\varepsilon+2)\, e^{\bar{\eta}(T)}.$$

Substituting the previous bound into (6.5), we obtain that

$$h(a_\sigma) \le \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log\left((2/\varepsilon+2)\, e^{\bar{\eta}(T)}\right) = \limsup_{T\to\infty} \frac{\bar{\eta}(T)}{T} = \limsup_{T\to\infty} \bar{a}(T).$$

Second, we consider the formula of topological entropy (6.8) in terms of separated sets, and prove the lower bound. Following (6.7), a finite set of points $E = \{x_1,\dots,x_N\} \subset K$ is (T,ε)-separated if and only if each pair $x_k, x_l \in E$ satisfies that

$$|x_k - x_l| \ge e^{-\bar{\eta}(T)}\varepsilon.$$

Recall that the initial set K is taken to be the closed unit ball (interval) centered at the origin. Consider the set defined by

$$E_2 := \left\{-1,\, -1+e^{-\bar{\eta}(T)}\varepsilon,\, \dots,\, -1+\left\lfloor 2e^{\bar{\eta}(T)}/\varepsilon \right\rfloor e^{-\bar{\eta}(T)}\varepsilon\right\}.$$

It is then straightforward to verify that E_2 is a (T,ε)-separated set. Hence the maximal cardinality of a (T,ε)-separated set satisfies that

$$N(a_\sigma,\varepsilon,T) \ge \left\lfloor 2e^{\bar{\eta}(T)}/\varepsilon \right\rfloor + 1 > (2/\varepsilon)\, e^{\bar{\eta}(T)}.$$

Substituting the previous bound into (6.8), we obtain that

$$h(a_\sigma) \ge \lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log\left((2/\varepsilon)\, e^{\bar{\eta}(T)}\right) = \limsup_{T\to\infty} \frac{\bar{\eta}(T)}{T} = \limsup_{T\to\infty} \bar{a}(T). \qquad \square$$

Remark 6.3. Alternatively, the lower bound $h(a_\sigma) \ge \max\{\hat{a}, 0\}$ can be proved directly using volume (Lebesgue measure) arguments. Due to the limit supremum in (6.14), for each $\delta > 0$, there exists an increasing sequence $(t_m)_{m\in\mathbb{N}}$ such that

$$\sum_{p\in\mathcal{P}} a_p \rho_p(t_m) > \hat{a} - \delta \qquad \forall\, m \in \mathbb{N}.$$

For each t_m, the solutions corresponding to initial points $x, y \in K$ satisfy that

$$|\xi_\sigma(y,t_m)-\xi_\sigma(x,t_m)| = e^{\sum_{p\in\mathcal{P}} a_p\tau_p(t_m)}|y-x| > e^{(\hat{a}-\delta)\,t_m}|y-x|.$$

Suppose that $E=\{x_1,\ldots,x_N\}\subset K$ is a (t_m,ε)-spanning set. For each $x_k\in E$, the open ball $B_{a_\sigma}(x_k,\varepsilon,t_m)$ satisfies that

$$\begin{aligned} B_{a_\sigma}(x_k,\varepsilon,t_m) &\subset \{x\in K : |x-x_k|<\varepsilon,\, |\xi_\sigma(x,t_m)-\xi_\sigma(x_k,t_m)|<\varepsilon\} \\ &\subset \left\{x\in K : e^{\max\{\hat{a}-\delta,\,0\}\,t_m}|x-x_k|<\varepsilon\right\}. \end{aligned}$$

Hence its volume is bounded by

$$\mu_L(B_{a_\sigma}(x_k,\varepsilon,t_m)) < 2e^{-\max\{\hat{a}-\delta,\,0\}\,t_m}\varepsilon.$$

Denote by $\mu_L(K)$ the volume of the initial set K, and by $|E|$ the cardinality of the set E. Then from (6.3), it follows that

$$\mu_L(K) \le \sum_{x_k\in E}\mu_L(B_{a_\sigma}(x_k,\varepsilon,t_m)) < 2|E|e^{-\max\{\hat{a}-\delta,\,0\}\,t_m}\varepsilon.$$

Hence the minimal cardinality of a (t_m,ε)-spanning set satisfies that

$$S(a_\sigma,\varepsilon,t_m) > \frac{\mu_L(K)}{2\varepsilon}e^{\max\{\hat{a}-\delta,\,0\}\,t_m}.$$

Substituting the previous estimate into (6.5), we obtain that

$$h(a_\sigma) \ge \lim_{\varepsilon\searrow 0}\limsup_{m\to\infty}\frac{1}{t_m}\log\left(\frac{\mu_L(K)}{2\varepsilon}e^{\max\{\hat{a}-\delta,\,0\}\,t_m}\right) = \max\{\hat{a}-\delta,\,0\}.$$

As $\delta>0$ is arbitrary, it follows that $h(a_\sigma)\ge\max\{\hat{a},\,0\}$. However, this proof cannot be extended to case of higher dimensions, even for a family of diagonal matrices $\{D_p=\mathrm{diag}(a_p^1,\ldots,a_p^n) : p\in\mathcal{P}\}$. This is due to the fact that, in general, there exists no sequence $(t_m)_{m\in\mathbb{N}}$ such that

$$\sum_{p\in\mathcal{P}} a_p^i\rho_p(t_m) > \left(\limsup_{t\to\infty}\sum_{p\in\mathcal{P}} a_p^i\rho_p(t)\right)-\delta \qquad \forall\, m\in\mathbb{N}$$

for all $i\in\{1,\ldots,n\}$.

Following the subadditivity of limit supremum, we obtain that

$$\begin{aligned}\hat{a} &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} a_p \rho_p(t) \\ &\le \sum_{p\in\mathcal{P}} \Big(\limsup_{t\to\infty} a_p \rho_p(t) \Big) \\ &\le \sum_{p\in\mathcal{P}} \Big(\max\{a_p,\, 0\} \limsup_{t\to\infty} \rho_p(t) \Big),\end{aligned}$$

and thus the following corollary of Theorem 6.4, which is more useful but in general more conservative.

Corollary 6.5. *Consider the switched scalar system* (6.17). *Its topological entropy satisfies that*

$$h(a_\sigma) \le \sum_{p\in\mathcal{P}} \max\{a_p \hat{\rho}_p,\, 0\}$$

with the asymptotic active rate $\hat{\rho}_p$ defined by (6.13).

We observe that

$$\hat{a} \le \Big(\max_{p\in\mathcal{P}} a_p \Big) \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \rho_p(t) = \max_{p\in\mathcal{P}} a_p,$$

while in general it is possible that

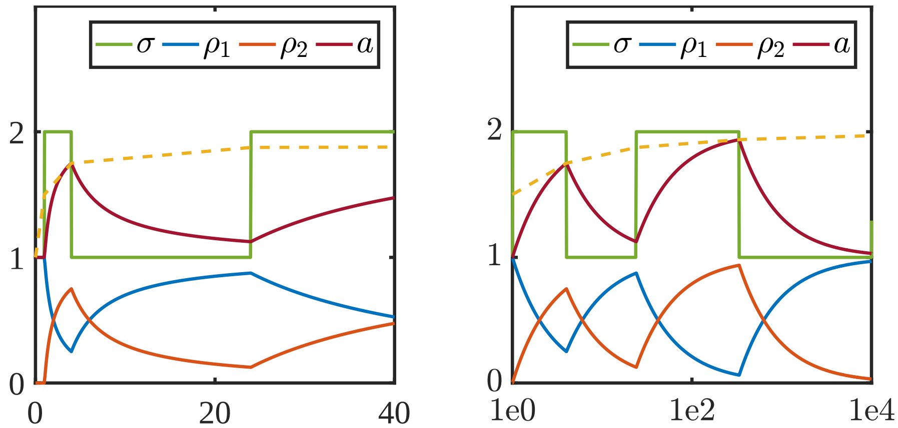
$$\sum_{p\in\mathcal{P}} a_p \hat{\rho}_p > \max_{p\in\mathcal{P}} a_p,$$

as demonstrated by the following example.

Example 6.1. Consider the switched scalar system (6.17) with the index set $\mathcal{P} = \{1,\, 2\}$, and the scalars $a_1 = 1$ and $a_2 = 2$. Construct a switching signal σ as follows:

- Let $(t_k)_{k\in\mathbb{N}}$ denote the sequence of switches with $\sigma \equiv 1$ on $[t_{2k}, t_{2k+1})$ and $\sigma \equiv 2$ on $[t_{2k+1}, t_{2k+2})$.
- Set $t_0 = 0$ and $t_1 = 1$. Then $\rho_1(t_1) = 1$ and $\rho_2(t_1) = 0$.
- For $k \ge 1$, set $t_{2k} := \operatorname{argmin}\{t > t_{2k-1} : \rho_2(t) \ge 1 - 2^{-2k}\}$ and $t_{2k+1} := \operatorname{argmin}\{t > t_{2k} : \rho_1(t) \ge 1 - 2^{-(2k+1)}\}$.

With some calculation, one can show that $t_k = 2^k \prod_{l=1}^{k-1}(2^l - 1)$ for $k \ge 2$. The switching signal σ, the active rates ρ_1 and ρ_2, and the average $a =$



(a) Linear scale on $[0, 100]$. (b) Logarithmic scale on $[1, 10^4]$.

Figure 6.1: The switching signal σ, active rates ρ_1, ρ_2, and function $\bar{a}_\sigma$ for Example 6.1.

$a_1\rho_1 + a_2\rho_2$ are plotted in Figure 6.1 (as the intervals between consecutive switches grow exponentially, for large time the functions are plotted using logarithmic scale). The asymptotic active rates are given by $\hat{\rho}_1 = \hat{\rho}_2 = \limsup_{k\to\infty} 1 - e^{-2k} = 1$. The topological entropy (the limit of the yellow dashed lines in Figure 6.1) satisfies that

$$h(a_\sigma) = \hat{a} = \limsup_{T\to\infty} (a_1\rho_1(t) + a_2\rho_2(t)) = 1 + \hat{\rho}_2 = 2 < 3 = \sum_{p\in\mathcal{P}} a_p\hat{\rho}_p.$$

The following result shows that the asymptotic average $\hat{a}$ in Theorem 6.4 can be used to establish GES of (6.17).

Proposition 6.6. *Consider the switched scalar system* (6.17)*. If the asymptotic average $\hat{a}$ defined by* (6.14) *is negative, then* (6.17) *is GES.*

Proof. For an initial point $x_0 \in \mathbb{R}$, the corresponding solution of (6.17) at time t satisfies that

$$|\xi_\sigma(x_0, t)| \leq e^{\sum_{p\in\mathcal{P}} a_p\tau_p(t)}|x_0|.$$

Due to the limit supremum in (6.14), for each $\lambda \in (0, -\hat{a}/2)$, there is a large enough $T_\lambda \geq 0$ such that

$$\sum_{p\in\mathcal{P}} a_p\tau_p(t) < (\hat{a} + \lambda)\, t < -\lambda t \qquad \forall\, t > T_\lambda.$$

Hence $|\xi_\sigma(x_0, t)| \leq e^{-\lambda t}|x_0|$ for all $t > T_\lambda$. Moreover,

$$|\xi_\sigma(x_0, t)| \leq e^{a_m T_\lambda}|x_0| \qquad \forall\, t \in [0, T_\lambda]$$

with $a_m := \max_{p\in\mathcal{P}} |a_p|$. Therefore,

$$|\xi_\sigma(x_0, t)| \leq e^{(a_m+\lambda)T_\lambda} e^{-\lambda t}|x_0| \qquad \forall\, t \geq 0,$$

that is, (6.17) is GES. □

6.5 Switched diagonal systems

Consider the switched linear system (6.10). Suppose that the matrices in $\{A_p : p \in \mathcal{P}\}$ are all diagonalizable, and they commute pairwise. Then there exists a (possibly complex) linear change of coordinates under which they are all diagonal (and vice versa) [104, Theorem 1.3.19].[39] In view of this result, we assume, without loss of generality, that every A_p is diagonal, and denote it by $D_p \in \mathbb{C}^{n\times n}$. Then (6.10) becomes a switched diagonal system modeled by

$$\dot{x} = D_\sigma x, \qquad x(0) \in K. \tag{6.18}$$

For $i = 1, \dots, n$, we denote by $a_p^i \in \mathbb{C}$ the i-th diagonal entry of D_p, that is, $a_p^i = \lambda_i(D_p)$, and write $D_p = \operatorname{diag}(a_p^1, \dots, a_p^n)$. The topological entropy of (6.18) is characterized by the following theorem.[40]

Theorem 6.7. *Consider the switched diagonal system* (6.18)*. Its topological entropy is given by*

$$h(D_\sigma) = \limsup_{T\to\infty} \sum_{i=1}^{n} \bar{a}_i(T) \tag{6.19}$$

where

$$\bar{a}_i(T) := \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \operatorname{Re}(a_p^i)\tau_p(t), \qquad i = 1, \dots, n \tag{6.20}$$

for a time horizon $T \geq 0$.

