ON THE THEORY OF ESTIMATION AND CONTROL WITH FINITE DATA-RATE

BY

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DISSERTATION

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ABSTRACT

In this dissertation, we discuss control and estimation problems for systems that operate with data-rate constraints. We start by studying the estimation entropy of switched linear systems. This quantity is the minimal data-rate we need to use to estimate the state of a switched linear system with an estimation error that decays with a prescribed exponential decay rate. We provide upper and lower bounds for the estimation entropy in terms of the Lyapunov exponents of the switched system. Also, we show that those bounds coincide with the entropy when the system is Lyapunov regular. We provide a coding scheme that solves the state reconstruction problem with the data rate as close as desired to the minimum. Under the regularity assumption, we show how to make that algorithm work causally. Next, we present sufficient conditions for a system to be Lyapunov regular and show that Markov Jump Linear Systems belong to that class. We also illustrate those theoretical results with simulations. Then, we switch subjects to the problem of defining controllability for linear time-varying systems with a finite data-rate. We explain why the usual notion of controllability is unfit when data-rate constraints are present. Then, we define a new controllability notion that makes sense in this scenario. We also justify why such a notion is natural. Next, we present a necessary condition and a sufficient condition for a system to be controllable with a finite data-rate. Finally, we revisit the topic of controllability with a finite data-rate, but we specialize our analysis to switched linear systems. Although this part of the work is more restrictive than the previous one, we show how the switched system structure allows us to derive sufficient conditions for our system to be controllable with a finite data-rate using information about the individual modes and some mild assumptions about the switching signal. In particular, when our switching signal satisfies an average dwell-time condition, we give a simple relation between the sampling time, chatter bound, and average-dwell time that guarantees that our system will be controllable with a finite data-rate. Then, we generalize our analysis by assuming we can have packet losses in our communication channel. We prove a sufficient condition for such a system to be controllable with a finite data-rate even when we might lose packets. We demonstrate this condition by constructing an algorithm, which makes this proof constructive.

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TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION **1**

CHAPTER 2 ESTIMATION ENTROPY, LYAPUNOV EXPONENTS, AND QUANTIZER DESIGN FOR SWITCHED LINEAR SYSTEMS **6**

2.1 Preliminaries 9

2.1.1 Example 11

2.2 Estimation Entropy 11

2.2.1 Connection with previous results 22

2.3 Quantization Algorithm 23

2.3.1 The Algorithm 24

2.3.2 Finding $(\mathcal{V}_j)_{j\in\mathbb{Z}_{\geq 0}}$ Online 34

2.4 Sufficient Conditions for Regularity 38

2.4.1 Sampled Continuous-time Regular Systems 38

2.4.2 Randomly Switched Systems 40

2.5 Conclusion 46

CHAPTER 3 CONTROLLABILITY FOR LINEAR TIME-VARYING SYSTEMS WITH A FINITE DATA-RATE **47**

3.1 Chapter Overview 47

3.2 Preliminaries 49

3.3 Controllability with Finite Data-Rate 52

3.3.1 Sufficient Condition 54

3.3.2 Necessary Condition 62

3.4 Conclusion 62

CHAPTER 4 CONTROLLABILITY AND CONTROLLER DESIGN FOR SWITCHED LINEAR SYSTEMS WITH A FINITE DATA-RATE **64**

4.1 Chapter Overview 64
4.2 Motivation 66
4.2.1 The Model 66
4.2.2 The Need for a New Controllability Notion 67
4.2.3 Motivating Example 71
4.3 Controllability with a Finite Data-Rate 72
4.3.1 The Necessary Condition 73
4.3.2 The Sufficient Condition: the case without packet losses 74
4.3.3 Applications 75
4.3.4 The Sufficient Condition: the general case 77
4.3.5 Understanding the General Sufficient Condition . . . 80
4.3.6 Quantized Control Concepts 85
4.3.7 The Gap Between the Conditions 87
4.4 Controller Design 89
4.4.1 Algorithm's Idea 89
4.4.2 Algorithm's Formal Description 93
4.4.3 Controller Analysis 97
4.4.4 Data-Rate Analysis 102
4.5 Conclusion 107

CHAPTER 5 FUTURE RESEARCH DIRECTIONS 108

REFERENCES 110

APPENDIX A 117

A.1 Proof of equivalence between the two definitions of complete controllability 117
A.2 Proof of Lemma 3.3.2 118
A.3 Proof of Lemma 3.3.4 118
A.4 Proof of Lemma 3.3.3 119
A.5 Proof of Proposition 4.2.2 121
A.6 Proof of Theorem 3.3.6 126
A.7 Proof of Theorem 4.3.2 130
A.8 Proof of Lemma 4.3.3 130
A.9 Proof of Proposition 4.3.5 134
A.10 Proof Corollary 4.3.6 136

A.11 Proof of Lemma 4.3.8 136
A.12 Proof of Lemma 4.3.9 141
A.13 Proof of Lemma 4.4.2 142
A.14 Proof of Lemma 4.4.3 143
A.15 Relation of Theorem 4.3.7 and the literature on entropy . . . 144

CHAPTER 1

INTRODUCTION

Control and communication theories have long histories. It is customary to trace the former field's beginning to Maxwell's paper [1], where he analyzed the stability of Watt's governor. For the latter, the seminal paper of Shannon [2] on the mathematical theory of communication formalized what would later be called information theory. It is less usual, however, to point out when these two subjects first appeared together. A good candidate for that first overlap is the seminal book by Norbert Wiener [3]. In that work, Wiener coined the term Cybernetics and started the homologous field of study. More relevant to our discussion is that that work is the starting point of the theory of control over communication channels. We see in the book [3] the connection between the transmission of information and the feedback loop. However, he only considered analog channels in that work. Since the transistor was just a proof of concept when Wiener wrote the reference [3], we can conjecture that that is why he neglected digital communication channels. Nowadays, however, digital communication is ubiquitous in most applications. As usual, new technologies bring unexplored technical challenges and theoretical problems.

To understand the new challenges that including a digital communication channel brings into the controller design, we need to take a step back and understand the concept of data-rate. In classical control, we have sensors that transmit perfect information to the controller. However, when this communication happens over a digital channel, we can only send packets of symbols from the transmitter to the receiver. Thus, we need to convert our measurement from the analog world into the digital one. We perform this latter part using a coding scheme, i.e., a coder and decoder pair. The purpose of the coder is to translate measurements from the analog world into the digital by encoding them as symbols before transmission. Once those symbols reach the receiver, the decoder converts them into measurement estimates.

Informally, the data-rate is the ratio between the packet's size, defined as the number of symbols it can contain, to the time elapsed to transmit it. It is well-known that a digital channel can only reliably communicate information if the data-rate is finite [2]. In the early days of control over digital communication channels, it was common to model the effect of that quantization error as an additive noise [4, 5] and use tools from probability theory to perform the analysis. Because of that, it was usual to treat both coding schemes and control designs separately, and this approximation gives satisfactory results for many systems.

The discovery of the Data Rate Theorem [5–8] changed that reality. This theorem proves the existence of a minimal data-rate below which we cannot control the plant. Moreover, we might need to design the controller and coding scheme together to control the plant when the system and the digital communication channel pose severe constraints. For instance, that theorem shows that, for unstable linear time-invariant systems, we need a positive minimal data-rate for stabilizing controllers to exist. Intuitively, this happens because we lose information about our measurement when we perform the quantization. When the quantization is too coarse, the value reconstructed at the decoder side might be too far from the original value. Over time, this error increases, and it will instabilize the plant. This latter comment shows that the usual approach we mentioned in the last paragraph, i.e., modeling the quantization error as additive noise and designing the controller, does not work when the data-rate is severely constrained. With this discussion, we make a case for studying what new constraints appear when our control or estimation tasks restrict us to work with a finite data-rate.

The reader who is used to lumped control systems might be asking what types of applications require the introduction of communication channels between the sensors and the controller. We notice that many practical control systems today have components distributed over space. Thus, performing a task that involves the joint operation of the system parts', such as synchronization, requires digital communication between its components. This latter remark justifies the claim: understanding what new constraints arise when the data-rate is limited is fundamental. We start by trying to find results similar to the Data Rate Theorem that works for other control and estimation tasks.

It is clear that the concept behind the Data Rate Theorem is that of entropy [9]. Entropy, loosely speaking, is the rate at which the system generates uncertainty. More precisely, it captures the rate at which two nearby trajectories of a dynamical system diverge from each other. The intuition behind why that concept is relevant becomes clear when we think of the system's behavior in the interval between the arrival of two consecutive data packages. In this interval, the control system behaves as an open-loop controlled system, and the feedback only happens when we receive an update. Therefore, it sounds natural to define and compute the entropy. We note, however, that the entropy notion we are interested in might be distinct for each specific control or estimation task. Because of that, several authors defined different entropies for distinct control and estimation tasks [9–13]. In the present work, we will focus on the estimation entropy [13]. The estimation entropy is related to the problem of reconstructing the state with a prescribed exponentially decreasing error bound with minimal data rate.

The first main goal of this document is to understand how we can compute the estimation entropy of switched linear systems and how to design a quantizer that operates with a data-rate close to the minimum possible. We start this part of the document by asking how to compute the estimation entropy of a switched linear system. More explicitly, we provide upper and lower bounds for the estimation entropy using the Lyapunov exponents of a Switched Linear System and prove that those bounds coincide for a large class of systems called Lyapunov regular. This so-called "regularity" property allows us to devise a causal algorithm that reconstructs the state with a data rate as close as desired to the estimation entropy. Next, we provide sufficient conditions for a Switched Linear System to be regular. In particular, we show that many Markov Jump Linear Systems [14] satisfy those conditions.

