Additional material to the paper 'Norm-controllability of nonlinear systems'

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Abstract

This technical report contains some additional calculations supplementing Example 6 in the paper *Norm-controllability of nonlinear systems* by M. A. Müller, D. Liberzon, and F. Allgöwer, *IEEE Transactions on Automatic Control*, 2015. References and labels in this technical report (in particular Equation labels (1)–(59) and all theorem numbers etc.) refer to those in that paper.

In the following, consider a fixed initial condition $x_0 = [x_1(0)\ x_2(0)]^T \in \mathbb{R}^2_{\geq 0} \setminus \mathcal{B}$, which means that $x_2(0) > (k/c)x_1(0)^2$. For a constant input $u = b > 0$, the (unique) equilibrium of system (34) is given by

$$\begin{aligned} x_{1,s}(b) &:= (c/2k)\big(-1 + \sqrt{1 + 4(k/c)b}\big) \\ x_{2,s}(b) &:= (k/c)x_{1,s}(b)^2. \end{aligned} \tag{60}$$

Now fix $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that $x_2(0) > (1-\delta)^2(k/c)x_1(0)^2$; note that such a δ exists due to the fact that $x_2(0) > (k/c)x_1(0)^2$. Apply the constant input $u(t) \equiv b$ to system (34) and distinguish the following three cases.

Case 1 (C1): $x_1(0) \geq x_{1,s}(b)$. In this case, it follows from (34) that $x_1(\cdot)$ is (strictly) decreasing with $\lim_{t\to\infty} x_1(t) = x_{1,s}(b)$; furthermore, $x_2(t) \geq (k/c)x_{1,s}(b)^2 =: \varphi_1(b)$ for all $t \geq 0$, as the solution cannot cross the curve $x_2 = (k/c)x_1^2$ from above if $x_1(\cdot)$ is strictly decreasing.

Case 2 (C2): $x_1(0) < x_{1,s}(b)$ and $x_2(0) \geq (1-\delta)^2(k/c)x_{1,s}(b)^2$. In this case, it follows from (34) that $x_1(\cdot)$ is (strictly) increasing with $\lim_{t\to\infty} x_1(t) = x_{1,s}(b)$. Define $\bar\tau := \inf\{\tau \geq 0 : x_1(\tau) \geq (1-\varepsilon)x_{1,s}(b)\}$. Then from (34) it follows that for all $0 \leq t \leq \bar\tau$ we have

$$\dot x_1 \geq -c(1-\varepsilon)x_{1,s}(b) - k(1-\varepsilon)^2 x_{1,s}(b)^2 + cb = c\varepsilon x_{1,s}(b) + k(2\varepsilon - \varepsilon^2)x_{1,s}(b)^2,$$

and hence

$$\bar\tau \leq \frac{\max\{(1-\varepsilon)x_{1,s}(b) - x_1(0), 0\}}{c\varepsilon x_{1,s}(b) + k(2\varepsilon-\varepsilon^2)x_{1,s}(b)^2} \leq \frac{(1-\varepsilon)x_{1,s}(b)}{c\varepsilon x_{1,s}(b) + k(2\varepsilon-\varepsilon^2)x_{1,s}(b)^2} = \frac{1-\varepsilon}{c\varepsilon + k(2\varepsilon-\varepsilon^2)x_{1,s}(b)} =: \bar T(b). \tag{61}$$

Furthermore, as $\dot x_2 \geq -cx_2$ according to (34), it follows that $x_2(t) \geq x_2(0)e^{-ct}$ for $0 \leq t \leq \bar\tau$. For $t > \bar\tau$, we have $x_2(t) \geq \min\{x_2(\bar\tau), (k/c)(1-\varepsilon)^2 x_{1,s}(b)^2\}$, which follows from the definition of $\bar\tau$, the fact that $x_1(\cdot)$ is strictly increasing, and the fact that $\dot x_2 \geq 0$ if the solution enters the region where $x_2 < (k/c)x_1^2$. Hence for all $t \geq 0$, we have

$$x_2(t) \geq \min\{x_2(0)e^{-c\bar\tau}, (k/c)(1-\varepsilon)^2 x_{1,s}(b)^2\} \geq \min\{x_2(0)e^{-c\bar T(b)}, (k/c)(1-\varepsilon)^2 x_{1,s}(b)^2\} =: \varphi_2(b). \tag{62}$$