[39] In the following analysis, we extend the definition of the switched system (6.1) to the complex-valued state space $\mathbb{C}^{n\times n}$.

[40] Compared with the results for switched scalar systems, Theorem 6.7 corresponds to Lemma 6.2, while Proposition 6.10 and Corollary 6.11 below correspond to Theorem 6.4 and Corollary 6.5, respectively; see also the discussion before Corollary 6.12.

Proof. For initial points $x, y \in K$ denoted by $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$, the corresponding solutions at time t satisfy that

$$\|\xi_\sigma(y,t) - \xi_\sigma(x,t)\| = \left\| e^{\sum_{p\in\mathcal{P}} D_p \tau_p(t)}(y-x) \right\| = \max_{1\le i\le n} e^{\sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t)} |y^i - x^i|.$$

For brevity, define

$$\bar{\eta}_i(T) := \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t) = \bar{a}_i(T)\,T, \qquad i = 1, \ldots, n$$

for a time horizon $T \ge 0$. Then

$$\max_{t\in[0,T]} \|\xi_\sigma(y,t) - \xi_\sigma(x,t)\| = \max_{1\le i\le n} e^{\bar{\eta}_i(T)} |y^i - x^i|.$$

First, we consider the formula of topological entropy (6.5) in terms of spanning sets, and prove the upper bound. Following (6.4), a finite set of points $E = \{x_1, \ldots, x_N\} \subset K$ with elements denoted by $x_k = (x_k^1, \ldots, x_k^n)$ is (T,ε)-spanning if and only if for each $x = (x^1, \ldots, x^n) \in K$, there is some $x_k \in E$ such that

$$|x^i - x_k^i| < e^{-\bar{\eta}_i(T)}\varepsilon \qquad \forall\, i \in \{1, \ldots, n\}.$$

Recall that the initial set K is taken to be the closed unit ball (cube) centered at the origin. Consider the sets defined by

$$E_1^i := \left\{-1,\, -1 + e^{-\bar{\eta}_i(T)}\varepsilon,\, \ldots,\, -1 + \left\lfloor 2e^{\bar{\eta}_i(T)}/\varepsilon \right\rfloor e^{-\bar{\eta}_i(T)}\varepsilon,\, 1\right\}$$

for $i = 1, \ldots, n$. It is then straightforward to verify that $E_1^1 \times \cdots \times E_1^n$ is a (T,ε)-spanning set. Hence the minimal cardinality of a (T,ε)-spanning set satisfies that

$$S(D_\sigma, \varepsilon, T) \le \prod_{i=1}^n \left(\left\lfloor 2e^{\bar{\eta}_i(T)}/\varepsilon \right\rfloor + 2 \right) \le (2/\varepsilon + 2)^n e^{\sum_{i=1}^n \bar{\eta}_i(T)}.$$

Substituting the previous bound into (6.5), we obtain that

$$\begin{aligned}
h(D_\sigma) &\le \lim_{\varepsilon \searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log\left((2/\varepsilon + 2)^n e^{\sum_{i=1}^n \bar{\eta}_i(T)}\right) \\
&= \limsup_{T\to\infty} \sum_{i=1}^n \frac{\bar{\eta}_i(T)}{T} \\
&= \limsup_{T\to\infty} \sum_{i=1}^n \bar{a}(T).
\end{aligned}$$

Second, we consider the formula of topological entropy (6.8) in terms of separated sets, and prove the lower bound. Following (6.7), a finite set of points $E = \{x_1, \dots, x_N\} \subset K$ with elements denoted by $x_k = (x_k^1, \dots, x_k^n)$ is (T, ε)-separated if and only if for each pair $x_k, x_l \in E$ satisfies that

$$|x_k^i - x_l^i| \ge e^{-\bar{\eta}_i(T)}\varepsilon$$

for some $i \in \{1, \dots, n\}$. Recall that the initial set K is taken to be the closed unit ball (cube) centered at the origin. Consider the sets defined by

$$E_2^i := \left\{-1,\ -1 + e^{-\bar{\eta}_i(T)}\varepsilon,\ \dots,\ -1 + \left\lfloor 2e^{\bar{\eta}_i(T)}/\varepsilon \right\rfloor e^{-\bar{\eta}_i(T)}\varepsilon\right\}$$

for $i = 1, \dots, n$. It is then straightforward to verify that $E_2^1 \times \cdots \times E_2^n$ is a (T, ε)-separated set. Hence the maximal cardinality of a (T, ε)-separated set satisfies that

$$N(D_\sigma, \varepsilon, T) \ge \prod_{i=1}^n \left(\left\lfloor 2e^{\bar{\eta}_i(T)}/\varepsilon \right\rfloor + 1\right) > (2/\varepsilon)^n e^{\sum_{i=1}^n \bar{\eta}_i(T)}.$$

Substituting the previous bound into (6.8), we obtain that

$$\begin{aligned}
h(D_\sigma) &\ge \lim_{\varepsilon \searrow 0} \limsup_{T\to\infty} \frac{1}{T} \log\left((2/\varepsilon)^n e^{\sum_{i=1}^n \bar{\eta}_i(T)}\right) \\
&= \limsup_{T\to\infty} \sum_{i=1}^n \frac{\bar{\eta}_i(T)}{T} \\
&= \limsup_{T\to\infty} \sum_{i=1}^n \bar{a}_i(T).
\end{aligned}$$ □

While the formula (6.19) in Theorem 6.7 is not very explicit, it gives the exact value of the topological entropy $h(D_\sigma)$. In the following, we use (6.19) to establish more useful but in general more conservative bounds of $h(D_\sigma)$.

First, we estimate $h(D_\sigma)$ based on the entropies $h(D_p)$ and traces $\mathrm{tr}(D_p)$ of individual modes.

Proposition 6.8. *Consider the switched diagonal system* (6.18)*.*

1. *Its topological entropy is upper bounded by the limit supremum of the asymptotic average of the topological entropies of individual modes, that is,*
$$h(D_\sigma) \leq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} h(D_p)\rho_p(t). \tag{6.21}$$
2. *Its topological entropy is lower bounded by the limit supremum of the asymptotic average of the traces of individual modes, that is,*
$$h(D_\sigma) \geq \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(D_p)\rho_p(t).$$

Proof. 1. Consider the auxiliary switched diagonal system

$$\dot{x} = R_\sigma x, \qquad x(0) \in K \tag{6.22}$$

with the family of diagonal matrices $\{R_p = \mathrm{diag}(r_p^1, \dots, r_p^n) : p \in \mathcal{P}\}$ defined by

$$r_p^i := \max\{\mathrm{Re}(a_p^i),\, 0\} \geq 0, \qquad i = 1, \dots, n.$$

Following the formula (6.12) for topological entropy of LTI systems, the entropies of corresponding individual modes of (6.18) and (6.22) are the same, that is,

$$h(R_p) = \sum_{i=1}^{n} r_p^i = \sum_{i=1}^{n} \max\{\mathrm{Re}(a_p^i),\, 0\} = h(D_p) \qquad \forall\, p \in \mathcal{P}.$$

Then (6.19) implies that[41]

$$\begin{aligned} h(R_\sigma) &= \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} r_p^i \tau_p(t) \\ &\geq \limsup_{T\to\infty} \sum_{i=1}^{n} \frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t) \\ &= h(D_\sigma). \end{aligned}$$

Moreover, for each $i \in \{1,\ldots,n\}$, the sum $\sum_{p\in\mathcal{P}} r_p^i \tau_p(t)$ is nondecreasing with respect to t; thus

$$\frac{1}{T} \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} r_p^i \tau_p(t) = \frac{1}{T} \sum_{p\in\mathcal{P}} r_p^i \tau_p(T) = \sum_{p\in\mathcal{P}} r_p^i \rho_p(T).$$

Therefore,

$$\begin{aligned} h(D_\sigma) \leq h(R_\sigma) &= \limsup_{T\to\infty} \sum_{i=1}^{n} \sum_{p\in\mathcal{P}} r_p^i \rho_p(T) \\ &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \left(\sum_{i=1}^{n} r_p^i \right) \rho_p(t) \\ &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} h(D_p)\rho_p(t). \end{aligned}$$

2. From (6.20), it follows that for each $i \in \{1,\ldots,n\}$,

$$\bar{a}_i(T) \geq \frac{1}{T} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(T) = \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(T) \qquad \forall\, T \geq 0.$$

Substituting the previous bound into (6.19), we obtain that

$$\begin{aligned} h(D_\sigma) &\geq \limsup_{T\to\infty} \sum_{i=1}^{n} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(T) \\ &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \left(\sum_{i=1}^{n} \mathrm{Re}(a_p^i) \right) \rho_p(t) \\ &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(D_p)\rho_p(t). \end{aligned}$$

□

[41] In particular, as the complex eigenvalues of D_p occur in complex conjugate pairs, the trace $\mathrm{tr}(D_p) = \sum_{i=1}^{n} a_p^i = \sum_{i=1}^{n} \mathrm{Re}(a_p^i) \in \mathbb{R}$.

From the formula (6.12) for topological entropy of LTI systems, for each individual mode p,

$$h(D_p) = \sum_{i=1}^{n} \max\{\mathrm{Re}(a_p^i), 0\} \geq \sum_{i=1}^{n} a_p^i = \mathrm{tr}(D_p).$$

Hence, in general, the upper and lower bounds in Proposition 6.8 coincide if and only if all diagonal entries in all modes have nonnegative real parts, that is, $\mathrm{Re}(a_p^i) \geq 0$ for all $i \in \{1, \dots, n\}$ and $p \in \mathcal{P}$.

Corollary 6.9. *Consider the switched diagonal system* (6.18)*. Suppose that* $\mathrm{Re}(a_p^i) \geq 0$ *for all* $p \in \mathcal{P}$ *and* $i \in \{1, \dots, n\}$*. Then the topological entropy of* (6.18) *is given by*

$$h(D_\sigma) = \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{tr}(D_p)\rho_p(t).$$

Second, we estimate $h(D_\sigma)$ based on the entropies of the switched scalar systems corresponding to individual scalar components.

Proposition 6.10. *Consider the switched diagonal system* (6.18)*. Its topological entropy is upper bounded by the sum of the entropies of the switched scalar systems corresponding to individual scalar components, that is,*

$$h(D_\sigma) \leq \sum_{i=1}^{n} \max\{\hat{a}_i,\, 0\} \tag{6.23}$$

with the asymptotic averages $\hat{a}_i$ *defined by*

$$\hat{a}_i := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(t), \qquad i = 1, \dots, n. \tag{6.24}$$

Proof. The proof follows from Theorem 6.7, the subadditivity of limit supremum, and Lemma 6.1. More specifically, from (6.19) and the subadditivity, it follows that

$$h(D_\sigma) = \limsup_{T\to\infty} \sum_{i=1}^{n} \bar{a}_i(T) \leq \sum_{i=1}^{n} \limsup_{T\to\infty} \bar{a}_i(T).$$

For $i = 1, \dots, n$, invoke Lemma 6.1 for the family of scalars $\{\mathrm{Re}(a_p^i) : p \in \mathcal{P}\}$.