Our second goal is to study what new constraints arise when we try to control linear time-varying systems with a finite data-rate. We argue that the usual notion of controllability for linear time-varying systems is unsuitable when the controller data-rate must be finite. Then, we introduce a new controllability concept that captures some of the properties of the usual controllability notion, which justifies calling it controllability with a finite data-rate. After that, we try to characterize that new controllability notion by giving a necessary condition and a suffcient condition for our system to be controllable with a finite data-rate. We prove the sufficient part by giving a constructive algorithm, which we can use to design controllers for such systems.

Our third and final goal is to study sufficient conditions for switched linear systems to be controllable with a finite data-rate. First, we specialize our previous sufficient condition for general linear time-varying systems to the case where our system is a switched one. By restricting our attention to this class of systems, we can derive much simpler sufficient conditions for controllability with a finite data-rate than the one we get for general linear time-varying systems. We take advantage of the switched system structure to derive conditions that only depend on the controllable subspaces of the individual modes and some mild properties of the switching signal. In particular, when the switching signal satisfies an average dwell-time condition, we get a straightforward relation between the sampling time, the chatter bound, and the average dwell-time. Next, we consider the possibility of packet losses, i.e., our communication between the sensor and controller might be faulty. We give a sufficient condition for such a system to be controllable with a finite data-rate. We prove this latter fact by providing a controller design technique.

The present document is organized as follows: in Chapter 2, we show our work on the estimation entropy of switched linear systems. We formalize all the concepts related to Oseledets' filtrations, Lyapunov exponents, and estimation entropy. Also, we provide proof for the upper and lower bounds for the estimation entropy and show that they are the same when regularity holds. Next, we present a coding scheme that operates with a data rate as close as desired to the minimum value with a suitable choice of parameters. After that, we show how to estimate those parameters online to make the scheme causal. Finally, we give sufficient conditions for regularity and prove that Markov Jump Linear Systems satisfy those with probability 1. In Chapter 3, we argue that we need to introduce a new controllability notion for systems that operate with a finite data-rate. We explain why that is the case and introduce a new controllability notion, which we justify why this extends the usual controllability notion to this case. Then, we prove a necessary condition and a sufficient condition for the system to be controllable with a finite data-rate. In Chapter 4, we address the same problem as in Chapter 3, but we focus on the case of switched linear systems. We start by motivating the study of such a problem again and give examples. Then, we derive sufficient conditions for a switched system to be controllable with a finite data-rate that only uses information about the individual modes and some mild conditions on the switching signal. In particular, we use one of those conditions to prove a simple condition for a system that satisfies an average dwell-time condition to be controllable with a finite data-rate. Next, we address the problem of controlling a switched system when packet losses occur. We give another sufficient condition for controllability for this case and prove it constructively by giving a controller design technique. Finally, in Chapter 5, we present some interesting future research directions.

CHAPTER 2

ESTIMATION ENTROPY, LYAPUNOV EXPONENTS, AND QUANTIZER DESIGN FOR SWITCHED LINEAR SYSTEMS

In this overview, we motivate the study of state estimation with finite data-rate. We start by noticing that many systems we find in practice require spatially distributed sensors, controllers, and actuators to operate. For these systems to function normally, their components must share data. Thus, we need communication channels to transmit information between the constituent pieces of our system. These channels, in their turn, constrain the data-rate we can use to send data from the source to its target. Consequently, this raises the question: what is the minimum data-rate for our system to work satisfactorily?

The answer to this question varies with the nature of the problem we want to solve. Nonetheless, the solution is often related to some entropy notion. Informally, our problem-associated entropy notion is the rate at which a system generates information related to the studied problem. This relationship between entropy and data-rate led many authors to propose distinct entropy notions for each task (see, e.g., [9, 11, 12, 15–17]). In this chapter, we study the minimum data rate to estimate the state of switched linear systems with a prescribed exponential decay rate of $\alpha \geq 0$ for the estimation error. This problem naturally guides us to the entropy concept called estimation entropy. Liberzon introduced this idea to solve state estimation problems for autonomous nonlinear systems in the work [18]. In some sense, we can understand estimation entropy as a rate at which the system generates uncertainty about the state. The estimation entropy value gives us the minimum data-rate associated with our problem. However, that is only part of the problem. To completely solve our task, we need to design a coder-estimator scheme. In this chapter, we address this issue. We construct a coding-estimator scheme that operates with an average data-rate arbitrarily close to the estimation entropy for switched linear systems. Another goal of this chapter is to show a relationship between the estimation entropy of a switched linear system and its Lyapunov exponents. It is worth mentioning that we previously showed the connection between those two results in the papers [19] and [20]. Nevertheless, we present a different proof of that fact, which makes it easier to see an interesting relationship between the Lyapunov exponents, geometric objects called filtrations (that play a role similar to eigenspaces in the linear time-invariant case), and the quantizer design.

We take this opportunity to give a brief overview of the literature on entropy notions for switched systems. We hope this contextualization will help the reader understand the present work's contributions. The first documents to explicitly present an entropy notion for switched systems, related to the estimation entropy defined in the article [18], were [21], [22], [23], [24], and [25]. Now, we explain the contribution of each one of those works to the current theory. The article [22] studied a nonstandard modification of the estimation entropy adapted to the analysis of switched nonlinear systems with unknown switching signals satisfying a minimum dwell-time restriction. The paper [23] extended those results by considering systems with inputs. The work [24] studied the topological entropy of switched linear systems, which equals the estimation entropy of [18] when $\alpha = 0$, by presenting upper and lower bounds under some structural assumptions on the modes. The papers [25] and [26] improved the bounds presented in [24]. All bounds we mentioned relied on individual modes and their active rates, and the authors did not use any other features of the switching signal. The authors of [25] concluded that we could not improve those bounds without further knowledge of the switching signal structure. After that, we presented tighter bounds for the class of regular switched systems in [19]. The tightness of these bounds relies on the knowledge of the entire switching signal. Another interesting aspect of [19] is that the upper bound provided for the estimation entropy of switched linear systems is related to the Lyapunov exponents. We mention that in [27], the authors obtained the same minimum average data-rate as in [19], with $\alpha = 0$, but for the mean square stabilization problem of scalar Markov Jump Linear Systems. We note that the relation between Lyapunov exponents and the entropy appears in several places in the dynamical systems literature, often under the name *Pesin entropy formula* [28–30], as well as in the invariance entropy formula for partially hyperbolic control systems [31], and in a lower bound for the estimation entropy of a class of differential dynamics on compact manifolds [32]. We remark that to obtain those relations, we need to assume differentiability of the flow and compactness of the state space, which is different from the technique used in [19], [20], and in the present work.

In more recent years, the work [33] provided a computation method for the maximum topological entropy over all possible switching signals. Further, those same authors showed in [34] that the topological entropy of a linear time-varying system equals the minimum average data-rate for the state observation with bounded estimation error. Finally, the reference [35] provided an algorithm stabilizing switched linear systems with an average data-rate arbitrarily close to the minimum. It is worth remarking that, although seemingly different, the minimum data-rate for stabilization obtained in [35], in terms of the Lyapunov exponent of exterior products, has the same value as the estimation entropy lower bound obtained in [19] utilizing the usual Lyapunov exponents of linear systems, with $\alpha = 0$ (see, e.g., Chapter 6 of [36]). Nevertheless, for the algorithm described in [35] to work, we must provide an upper bound for the topological entropy at the start of the operation, which might not be a realistic assumption, as discussed in [20].

In this chapter, we present the works published in [19], [37], and [20]. Here, we connect Lyapunov exponents and estimation entropy. Also, we give a constructive algorithm that builds a state estimate for a switched linear system with a prescribed exponential decay rate $\alpha \geq 0$ for the estimation error with an average data-rate as close as desired to the estimation entropy. Interestingly, assuming that our system is regular, we can make the previous algorithm causal. We naturally ask if regularity is a mild assumption, which we answer affirmatively by showing that many practical systems fall into that category. For example, with probability one, the realizations of Markov Jump Linear Systems are regular. We also describe how to use regularity to make our quantization algorithm work causally. Remarkably, we can ensure the exponential decay of the error without knowing the entire switching signal, as required in [35].

This chapter has the following structure: in Section 2.1, we describe the problem we want to study. Also, we present a motivating example that we revisit throughout the chapter. In Section 2.2, we examine the notions of Lyapunov exponents, Oseledets' filtration, and estimation entropy. In our analysis of the estimation entropy of switched linear systems, we provide an upper bound for it and show that, under the assumption of Lyapunov regularity, that upper bound becomes equality. We introduce our quantization algorithm in Section 2.3. Then, we use the concepts presented in the previous section to show that this algorithm can operate with an average data-rate close to the estimation entropy. After that, in Section 2.4, we show that the Lyapunov regularity condition is natural for many systems. Finally, in Section 2.5, we conclude the contents of this chapter. It is worth remarking that one can find the contents of this chapter presented in a slightly different form in the works [19], [37], and [20].