Case 3 (C3): $x_1(0) < x_{1,s}(b)$ and $x_2(0) < (1-\delta)^2(k/c)x_{1,s}(b)^2$. Again, it follows from (34) that $x_1(\cdot)$ is (strictly) increasing with $\lim_{t\to\infty} x_1(t) = x_{1,s}(b)$. Define $\hat\tau := \inf\{\tau \geq 0 : x_2(\tau) = (k/c)x_1(\tau)^2\}$ and $\tau' := \inf\{\tau \geq 0 : x_2(0) = (k/c)x_1(\tau)^2\}$. Note that $\hat\tau \leq \tau'$ due to the fact that $x_1(\cdot)$ is (strictly) increasing and $x_2(\cdot)$ is

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(strictly) decreasing for $0 \leq t \leq \hat{\tau}$ according to (34) and hence $x_2(\hat{\tau}) \leq x_2(0)$. The definition of τ' implies that $x_1(t) \leq ((c/k)x_2(0))^{1/2}$ for all $0 \leq t \leq \tau'$, and hence from (34) it follows that during this time interval

$$\begin{aligned}\dot{x}_1 &\geq -c\sqrt{\frac{c}{k}x_2(0)} - k\frac{c}{k}x_2(0) + cb \\ &> -c(1-\delta)x_{1,s}(b) - k(1-\delta)^2 x_{1,s}(b)^2 + cb = c\delta x_{1,s}(b) + k(2\delta-\delta^2)x_{1,s}(b)^2,\end{aligned}$$

where the second inequality follows from the fact that $x_2(0) < (1-\delta)^2(k/c)x_{1,s}(b)^2$. Hence we obtain that

$$\hat{\tau} \leq \tau' \leq \frac{\sqrt{\frac{c}{k}x_2(0)} - x_1(0)}{c\delta x_{1,s}(b) + k(2\delta-\delta^2)x_{1,s}(b)^2} < \frac{(1-\delta)x_{1,s}(b)}{c\delta x_{1,s}(b) + k(2\delta-\delta^2)x_{1,s}(b)^2} = \frac{1-\delta}{c\delta + k(2\delta-\delta^2)x_{1,s}(b)} =: \hat{T}(b). \tag{63}$$

Furthermore, as $\dot{x}_2 \geq -cx_2$ according to (34), it follows that $x_2(t) \geq x_2(0)e^{-ct}$ for $0 \leq t \leq \hat{\tau}$. For $t > \hat{\tau}$, we obtain $y(t) = qx_2(t) \geq q\min\{\Psi(t-\hat{\tau}, b) + x_2(\hat{\tau}), \rho(b)\}$, which follows as $x_2(\hat{\tau}) \in \mathcal{B}$ by definition of $\hat{\tau}$. Hence, as $\Psi(0,\cdot) \equiv 0$ according to (28), we obtain that for all $t \geq 0$

$$\begin{aligned}x_2(t) &\geq \min\{\Psi(\max\{t-\hat{\tau}, 0\}, b) + x_2(0)e^{-c\hat{\tau}}, \rho(b)\} \\ &\geq \min\{\Psi(\max\{t-\hat{T}(b), 0\}, b) + x_2(0)e^{-c\hat{T}(b)}, \rho(b)\} =: \varphi_3(t,b),\end{aligned} \tag{64}$$

where the second inequality follows from (63) and the fact that $\Psi(\cdot, b)$ is nondecreasing.