Then (6.16) holds with $\bar{a} \equiv \bar{a}_i$ and $\hat{a} = \hat{a}_i$, that is,

$$\limsup_{T\to\infty} \bar{a}_i(T) = \max\{\hat{a}_i,\, 0\}.$$

Combining the results above gives (6.23). $\square$

Following the subadditivity of limit supremum, we obtain that

$$\begin{aligned}
\hat{a}_i &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\rho_p(t) \\
&\le \sum_{p\in\mathcal{P}} \Big(\limsup_{t\to\infty} \mathrm{Re}(a_p^i)\rho_p(t) \Big) \\
&\le \sum_{p\in\mathcal{P}} \Big(\max\{\mathrm{Re}(a_p^i),\, 0\} \limsup_{t\to\infty} \rho_p(t) \Big),
\end{aligned}$$

and thus the following corollary of Proposition 6.10, which is more useful but in general more conservative.[42]

Corollary 6.11. *Consider the switched diagonal system* (6.18). *Its topological entropy satisfies that*

$$h(D_\sigma) \le \sum_{i=1}^{n} \sum_{p\in\mathcal{P}} \max\{\mathrm{Re}(a_p^i)\hat{\rho}_p,\, 0\} = \sum_{p\in\mathcal{P}} h(D_p)\hat{\rho}_p$$

with the asymptotic active rate $\hat{\rho}_p$ defined by (6.13).

We observe from (6.21) that

$$h(D_\sigma) \le \Big(\max_{p\in\mathcal{P}} h(D_p) \Big) \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \rho_p(t) = \max_{p\in\mathcal{P}} h(D_p),$$

while in general it is possible that

$$\sum_{p\in\mathcal{P}} h(D_p)\hat{\rho}_p > \max_{p\in\mathcal{P}} h(D_p).$$

Also, due to the subadditivity of limit supremum, Proposition 6.10 only gives an upper bound of the topological entropy, instead of its exact value as in Theorem 6.4 for switched scalar systems. In general, (6.23) holds with

[42] Corollary 6.11 can also be established based on (6.21) and the subadditivity of limit supremum via a similar argument.

equality if and only if the limits $\lim_{t\to\infty} \rho_p(t)$ exist for all $p \in \mathcal{P}$ (e.g., when the switching signal σ is periodic).

Corollary 6.12. *Consider the switched diagonal system* (6.18)*. If the active rate $\rho_p(t)$ converges as $t \to \infty$ for each mode p, then the asymptotic active rate satisfies that $\hat{\rho}_p = \lim_{t\to\infty} \rho_p(t)$; thus the topological entropy of* (6.18) *is given by*

$$h(D_\sigma) = \sum_{i=1}^{n} \max\{\hat{a}_i, 0\},$$

where the asymptotic averages $\hat{a}_i$ are defined by (6.24) *and now satisfy that $\hat{a}_i = \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\hat{\rho}_p$. Equivalently, $h(D_\sigma)$ equals the topological entropy of the LTI system* (6.11) *with the matrix $A = \sum_{p\in\mathcal{P}} D_p\hat{\rho}_p$.*

The following result shows that the asymptotic averages $\hat{a}_1, \ldots, \hat{a}_n$ defined in Proposition 6.10 can be used to establish GES of (6.18).

Proposition 6.13. *Consider the switched diagonal system* (6.18)*. If the asymptotic averages $\hat{a}_1, \ldots, \hat{a}_n$ defined by* (6.24) *are all negative, then* (6.18) *is GES.*

Proof. The proof is along the lines of the proof of Proposition 6.6, and is omitted here. □

6.6 Switched triangular systems

Consider the switched linear system (6.10), and the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ generated by the family of matrices $\{A_p : p \in \mathcal{P}\}$ with respect to the standard Lie bracket $[\cdot,\cdot]$ defined by

$$[A_p, A_q] := A_pA_q - A_qA_p.$$

Suppose that $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable. Then there exists a (possibly complex) linear change of coordinates under which the matrices in $\{A_p : p \in \mathcal{P}\}$ are all upper-triangular (Lie's theroem [6, Proposition 2.8]). In view of this result, we assume, without loss of generality, that every A_p is upper-triangular, and denote it by

$$U_p = \begin{bmatrix} a_p^1 & b_p^{1,2} & \cdots & b_p^{1,n-1} & b_p^{1,n} \\ 0 & a_p^2 & \cdots & b_p^{2,n-1} & b_p^{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_p^{n-1} & b_p^{n-1,n} \\ 0 & 0 & \cdots & 0 & a_p^n \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Then (6.10) becomes a switched triangular system modeled by

$$\dot{x} = U_\sigma x, \qquad x(0) \in K. \tag{6.25}$$

As in the case with diagonal matrices, $a_p^i = \lambda_i(U_p)$ for all $i \in \{1, \dots, n\}$. The topological entropy of (6.25) is characterized by the following theorem.

Theorem 6.14. *Consider the switched triangular system* (6.25)*. Its topological entropy satisfies that*

$$h(U_\sigma) \le n \max\{\hat{a}_1, 0\} + \sum_{i=2}^{n} (n+1-i) \max\{\tilde{a}_i, 0\} \tag{6.26}$$

with the asymptotic averages

$$\hat{a}_1 := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^1)\rho_p(t) \tag{6.27}$$

and

$$\tilde{a}_i := \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} (\mathrm{Re}(a_p^i) - \mathrm{Re}(a_p^{i-1}))\rho_p(t), \qquad i = 2, \dots, n. \tag{6.28}$$

Proof. First, for an initial point $x_0 \in K$ denoted by $x_0 = (x_0^1, \dots, x_0^n)$, we derive a formula of the corresponding solution $\xi_\sigma(x_0, t)$ at time t. For brevity, define

$$\eta_i(t) := \sum_{p\in\mathcal{P}} a_p^i \tau_p(t), \qquad i = 1, \dots, n$$

and

$$\nu_{i,j}(t) := \sum_{p\in\mathcal{P}} (a_p^j - a_p^i)\tau_p(t) = \eta_j(t) - \eta_i(t), \qquad 1 \le i < j \le n$$

for $t \geq 0$.

Lemma 6.3. *For $k = 0, \ldots, n-1$, the $(n-k)$-th component of the solution $\xi_\sigma(x_0, t)$ at time t is given by*

$$\xi_\sigma^{n-k}(x_0,t) = e^{\eta_{n-k}(t)}\Bigg(x_0^{n-k} + \sum_{l=0}^{k-1}\Bigg(x_0^{n-l} \\ \times \sum_{i=1}^{k-l}\sum_{(c_0,\ldots,c_i)\in\mathcal{C}_{k,l,i}} \int_0^t\int_0^{s_1}\cdots\int_0^{s_{i-1}}\prod_{j=1}^{i} b_{\sigma(s_j)}^{c_{j-1},c_j} e^{\nu_{c_{j-1},c_j}(s_j)}\mathrm{d}s_j\Bigg)\Bigg), \quad (6.29)$$

with the sets

$$\mathcal{C}_{k,l,i} := \{(c_0,\ldots,c_i)\in\mathbb{Z}^{i+1} : n-k = c_0 < c_1 < \cdots < c_{i-1} < c_i = n-l\} \quad (6.30)$$

for $l = 0, \ldots, k-1$ and $i = 1, \ldots, k-l$.

Proof. See Appendix A.11. □

Second, consider initial points $x, y \in K$ denoted by $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$. For a time horizon $T \geq 0$, we estimate $\|\xi_\sigma(y,t) - \xi_\sigma(x,t)\|$ over $t \in [0,T]$ based on the formula (6.29). For brevity, let $b_m := \max_{p\in\mathcal{P},\,1\leq i<j\leq n} |b_p^{i,j}|$, and define

$$\bar{\eta}_i(T) := \max_{t\in[0,T]} \mathrm{Re}(\eta_i(t)) = \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^i)\tau_p(t), \qquad i = 1,\ldots,n$$

and

$$\bar{\nu}_{i,j}(T) := \max_{t\in[0,T]} \mathrm{Re}(\nu_{i,j}(t)) = \max_{t\in[0,T]} \sum_{p\in\mathcal{P}} (\mathrm{Re}(a_p^j) - \mathrm{Re}(a_p^i))\tau_p(t)$$

for $1 \leq i < j \leq n$.

Lemma 6.4. *The solutions corresponding to $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ satisfy that*

$$\max_{t\in[0,T]} \|\xi_\sigma(y,t) - \xi_\sigma(x,t)\| \leq \sum_{i=1}^{n}\left(|x^i - y^i|\, e^{\bar{\eta}_1(T) + \sum_{j=1}^{i-1}\bar{\nu}_{j,j+1}(T)}\hat{P}_i(T)\right) \quad (6.31)$$

with a family of positive and increasing polynomials $\hat{P}_i(T)$, $i = 1, \ldots, n$.

Proof. See Appendix A.12. □

Third, we consider the formula of entropy (6.5) in terms of spanning sets, and establish an upper bound of $h(U_\sigma)$. Recall that the initial set K is taken to be the closed unit ball (cube) centered at the origin. Consider the sets defined by

$$E^i := \{-1,\, -1 + \varepsilon/\theta_i(T),\, \ldots,\, -1 + \lfloor 2\theta_i(T)/\varepsilon \rfloor \varepsilon/\theta_i(T),\, 1\}$$

for $l = 1, \ldots, n$ with

$$\theta_i(T) := n e^{\bar{\eta}_1(T) + \sum_{j=1}^{i-1} \bar{\nu}_{j,j+1}(T)} \hat{P}_i(T).$$

It is then straightforward to verify that $E^1 \times \cdots \times E^n$ is a (T, ε)-spanning set. Hence the minimal cardinality of a (T, ε)-spanning set satisfies that

$$\begin{aligned} S(U_\sigma, \varepsilon, T) &\leq \prod_{i=1}^{n} \left(\lfloor 2\theta_i(T)/\varepsilon \rfloor + 2 \right) \\ &\leq e^{\sum_{i=1}^{n} (\bar{\eta}_1(T) + \sum_{j=1}^{i-1} \bar{\nu}_{j,j+1}(T))} \prod_{i=1}^{n} (2n\hat{P}_i(T)/\varepsilon + 2) \end{aligned}$$

Substituting the previous bound into (6.5), we obtain that

$$\begin{aligned} h(U_\sigma) &\leq \lim_{\varepsilon \searrow 0} \limsup_{T \to \infty} \frac{1}{T} \log \left(e^{\sum_{i=1}^{n} (\bar{\eta}_1(T) + \sum_{j=1}^{i-1} \bar{\nu}_{j,j+1}(T))} \prod_{i=1}^{n} (2n\hat{P}_i(T)/\varepsilon + 2) \right) \\ &= \limsup_{T \to \infty} \sum_{i=1}^{n} \frac{1}{T} \left(\bar{\eta}_1(T) + \sum_{j=1}^{i-1} \bar{\nu}_{j,j+1}(T) \right). \end{aligned}$$

Finally, we establish the upper bound (6.26) based on the subadditivity of limit supremum and Lemma 6.1. Following the subadditivity of limit supremum, we obtain that

$$h(U_\sigma) \leq n \limsup_{T \to \infty} \frac{\bar{\eta}_1(T)}{T} + \sum_{j=1}^{n-1} \left((n-j) \limsup_{T \to \infty} \frac{\bar{\nu}_{j,j+1}(T)}{T} \right).$$

Invoke Lemma 6.1 for the family of scalars $\{\mathrm{Re}(a_p^1) : p \in \mathcal{P}\}$. Then (6.16) holds with $\bar{a}(T) \equiv \bar{\eta}_1(T)/T$ and $\hat{a} = \hat{a}_1$ for the constant $\hat{a}_1$ defined by (6.27), that is,

$$\limsup_{T\to\infty} \frac{\bar{\eta}_1(T)}{T} = \max\{\hat{a}_1,\, 0\}.$$

Also, for $i = 2, \dots, n$, invoke Lemma 6.1 for the family of scalars $\{\mathrm{Re}(a_p^i) - \mathrm{Re}(a_p^{i-1}) : p \in \mathcal{P}\}$. Then (6.16) holds with $\bar{a}(T) \equiv \bar{\nu}_{i-1,i}(T)/T$ and $\hat{a} = \tilde{a}_i$ for the constant $\tilde{a}_i$ defined by (6.28), that is,

$$\limsup_{T\to\infty} \frac{\bar{\nu}_{i-1,i}(T)}{T} = \max\{\tilde{a}_i,\, 0\}.$$

Combining the results above gives (6.26). $\square$

Following the subadditivity of limit supremum, we obtain that

$$\begin{aligned}
\hat{a}_1 &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} \mathrm{Re}(a_p^1)\rho_p(t) \\
&\le \sum_{p\in\mathcal{P}} \Big(\limsup_{t\to\infty} \mathrm{Re}(a_p^1)\rho_p(t) \Big) \\
&\le \sum_{p\in\mathcal{P}} \Big(\max\{\mathrm{Re}(a_p^1),\, 0\} \limsup_{t\to\infty} \rho_p(t) \Big)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{a}_i &= \limsup_{t\to\infty} \sum_{p\in\mathcal{P}} (\mathrm{Re}(a_p^i) - \mathrm{Re}(a_p^{i-1}))\rho_p(t) \\
&\le \sum_{p\in\mathcal{P}} \Big(\max\{\mathrm{Re}(a_p^i) - \mathrm{Re}(a_p^{i-1}),\, 0\} \limsup_{t\to\infty} \rho_p(t) \Big)
\end{aligned}$$

for $i = 2, \dots, n$. Hence we obtain the following corollary of Theorem 6.14, which is more useful but in general more conservative.

