Additional material to the paper
’Norm-controllability of nonlinear systems’

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Abstract

This technical report contains some additional calculations supplementing Example 6 in the paper Norm-controllability of nonlinear systems by M. A. Müller, D. Liberzon, and F. Allgöwer, IEEE Transactions on Automatic Control, 2015. References and labels in this technical report (in particular Equation labels (1)–(59) and all theorem numbers etc.) refer to those in that paper.

In the following, consider a fixed initial condition \( x_0 = [x_1(0)\ x_2(0)]^T \in \mathbb{R}_{\geq 0}^2 \setminus \mathcal{B} \), which means that \( x_2(0) > (k/c)x_1(0)^2 \). For a constant input \( u = b > 0 \), the (unique) equilibrium of system (34) is given by

\[
\begin{align*}
    x_{1,s}(b) &:= \frac{c}{2k}(1 + \sqrt{1 + 4(k/c)b}) \\
    x_{2,s}(b) &:= \frac{k}{c}x_{1,s}(b)^2.
\end{align*}
\]

(60)

Now fix \( 0 < \varepsilon < 1 \) and \( 0 < \delta < 1 \) such that \( x_2(0) > (1 - \delta)^2(k/c)x_1(0)^2 \); note that such a \( \delta \) exists due to the fact that \( x_2(0) > (k/c)x_1(0)^2 \). Apply the constant input \( u(t) \equiv b \) to system (34) and distinguish the following three cases.

Case 1 (C1): \( x_1(0) \geq x_{1,s}(b) \). In this case, it follows from (34) that \( x_1(\cdot) \) is (strictly) decreasing with \( \lim_{t \to \infty} x_1(t) = x_{1,s}(b) \); furthermore, \( x_2(t) \geq (k/c)x_{1,s}(b)^2 =: \varphi_1(b) \) for all \( t \geq 0 \), as the solution cannot cross the curve \( x_2 = (k/c)x_1^2 \) from above if \( x_1(\cdot) \) is strictly decreasing.

Case 2 (C2): \( x_1(0) < x_{1,s}(b) \) and \( x_2(0) \geq (1 - \delta)^2(k/c)x_{1,s}(b)^2 \). In this case, it follows from (34) that \( x_1(\cdot) \) is (strictly) increasing with \( \lim_{t \to \infty} x_1(t) = x_{1,s}(b) \). Define \( \bar{\tau} := \inf \{ \tau \geq 0 : x_1(\tau) \geq (1 - \varepsilon)x_{1,s}(b) \} \). Then from (34) it follows that for all \( 0 \leq t \leq \bar{\tau} \) we have

\[
\dot{x}_1 \geq -c(1 - \varepsilon)x_{1,s}(b) - k(1 - \varepsilon)^2x_{1,s}(b)^2 + cb = c\varepsilon x_{1,s}(b) + k(2\varepsilon - \varepsilon^2)x_{1,s}(b)^2,
\]

and hence

\[
\bar{\tau} \leq \frac{\max\{1 - \varepsilon\}x_{1,s}(b) - x_1(0),0\}}{c\varepsilon x_{1,s}(b) + k(2\varepsilon - \varepsilon^2)x_{1,s}(b)^2} \leq \frac{1 - \varepsilon}{c\varepsilon + k(2\varepsilon - \varepsilon^2)x_{1,s}(b)} =: \bar{T}(b).
\]

(61)

Furthermore, as \( \dot{x}_2 \geq -c x_2 \) according to (34), it follows that \( x_2(t) \geq x_2(0)e^{-ct} \) for \( 0 \leq t \leq \bar{\tau} \). For \( t > \bar{\tau} \), we have \( x_2(t) \geq \min\{x_2(\bar{\tau}), (k/c)(1 - \varepsilon)^2x_{1,s}(b)^2\} \), which follows from the definition of \( \bar{\tau} \), the fact that \( x_1(\cdot) \) is strictly increasing, and the fact that \( \dot{x}_2 \geq 0 \) if the solution enters the region where \( x_2 < (k/c)x_1^2 \). Hence for all \( t \geq 0 \), we have

\[
\begin{align*}
    x_2(t) &\geq \min\{x_2(0)e^{-ct}, (k/c)(1 - \varepsilon)^2x_{1,s}(b)^2\} \\
    &\geq \min\{x_2(0)e^{-cT(b)}, (k/c)(1 - \varepsilon)^2x_{1,s}(b)^2\} =: \varphi_2(b).
\end{align*}
\]