Chapter notations: We denote by $||\cdot||_P$ the norm in $\mathbb{R}^d$ induced by the inner product $\langle x, y\rangle_P = x^\top P y$, with P positive definite. We denote by $||\cdot||$ the infinity-norm in a finite dimensional vector space. Let $\mathbb{R} = (-\infty, \infty)$, let $\mathbb{Z}_{\geq 0} = \{0, 1, \ldots\}$ the nonnegative integers, and let $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers. We denote by $[m] := \{1, \ldots, m\}$ for any $m \in \mathbb{N}$. For a real number x, we denote by $\lceil x \rceil$ the smallest integer number y such that $x \leq y$. For any set E, we denote by $\#E$ its cardinality. For subsets of $\mathbb{R}^d$ we denote $\mathrm{vol}(E)$ the volume of the set (its Lebesgue measure). Further, we denote by $\mathrm{diam}(E)$, where $E \subset \mathbb{R}^d$ the set's diameter according to the metric induced by the norm $||\cdot||$. We also denote by $\dim(V)$ the dimension of a linear vector space V. Also, for any $x > 0$, $\log x$ is the logarithm with base e. Furthermore, we define by $\mathbb{B}(x, r)$ with $x \in \mathbb{R}^d$ and $r > 0$ the infinity-norm ball (hypercube) with center x and radius r.

We denote by $\mathcal{M}(d, \mathbb{R})$ the set of all $d \times d$ matrices over the reals, and we denote by $\mathrm{GL}(\mathbb{R}, d)$ the set of all $d \times d$ invertible matrices over the reals. We denote $\det(A)$ and $\mathrm{tr}(A)$ the determinant and the trace of the matrix A, respectively. Further, $I_d \in \mathcal{M}(d, \mathbb{R})$ is the identity matrix.

Additionally, consider the parallelepiped defined by $\{\kappa_i v_i : \kappa_i \in [0, 1]\}$, where $\{v_i\}_{i=1}^k \subset \mathbb{R}^d$ is a linearly independent set of vectors. We denote the k-th volume of the parallelepiped by $\mathrm{vol}\left(\{v_1, \cdots, v_k\}\right)$ and its numerical value is given by $\sqrt{\det\left(V^\top V\right)}$, where V is the $d \times k$ matrix with columns v_i.[1]

2.1 Preliminaries

In the chapter, we consider the following model

$$\dot{x}(t) = \mathcal{A}_{\sigma(t)} x(t), \tag{2.1}$$

[1]Notice that interchanging the order of the columns does not change the k-th volume.

where $x(t) \in \mathbb{R}^d$, $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is a switching signal and Σ is a finite cardinality set, and $\mathcal{A}_{\sigma(t)} \in \mathcal{M}(d, \mathbb{R})$. We denote by $\Phi(t, t_0)$ the *state-transition matrix* of (2.1), i.e. the solution of the ODE $\frac{d}{dt}\Phi(t, t_0) = \mathcal{A}_{\sigma(t)}\Phi(t, t_0)$ with $\Phi(t_0, t_0) = I_d$ and t_0 being the initial time. Furthermore, we will make the assumption that σ is constant on intervals of the type $[t_i, t_{i+1})$ for $i \in \mathbb{Z}_{\geq 0}$, where $(t_i)_{i \in \mathbb{Z}_{\geq 0}}$ is a strictly increasing sequence of positive times such that $\lim_{i\to\infty} t_i = \infty$. The elements of the sequence $(t_i)_{i\in\mathbb{N}}$ are called *switching times*. We also need to define an increasing sequence of *sampling times* $(\tau_k)_{k\in\mathbb{Z}_{\geq 0}}$, with $\tau_k = kT_p$ for all $k \in \mathbb{Z}_{\geq 0}$ and some $T_p > 0$.

Then, we can rewrite the model described in equation (2.1) using its *exact discrete-time model*, defined by:

$$x_{k+1} = \tilde{A}_k x_k, \tag{2.2}$$

where $(x_k)_{k\in\mathbb{Z}_{\geq 0}}$ is the state at the sampling times τ_k, i.e. $x_k = x(\tau_k)$ for $k \in \mathbb{Z}_{\geq 0}$, and $\tilde{A}_k \coloneqq \Phi(\tau_{k+1}, \tau_k)$ for $k \in \mathbb{Z}_{\geq 0}$.

Consider the following definitions of coder-estimator scheme (see, e.g., [9,15]). Let $\{\tau_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be the previously described sequence of sampling times. Also, let $\{\mathcal{C}^n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of alphabets with uniformly bounded cardinality, i.e. $\exists M > 0$, $\#\mathcal{C}^i < M, \forall i \in \mathbb{Z}_{\geq 0}$. We call the elements q of a finite alphabet *symbols*. Furthermore, let $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of functions such that $\gamma_n : \prod_{i=0}^{n-1} \mathcal{C}^i \times \mathbb{R}^{d(n+1)} \to \mathcal{C}^n$, where γ_n is called the *coder mapping at time n*. We can write the coder mapping in the following more explicit way

$$\begin{aligned} &\gamma_0 : x(\tau_0) \mapsto q_0, \\ &\gamma_n : (q_0, \ldots, q_{n-1}, x(\tau_0), \ldots, x(\tau_n)) \mapsto q_n, \end{aligned}$$

where $q_n \in \mathcal{C}^n$ for all $n \in \mathbb{Z}_{\geq 0}$.

The *average data-rate* of a coder-estimator scheme is defined as

$$b \coloneqq \limsup_{j\to\infty} \frac{1}{t_j} \sum_{i=0}^{j} \log\left(\#\mathcal{C}^i\right). \tag{2.3}$$

2.1.1 Example

Now, we present an example that motivates our study. This example considers a randomly switched system. Our goal with this example is to show that we can use information about the switching signal to design a quantization algorithm that outperforms methods that treat the individual modes separately in terms of average data-rate. We revisit this example several times in this chapter as we develop our theory.

Example 2.1.1. Let $B_1 = \begin{pmatrix} 0.9 & 0.03 \\ 0 & 1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1.1 & 0.02 \\ 0 & 1 \end{pmatrix}$ be the modes of our discrete-time switched system. This implies that $\tilde{A}_k \in \{B_1, B_2\}$ for $k \in \mathbb{Z}_{\geq 0}$. We see that mode B_2 is unstable. Thus, using the standard quantization scheme [7], which guarantees a uniformly bounded estimation error using the minimum average data-rate for each mode separately, will force us to use a positive average data-rate whenever that mode is active. Nevertheless, in this chapter, we show that if our switching signal comes from a Markov chain defined by the matrix of transition probabilities $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$, where P_{ij} is the transition probability from mode i to mode j, then, we can design an algorithm that estimates the state with a uniformly bounded estimation error and that operates with an average data-rate arbitrarily close to zero, with probability one.

Later in this document, we describe a quantization scheme that reconstructs the state using an average data-rate arbitrarily close to the estimation entropy for a large class of switched systems, which we call regular systems. It is remarkable that, with probability one, Markov Jump Linear Systems, like the one in this example, are in this class.

2.2 Estimation Entropy

In the present section, we introduce several instrumental concepts, including estimation entropy and Lyapunov exponents. We use those concepts to derive an upper bound for the estimation entropy discrete-time switched systems. More explicitly, we give an upper bound for the estimation entropy using the Lyapunov exponents. Furthermore, when we assume that our system is Lyapunov regular, we prove that that upper bound coincides with the estimation entropy's value. We need to mention that the definitions presented in this section are adaptations of definitions provided in other works, namely: [13], Chapter 2 of [36], and Chapter 3 of [38].

Throughout this document, for a sequence of invertible matrices $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}(d,\mathbb{R})$, we denote the *discrete-time state-transition matrix* of the system (2.2) by

$$\Phi_n := A_n \cdots A_1. \tag{2.4}$$

Further, we define $A_n := \tilde{A}_k$ for $n = k+1$ with[2] $k \in \mathbb{Z}_{\geq 0}$. We assume that $K \subset \mathbb{R}^d$, the set of possible initial conditions, is a compact set with a nonempty interior. Further, the solution of (2.2) at time step n with initial condition $x \in \mathbb{R}^d$ is given by $\xi(x,n) = \Phi_n x$, where the matrix sequence comes from the right-hand side of (2.2).

For the next two definitions, pick an $\alpha \geq 0$, and let $T \in \mathbb{N}$ be the time horizon.

Definition 2.2.1. For every $\epsilon > 0$, we call a finite set of functions $\hat{X} = \{\hat{x}_1(\cdot),\ldots,\hat{x}_N(\cdot)\}$, from $\{0,\ldots,T-1\}$ to $\mathbb{R}^d$, a (T,ϵ,α,K)-*approximating set* if for every initial condition $x \in K$, there exists $\hat{x}_i \in \hat{X}$ such that $||\xi(x,n) - \hat{x}_i(n)|| < \epsilon e^{-\alpha n}, \quad \forall n \in \{0,\ldots,T-1\}$. Let $s_{\text{est}}(T,\epsilon,\alpha,K)$ be the minimum cardinality of a (T,ϵ,α,K)- approximating set. We define the estimation entropy as

$$h_{\text{est}}(\alpha,K) := \lim_{\epsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log s_{\text{est}}(T,\epsilon,\alpha,K).$$

Definition 2.2.2. For every $\epsilon > 0$, we call a finite set of points $S = \{x_1,\ldots,x_N\} \subset K$ a (T,ϵ,α,K)-*spanning set* if for every initial state $x \in K$, there exists $x_i \in S$ such that $||\xi(x,n) - \xi(x_i,n)|| < \epsilon e^{-\alpha n}, \quad \forall n \in \{0,\ldots,T-1\}$. Let $s^*_{\text{est}}(T,\epsilon,\alpha,K)$ be the minimum cardinality over all possible (T,ϵ,α,K)- spanning set.