Corollary 6.15. *Consider the switched triangular system* (6.25)*. Its topological entropy satisfies that*

$$\begin{aligned}
h(D_\sigma) \le \sum_{p\in\mathcal{P}} \Bigg(\hat{\rho}_p \Bigg(& n \max\{\mathrm{Re}(a_p^1),\, 0\} \\
& + \sum_{i=2}^{n} (n+1-i) \max\{(\mathrm{Re}(a_p^i) - \mathrm{Re}(a_p^{i-1})),\, 0\} \Bigg) \Bigg)
\end{aligned}$$

with the asymptotic active rate $\hat{\rho}_p$ *defined by* (6.13).

6.7 Future work

The entropy notion in this chapter was formulated for switched systems with a known switching signal. For switched systems with an unknown switching signal, a different notion of entropy is needed to capture the additional uncertainty about the state, and to model the extra information necessary for feedback stabilization. In Chapter 5, such extra information were collected by monitoring the active modes at sampling times, and our communication and control strategy achieved feedback stabilization. Hence the sufficient data rate (5.3) should give an upper bound of the entropy notion to be defined.

Another topic for future research is to reconcile the switching characterizations for entropy computation and for control design. More specifically, the entropy computation in this chapter is based on the notion of active rates (i.e., the proportion of time in which each mode is active). Such a property is rarely seen in the literature of switched control systems, and incorporating it into the control design procedure may lead to more precise data-rate bounds.

Appendix A

Technical proofs

A.1 Proof of Lemma 3.1

We first construct a hybrid arc z_i and a hybrid input $\tilde{d}_i$ in a recursive manner, and then prove that $(z_i, \tilde{d}_i)$ is a complete solution pair of (3.14). Denote by $\{t_{i,k} : k \in \mathbb{N}\}$ with $t_{i,0} := 0$ the set of switches of σ_i. For $T \geq 0$, define $K_{i,T} := \max\{k \in \mathbb{N} : t_{i,k} \leq T\}$, and

$$E_{i,T} := \left(\bigcup_{k=0}^{K_{i,T}-1} ([t_{i,k}, t_{i,k+1}], k) \right) \cup ([t_{i,K_{i,T}}, T], K_{i,T}).$$

Then $E_{i,T}$ is a compact hybrid time domain. Consider the hybrid arc $z_i = (\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ and the hybrid input $\tilde{d}_i = (\tilde{u}_i, \tilde{w}_i)$ defined so that $\operatorname{dom} z_i = \operatorname{dom} \tilde{d}_i$, and for each $T \geq 0$,

- $\operatorname{dom} z_i \cap ([0, T] \times \{0, 1, \ldots, K_{i,T}\}) = E_{i,T}$;
- for each $(t, k) \in E_{i,T}$, $\tilde{x}_i(t, k) = x_i(t)$, $\tilde{u}_i(t, k) = x_j(t)$, $\tilde{w}_i(t, k) = w_i(t)$, and $\tilde{\sigma}_i(t, k) = \sigma_i(t_{i,k})$;
- for each $(t, k) \in E_{i,T}$,

$$\tau_i(t, k) = \begin{cases} \Theta_i, & \text{if } k = 0; \\ \min\{\Theta_i,\ \bar{\tau}_{s,i}(t, k)\}, & \text{if } k > 0,\ \sigma_i(t_{i,k}) \in \mathcal{P}_{s,i}; \\ \bar{\tau}_{u,i}(t, k), & \text{if } k > 0,\ \sigma_i(t_{i,k}) \in \mathcal{P}_{u,i}, \end{cases} \tag{A.1}$$

  where

$$\begin{aligned} \bar{\tau}_{s,i}(t, k) &:= \tau_i(t_{i,k}, k-1) - \ln \mu_i + \theta_i(t - t_{i,k}), \\ \bar{\tau}_{u,i}(t, k) &:= \tau_i(t_{i,k}, k-1) - \ln \mu_i + (\theta_i - (\lambda_{s,i} + \lambda_{u,i}))(t - t_{i,k}). \end{aligned} \tag{A.2}$$

Based on Assumption 3.1–3.3 and the inequality (3.6), in the following we show that $(z_i, \tilde{d}_i)$ is a complete solution pair of the hybrid system (3.14). Clearly, z_i and $\tilde{d}_i$ are defined on the same hybrid time domain, and satisfy the inclusions in (3.14) by construction. Hence it remains to prove that z_i is complete and $z_i(t,k) \in \mathcal{C}_i \cup \mathcal{D}_i = \mathcal{Z}_i$ for all $(t,k) \in \operatorname{dom} z_i$, which amounts to showing the following properties.

1. From Assumption 3.1, it follows that the subsystem Σ_i in (3.2) is forward complete. Thus its solution is defined for all $t \geq 0$. Consequently, $\operatorname{dom} z_i$ is unbounded in the t-direction, and $\tilde{x}_i(t,k) \in \mathbb{R}^{n_i}$ for all $(t,k) \in \operatorname{dom} z_i$.
2. As the range of the switching signal σ_i is $\mathcal{P}_i$, it follows that $\tilde{\sigma}_i(t,k) \in \mathcal{P}_i$ for all $(t,k) \in \operatorname{dom} z_i$.
3. From (A.1) and (A.2), it follows that $\tau_i(t,k) \leq \Theta_i$ for all $(t,k) \in \operatorname{dom} z_i$ (in particular, the inequality in (3.16) implies that $\tau_i(\cdot,k)$ is decreasing on $[t_{i,k}, t_{i,k+1}]$ if $\sigma_i(t_{i,k}) \in \mathcal{P}_{u,i}$). Meanwhile, for each $(t,k) \in \operatorname{dom} z_i$, let[43]

$$(t_0, k_0) := \operatorname*{argmax}_{(s,l) \in \operatorname{dom} z_i,\, (s,l) \preceq (t,k)} \{s + l : \tau_i(s,l) = \Theta_i\}.$$

   Substituting (A.2) into (A.1), we obtain that

$$\begin{aligned}
\tau_i(t,k) &= \tau_i(t_0,k_0) - N(t,t_0)\ln\mu_i + \theta_i(t-t_0) + (\lambda_{s,i} + \lambda_{u,i})T_{u,i}(t,t_0) \\
&\geq \Theta_i - (N_{0,i} + (t-t_0)/\tau_{a,i})\ln\mu_i \\
&\quad + \theta_i(t-t_0) - (\lambda_{s,i} + \lambda_{u,i})(T_{0,i} + \rho_i(t-t_0)) \\
&= (\Theta_i - N_{0,i}\ln\mu_i - T_{0,i}(\lambda_{s,i} + \lambda_{u,i})) \\
&\quad + (\theta_i - \ln\mu_i/\tau_{a,i} - \rho_i(\lambda_{s,i} + \lambda_{u,i}))(t-t_0) \\
&= 0,
\end{aligned}$$

   where the inequality follows from Assumptions 3.2–3.3, and the last equality follows from the definitions (3.8) of Θ_i and (3.16) of θ_i. Thus $\tau_i(t,k) \in [0, \Theta_i]$ for all $(t,k) \in \operatorname{dom} z_i$.

Hence $(z_i, \tilde{d}_i)$ is a complete solution pair of the hybrid system (3.14).[44] $\square$

[43] As $\tau_i(0,0) = \Theta_i$, the hybrid time (t_0, k_0) defined here always exists.

[44] According to [3, Proposition 2.10], for a hybrid system with local existence of solutions, a solution is complete if it has no finite escape time and does not jump out of the union of the jump set and the closure of the flow set. Unfortunately, we cannot apply this result directly since in the hybrid system (3.14), the local existence of solutions is not satisfied everywhere. In particular, for all points $z_i = (\tilde{x}_i, \tilde{\sigma}_i, 0)$ with $\tilde{\sigma}_i \in \mathcal{P}_{u,i}$, the condition (VC) in [3, Proposition 2.10] does not hold. However, the hybrid arcs we constructed will never arrive at such points.

A.2 Proof of Lemma 3.2

We establish the properties in Lemma 3.2 based on the corresponding properties in Assumption 3.1.

First, the fact that $\psi_{1,i}, \psi_{2,i} \in \mathcal{K}_\infty$ implies that $\tilde{\psi}_{1,i}, \tilde{\psi}_{2,i} \in \mathcal{K}_\infty$, and (3.19) follows from (3.3) and the fact that $|z_i|_{\mathcal{A}_i} = |\tilde{x}_i|$ and $\tau_i \in [0, \Theta_i]$.

Second, $\lambda_i > 0$ follows from the inequality in (3.16). As the function $V_i(z_i) = V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i)$ is continuously differentiable in $\tilde{x}_i$ and τ_i, and $\dot{\tilde{\sigma}}_i \equiv 0$, the gradient ∇V_i is well-defined on $\mathcal{Z}_i$. Consider an arbitrary $(z_i, \tilde{u}_i, \tilde{w}_i) \in \mathcal{C}_i$ such that

$$|\tilde{x}_i| = |z_i|_{\mathcal{A}_i} \geq \max\{\phi_i(|\tilde{u}_i|), \phi_i^w(|\tilde{w}_i|), \delta_i\}.$$

Regarding $\tilde{\sigma}_i$, there are two possibilities.

1. If $\tilde{\sigma}_i \in \mathcal{P}_{s,i}$, then (3.4) implies that for all $v_i \in F_i(z_i, \tilde{u}_i, \tilde{w}_i)$,

$$\begin{aligned}
\nabla V_i(z_i) \cdot y_i &\leq \nabla_{\tilde{x}_i} V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \cdot f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{u}_i, \tilde{w}_i) + \nabla_{\tilde{\tau}_i} V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \cdot \theta_i \\
&= \nabla V_{i,\tilde{\sigma}_i}(\tilde{x}_i) \cdot f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{u}_i, \tilde{w}_i) e^{\tau_i} + \theta_i V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i} \\
&\leq -(\lambda_{s,i} - \theta_i) V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i} \\
&= -\lambda_i V_i(z_i).
\end{aligned}$$

2. If $\tilde{\sigma}_i \in \mathcal{P}_{u,i}$, then (3.4) implies that for all $v_i \in F_i(z_i, \tilde{u}_i, \tilde{w}_i)$,

$$\begin{aligned}
\nabla V_i(z_i) \cdot y_i &= \nabla_{\tilde{x}_i} V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \cdot f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{u}_i, \tilde{w}_i) \\
&\quad + \nabla_{\tilde{\tau}_i} V_i(\tilde{x}_i, \tilde{\sigma}_i, \tau_i) \cdot (\theta_i - (\lambda_{s,i} + \lambda_{u,i})) \\
&= \nabla V_{i,\tilde{\sigma}_i}(\tilde{x}_i) \cdot f_{i,\tilde{\sigma}_i}(\tilde{x}_i, \tilde{u}_i, \tilde{w}_i) e^{\tau_i} + (\theta_i - (\lambda_{s,i} + \lambda_{u,i})) V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i} \\
&\leq (\lambda_{u,i} + \theta_i - (\lambda_{s,i} + \lambda_{u,i})) V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i} \\
&= -\lambda_i V_i(z_i).
\end{aligned}$$

Therefore, (3.21) holds.