(62)

Case 3 (C3): \( x_1(0) < x_{1,s}(b) \) and \( x_2(0) < (1 - \delta)^2(k/c)x_{1,s}(b)^2 \). Again, it follows from (34) that \( x_1(\cdot) \) is (strictly) increasing with \( \lim_{t \to \infty} x_1(t) = x_{1,s}(b) \). Define \( \tilde{\tau} := \inf\{\tau \geq 0 : x_2(\tau) = (k/c)x_1(\tau)^2\} \) and \( \tilde{\tau}' := \inf\{\tau \geq 0 : x_2(0) = (k/c)x_1(\tau)^2\} \). Note that \( \tilde{\tau} \leq \tilde{\tau}' \) due to the fact that \( x_1(\cdot) \) is (strictly) increasing and \( x_2(\cdot) \) is...
(strictly) decreasing for $0 \leq t \leq \hat{\tau}$ according to (34) and hence $x_2(\hat{\tau}) \leq x_2(0)$. The definition of $\tau'$ implies that $x_1(t) \leq ((c/k)x_2(t))^{1/2}$ for all $0 \leq t \leq \tau'$, and hence from (34) it follows that during this time interval
\[
\dot{x}_1 \geq -c\sqrt{\frac{c}{k}x_2(0) - k\frac{c}{k}x_2(0) + cb} \\
> -c(1 - \delta)x_1,s(b) - k(1 - \delta)^2x_1,s(b)^2 + cb = c\delta x_1,s(b) + k(2\delta - \delta^2)x_1,s(b)^2;
\]
where the second inequality follows from the fact that $x_2(0) < (1 - \delta)^2(k/c)x_1,s(b)^2$. Hence we obtain that
\[
\hat{\tau} \leq \tau' \leq \frac{\sqrt{k/k}x_2(0) - x_1(0)}{c\delta x_1,s(b) + k(2\delta - \delta^2)x_1,s(b)^2} < \frac{(1 - \delta)x_1,s(b)}{c\delta x_1,s(b) + k(2\delta - \delta^2)x_1,s(b)^2} = \frac{1 - \delta}{c\delta + k(2\delta - \delta^2)x_1,s(b)} =: \hat{T}(b),
\]
(63)

Furthermore, as $\dot{x}_2 \geq -cx_2$ according to (34), it follows that $x_2(t) \geq x_2(0)e^{-ct}$ for $0 \leq t \leq \hat{\tau}$. For $t > \hat{\tau}$, we obtain $y(t) = qx_2(t) \geq q \min \{\Psi(t - \hat{\tau}, b) + x_2(\hat{\tau}), \rho(b)\}$, which follows as $x_2(\hat{\tau}) \in B$ by definition of $\hat{\tau}$. Hence, as $\Psi(0, \cdot) \equiv 0$ according to (28), we obtain that for all $t \geq 0$
\[
x_2(t) \geq \min \{\Psi(\max\{t - \hat{\tau}, 0\}, b) + x_2(0)e^{-ct}, \rho(b)\} \\
\geq \min \{\Psi(\max\{t - \hat{T}(b), 0\}, b) + x_2(0)e^{-c\hat{T}(b)}, \rho(b)\} =: \varphi_3(t, b),
\]
(64)
where the second inequality follows from (63) and the fact that $\Psi(\cdot, b)$ is nondecreasing.