Theorem 1 from [13] tells us that

$$h_{\text{est}}(\alpha,K) = \lim_{\epsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log s^*_{\text{est}}(T,\epsilon,\alpha,K).$$

We use this characterization of the estimation entropy in the proof of Theorem 2.2.6.

[2]Notice that $A_1 = \tilde{A}_0$. Thus, $A_n = \Phi(T_p n, T_p(n-1))$ for $n \in \mathbb{N}$. We make this choice to be consistent with the literature.

Definition 2.2.3. A *Lyapunov index* is a function $\lambda : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ with the following properties:

- $\lambda(\kappa v) = \lambda(v)$, for every real $\kappa \neq 0$
- $\lambda(v + w) \leq \max\{\lambda(v), \lambda(w)\}$
- $\lambda(0) = -\infty$

A Lyapunov index $\lambda(\cdot)$ can take at most d distinct real values (see, e.g., [36]). We recall that the value $-\infty$, corresponding to $\lambda(0)$, is not a real value.

Definition 2.2.4. The *Lyapunov exponent* associated with a sequence of matrices $(A_n)_{n\in\mathbb{N}}$ is the following Lyapunov index[3]:

$$\lambda(v) := \limsup_{n\to\infty} \frac{1}{n} \log\left(||\Phi_n v||\right),$$

for $v \in \mathbb{R}^d \setminus \{0\}$. Also, we define $\lambda(0) := -\infty$.

The Lyapunov exponent, $\lambda(\cdot)$ is a special Lyapunov index (see, e.g., [36]). Thus, we note that its image has at most d distinct values. We denote these values by χ_i, for $i \in [q]$, where $q \leq d$, and we index them according to the increasing order for real numbers, i.e., $\chi_1 < \cdots < \chi_q$. We call χ_i, $i = 1, \ldots, q$ the *Lyapunov exponent values.*

Definition 2.2.5. *A filtration (or flag) on $\mathbb{R}^d$ is a family of vector subspaces $\mathbb{V} = (E_i)_{i=0}^q$, with $q \leq d$, such that $\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_q = \mathbb{R}^d$. Further, we call $\mathcal{V} = \{v_i\}_{i=1}^d$ a normal basis of the filtration $\mathbb{V}$ if it is a basis for $\mathbb{R}^d$, and for every $j \geq 1$, the subset of $\mathbb{V}$ given by $\{v_i\}_{i=1}^{\dim(E_j)}$ is a basis for E_j.*

In this document, we are interested in a specific type of filtration: the Oseledets' filtration.

Definition 2.2.6. A filtration $\mathcal{V}_\lambda$ associated with the sequence of invertible matrices $(A_n)_{n\in\mathbb{N}}$ such that $E_i = \left\{v \in \mathbb{R}^d : \lambda(v) \leq \chi_i\right\}$, where $\lambda(\cdot)$ is the Lyapunov exponent for the sequence, and χ_i are the Lyapunov exponent values of the sequence previously defined, is called an *Oseledets' filtration.* Also, the subspaces $E_i \in \mathcal{V}_\lambda$ are called *Oseledets' subspaces.* In addition, the following $\dim(E_i) - \dim(E_{i-1})$ is called the *multiplicity of the Lyapunov*

[3]Note that the function does not change if we change the norm.

exponent value χ_i. If[4] $\dim(E_i) - \dim(E_{i-1}) = 1$ for every $i \in \{1, \dots, q\}$, we say that the Lyapunov exponents are *simple*. Finally, define $\Lambda = \{\lambda_j\}_{j=1}^{d}$ as an ordered list with repetition where for every $j = 1, \dots, d$, there exists some $i \in \{1, \dots, q\}$ such that $\lambda_j = \chi_i$, and for every $i = 1, \dots, q$, χ_i appears $\dim(E_i) - \dim(E_{i-1})$ times in Λ. The order in Λ can be any total order relation in the set Λ chosen among those for which $\lambda_1 \leq \cdots \leq \lambda_d$. We call the elements $\lambda_i \in \Lambda$ the *Lyapunov exponents with multiplicity* of $(A_n)_{n\in\mathbb{N}}$.

The Oseledets' filtration depends on the entire sequence $(A_n)_{n\in\mathbb{N}}$. We illustrate that fact with the next example.

Example 2.2.1. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that the sequences $A'_n = A$ for all $n \in \mathbb{N}$, and the sequence $A_n = A$ for $n \in \mathbb{N} \setminus \{N\}$ and $A_N = B$ for some $N \in \mathbb{N}$, have the same Lyapunov exponents. However, these sequences have different Oseledets filtrations. We can see that the Oseledets' filtration of the first sequence is $E_1 = \operatorname{span}\left\{\begin{pmatrix} 1 & 0 \end{pmatrix}^\top\right\} \subsetneq E_2 = \mathbb{R}^2$. The second sequence, on the other hand, has an Oseledets filtration given by $E_1 = \operatorname{span}\left\{\begin{pmatrix} 0 & 1 \end{pmatrix}^\top\right\} \subsetneq E_2 = \mathbb{R}^2$.

Definition 2.2.7. A sequence $(A_n)_{n\in\mathbb{N}}$ is called *tempered* if

$$\lim_{n\to\infty} \frac{1}{n} \log ||A_n|| = 0.$$

If the sequence $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded, then it is tempered. In particular, if $(A_n)_{n\in\mathbb{N}}$ has finitely many values, then it is temperedness. Remarkably, temperedness does not imply a sub-exponential growth rate for Φ_n. One counter-example is the sequence $A_n = n$, which is tempered because $\lim_{n\to\infty} \frac{\log(n)}{n} = 0$, but $\Phi_n = n!$ grows faster than any exponential.

Example 2.2.2 (Example 2.1.1 revisited.)**.** We take this opportunity to revisit Example 2.1.1. Let $\{e_1, e_2\}$ be the canonical basis for $\mathbb{R}^2$. Also, we denote by $a_{ij}(n)$ the element in the i-th row and j-th column of the matrix A_n. Further, we denote by $\phi_{ij}(n)$ the elements of the matrix Φ_n. Furthermore, denote by $m_i(n) = \sum_{k=1}^{n} \mathbb{I}_{\left\{(A_n)_{n\in\mathbb{N}}: A_k = B_i\right\}}((A_n)_{n\in\mathbb{N}})$, where $\mathbb{I}_A(\cdot)$ is the indicator function, i.e., $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$, otherwise. The quantity $m_i(n)$ counts how many times mode i was active until time n.

[4]Equivalently, we could say that $d = q$.

A simple computation shows us that $\phi_{11}(n) = 0.9^{m_1(n)}1.1^{m_2(n)}$, $\phi_{22}(n) = 1$, and $\phi_{12}(n) = a_{11}(n)\phi_{12}(n-1) + a_{12}(n)$ for $n \geq 1$ with initial conditions $\phi_{ii} = 1$ and $\phi_{ij} = 0$ if $i \neq j$. Then, the Lyapunov exponents of the sequence $(A_n)_{n\in\mathbb{N}}$ are given by

$$\begin{aligned}\lambda(e_1) &= \limsup_{n\to\infty} \frac{1}{n}\log(\|\Phi_n e_1\|)\\ &= \limsup_{n\to\infty} \frac{1}{n}\log\bigl(0.9^{m_1(n)}1.1^{m_2(n)}\bigr)\\ &= \limsup_{n\to\infty} \frac{m_1(n)}{n}\log(0.9) + \frac{m_2(n)}{n}\log(1.1).\end{aligned}$$

Now, note that the average time an ergodic Markov chain remains on mode i is, with probability one, π_i, where π_i comes from the solution to the system of equations $\pi = \pi P$ and $\sum_{i=1}^{2}\pi_i = 1$, where $(\pi_1, \pi_2) = \pi$. This latter fact leads us to the conclusion that, with probability one, realizations of our Markov process will be such that the fractions $\frac{m_i(n)}{n}$ converge to π_i for $i \in [2]$. Thus, $\lambda(e_1) = \frac{1}{2}\log(0.99) < 0$. We further note that we can interpret $\phi_{12}(n) = a_{11}(n)\phi_{12}(n-1) + a_{12}(n)$ as a scalar linear time-varying system with an input $a_{12}(n)$. Therefore, if $\prod_{j=1}^{n} a_{11}(j) < 1$ and $a_{12}(n)$ are bounded, we conclude that $\phi_{12}(n)$ is bounded. Simple calculations show us that $a_{12}(n)$ and $\prod_{j=1}^{n} a_{11}(j) = 0.9^{m_1(n)}1.1^{m_2(n)}$ are bounded. We can now compute $\lambda(e_2)$ by noticing that $\|\Phi_n e_2\| = \max\{\phi_{12}(n), 1\}$ is bounded, hence $\lambda(e_2) = 0$ with probability 1.
Finally, note that the filtration $E_1 = \operatorname{span}\{e_1\} \subsetneq E_2 = \mathbb{R}^2$ is the Oseledets' filtration and that $\{e_1, e_2\}$ forms a normal basis for it.