Last, consider an arbitrary $z_i \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln \mu_i, \Theta_i]$. For each $z_i^+ \in G_i(z_i)$, it holds that $z_i^+ = (\tilde{x}_i, \tilde{p}_i, \tau_i - \ln \mu_i)$ for some $\tilde{p}_i \in \mathcal{P}_i \backslash \{\tilde{\sigma}_i\}$, and (3.5) implies that

$$V_i(y_i) = V_{i,\tilde{p}_i}(\tilde{x}_i) e^{\tau_i - \ln \mu_i} \leq \mu_i V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i - \ln \mu_i} = V_{i,\tilde{\sigma}_i}(\tilde{x}_i) e^{\tau_i} = V_i(z_i),$$

that is, (3.23) holds. □

A.3 Proof of Lemma 3.5

By definition, for each fixed $k \in \mathbb{Z}_+$, the function $z(\cdot, k)$ is absolutely continuous on $\{t : (t, k) \in \operatorname{dom} z\}$. Recall that V is differentiable almost everywhere, and its Clarke derivative V° satisfies (3.29). Following the arguments in [105, p. 99], for each fixed $l \in \mathbb{Z}_+$ satisfying $k_1 \leq l \leq k_2$, the function $V(z(\cdot, l))$ is absolutely continuous on $\bar{I}_l := \{s : (s, l) \in \operatorname{dom} z, (t_1, k_1) \preceq (s, l) \preceq (t_2, k_2)\}$, and satisfies that

$$\frac{\mathrm{d}V(z(s,l))}{\mathrm{d}s} \leq -h(V(z(s,l))) \qquad \text{a.e. on } \bar{I}_l. \tag{A.3}$$

More precisely, for each fixed $l \in \mathbb{Z}_+$, as $z(\cdot, l)$ is absolutely continuous on $\bar{I}_l$ and V is locally Lipschitz, it follows that for almost all $t \in \bar{I}_l$ and all $v = \dot{z}(s, l) \in F(z(s, l), \tilde{w}(s, l))$,

$$\begin{aligned}
\frac{\mathrm{d}V(z(s,l))}{\mathrm{d}s} &= \lim_{t \to 0^+} \frac{V(z(s,l) + tv) - V(z(s,l))}{t} \\
&\leq \limsup_{t \to 0^+,\, y \to z(s,l)} \frac{V(y + tv) - V(y)}{t} \\
&= V^\circ(z(s,l); v) \\
&\leq -h(V(z(s,l))),
\end{aligned}$$

where the last inequality is due to (3.29).

Remark A.1. Consider the case where for at least one $i \in \{1, 2\}$, the function V_i is not constant in any open subset of $\mathcal{Z}_i$, which is rather common in Lyapunov analysis. Then the set $\{z = (z_1, z_2) : \chi(V_1(z_1)) = V_2(z_2)\}$ has zero Lebesgue measure. (If not, consider an open ball contained in this set centered at (z_1^*, z_2^*). Then $\chi(V_1(z_1^*)) = V_2(z_2^*)$. Without loss of generality, assume that V_2 is not constant in any open subset of $\mathcal{Z}_2$. Then there exists another point (z_1^*, z_2') in this open ball such that $z_2' \neq z_2^*$ and $V_2(z_2') \neq V_2(z_2^*) = \chi(V_1(z_1^*))$, which contradicts the assumption that this open ball is contained in the aforementioned set.) Consequently, by virtue of [51, Lemma 1], it is sufficient to conclude (A.3) from properties of V outside this set, and the corresponding analysis in the proof of item 2 in Lemma 3.4 becomes unnecessary.

The remaining proof is along the lines of the proof of the comparison principle for hybrid systems in [23, Lemma C.1]. The only significant difference is that, at jump times, the function V here satisfies (3.33), which is weaker than the second condition in [23, Lemma C.1], but sufficient for the estimate (3.41) using class $\mathcal{KL}$ function. Indeed, if the hybrid arc z jumps at $(t,k) \in \operatorname{dom} z$ (i.e., $(t,k+1) \in \operatorname{dom} z$), then (3.33) implies (recall that $h \in \mathcal{PD}$)

$$\int_{V(z(t,k))}^{V(z(t,k+1))} \frac{\mathrm{d}\tau}{h(\tau)} \leq 0.$$

Following the proof of [23, Lemma C.1], we see that (A.3) implies

$$\int_{V(z(t_1,k_1))}^{V(z(t_2,k_2))} \frac{\mathrm{d}\tau}{h(\tau)} \leq -(t_2 - t_1),$$

from which our claim follows exactly as in the proof of [87, Lemma 4.4]. □

A.4 Proof of Lemma 5.1

For all square matrices X and Y,

$$\begin{aligned}
&\|e^{X+Y} - e^X\| \\
&= \left\| \sum\nolimits_{m=0}^{\infty} \frac{1}{m!}((X+Y)^m - X^m) \right\| \\
&= \left\| \sum\nolimits_{m=0}^{\infty} \frac{1}{m!}\left(\sum\nolimits_{i=0}^{m} \binom{m}{i} X^{m-i}Y^i - X^m \right) \right\| \\
&= \left\| \sum\nolimits_{m=1}^{\infty} \frac{1}{m!} \sum\nolimits_{i=1}^{m} \binom{m}{i} X^{m-i}Y^i \right\| \\
&= \left\| \sum\nolimits_{m=1}^{\infty} \frac{1}{(m-1)!} \sum\nolimits_{j=0}^{m-1} \binom{m-1}{j} \frac{X^{m-1-j}Y^{j+1}}{j+1} \right\| \\
&\leq \sum\nolimits_{m=1}^{\infty} \frac{1}{(m-1)!} \sum\nolimits_{j=0}^{m-1} \binom{m-1}{j} \|X\|^{m-1-j}\|Y\|^{j+1} \\
&= \sum\nolimits_{l=0}^{\infty} \frac{1}{l!}(\|X\| + \|Y\|)^l \|Y\| \\
&= e^{\|X\|+\|Y\|}\|Y\|.
\end{aligned}$$

□

A.5 Proof of Lemma 5.2

We first recall the following useful facts from linear algebra. From the definition of ∞-norm, it follows that

$$\|v\|^2 \leq v^\top v, \quad v_1^\top v_2 \leq n\|v_1\|\|v_2\| \tag{A.4}$$

for all vectors $v, v_1, v_2 \in \mathbb{R}^n$. Also,

$$\underline{\lambda}(S) \leq \frac{v^\top S v}{v^\top v} \leq \overline{\lambda}(S) \qquad \forall\, v \in \mathbb{R}^n \backslash \{0\} \tag{A.5}$$

for all symmetric matrices $S \in \mathbb{R}^{n\times n}$ (i.e., $S^\top = S$).

At t_{k+1}, the condition (5.19) implies that

$$V_p(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_p x_{k+1}^* + \rho_p E_{k+1}^2$$

with x_{k+1}^* and E_{k+1} given by (5.22) and (5.21), respectively. First, (5.22) can be rewritten as

$$x_{k+1}^* = S_p c_k = S_p(x_k^* + \Delta_k)$$

with $\Delta_k := c_k - x_k^*$. Then

$$\begin{aligned}
&(x_{k+1}^*)^\top P_p x_{k+1}^* \\
&= (x_k^* + \Delta_k)^\top S_p^\top P_p S_p (x_k^* + \Delta_k) \\
&= (x_k^*)^\top S_p^\top P_p S_p x_k^* + 2(x_k^*)^\top S_p^\top P_p S_p \Delta_k + \Delta_k^\top S_p^\top P_p S_p \Delta_k \\
&\leq (x_k^*)^\top (P_p - Q_p) x_k^* + 2n_x \|x_k^*\| \|S_p^\top P_p S_p\| \|\Delta_k\| + n_x \|S_p^\top P_p S_p\| \|\Delta_k\|^2,
\end{aligned}$$

where the last inequality follows from (5.39) and (A.4). Moreover, (A.4) and (A.5) imply that

$$\begin{aligned}
(x_k^*)^\top Q_p x_k^* &\geq \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \geq \frac{\underline{\lambda}(Q_p)}{\overline{\lambda}(P_p)} (x_k^*)^\top P_p x_k^*, \\
(x_k^*)^\top Q_p x_k^* &\geq \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \geq \underline{\lambda}(Q_p)\|x_k^*\|^2.
\end{aligned}$$

Combining the inequalities above and completing the square, we obtain that

$$
\begin{aligned}
(x_{k+1}^*)^\top P_p x_{k+1}^* \leq{}& \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* - \frac{1}{2}\underline{\lambda}(Q_p)\|x_k^*\|^2 \\
&+ 2n_x\|x_k^*\|\|S_p^\top P_p S_p\|\|\Delta_k\| + n_x\|S_p^\top P_p S_p\|\|\Delta_k\|^2 \\
\leq{}& \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \chi_p\|\Delta_k\|^2 \\
&- \left(\sqrt{\frac{1}{2}\underline{\lambda}(Q_p)}\|x_k^*\| - \frac{\sqrt{2}n_x\|S_p^\top P_p S_p\|}{\sqrt{\underline{\lambda}(Q_p)}}\|\Delta_k\|\right)^2 \\
\leq{}& \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2}\chi_p E_k^2,
\end{aligned}
$$

where the last inequality follows partially from (5.11). Second, from (5.21) and Young's inequality with ϕ_1, it follows that

$$
E_{k+1}^2 = \left(\frac{\Lambda_p}{N}E_k + \Phi_p(\tau_s)\delta_k\right)^2 \leq \frac{(1+\phi_1)\Lambda_p^2}{N^2}E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\Phi_p(\tau_s)^2\delta_k^2.
$$

Therefore,

$$
\begin{aligned}
V_p(x_{k+1}^*, E_{k+1}) \leq{}& \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \left(\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p}\right. \\
&\left. + \frac{(1+\phi_1)\Lambda_p^2}{N^2}\right)\rho_p E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\rho_p\Phi_p(\tau_s)^2\delta_k^2,
\end{aligned}
$$

which in turn implies (5.43). □

A.6 Proof of Lemma 5.3

At t_{k+1}, the condition (5.23) implies that

$$
V_q(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_q x_{k+1}^* + \rho_q E_{k+1}^2
$$

with x_{k+1}^* and E_{k+1} given by (5.29) and (5.28), respectively. First, (5.29) can be rewritten as

$$
x_{k+1}^* = H_{pq}c_k = H_{pq}(x_k^* + \Delta_k)
$$

with $\Delta_k = c_k - x_k^*$. Then

$$\begin{aligned}
(x_{k+1}^*)^\top P_q x_{k+1}^* &= (x_k^* + \Delta_k)^\top H_{pq}^\top P_q H_{pq}(x_k^* + \Delta_k) \\
&\le \overline{\lambda}(P_q) h_{pq}^2 (x_k^* + \Delta_k)^\top (x_k^* + \Delta_k) \\
&\le \overline{\lambda}(P_q) h_{pq}^2 (2(x_k^*)^\top x_k^* + 2\Delta_k^\top \Delta_k) \\
&\le 2\overline{\lambda}(P_q) h_{pq}^2 (x_k^*)^\top x_k^* + 2n_x \overline{\lambda}(P_q) h_{pq}^2 \|\Delta_k\|^2 \\
&\le \frac{2\overline{\lambda}(P_q) h_{pq}^2}{\underline{\lambda}(P_p)} (x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2} 2n_x \overline{\lambda}(P_q) h_{pq}^2 E_k^2,
\end{aligned}$$

where the inequalities follows from (5.11), (A.4), (A.5), and Young's inequality. Second, from (5.30) and Young's inequality with ϕ_2, it follows that

$$\begin{aligned}
E_{k+1}^2 &\le (\alpha_{pq}\|x_k^*\| + \beta_{pq} E_k + \gamma_{pq}\delta_k)^2 \\
&\le (2+\phi_2)(\alpha_{pq}^2 \|x_k^*\|^2 + \beta_{pq}^2 E_k^2) + \left(1 + \frac{2}{\phi_2}\right)\gamma_{pq}^2 \delta_k^2
\end{aligned}$$

for every $\phi_2 > 0$, in which

$$\|x_k^*\|^2 \le (x_k^*)^\top x_k^* \le \frac{1}{\underline{\lambda}(P_p)} (x_k^*)^\top P_p x_k^*$$

due to (A.4) and (A.5). Therefore,

$$\begin{aligned}
V_q(x_{k+1}^*, E_{k+1}) \le &\left(\frac{2\overline{\lambda}(P_q) h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2 \rho_q}{\underline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* \\
&+ \left(\frac{(N-1)^2}{N^2} \frac{2n_x \overline{\lambda}(P_q) h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2 \rho_q}{\rho_p}\right)\rho_p E_k^2 \\
&+ \left(1 + \frac{2}{\phi_2}\right)\rho_q \gamma_{pq}^2 \delta_k^2,
\end{aligned}$$

which in turn implies (5.45). □

A.7 Proof of Lemma 5.4

First, consider the function $\zeta : [0, 1) \to \mathbb{R}$ defined by

$$\zeta(s) = 1 + \frac{\ln(\mu + s(1-\nu)\mu_d/\nu_d)}{\ln(1/(\nu + s(1-\nu)))}.$$