Combining the above three cases, there exist constants $0 \leq b' < b''$ such that we have case C1 for $0 \leq b \leq b'$, case C2 for $b' < b \leq b''$, and case C3 for $b > b''$. Now define the function
\[
\varphi(a, b) := \begin{cases} 
q\varphi_1(b) & 0 \leq b \leq b' \\
q\varphi_2(b) & b' < b \leq b'' \\
q\varphi_3(a, b) & b > b''
\end{cases}
\]
We have shown above that for each $a, b > 0$, by applying the constant input $u \equiv b$ it follows that $y(a) = qx_2(a) \geq \varphi(a, b)$. Hence in order to conclude that the system (34) is norm-controllable from $x_0$, it remains to show that $\varphi(a, b) \geq \gamma(a, b)$ for some function $\gamma$ satisfying the properties of Definition 1. To this end, note the following.

By (60) and the definition of $\varphi_1$, it follows that $\varphi_1 \in K_{\infty}$. Furthermore, by (61) we have that $\hat{T}(\cdot)$ is continuous and strictly decreasing, and hence by definition of $\varphi_2$ in (62) it follows that $\varphi_2 \in K$. Finally, from (63) it follows that $\hat{T}(\cdot)$ is continuous and strictly decreasing with $\lim_{b \to \infty} \hat{T}(b) = 0$. Using this together with the fact that $\Psi(\cdot, \cdot) \in K_{\infty}$ for each $a > 0$ according to (28) and $\rho \in K_{\infty}$, we obtain from the definition of $\varphi_3$ in (64) that $\varphi_3(a, \cdot) \in K_{\infty}$ for each fixed $a > 0$, and $\varphi_3(\cdot, b)$ is nondecreasing for each fixed $b > 0$.

Now fix some $b'' > b''$, let $\bar{\tau} := \min\{\varphi_1(b''), \varphi_2(b'')\}$, and define the function $\gamma$ as
\[
\gamma(a, b) := \begin{cases} 
q \min\{\varphi_1(b), \varphi_2(b), \varphi_3(a, b)\} & 0 \leq b \leq b'' \\
q \min\{\varphi_3(a, b), \frac{\varphi_3(a, b'') - \bar{\varphi}}{b'' - b} + \bar{\varphi}\} & b'' < b \leq b'' \\
q \varphi_3(a, b) & b > b''
\end{cases}
\]
(65)

By definition, we have that $\varphi(a, b) \geq \gamma(a, b)$ for all $a, b > 0$, and from the above considerations, it follows that $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot) \in K_{\infty}$ for each fixed $a > 0$. By Definition 1, this means that system (34) is norm-controllable from $x_0$ with gain function $\gamma$ given by (65). Since $x_0 \in \mathbb{R}^2_0 \setminus B$ was arbitrary, we conclude that system (34) is also norm-controllable from all $x_0 \in \mathbb{R}^2_0 \setminus B$.

An interpretation of this fact is as follows. While as discussed in Example 6, the amount of product $B$ inside the reactor will first decrease (due to the outlet stream) if $x_2 > (k/c)x_2^2$, the time during which it decreases goes to zero as $b$, i.e., the concentration of $A$ in the inlet stream, increases. Hence still for each fixed time $a > 0$, the amount of product can be made large by increasing the concentration of $A$ in the inlet stream. On the other hand, the conditions of Theorem 3 cannot be satisfied with $V(x) = |x_2|$, as their satisfaction would imply that the amount of product cannot be increased from the beginning on.

Finally, we remark that for all $x \in B$, i.e., on the set where Theorem 3 applies, a uniform (with respect to the initial condition $x_0$) gain function $\gamma$ can be obtained; namely, replacing $x_2(0)$ in (36) by $0$ results in a gain function $\gamma$ which is independent of $x_0$. On the other hand, the function $\gamma$ obtained in (65) for the case that $x_0 \notin B$ is not independent of $x_0$, and it is not clear whether a uniform (with respect to $x_0$) lower bound for $\gamma$ can be found in this case.