We remark that, although the sequence $\bigl(A_n\bigr)_{n\in\mathbb{N}}$ comes from a stochastic process, we calculated the values of the Lyapunov exponents for a generic realization. Thus, we always choose a specific realization, as in the deterministic case. Nonetheless, we use the Markov chain's properties to show that our result holds for almost all realizations of the random process.

Definition 2.2.8. A sequence $(A_n)_{n\in\mathbb{N}}$ is called (Lyapunov) *regular* if

$$\lim_{n\to\infty} \frac{1}{n}\log\left(|\det\left(\Phi_n\right)|\right) = \sum_{i=1}^{d}\lambda_i.$$

We call a system given by Equation (2.2) *regular*, if its associated matrix sequence is regular.

We use the following two examples to illustrate the notion of regularity.

Example 2.2.3. In this example, we denote by $\{e_1, e_2\}$ the canonical basis in $\mathbb{R}^2$. Let $\rho > 1$, $B_1 = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$, and $B_2 = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix}$. Now, we build a matrix sequence in the following way: let $A_n = B_1$, when $n \in \{2^i, \cdots, 2^{i+1}-1\}$ for $i \in \mathbb{N}$ odd, and let $A_n = B_2$, otherwise. We notice that $\det(|\Phi_n|) = 1$ regardless of the sequence $(A_n)_{n\in\mathbb{N}}$ since $\det(B_1) = \det(B_2) = 1$. Consider the subsequence with indices $n_k = 2^k$ for $k \in \mathbb{N}$. Then, using induction, we see that $\|\Phi_{n_k}(e_1)\| = \rho^{-\sum_{i=1}^{k}(-2)^{i-1}+(-1)^k}$. Thus, $\frac{\log(\|\Phi_{n_k}(e_1)\|)}{2^k} = \sum_{\ell=1}^{k}\left((-1)^{\ell+1}(2)^{-\ell} + (-1)^k 2^{-k}\right)\log(\rho)$, after the change of varibles $\ell = -i + k + 1$. Further restricting ou analizys to the subsequence with indices $n_k = 2^k$ for k even, we show that $\lim_{k\to\infty}\sum_{\ell=1}^{k}(-1)^{\ell+1}(2)^{-\ell}$ $\log(\rho) + (-1)^k 2^{-k}\log(\rho) = \frac{1}{3}\log(\rho) > 0$. Therefore, the largest Lyapunov exponent must be positive since $\lambda(e_1) > 0$. However, we have $\det(|\Phi_n|) = 1$, which implies that the sequence cannot be regular.

Example 2.2.4. Now, we use the same matrices B_1 and B_2 as in Example 2.2.3, but we consider a different sequence. Let $A_n = B_1$, whenever n is divisible by 4, and $A_n = B_2$, otherwise. Further, denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. Thus, simple calculations show us that $\lambda(e_1) = -\frac{1}{2}\log\rho$ and $\lambda(e_2) = \frac{1}{2}\log\rho$. Hence, we conclude two things: first, the sequence is regular. Second, the canonical basis $\{e_1, e_2\}$ is a normal basis for the Oseledets' filtration.

Note that, in Example 2.2.3, we cannot replace the limit superior that appears in Definition 2.2.4 by a usual limit. However, we can replace it in Example 2.2.4. One might wonder if regularity is related to this behavior since the sequence from Example 2.2.4 is regular while the one in Example 2.2.3 is not. Interestingly, that is precisely the case. Lemma 2.2.5, which presents equivalent characterizations for regularity, helps us understand that better. We readily see that the second bullet of Lemma 2.2.5, applied when $\mathcal{I}$ is a singleton, implies the claim we just made, i.e., we can only replace the limit superior by a limit when the sequence is regular.

We remark that Lemma 2.2.5 is a classical result from the dynamical system's literature. We refer to Chapters 3 and 7 of [36] for a proof.

Lemma 2.2.5. Given a tempered sequence $(A_n)_{n\in\mathbb{N}}$ of invertible matrices, let $\{v_1,\dots,v_d\}$ be any normal basis for the Oseledets' filtration of the sequence $(A_n)_{n\in\mathbb{N}}$, and let $\mathcal{I}\subset\{1,\cdots,d\}$ be any set of indices. Further, let λ_i be the Lyapunov exponents with multiplicity of the sequence $(A_n)_{n\in\mathbb{N}}$. Then, the following conditions are equivalent

- $\lim_{n\to\infty}\frac{1}{n}\log\left(\left|\det\left(\Phi_n\right)\right|\right)=\sum_{i=1}^{d}\lambda_i$;
- $\lim_{n\to\infty}\frac{1}{n}\log\left(\mathrm{vol}\left(\{\Phi_n v_i : i\in\mathcal{I}\}\right)\right)=\sum_{i\in\mathcal{I}}\lambda_i$.
- The matrix $\lim_{n\to\infty}\left(\Phi_n^\top\Phi_n\right)^{\frac{1}{2n}}$ exists.

Now, we state the section's main Theorem. The proof of this theorem appears in the work [20].

Theorem 2.2.6. Let $\alpha\geq 0$. Let $(A_n)_{n\in\mathbb{N}}$ be a tempered sequence of invertible matrices. Let $K\subset\mathbb{R}^d$ be a compact set of possible initial conditions with a nonempty interior. Denote by λ_i, with $i=1,\cdots,d$, the Lyapunov exponents with multiplicity of $(A_n)_{n\in\mathbb{N}}$. Then, the estimation entropy of the discrete switched system (2.2) satisfies:

$$h_{\mathrm{est}}(\alpha,K)\leq\sum_{i=1}^{d}\max\{0,\lambda_i+\alpha\}\,, \tag{2.5}$$

with equality if the system is regular.

Proof. For the proof of the upper bound, we build a (T,ϵ,α,K) - approximating set $\mathcal{F}_T$ and calculate its cardinality. First, denote by $\{v_1,\dots,v_d\}$ a normal basis for the Oseledets' filtration associated with the sequence $(A_n)_{n\in\mathbb{N}}$. Then, pick an $\epsilon>0$. Further, choose an arbitrary block length $\ell\in\mathbb{N}$ and a time horizon $T\in\mathbb{N}$ such that $T>\ell$. Also, for a fixed but arbitrary $\delta>0$, we define

$$\Gamma_i^j:=\max\left\{\max_{k\in\{0,\dots,\ell-1\}}\left\|\Phi_{j\ell-k}v_i\right\|,e^{(\lambda_i+\delta)j\ell},e^{(\lambda_i+\delta)((j-1)\ell+1)}\right\} \tag{2.6}$$

for $i\in\{1,\dots,d\}$ and $j\in\{1,\dots,\lceil(T-1)/\ell\rceil\}$, and $\Gamma_i^0:=1$ for $i\in\{1,\dots,d\}$. Consider the box $B^0:=\left\{\sum_{i=1}^{d}\gamma_i v_i:\underline{\kappa}_i^0\leq\gamma_i<\overline{\kappa}_i^0\right\}$, where $\underline{\kappa}_i^0$ and $\overline{\kappa}_i^0$ are such that $K\subset B^0$ and $\mathrm{diam}\left(B^0\right)<\infty$. Further, consider the following sets: $\mathcal{C}_i^0:=\left\{1,\cdots,\left\lceil d\frac{\overline{\kappa}_i^0-\underline{\kappa}_i^0}{\epsilon}\right\rceil\right\}$ for $i\in\left\{1,\dots,d\right\}$, and $\mathcal{C}_i^{j+1}:=\left\{1,\cdots,\left\lceil\frac{\Gamma_i^{j+1}}{\Gamma_i^j}e^{\alpha\ell}\right\rceil\right\}$ for $i\in\{1,\dots,d\}$ and $j\in\{0,\dots,\lceil(T-1)/\ell\rceil\}$. Now, define the set $\mathcal{Q}$ of all ordered tuples $\left(q^0, \ldots, q^{\lceil (T-1)/\ell \rceil}\right)$ with $q^j = \left(q^j_1, \ldots, q^j_d\right)$ and $q^j_i \in \mathcal{C}^j_i$. For a given $q = \left(q^0, \ldots, q^{\lceil (T-1)/\ell \rceil}\right) \in \mathcal{Q}$, we build a function $\hat{x}_q(\cdot)$ such that the value of the function at time $t \in \{0, \ldots, T-1\}$, i.e. $\hat{x}_q(t)$, depends only on $\left(q^0, \ldots, q^{\lceil t/\ell \rceil}\right)$. Before presenting the function's construction, consider the following recursive definitions:

$$\underline{\kappa}^{j+1}_i(q) \coloneqq \underline{\kappa}^j_i(q) + \frac{\epsilon}{d}\left(\Gamma^j_i e^{\alpha j \ell}\right)^{-1}\left(q^j_i - 1\right), \tag{2.7}$$

$$\overline{\kappa}^{j+1}_i(q) \coloneqq \underline{\kappa}^j_i(q) + \frac{\epsilon}{d}\left(\Gamma^j_i e^{\alpha j \ell}\right)^{-1} q^j_i \tag{2.8}$$

where $i \in \{1, \ldots, d\}$, $j \in \{0, \ldots, \lceil (T-1)/\ell \rceil\}$, and $q \in \mathcal{Q}$.
Now, define for $j \in \{0, \ldots, \lceil (T-1)/\ell \rceil\}$, $i \in \{1, \ldots, d\}$, and $q \in \mathcal{Q}$ the quantity

$$\hat{\beta}^j_i(q) \coloneqq \underline{\kappa}^j_i(q) + \frac{\epsilon}{d}\left(\Gamma^j_i e^{\alpha j \ell}\right)^{-1}(q^j_i - 1/2). \tag{2.9}$$