From (5.47) and (5.48), it follows that $\Theta > 1$ in (5.51), and that ζ is continuous and increasing. Moreover, as

$$\zeta(0) = 1 + \frac{\ln \mu}{\ln(1/\nu)} < \frac{\tau_a}{\tau_s}$$

due to (5.49), there exists a small enough constant $\phi_3 \in (0, 1)$ such that $\zeta(\phi_3) < \tau_a/\tau_s$; thus $\theta < 1$ in (5.51).

The remaining proof follows in principle from the arguments in [27, 21]. If there exists an integer $l \in \{i, \ldots, k-1\}$ such that

$$V_{\sigma(t_l)}(x_l^*, E_l) > \frac{1}{\phi_3(1-\nu)} \nu_d \delta_l^2, \tag{A.6}$$

then (5.43) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\nu + \phi_3(1-\nu)) V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) = \sigma(t_l)$; whereas (5.45) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\mu + \phi_3(1-\nu)\mu_d/\nu_d) V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) \neq \sigma(t_l)$. Hence for two integers l', l'' such that $i \leq l' < l'' \leq k$ and that (A.6) holds for all $l \in \{l', \ldots, l''-1\}$,

$$\begin{aligned}
V_{\sigma(t_{l''})}(x_{l''}^*, E_{l''}) &< (\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{N_\sigma(t_{l''}, t_{l'})} \\
&\quad \times (\nu + \phi_3(1-\nu))^{l''-l'-N_\sigma(t_{l''}, t_{l'})} V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}) \\
&= (\nu + \phi_3(1-\nu))^{l''-l'} \Theta^{N_\sigma(t_{l''}, t_{l'})} V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}), \\
&< (\nu + \phi_3(1-\nu))^{l''-l'} \Theta^{N_0 + (l''-l')\tau_s/\tau_a} V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}) \\
&= \theta^{l''-l'} \Theta^{N_0} V_{\sigma(t_{l'})}(x_{l'}^*, E_{l'}),
\end{aligned}$$

where $N_\sigma(t_{l''}, t_{l'})$ denotes the number of switches on $(t_{l'}, t_{l''}]$, and the last inequality follows from $\Theta > 1$ and Assumption 5.1. Therefore, if (A.6) holds for all $l \in \{i, \ldots, k-1\}$, then

$$V_{\sigma(t_k)}(x_k^*, E_k) < \theta^{k-i} \Theta^{N_0} V_{\sigma(t_i)}(x_i^*, E_i).$$

Otherwise, let

$$k' := \max\left\{l \leq k-1 : V_{\sigma(t_l)}(x_l^*, E_l) \leq \frac{1}{\phi_3(1-\nu)}\nu_d \delta_l^2\right\}.$$

Then

$$\begin{aligned} V_{\sigma(t_{k'+1})}(x_{k'+1}^*, E_{k'+1}) &\leq \mu V_{\sigma(t_{k'})}(x_{k'}^*, E_{k'}) + \mu_d \delta_{k'}^2 \\ &\leq \frac{\mu}{\phi_3(1-\nu)}\nu_d \delta_{k'}^2 + \mu_d \delta_{k'}^2 \\ &= \Theta_d \delta_{k'}^2 \end{aligned}$$

(see also Remark 5.3); thus

$$V_{\sigma(t_k)}(x_k^*, E_k) < \theta^{k-k'-1}\Theta^{N_0} V_{\sigma(t_{k'+1})}(x_{k'+1}^*, E_{k'+1}) \leq \Theta^{N_0}\Theta_d \delta_{k'}^2$$

as (A.6) holds for all $l \in \{k'+1, \ldots, k-1\}$. The proof of Lemma 5.4 is completed by combining the bounds for the two cases and noticing that $\delta_l = \delta_i$ for all $l \in \{i, \ldots, k-1\}$. □

A.8 Proof of Lemma 5.5

Let $p = \sigma(t_j)$ and $q = \sigma(t_i)$. At the sampling time of recovery t_i,

$$V_q(x_i^*, E_i) = (x_i^*)^\top P_q x_i^* + \rho_q E_i^2$$

with $x_i^* = x_j^*$ and E_i bounded by (5.54). First, (A.5) implies that

$$(x_i^*)^\top P_q x_i^* \leq \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)}(x_j^*)^\top P_p x_j^*.$$

Second, following (5.54) and Young's inequality with ϕ_4, we obtain that

$$\begin{aligned} E_i^2 &\leq \hat{\beta}^{2\eta(\delta_d/\delta_j)}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right)^2 \\ &\leq \hat{\beta}^{2\eta(\delta_d/\delta_j)}\left((2+\phi_4)\left(\frac{\bar{\alpha}^2}{(\hat{\beta}-1)^2}\|x_j^*\|^2 + E_j^2\right) + \left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2}{(\hat{\beta}-1)^2}\delta_j^2\right) \end{aligned}$$

for every $\phi_4 > 0$, in which

$$\|x_j^*\|^2 \le (x_j^*)^\top x_j^* \le \frac{1}{\underline{\lambda}(P_p)}(x_j^*)^\top P_p x_j^*$$

due to (A.4) and (A.5). Therefore,

$$\begin{aligned} V_q(x_i^*, E_i) \le \hat{\beta}^{2\eta(\delta_d/\delta_j)} \Bigg(\bigg(\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2 \rho_q}{(\hat{\beta}-1)^2 \underline{\lambda}(P_p)} \bigg) (x_j^*)^\top P_p x_j^* \\ + \frac{(2+\phi_4)\rho_q}{\rho_p} \rho_p E_k^2 + \bigg(1 + \frac{2}{\phi_4}\bigg) \frac{\bar{\gamma}^2 \rho_q}{(\hat{\beta}-1)^2} \delta_j^2 \Bigg), \end{aligned}$$

which in turn implies (5.55). □

A.9 Proof of Lemma 5.7

Let $[t_{i_m}, t_{j_{m+1}})$ be the stabilizing stage containing t_k, that is, $i_m \le k \le j_{m+1} - 1$. Substituting (5.50) with $i = i_m$ and $k = j_{m+1}$ into (5.55) with $j = j_{m+1}$ and $i = i_{m+1}$, we obtain that

$$\begin{aligned} & V_{\sigma(t_{i_{m+1}})}(x_{i_{m+1}}^*, E_{i_{m+1}}) \\ \le\; & \hat{\beta}^{2\eta(\delta_d/\delta_{j_{m+1}})} \big(\omega V_{\sigma(t_{j_{m+1}})}(x_{j_{m+1}}^*, E_{j_{m+1}}) + \omega_d \delta_{j_{m+1}}^2 \big) \\ <\; & \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\omega \Theta^{N_0} \big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x_{i_m}^*, E_{i_m}) \\ & + \Theta_d (1+\varepsilon_\delta)^{2m} \delta_0^2 \big) + \omega_d (1+\varepsilon_\delta)^{2(m+1)} \delta_0^2 \big) \\ =\; & \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x_{i_m}^*, E_{i_m}) + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big), \end{aligned}$$

in which

$$\theta^{j_{m+1}-i_m} \le \theta^{i_{m+1}-i_m} \theta^{-\eta(\delta_d/\delta_{j_{m+1}})}$$

due to (5.53) and $\theta < 1$. Hence

$$\begin{aligned} & V_{\sigma(t_{i_{m+1}})}(x_{i_{m+1}}^*, E_{i_{m+1}}) \\ <\; & \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{i_{m+1}-i_m} \theta^{-\eta(\delta_d/\delta_{j_m})} V_{\sigma(t_{i_m})}(x_{i_m}^*, E_{i_m}) \\ & + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big) \\ \le\; & \Psi \psi^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{i_{m+1}-i_m} V_{\sigma(t_{i_m})}(x_{i_m}^*, E_{i_m}) + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big). \end{aligned}$$

Based on this recursive bound, it is straightforward to derive that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&< \Psi\psi^{2\eta(\delta_d/\delta_{i_m})}\Big(\theta^{i_m-i_{m-1}}\Psi\psi^{2\eta(\delta_d/\delta_{i_{m-1}})}\Big(\theta^{i_{m-1}-i_{m-2}}V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2(m-2)}\delta_0^2\Big) + (\Theta_d + \psi_d\omega_d)(1+\varepsilon_\delta)^{2(m-1)}\delta_0^2\Big) \\
&\leq \Psi^2\psi^{2(\eta(\delta_d/\delta_{i_m})+\eta(\delta_d/\delta_{i_{m-1}}))}\Big(\theta^{i_m-i_{m-2}}V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\psi_d)(1+\varepsilon_\delta)^{2(m-2)}\delta_0^2\Big) \\
&< \cdots \\
&< \Psi^m\psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m-i_0}V_{\sigma(t_{i_0})}(0, E_{i_0}) \\
&\quad + (\Theta_d + \psi_d\omega_d)(1+\psi_d+\cdots+\psi_d^{m-1})\delta_0^2\Big), \\
&< \Psi^m\psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m-i_0}V_{\sigma(t_{i_0})}(0, E_{i_0}) + (\Theta_d + \psi_d\omega_d)\delta_0^2\sum_{l=0}^{m-1}\psi_d^l\Big),
\end{aligned}$$

which, combined with (5.57) and (5.58), implies that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\leq \Psi^m\psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m-i_0}\hat{\beta}^{2(\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0))}\big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d\delta_0^2\big) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2\sum_{l=0}^{m-1}\psi_d^l\Big) \\
&\leq \Psi^m\psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m}\psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2\sum_{l=0}^{m-1}\psi_d^l\Big).
\end{aligned}$$

Finally, substituting the previous bound into (5.50) with $i = i_m$, we obtain

$$\begin{aligned}
&V_{\sigma(t_k)}(x^*_k, E_k) \\
&< \Theta^{N_0}\big(\theta^{k-i_m}V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) + \Theta_d\delta_{i_m}^2\big) \\
&< \Theta^{N_0}\Big(\theta^{k-i_m}\Psi^m\psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m}\psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) \\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2\sum_{l=0}^{m-1}\psi_d^l\Big) + \Theta_d(1+\varepsilon_\delta)^{2m}\delta_0^2\Big) \\
&\leq \Theta^{N_0}\Psi^m\psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^k\psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) \\
&\quad + \Big(\Theta_d\sum_{l=0}^m\psi_d^l + \omega_d\sum_{l=1}^m\psi_d^l\Big)\delta_0^2\Big).
\end{aligned}$$

The proof of Lemma 5.7 is completed by replacing m with its upper bound $N_d(\delta_d)$. □