Combining the above three cases, there exist constants $0 \leq b' < b''$ such that we have case C1 for $0 \leq b \leq b'$, case C2 for $b' < b \leq b''$, and case C3 for $b > b''$. Now define the function

$$\varphi(a,b) := \begin{cases} q\varphi_1(b) & 0 \leq b \leq b' \\ q\varphi_2(b) & b' < b \leq b'' \\ q\varphi_3(a,b) & b > b'' \end{cases}$$

We have shown above that for each $a, b > 0$, by applying the constant input $u \equiv b$ it follows that $y(a) = qx_2(a) \geq \varphi(a,b)$. Hence in order to conclude that the system (34) is norm-controllable from x_0, it remains to show that $\varphi(a,b) \geq \gamma(a,b)$ for some function γ satisfying the properties of Definition 1. To this end, note the following. By (60) and the definition of φ_1, it follows that $\varphi_1 \in \mathcal{K}_\infty$. Furthermore, by (61) we have that $\bar{T}(\cdot)$ is continuous and strictly decreasing, and hence by definition of φ_2 in (62) it follows that $\varphi_2 \in \mathcal{K}$. Finally, from (63) it follows that $\hat{T}(\cdot)$ is continuous and strictly decreasing with $\lim_{b\to\infty}\hat{T}(b) = 0$. Using this together with the fact that $\Psi(a,\cdot) \in \mathcal{K}_\infty$ for each $a > 0$ according to (28) and $\rho \in \mathcal{K}_\infty$, we obtain from the definition of φ_3 in (64) that $\varphi_3(a,\cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$, and $\varphi_3(\cdot, b)$ is nondecreasing for each fixed $b > 0$.

Now fix some $b''' > b''$, let $\bar{\varphi} := \min\{\varphi_1(b''), \varphi_2(b'')\}$, and define the function γ as

$$\gamma(a,b) = \begin{cases} q\min\{\varphi_1(b), \varphi_2(b), \varphi_3(a,b)\} & 0 \leq b \leq b'' \\ q\min\{\varphi_3(a,b), \frac{\varphi_3(a,b''')-\bar{\varphi}}{b'''-b''}(b-b'') + \bar{\varphi}\} & b'' < b \leq b''' \\ q\varphi_3(a,b) & b > b''' \end{cases} \tag{65}$$

By definition, we have that $\varphi(a,b) \geq \gamma(a,b)$ for all $a, b > 0$, and from the above considerations, it follows that $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a,\cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$. By Definition 1, this means that system (34) is norm-controllable from x_0 with gain function γ given by (65). Since $x_0 \in \mathbb{R}^2_{\geq 0} \setminus \mathcal{B}$ was arbitrary, we conclude that system (34) is also norm-controllable from all $x_0 \in \mathbb{R}^2_{\geq 0} \setminus \mathcal{B}$.

An interpretation of this fact is as follows. While as discussed in Example 6, the amount of product B inside the reactor will first decrease (due to the outlet stream) if $x_2 > (k/c)x_1^2$, the time during which it decreases goes to zero as b, i.e., the concentration of A in the inlet stream, increases. Hence still for each fixed time $a > 0$, the amount of product can be made large by increasing the concentration of A in the inlet stream. On the other hand, the conditions of Theorem 3 cannot be satisfied with $V(x) = |x_2|$, as their satisfaction would imply that the amount of product can be increased from the beginning on.

Finally, we remark that for all $x \in \mathcal{B}$, i.e., on the set where Theorem 3 applies, a uniform (with respect to the initial condition x_0) gain function γ can be obtained; namely, replacing $x_2(0)$ in (36) by 0 results in a gain function γ which is independent of x_0. On the other hand, the function γ obtained in (65) for the case that $x_0 \notin \mathcal{B}$ is not independent of x_0, and it is not clear whether a uniform (with respect to x_0) lower bound for γ can be found in this case.