Finally, for $t \in \{0, \ldots, T-1\}$ and a given $q \in \mathcal{Q}$, define the function $\hat{x}_q(t) \coloneqq \sum_{i=1}^d \hat{\beta}^j_i \Phi_t v_i$, where $j = \lceil t/\ell \rceil - 1$, i.e. j is such that $j\ell + 1 \le t \le (j+1)\ell$. In words, we are using the same β_i estimate $\hat{\beta}^j_i$ for all t such that $j = \lceil t/\ell \rceil - 1$ holds true. Further note that any such t satisfy $t = (j+1)\ell - k$ for some $k \in \{0, \ldots, \ell - 1\}$.
Notice that, for given $q \in \mathcal{Q}$, $i \in \{1, \ldots, d\}$, and $j \in \{1, \ldots, \lceil (T-1)/\ell \rceil\}$ the estimate $\hat{\beta}^j_i(q)$ is the midpoint of $\left[\underline{\kappa}^{j+1}_i(q), \overline{\kappa}^{j+1}_i(q)\right)$ by Equations (2.9), (2.7), and (2.8). Also, note that for any given $\beta \in \left[\underline{\kappa}^{j+1}_i(q), \overline{\kappa}^{j+1}_i(q)\right)$, we have that $\left|\hat{\beta}^j_i(q) - \beta\right| < \frac{\epsilon}{2d}\left(\Gamma^j_i e^{\alpha j \ell}\right)^{-1}$ again by Equations (2.9), (2.7), and (2.8). Now, let $\mathcal{F}_T$ be the set of functions $\hat{x}_q(\cdot)$ for $q \in \mathcal{Q}$.
We claim that $\mathcal{F}_T$ is a (T, ϵ, α, K)-approximating set. To see that, let $x \in K$ and write it as $x = \sum_{i=1}^d \beta_i v_i$. We proceed by induction over $j \in \{0, \ldots, \lceil (T-1)/\ell \rceil\}$ to show that there exists a $q \in \mathcal{Q}$ such that the corresponding $\hat{\beta}^j_i(q)$ satisfies[5] $\left|\hat{\beta}^j_i(q) - \beta_i\right| < \frac{\epsilon}{2d}\left(\Gamma^j_i e^{\alpha j \ell}\right)^{-1}$. Consequently, we conclude that the corresponding $\hat{x}_q(\cdot)$ satisfies $\|\hat{x}_q(t) - \xi(x,t)\| < \epsilon e^{-\alpha t}$ for $t \in \{0, \ldots, T-1\}$.
Step 0:
We have that $\beta_i \in [\underline{\kappa}^0_i, \overline{\kappa}^0_i)$ for $i \in \{1, \ldots, d\}$ by definition of B^0. Let $q^0 = (q^0_1, \ldots, q^0_d)$, with $q^0_i \in \mathcal{C}^0_i$, be such that $\beta_i \in [\underline{\kappa}^1_i(q), \overline{\kappa}^1_i(q))$ for every $i \in \{1, \ldots, d\}$. Notice that $\underline{\kappa}^1_i(q)$ and $\overline{\kappa}^1_i(q)$ depend only on $\underline{\kappa}^0_i$, $\overline{\kappa}^0_i$, and q^0_i. By

[5] Notice that this is equivalent to $\beta_i \in \left[\underline{\kappa}^{j+1}_i(q), \overline{\kappa}^{j+1}_i(q)\right)$.

Equations (2.9), (2.7), and (2.8), we have that $\left|\beta_i - \hat{\beta}_i^0(q)\right| \leq \frac{\epsilon}{2d}$. Thus, for any $\hat{x}_q(\cdot) \in \mathcal{F}_T$, with $q \in \mathcal{Q}$ and q^0 as the one described here, we have that $\|\hat{x}_q(0) - \xi(x,0)\| = \left\|\sum_{i=1}^d \left(\beta_i - \hat{\beta}_i^0(q)\right) v_i\right\| \leq \frac{\epsilon}{2d}\left\|\sum_{i=1}^d v_i\right\| \leq \frac{\epsilon}{2}$, where the last inequality comes from the fact that $\|v_i\| = 1$.

Step j+1:

From our induction hypothesis, $\beta_i \in \left[\underline{\kappa}_i^j(q), \overline{\kappa}_i^j(q)\right)$ for $i \in \{1,\ldots,d\}$. Now, let $q^j = \left(q_1^j,\ldots,q_d^j\right)$, with $q_i^j \in \mathcal{C}_i^j$, be such that $\beta_i \in \left[\underline{\kappa}_i^{j+1}(q), \overline{\kappa}_i^{j+1}(q)\right)$ for every $i \in \{1,\ldots,d\}$. Notice that $\underline{\kappa}_i^{j+1}(q)$ and $\overline{\kappa}_i^{j+1}(q)$ depend only on $\underline{\kappa}_i^j(q)$, $\overline{\kappa}_i^j(q)$, and q_i^j. By Equations (2.9), (2.7), and (2.8), we have that $\left|\beta_i - \hat{\beta}_i^j(q)\right| \leq \frac{\epsilon}{2d}\left(\Gamma_i^j e^{\alpha j \ell}\right)^{-1}$. Thus, for $(j-1)\ell + 1 \leq t \leq j\ell$ and for any $\hat{x}_q(\cdot) \in \mathcal{F}_T$, with $q \in \mathcal{Q}$ and $(q^0,\ldots,q^j)$ as the one inductively described here, we have that

$$
\begin{aligned}
||\hat{x}_q(t) - \xi(x,t)|| &= \left\|\sum_{i=1}^d \left(\hat{\beta}_i^j(q) - \beta_i\right)\Phi_t v_i\right\| \\
&\leq \frac{\epsilon}{2d} e^{-\alpha j \ell}\left\|\sum_{i=1}^d \frac{\Phi_t v_i}{\Gamma_i^j}\right\| \leq \frac{\epsilon}{2} e^{-\alpha t},
\end{aligned}
$$

where the last inequality comes from the facts that[6] $\|\Phi_t v_i\| \leq \Gamma_i^j$ and $e^{-\alpha j \ell} \leq e^{-\alpha t}$ for $t \in \{(j-1)\ell + 1, \ldots, j\ell\}$. With this, we conclude the induction.[7]

Since there exists a one-to-one correspondence between elements of $\mathcal{Q}$ and $\mathcal{F}_T$, the cardinality of $\mathcal{F}_T$ is given by $\prod_{j=0}^{\lceil (T-1)/\ell \rceil}\left(\prod_{i=1}^d \#\mathcal{C}_i^j\right)$. Also, because $\mathcal{F}_T$ is a (T,ϵ,α,K)-approximating set, its cardinality is an upper bound for $s_{\text{est}}(T,\epsilon,\alpha,K)$, the minimum cardinality of any (T,ϵ,α,K)-approximating set. Therefore, we conclude that

$$
\frac{1}{T}\log s_{\text{est}}(T,\epsilon,\alpha,K) \leq \frac{1}{T}\sum_{j=0}^{\lceil (T-1)/\ell \rceil}\sum_{i=1}^d \log\left(\#\mathcal{C}_i^j\right).
$$

Recall that, by the definition 2.2.4 of Lyapunov exponent, for any given $\delta > 0$, $\exists N_i \in \mathbb{N}$ such that $\forall t \geq N_i$ we have that $\frac{1}{t}\log\left(\|\Phi_t v_i\|\right) \leq \lambda_i + \delta$ for a given $i \in \{1,\ldots,d\}$, from which we get that $\|\Phi_t v_i\| \leq e^{(\lambda_i+\delta)t}$ for all $t \geq N_i$.

[6] An immediate consequence of Equation (2.6).

[7] We need a minor change for the final step, i.e. $j = \lceil (T-1)/\ell \rceil$, in the induction process. Because of the domain of $\hat{x}_q(\cdot)$, we have to consider $(\lceil (T-1)/\ell \rceil - 1)\ell \leq t \leq T$ instead of $(\lceil (T-1)/\ell \rceil - 1)\ell \leq t \leq \lceil (T-1)/\ell \rceil \ell$. This is the only change needed to prove the induction.