A.10 Proof of Lemma 6.1

Following the discussion before Lemma 6.1, it suffices to prove that

$$\limsup_{T\to\infty} \bar{a}(T) \le \max\{\hat{a},\, 0\}.$$

For brevity, let $a_m := \max_{p\in\mathcal{P}} |a_p| \ge 0$. Due to the limit supremum in (6.14), for each $\delta > 0$, there is a large enough $T'_\delta \ge 0$ such that

$$\sum_{p\in\mathcal{P}} a_p \rho_p(t) < \hat{a} + \delta \qquad \forall\, t > T'_\delta.$$

However, in general it is possible that $\bar{a}(T) \ge \hat{a} + \delta$ for some $T > T'_\delta$ due to the maximum in (6.15), if there is at least one scalar $a_p < 0$. For a time horizon $T > T'_\delta$, let

$$t^* := \operatorname*{argmax}_{t\in[0,T]} \sum_{p\in\mathcal{P}} a_p \tau_p(t).$$

If $t^* \in (T'_\delta, T]$, then

$$\bar{a}(T) = \frac{1}{T}\sum_{p\in\mathcal{P}} a_p \tau_p(t^*) \le \frac{1}{t^*}\sum_{p\in\mathcal{P}} a_p \tau_p(t^*) < \hat{a} + \delta;$$

otherwise $t^* \in [0, T'_\delta]$, and

$$\bar{a}(T) = \frac{1}{T}\sum_{p\in\mathcal{P}} a_p \tau_p(t^*) \le \frac{a_m t^*}{T} \le \frac{a_m T'_\delta}{T}.$$

Hence

$$\bar{a}(T) \le \max\{\hat{a} + \delta,\, a_m T'_\delta / T\} \qquad \forall\, T > T'_\delta.$$

Consequently, there is a large enough $T_\delta \ge 0$ (e.g., $T_\delta = \max\{T'_\delta,\, a_m T'_\delta/\delta\}$) such that

$$\bar{a}(T) \le \max\{\hat{a},\, 0\} + \delta \qquad \forall\, T > T_\delta.$$

As $\delta > 0$ is arbitrary, it follows that (6.16) holds. $\square$

A.11 Proof of Lemma 6.3

We regard (6.25) with the state denoted by $x = (x^1, \ldots, x^n)$ as a family of scalar differential equations

$$
\begin{aligned}
\dot{x}^1 &= a_\sigma^1 x^1 + b_\sigma^{1,2} x^2 + \cdots + b_\sigma^{1,n} x^n, \\
\dot{x}^2 &= a_\sigma^2 x^2 + b_\sigma^{2,3} x^3 + \cdots + b_\sigma^{2,n} x^n, \\
&\vdots \\
\dot{x}^{n-1} &= a_\sigma^{n-1} x^{n-1} + b_\sigma^{n-1,n} x^n, \\
\dot{x}^n &= a_\sigma^n x^n,
\end{aligned}
$$

and prove Lemma 6.3 by mathematical induction.

A.11.1 The base case (the n-th component)

For

$$\dot{x}^n = a_\sigma^n x^n,$$

the state-transition function is given by

$$\phi_\sigma^n(t, s) = e^{\eta_n(t) - \eta_n(s)}$$

for $t, s \geq 0$. Then the n-th component of $\xi_\sigma(x_0, t)$ at time t is given by

$$\xi_\sigma^n(x_0, t) = e^{\eta_n(t)} x_0^n,$$

that is, (6.29) holds for $k = 0$.

A.11.2 The inductive step

Consider an arbitrary $m \in \{1, \ldots, n-1\}$. Suppose that for $k = 0, \ldots, m-1$, the $(n-k)$-th component of $\xi_\sigma(x_0, t)$ at time t is given by (6.29). For

$$\dot{x}^{n-m} = a_\sigma^n x^n + \sum_{k=0}^{m-1} b_\sigma^{n-m, n-k} x^{n-k},$$

the state-transition function is given by

$$\phi_\sigma^{n-m}(t,s) = e^{\eta_{n-m}(t)-\eta_{n-m}(s)}$$

for $t, s \geq 0$. Then by variation of constants, the $(n-m)$-th component of $\xi_\sigma(x_0,t)$ at time t is given by

$$\begin{aligned}
&\xi_\sigma^{n-m}(x_0,t)\\
&= e^{\eta_{n-m}(t)}\Bigg(x_0^{n-m} + \sum_{k=0}^{m-1}\int_0^t e^{-\eta_{n-m}(s_1)} b_{\sigma(s_1)}^{n-m,n-k}\xi_\sigma^{n-k}(x_0,s_1)\,\mathrm{d}s_1\Bigg)\\
&= e^{\eta_{n-m}(t)}\Bigg(x_0^{n-m} + \sum_{k=0}^{m-1}\int_0^t b_{\sigma(s_1)}^{n-m,n-k} e^{\nu_{n-m,n-k}}(s_1)\Bigg(x_0^{n-k} + \sum_{l=0}^{k-1}\Bigg(x_0^{n-l}\\
&\quad\times\sum_{i=1}^{k-l}\sum_{(c_0,\dots,c_i)\in\mathcal{C}_{k,l,i}}\int_0^{s_1}\cdots\int_0^{s_i}\prod_{j=1}^{i} b_{\sigma(s_{j+1})}^{c_{j-1},c_j} e^{\nu_{c_{j-1},c_j}(s_{j+1})}\mathrm{d}s_{j+1}\Bigg)\Bigg)\mathrm{d}s_1\Bigg)\\
&= e^{\eta_{n-m}(t)}\Bigg(x_0^{n-m} + \sum_{k=0}^{m-1}\Bigg(x_0^{n-k}\int_0^t b_{\sigma(s_1)}^{n-m,n-k} e^{\nu_{n-m,n-k}}(s_1)\,\mathrm{d}s_1\Bigg) +\\
&\quad\sum_{k=0}^{m-1}\sum_{l=0}^{k-1}\Bigg(x_0^{n-l}\sum_{i=1}^{k-l}\sum_{(c_0,\dots,c_i)\in\mathcal{C}_{k,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_i} b_{\sigma(s_1)}^{n-m,n-k} e^{\nu_{n-m,n-k}}(s_1)\\
&\quad\times\Bigg(\prod_{j=1}^{i} b_{\sigma(s_{j+1})}^{c_{j-1},c_j} e^{\nu_{c_{j-1},c_j}(s_{j+1})}\mathrm{d}s_{j+1}\Bigg)\mathrm{d}s_1\Bigg)\Bigg).
\end{aligned}$$

First, (6.30) implies that $\mathcal{C}_{m,l,1} = \{(n-m, n-l)\}$; thus

$$\begin{aligned}
&\sum_{k=0}^{m-1}\Bigg(x_0^{n-k}\int_0^t b_{\sigma(s_1)}^{n-m,n-k} e^{\nu_{n-m,n-k}}(s_1)\,\mathrm{d}s_1\Bigg)\\
&= \sum_{l=0}^{m-1}\Bigg(x_0^{n-l}\sum_{(c_0,c_1)\in\mathcal{C}_{m,l,1}}\int_0^t b_{\sigma(s_1)}^{c_0,c_1} e^{\nu_{c_0,c_1}}(s_1)\mathrm{d}s_1\Bigg).
\end{aligned}$$

Second, we change the order of summation and obtain that

$$\begin{aligned}
&\sum_{k=0}^{m-1}\sum_{l=0}^{k-1}\Bigg(x_0^{n-l}\sum_{i=1}^{k-l}\sum_{(c_0,\ldots,c_i)\in\mathcal{C}_{k,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_i} b_{\sigma(s_1)}^{n-m,n-k}e^{\nu_{n-m,n-k}}(s_1)\\
&\times\Bigg(\prod_{j=1}^{i} b_{\sigma(s_{j+1})}^{c_{j-1},c_j}e^{\nu_{c_{j-1},c_j}(s_{j+1})}\mathrm{d}s_{j+1}\Bigg)\mathrm{d}s_1\Bigg)\\
=&\sum_{l=0}^{m-2}\Bigg(x_0^{n-l}\sum_{k=l+1}^{m-1}\sum_{i=1}^{k-l}\sum_{(c_0,\ldots,c_i)\in\mathcal{C}_{k,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_i} b_{\sigma(s_1)}^{n-m,n-k}e^{\nu_{n-m,n-k}}(s_1)\\
&\times\Bigg(\prod_{j=1}^{i} b_{\sigma(s_{j+1})}^{c_{j-1},c_j}e^{\nu_{c_{j-1},c_j}(s_{j+1})}\mathrm{d}s_{j+1}\Bigg)\mathrm{d}s_1\Bigg)\\
=&\sum_{l=0}^{m-2}\Bigg(x_0^{n-l}\sum_{i=1}^{m-l-1}\sum_{k=l+i}^{m-1}\sum_{(c_0,\ldots,c_i)\in\mathcal{C}_{k,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_i} b_{\sigma(s_1)}^{n-m,n-k}e^{\nu_{n-m,n-k}}(s_1)\\
&\times\Bigg(\prod_{j=1}^{i} b_{\sigma(s_{j+1})}^{c_{j-1},c_j}e^{\nu_{c_{j-1},c_j}(s_{j+1})}\mathrm{d}s_{j+1}\Bigg)\mathrm{d}s_1\Bigg)\\
=&\sum_{l=0}^{m-2}\Bigg(x_0^{n-l}\\
&\times\sum_{i'=2}^{m-l}\sum_{k=l+i'-1}^{m-1}\sum_{(c'_0,\ldots,c'_{i'})\in\mathcal{C}'_{k,l,i'}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{i'-1}}\prod_{j'=1}^{i'} b_{\sigma(s_j)}^{c'_{j'-1},c'_{j'}}e^{\nu_{c'_{j'-1},c'_{j'}}(s_{j'})}\mathrm{d}s_{j'}\Bigg),
\end{aligned}$$

where in the last step we change the indices of summation by letting $i'=i+1$, $j'=j+1$, $c'_{j'}=c_j$, and $c'_0=n-m$. Hence the set $\mathcal{C}'_{k,l,i'}$ is given by

$$\mathcal{C}'_{k,l,i'}=\{(c'_0,\ldots,c'_{i'})\in\mathbb{Z}^{i'+1}:c'_0=n-m,\ n-k=c'_1<\cdots<c'_{i'}=n-l\}.$$

Consider the set $\mathcal{C}_{m,l,i'}$ defined according to (6.30), that is,

$$\mathcal{C}_{m,l,i'}=\{(c_0,\ldots,c_{i'})\in\mathbb{Z}^{i'+1}:n-m=c_0<\cdots<c_{i'}=n-l\}.$$

In the following, we prove that the family of sets $\{\mathcal{C}'_{k,l,i'}:k=l+i'-1,\ldots,m-1\}$ forms a partition of $\mathcal{C}_{m,l,i'}$.

- For each $(c_0^1,\ldots,c_{i'}^1)\in\mathcal{C}'_{k_1,l,i'}$ and $(c_0^2,\ldots,c_{i'}^2)\in\mathcal{C}'_{k_2,l,i'}$ with $k_1\neq k_2$, as $c_1^1=n-k_1\neq n-k_2=c_1^2$, it follows that $(c_0^1,\ldots,c_{i'}^1)\neq(c_0^2,\ldots,c_{i'}^2)$. Hence the sets in $\{\mathcal{C}'_{k,l,i'}:k=l+i'-1,\ldots,m-1\}$ are pairwise disjoint.
- For each $(c'_0,\ldots,c'_{i'})\in\mathcal{C}'_{k,l,i'}$, as $c'_1=n-k\geq n-m+1$ and $c'_{i'}=n-l$, it

follows that $(c'_0,\dots,c'_{i'})\in\mathcal{C}_{m,l,i'}$. Hence $\cup_{k=l+i'-1}^{m-1}\mathcal{C}'_{k,l,i'}\subset\mathcal{C}_{m,l,i'}$.

- For each $(c_0,\dots,c_{i'})\in\mathcal{C}_{m,l,i'}$, as $c_1\geq c_0+1=n-m+1$ and $c_1\leq c_{i'}-(i'-1)=n-l-i'+1$, it follows that $k':=n-c_1$ satisfies $l+i'-1\leq k'\leq m-1$; thus $(c_0,\dots,c_{i'})\in\mathcal{C}'_{k',l,i'}$. Hence $\mathcal{C}_{m,l,i'}\subset\cup_{k=l+i'-1}^{m-1}\mathcal{C}'_{k,l,i'}$.