We restrict our choice of δ to be such that $\lambda_i + \delta < 0$ for all $\lambda_i < 0$ with $i \in \{1, \ldots, d\}$. However, we can choose $\delta > 0$ arbitrarilly small. Now, from Equation (2.6), we have that $\Gamma_i^j = e^{(\lambda_i+\delta)j\ell}$ if $\lambda_i \geq 0$, and $\Gamma_i^j = e^{(\lambda_i+\delta)((j-1)\ell+1)}$ if $\lambda_i < 0$, with both equalities being valid for all j such that $(j-1)\ell+1 \geq N_i$. For simplicity denote $M \coloneqq \max\left\{\left\lceil \frac{N_i-1}{\ell} + 1\right\rceil, i = 1, \ldots, d\right\}$. Therefore, it is true that $\frac{\Gamma_i^{j+1}}{\Gamma_i^j} = e^{(\lambda_i+\delta)\ell}$ for $j \geq M$ and $i \in \{1, \ldots, d\}$. From our previous discussion, with our previously fixed δ, we have the following inequality

$$\begin{aligned} h_{\text{est}}(\alpha, K) \leq & \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \sum_{j=0}^{M} \sum_{i=1}^{d} \log\left(\#\mathcal{C}_i^j\right) \\ & - \frac{(M+1)}{T} \sum_{i=1}^{d} \log\left(\left\lceil e^{(\lambda_i+\alpha+\delta)\ell}\right\rceil\right) + \frac{1}{\ell} \sum_{i=1}^{d} \log\left(\left\lceil e^{(\lambda_i+\alpha+\delta)\ell}\right\rceil\right) \end{aligned}$$

where we notice that the first two terms on the right hand side vanish when T goes to infinity. Thus, we have that $h_{\text{est}}(\alpha, K) \leq \frac{1}{\ell}\sum_{i=1}^{d} \log\left(\left\lceil e^{(\lambda_i+\alpha+\delta)\ell}\right\rceil\right)$. Since $\delta > 0$ can be arbitrarily small, this shows that

$$h_{\text{est}}(\alpha, K) \leq \frac{1}{\ell} \sum_{i=1}^{d} \log\left(\left\lceil e^{(\lambda_i+\alpha)\ell}\right\rceil\right).$$

Finally, because ℓ can be made arbitrarily large, we get inequality (2.5). Here, we used the fact that $\lim_{\ell\to\infty} \frac{1}{\ell}\log\left(\left\lceil e^{y\ell}\right\rceil\right) = \max\{y, 0\}$ for $y \in \mathbb{R}$. To see this, note that we have $\left\lceil e^{y\ell}\right\rceil = 1$ for $y \leq 0$, so $\log\left(\left\lceil e^{y\ell}\right\rceil\right) = 0$, and we have that $y \leq \frac{1}{\ell}\log\left(\left\lceil e^{y\ell}\right\rceil\right) \leq \frac{1}{\ell}\log\left(e^{y\ell}(1+e^{-y\ell})\right) = y + \frac{1}{\ell}\log\left(1+e^{-y\ell}\right)$ for $y > 0$, from which we conclude that the limit equals $\max\{y, 0\}$.

For the lower bound, assume that $(A_n)_{n\in\mathbb{N}}$ is regular. Let $\{v_1, \ldots, v_d\}$ be a normal basis for the Oseledets' filtration associated with $(A_n)_{n\in\mathbb{N}}$. Further, fix an arbitrary $\delta > 0$ and pick an $\epsilon > 0$. Define $\mathcal{I} \coloneqq \left\{i \in \{1, \ldots, d\} : \lambda_i + \alpha + \delta > 0\right\}$ a set of indices and $U \coloneqq \left\{\sum_{i\in\mathcal{I}} \gamma_i v_i : \underline{\kappa}_i \leq \gamma_i \leq \overline{\kappa}_i\right\}$, with $\underline{\kappa}_i$ and $\overline{\kappa}_i$ such that $U \subset K$, which is always possible because K has nonempty interior. For simplicity, assume that $\underline{\kappa}_i = 0$ and $\overline{\kappa}_i = \overline{\kappa}$ for all $i \in \mathcal{I}$. If this is not the case, translate the set K so that the origin will be in its interior, a transformation that does not change the $\#\mathcal{I}$-th volume. Therefore, U is the parallelepiped $\{\kappa_i v_i : \kappa_i \in [0, \overline{\kappa}]\}$ with $\#\mathcal{I}$-th volume given by $\text{vol}(U) = (\overline{\kappa})^{\#\mathcal{I}}\text{vol}(\{v_i : i \in \mathcal{I}\})$.

Now, from the regularity hypothesis and the second bullet in Lemma 2.2.5, for our $\delta > 0$, $\exists N \in \mathbb{N}$ such that $\forall j > N$ we have that

$$\left| \frac{1}{j} \log \operatorname{vol}\left(\{\Phi_j v_i : i \in \mathcal{I}\}\right) - \sum_{i \in \mathcal{I}} \lambda_i \right| \leq \delta \# \mathcal{I},$$

which implies that $\operatorname{vol}\left(\{\Phi_j v_i : i \in \mathcal{I}\}\right) \geq e^{\sum_{i \in \mathcal{I}} (\lambda_i - \delta) j}$.
Notice that the parallelepiped $\Phi_j U = \left\{ \sum_{i \in \mathcal{I}} \gamma_i \Phi_j v_i : 0 \leq \gamma_i \leq \overline{\kappa} \right\}$ has the $\#\mathcal{I}$-th volume equal to $\operatorname{vol}(\Phi_j U) = (\overline{\kappa})^{\#\mathcal{I}} \operatorname{vol}\left(\{\Phi_j v_i : i \in \mathcal{I}\}\right)$. Now, let $C = \{x_1, \cdots, x_N\}$ be an (T, ϵ, α, U)-spanning set. We show how the cardinality of C compares with the minimum cardinality, s^*_{est}, of a (T, ϵ, α, U)-spanning set.
First, recall that $\mathbb{B}\left(x, r\right)$ is the infinity-norm ball (hypercube) centered at x with radius r. Define $\mathbb{B}^{(j,\mathcal{I})}(x, r) \coloneqq \mathbb{B}(x, r) \cap \left\{ \sum_{i \in \mathcal{I}} \gamma_i \Phi_j v_i : \gamma_i \in \mathbb{R} \right\}$, i.e. the intersection of the ball with the subspace spanned by the vectors $\Phi_j v_i$ for $i \in \mathcal{I}$. Now, since C is (T, ϵ, α, U)-spanning, we cover $\Phi_T U$ with balls of radius $\epsilon e^{-\alpha T}$ centered at $\Phi_T x_i$ for $x_i \in C$. Because the balls $\mathbb{B}^{(j,\mathcal{I})}\left(\Phi_T x_i, \epsilon e^{-\alpha T}\right)$ cover $\Phi_T U$, the sum of their $\#\mathcal{I}$-th volumes, i.e. the cardinality of C times the $\#\mathcal{I}$-th volume of a single ball, is larger than or equal to the $\#\mathcal{I}$-th volume of $\Phi_T U$. From this, we conclude that $\#C \geq \frac{\operatorname{vol}(\Phi_T U)}{\operatorname{vol}(\mathbb{B}(\Phi_T x_i, \epsilon e^{-T\alpha}))}$. Lastly, because $s^*_{\text{est}}(T, \epsilon, \alpha, U)$ is the lowest value for the cardinality of any (T, ϵ, α, U)-spanning set, we conclude that

$$s^*_{\text{est}}(T, \epsilon, \alpha, U) \geq \frac{\operatorname{vol}(\Phi_T U)}{\operatorname{vol}(\mathbb{B}(\Phi_T x_i, \epsilon e^{-T\alpha}))} = \left(\frac{\overline{\kappa}}{2\epsilon e^{-T\alpha}} \right)^{\#\mathcal{I}} \operatorname{vol}\left(\{\Phi_T v_i : i \in \mathcal{I}\}\right).$$

It is straightforward to see that $s^*_{\text{est}}(T, \epsilon, \alpha, K) \geq s^*_{\text{est}}(T, \epsilon, \alpha, U)$, by the fact that any (T, ϵ, α, K)-spanning set is also a (T, ϵ, α, U)-spanning set. Thus, we arrive at $s^*_{\text{est}}(T, \epsilon, \alpha, K) \geq \left(\frac{\overline{\kappa}}{2\epsilon e^{-T\alpha}} \right)^{\#\mathcal{I}} \operatorname{vol}\left(\{\Phi_T v_i : i \in \mathcal{I}\}\right)$. Furthermore, we have that

$$h_{\text{est}}(K, \alpha) \geq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \Bigg(\log \left(\frac{\overline{\kappa}}{2\epsilon} \right)^{\#\mathcal{I}} + \log \operatorname{vol}\left(\{\Phi_T v_i : i \in \mathcal{I}\}\right) \Bigg) + \alpha \#\mathcal{I},$$

and, since T can be taken to be larger than N, we derive that $h_{\text{est}}(K, \alpha) \geq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \Bigg(\log \left(\frac{\overline{\kappa}}{2\epsilon} \right)^{\#\mathcal{I}} \Bigg) + \sum_{i \in \mathcal{I}} (\lambda_i + \alpha - \delta)$, and we conclude that $h_{\text{est}}(\alpha, K) \geq \sum_{i \in \mathcal{I}} (\lambda_i + \alpha - \delta) = \sum_{i=1}^{d} \max\left\{\lambda_i + \alpha - \delta, 0\right\}$, where the last equality comes from the definition of $\mathcal{I}$. Finally, by the fact that $\delta > 0$ was arbitrary, we have that $h_{\text{est}}(\alpha, K) \geq \sum_{i \in \mathcal{I}} \max\left\{\lambda_i + \alpha, 0\right\}$. □

Example 2.2.7 (Example 2.1.1 revisited.). We can finally resume the analysis of Example 2.1.1. By our previous computations in Section 2.1, we concluded that our system's Lyapunov exponents are $\lambda(e_1) = \frac{1}{2}\log(0.99) < 0$ and $\lambda(e_2) = 0$ with probability one. Thus, we deduce that the system's estimation entropy satisfies the inequality

$$h_{\text{est}}(\alpha, K) \leq \max\left\{\frac{1}{2}\log(0.99) + \alpha, 0\right\} + \max\{\alpha, 0\} \tag{2.10}$$

with probability 1.