Therefore,

$$\begin{aligned}&\sum_{l=0}^{m-2}\Bigg(x_0^{n-l}\\&\quad\times\sum_{i'=2}^{m-l}\sum_{k=l+i'-1}^{m-1}\sum_{(c'_0,\dots,c'_{i'})\in\mathcal{C}'_{k,l,i'}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{i'-1}}\prod_{j'=1}^{i'}b_{\sigma(s_j)}^{c'_{j'-1},c'_{j'}}e^{\nu_{c'_{j'-1},c'_{j'}}(s_{j'})}\mathrm{d}s_{j'}\Bigg)\\&=\sum_{l=0}^{m-2}\Bigg(x_0^{n-l}\sum_{i=2}^{m-l}\sum_{(c_0,\dots,c_i)\in\mathcal{C}_{m,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{i-1}}\prod_{j=1}^{i}b_{\sigma(s_j)}^{c_{j-1},c_j}e^{\nu_{c_{j-1},c_j}(s_j)}\mathrm{d}s_j\Bigg).\end{aligned}$$

Combing the results above, we obtain that

$$\begin{aligned}\xi_\sigma^{n-m}(x_0,t)=e^{\eta_{n-m}(t)}\Bigg(x_0^{n-m}+\sum_{l=0}^{m-1}\Bigg(x_0^{n-l}\\\times\sum_{i=1}^{m-l}\sum_{(c_0,\dots,c_i)\in\mathcal{C}_{m,l,i}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{i-1}}\prod_{j=1}^{i}b_{\sigma(s_j)}^{c_{j-1},c_j}e^{\nu_{c_{j-1},c_j}(s_j)}\mathrm{d}s_j\Bigg)\Bigg),\end{aligned}$$

that is, (6.29) holds for $k=m$.

Therefore, by mathematical induction, the formula (6.29) holds for all $k\in\{1,\dots,n-1\}$. $\square$

A.12 Proof of Lemma 6.4

For all $1\leq i<j\leq n$, the quantities $\bar{\eta}_i(T)$ and $\bar{\nu}_{i,j}(T)$ for a time horizon $T\geq 0$ satisfies that $\bar{\eta}_i(T),\bar{\nu}_{i,j}(T)\geq 0$, that $\bar{\eta}_j(T)\leq\bar{\nu}_{i,j}(T)+\bar{\eta}_i(T)$, and that $\bar{\nu}_{i,j}(T)\leq\bar{\nu}_{i,k}(T)+\bar{\nu}_{k,j}(T)$ for all $i<k<j$. Hence

$$\bar{\eta}_i(T)\leq\bar{\eta}_1(T)+\sum_{i'=1}^{i-1}\bar{\nu}_{i',i'+1}(T)\qquad\forall\,T\geq 0,\tag{A.7}$$

and

$$\bar{\nu}_{i,j}(T) \leq \sum_{j'=i}^{j-1} \bar{\nu}_{j',j'+1}(T) \qquad \forall\, T \geq 0. \tag{A.8}$$

Following (6.29) and the triangle inequality, for each $k \in \{0, \dots, n-1\}$, the $(n-k)$-th components of the solutions $\xi_\sigma(x,t)$ and $\xi_\sigma(y,t)$ at time t satisfy that

$$\begin{aligned}
&|\xi_\sigma^{n-k}(y,t) - \xi_\sigma^{n-k}(x,t)| \\
&\leq e^{\mathrm{Re}(\eta_{n-k}(t))}\Bigg(|y^{n-k} - x^{n-k}| + \sum_{l=0}^{k-1}\Bigg(|x^{n-l} - y^{n-1}| \\
&\quad \times \sum_{i=1}^{k-l} \sum_{(c_0,\dots,c_i)\in\mathcal{C}_{k,l,i}} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \prod_{j=1}^{i} |b_{\sigma(s_j)}^{c_{j-1},c_j}| e^{\mathrm{Re}(\nu_{c_{j-1},c_j}(s_j))} \mathrm{d}s_j\Bigg)\Bigg).
\end{aligned}$$

First, in each integral, we replace $|b_\sigma^{c_{j-1},c_j}|$ and $\mathrm{Re}(\nu_{c_{j-1},c_j}(s_j))$ with b_m and $\bar{\nu}_{c_{j-1},c_j}$, respectively,

$$\begin{aligned}
&\int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \prod_{j=1}^{i} |b_{\sigma(s_j)}^{c_{j-1},c_j}| e^{\mathrm{Re}(\nu_{c_{j-1},c_j}(s_j))} \mathrm{d}s_j \\
&= \int_0^t |b_{\sigma(s_1)}^{c_0,c_1}| e^{\mathrm{Re}(\nu_{c_0,c_1}(s_1))} \Bigg(\int_0^{s_1} |b_{\sigma(s_2)}^{c_1,c_2}| e^{\mathrm{Re}(\nu_{c_1,c_2}(s_2))} \\
&\quad \times \Bigg(\cdots\Bigg(\int_0^{s_{i-1}} |b_{\sigma(s_i)}^{c_{i-1},c_i}| e^{\mathrm{Re}(\nu_{c_{i-1},c_i}(s_i))} \mathrm{d}s_i\Bigg)\cdots\Bigg)\mathrm{d}s_2\Bigg)\mathrm{d}s_1 \\
&\leq b_m \int_0^t |b_{\sigma(s_1)}^{c_0,c_1}| e^{\mathrm{Re}(\nu_{c_0,c_1}(s_1))} \Bigg(\int_0^{s_1} |b_{\sigma(s_2)}^{c_1,c_2}| e^{\mathrm{Re}(\nu_{c_1,c_2}(s_2))} \Bigg(\cdots\Bigg(\int_0^{s_{i-2}} |b_{\sigma(s_{i-1})}^{c_{i-2},c_{i-1}}| \\
&\quad \times e^{\mathrm{Re}(\nu_{c_{i-2},c_{i-1}}(s_{i-1})) + \bar{\nu}_{c_{i-1},c_i}(s_{i-1})} s_{i-1} \mathrm{d}s_{i-1}\Bigg)\cdots\Bigg)\mathrm{d}s_2\Bigg)\mathrm{d}s_1 \\
&\leq \frac{b_m^2}{2!} \int_0^t |b_{\sigma(s_1)}^{c_0,c_1}| e^{\mathrm{Re}(\nu_{c_0,c_1}(s_1))} \Bigg(\int_0^{s_1} |b_{\sigma(s_2)}^{c_1,c_2}| e^{\mathrm{Re}(\nu_{c_1,c_2}(s_2))} \Bigg(\cdots\Bigg(\int_0^{s_{i-3}} |b_{\sigma(s_{i-2})}^{c_{i-3},c_{i-2}}| \\
&\quad \times e^{\mathrm{Re}(\nu_{c_{i-3},c_{i-2}}(s_{i-2})) + \bar{\nu}_{c_{i-2},c_{i-1}}(s_{i-2}) + \bar{\nu}_{c_{i-1},c_i}(s_{i-2})} s_{i-2}^2 \mathrm{d}s_{i-2}\Bigg)\cdots\Bigg)\mathrm{d}s_2\Bigg)\mathrm{d}s_1 \\
&\leq \cdots \\
&\leq \frac{(b_m t)^i}{i!} e^{\sum_{j=1}^{i} \bar{\nu}_{c_{j-1},c_j}(t)} \\
&\leq \frac{(b_m t)^i}{i!} e^{\sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(t)},
\end{aligned}$$

where the last inequality follows from (A.8) and the fact that $c_0 = n-k$

and $c_i = n-l$ for $(c_0,\dots,c_i) \in \mathcal{C}_{k,l,i}$. As $\sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(t)$ is independent of $(c_0,\dots,c_i) \in \mathcal{C}_{k,l,i}$ and $i \in \{1,\dots,k-l\}$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{k-l} \sum_{(c_0,\dots,c_i)\in\mathcal{C}_{k,l,i}} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \prod_{j=1}^{i} b_{\sigma(s_j)}^{c_{j-1},c_j} e^{\nu_{c_{j-1},c_j}(s_j)} \mathrm{d}s_j \\
&\quad \le e^{\sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(t)} P_{k,l}(t)
\end{aligned}$$

with a positive and increasing polynomial of $t \ge 0$ defined by

$$P_{k,l}(t) := \sum_{i=1}^{k-l} \frac{|\mathcal{C}_{k,l,i}|(b_{\max}t)^i}{i!} = \sum_{i=1}^{k-l} \left(\binom{k-l-1}{i-1} \frac{(b_{\max}t)^i}{i!} \right),$$

where the equality follows from fact that the set $\mathcal{C}_{k,l,i}$ can be characterized by the combinations of $i-1$ increasing integers from $n-k+1$ to $n-l-1$ (in particular, in the definition (6.30) of $\mathcal{C}_{k,l,i}$, the first point $c_0 = n-k$ and the last point $c_i = n-l$ are fixed). Hence

$$\begin{aligned}
&|\xi_\sigma^{n-k}(y,t) - \xi_\sigma^{n-k}(x,t)| \\
&\le e^{\mathrm{Re}(\eta_{n-k}(t))} \left(|y^{n-k} - x^{n-k}| + \sum_{l=0}^{k-1} \left(|x^{n-l} - y^{n-l}|\, e^{\sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(t)} P_{k,l}(t) \right) \right).
\end{aligned}$$

Second, taking the maximum over $t \in [0,T]$, we obtain that

$$\begin{aligned}
&\max_{t\in[0,T]} |\xi_\sigma^{n-k}(y,t) - \xi_\sigma^{n-k}(x,t)| \\
&\le e^{\bar{\eta}_{n-k}(T)} \left(|y^{n-k} - x^{n-k}| + \sum_{l=0}^{k-1} \left(|x^{n-l} - y^{n-l}|\, e^{\sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(T)} P_{k,l}(T) \right) \right) \\
&= e^{\bar{\eta}_{n-k}(T)} |y^{n-k} - x^{n-k}| \\
&\quad + \sum_{l=0}^{k-1} \left(|x^{n-l} - y^{n-l}|\, e^{\bar{\eta}_{n-k}(T) + \sum_{j'=n-k}^{n-l-1} \bar{\nu}_{j',j'+1}(T)} P_{k,l}(T) \right) \\
&\le e^{\bar{\eta}_1(T) + \sum_{j'=1}^{n-k-1} \bar{\nu}_{j',j'+1}(T)} |y^{n-k} - x^{n-k}| \\
&\quad + \sum_{l=0}^{k-1} \left(|x^{n-l} - y^{n-l}|\, e^{\bar{\eta}_1(T) + \sum_{j'=1}^{n-l-1} \bar{\nu}_{j',j'+1}(T)} P_{k,l}(T) \right),
\end{aligned}$$

where the last inequality follows from (A.7).

Finally, we derive an upper bound of $\max_{t\in[0,T]} \|\xi_\sigma(y,t) - \xi_\sigma(x,t)\|$. Consider the positive and increasing polynomials of $T \geq 0$ defined by

$$\hat{P}_{n-l}(T) := \sum_{i=0}^{n-l-1} \left(\binom{n-l-1}{i} \frac{(b_{\max} T)^i}{i!} \right), \qquad l = 0, \dots, n-1.$$

Then $\hat{P}_1(T) \equiv 1$, and for each $l < k < n-1$,

$$\hat{P}_{n-l}(T) \geq P_{k,l}(T) + 1 \qquad \forall\, T \geq 0.$$

Hence

$$\begin{aligned} &\max_{t \in [0,T]} |\xi_\sigma^{n-k}(y,t) - \xi_\sigma^{n-k}(x,t)| \\ &\leq \sum_{l=0}^{k} \left(|x^{n-l} - y^{n-l}|\, e^{\bar{\eta}_1(T) + \sum_{j'=1}^{n-l-1} \bar{\nu}_{j',j'+1}(T)} \hat{P}_{n-l}(T) \right). \end{aligned}$$

Taking maximum over $k \in \{0, \dots, n-1\}$ and changing the indices of summation by $i = n-l$ and $j = j'$ gives (6.31). $\square$

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