2.2.1 Connection with previous results

We take this opportunity to draw a connection between the results from [26] and those from this chapter. We note that [26] obtained bounds that depend solely on the individual modes and their respective active rates. The first important remark we must make is that we cannot deduce the results we presented so far cannot from the results from [26]. To understand why, we need a few definitions: denote by $\mu(A) = \lim_{t\downarrow 0}\frac{\|I+tA\|-1}{t}$ the *matrix measure* of the matrix A, and by $\mathbb{1}_p(\sigma)$ the *indicator function* of mode p, i.e. $\mathbb{1}_p(q) = 1$ if $p = q$ and $\mathbb{1}_p(q) = 0$, otherwise. Finally, define the *active rate* of mode p as $\rho_p(t) = \frac{1}{t}\int_0^t \mathbb{1}_p(\sigma(\tau))d\tau$. The upper bound for the topological entropy obtained in [26] is

$$h_{\text{est}}(0, K) \leq \max\Big\{\limsup_{t\to\infty}\sum_{p\in\Sigma}\mu(\mathcal{A}_{\sigma(t)})\rho_p(t)d, 0\Big\}.$$

We can easily see that that bound is conservative. For example, consider a system that never switches and for which its only active mode has a unique unstable eigenvalue. We can take $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ to make our example more concrete. We easily see that, by the properties of the matrix measure, we must have that $\mu(A) \geq 2$. Also, the topological entropy is 2, but the upper bound is greater or equal to 4. This example shows that we must consider the eigenstructure of the mode in our analysis. That is precisely the role of the Oseledets' filtration. We can divide a normal basis for this filtration into vectors corresponding to nonnegative Lyapunov exponents and the ones corresponding to negative ones. Analyzing only the former ones, we focus our attention on the directions where our system does not contract, which allows us to arrive at the correct value for the topological entropy in this case. In this manner, we avoid the conservative bound from [26].

The lower bound obtained in [26] was the following:

$$h_{\text{est}}(0, K) \geq \Big\{ \limsup_{t \to 0} \sum_{p \in \Sigma} \text{tr}(\mathcal{A}_p)\rho_p(t) \Big\}.$$

The proof of that inequality relied on the classical volume counting argument, as in Theorem 2.2.6. As expected, geometric reasons prevented this lower bound from being tight. More explicitly, the volume considered in [26] was $d-$dimensional. Thus, if there is a direction in the state space where the state is contracting, then the volume will decrease. Consider, for example, the mode $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$. It has a negative trace but a positive topological entropy. The two-dimensional volume of the set of initial conditions will decrease, but a one-dimensional subspace will grow. In Theorem 2.2.6, we deal with that issue by removing those shrinking directions and only looking at the expansive ones. Once again, the restrictive setting of only knowing the active rate prevents the bound from being tight.

It is worth mentioning that knowing the Oseledets' filtration requires complete knowledge of the whole switching signal, even at future times. Hence, although the result from Theorem 2.2.6 is tighter, it also requires much more information than the bounds obtained in [26].

We remark here that regularity, informally, means that the geometric notion of expanding directions and the notion of an expanding volume form are compatible. This latter fact allows us to get the identity in Theorem 2.2.6.

2.3 Quantization Algorithm

The goal of this section is to describe the quantization algorithm. This algorithm's purpose is to reconstruct the state of system (2.2) with an estimation error that decays with a prescribed exponential rate using only quantized measurements. The algorithm produces over-approximations to the reachable set that depend on some parameter choices, the switching signal, and the desired exponential decay for the estimation error. One important parameter is a family of bases $\mathcal{V}_j = \{v_1^j, \ldots, v_d^j\}$, $j \in \mathbb{Z}_{\geq 0}$ for $\mathbb{R}^d$. How we choose such a family affects the algorithm's average data-rate. We show that there are choices of families that allow our algorithm to operate with an average data-rate as close to the estimation entropy of our system as desired. More than that, we show how we can build such a family online, assuming that the switching signal is known.

We recall that as in Section 2.2, $T_p > 0$ is a sampling time, and the sequence $(A_n)_{n\in\mathbb{N}}$ corresponds to[8] the exact discrete-time model of some continuous-time model described by equation (2.1), i.e., $A_n = \Phi(T_p n, T_p(n-1))$.

2.3.1 The Algorithm

Now, we are ready to define our quantization scheme for switched linear systems. We assume that the switching signal is known to us, i.e., $\sigma(t)$ for all values of $t \in \mathbb{R}_{\geq 0}$. Note that this implies that we must know the entire sequence $(A_n)_{n\in\mathbb{N}}$, where A_n's appear in Equation (2.2). Further, we assume that we have a family of orthonormal bases $\mathcal{V}^j$ for $\mathbb{R}^d$. By appropriately choosing that family, we can make the average data-rate used by our algorithm arbitrarily close to the upper bound for the estimation entropy obtained in Theorem 2.2.6, namely $\sum_{i=1}^{d}\{\lambda_i + \alpha, 0\}$. Consequently, under the assumption of regularity, as we proved in Theorem 2.2.6, the algorithm will operate with an average data-rate arbitrarily close to the estimation entropy.

Before we proceed, we need some additional concepts. Let $\ell \in \mathbb{N}$ be the *block length*, and let $j \in \mathbb{N}$ be a number that indexes our algorithm's iteration. We also need to mention that the following description is only valid for positive times because the case of time equal zero is slightly different because of how we initialize our algorithm. Nevertheless, the idea behind the algorithm mechanism is essentially the same for all time indexes. Informally, the algorithm operates in the following manner: Let the initial state x be inside the region $\bar{B}^{j-1}$, a parallelepiped in $\mathbb{R}^d$. Given a basis $\{v_i^j\}_{i=1}^d$ from the family $\mathcal{V}^j$, build a new parallelepiped $\tilde{B}^j$ with sides parallel to the vectors v_i^j's that contains $\bar{B}^{j-1}$. Now, we flow $\tilde{B}^j$ forward using $\Phi_{j\ell+1}$ and denote it by B^j. More precisely, we define $B^j = \Phi_{j\ell+1}(\tilde{B}^j)$. Note that, since x belongs to $\bar{B}^{j-1}$ and $\bar{B}^{j-1} \subset \tilde{B}^j$, we have that the state at the current time $j\ell + 1$,

[8] As described after equation (2.4), i.e., $A_n = \tilde{A}_k$ for $n = k + 1$ and $k \in \mathbb{Z}_{\geq 0}$.

i.e. $\xi(x, j\ell + 1)$, belongs to B^j. We have quantization subregions inside the set B^j, each corresponding to a distinct quantization symbol. We denote by q^j the quantization symbol corresponding to the quantization subregion that contains $\xi(x, j\ell+1)$. Next, we flow the previous quantization subregion, which corresponds to the symbol q^j, backward by $\Phi_{j\ell+1}$ and define the result to be $\bar{B}^j$. Finally, we repeat the procedure.

We must stress that our algorithm will work for an arbitrary choice of bases $\left\{v_i^j\right\}_{i=1}^d$ with $j \in \mathbb{Z}_{\geq 0}$. However, the choice of bases affects the average data-rate used by our algorithm. We show in Corollary 2.3.2 and Theorem 2.3.4 how to choose bases that guarantee that the average data-rate will approach the estimation entropy asymptotically. Finally, we note that we construct our estimates using only measurements that happen at time instants of the form $t = j\ell + 1$ with $j \in \mathbb{Z}_{\geq 0}$ and at the initial time $t = 0$. The reason why we only use measurements at those times is related to the idea of block coding (see, e.g., Chapter 5 of [39]). As we will see later, this idea allows the algorithm's average data-rate to approach the estimation entropy arbitrarily close in some cases.

In what follows, we assume that $\mathbb{R}^d$ is endowed with the canonical inner product $\langle \cdot, \cdot \rangle$. The following algorithm, proof of correctness, and corollary, were first presented in [20].

Quantizer algorithm

Initialization: Let K be the set of possible initial conditions, $x \in K$ be the true initial condition, $\epsilon > 0$ a prescribed precision, $T_p > 0$ the sampling time, and $\ell \in \mathbb{N}$ be the block length. Also, consider the sequence $(A_n)_{n\in\mathbb{N}}$, where[9] $A_n = \Phi(T_p n, T_p(n-1))$ and $\Phi_n = A_n \dots A_1$. Further, let $\mathcal{V}_j = \left\{v_1^j, \dots, v_d^j\right\}$, $j \in \mathbb{Z}_{\geq 0}$ be a family of orthonormal bases for $\mathbb{R}^d$. We define $\Gamma_i^0 = 1$ for all $i \in \{1, \dots, d\}$. If the system is known to be regular, set

$$\Gamma_i^j := \max_{k\in\{0,\dots,\ell-1\}} \left\|\Phi_{j\ell-k} v_i^j\right\|, \tag{2.11}$$

otherwise

$$\Gamma_i^j := \max\left\{ \max_{k\in\{0,\dots,\ell-1\}} \left\|\Phi_{j\ell-k} v_i^j\right\|, e^{T_p(\lambda_i+\delta)j\ell}, e^{T_p(\lambda_i+\delta)((j-1)\ell+1)} \right\} \tag{2.12}$$

[9]Note that $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}(d, \mathbb{R})$ might be an infinite set in general.

for a prescribed $\delta > 0$ and[10] $\lambda_i \coloneqq \limsup_{j\to\infty} \frac{1}{j} \log\left(||\Phi_j v_i^j||\right)$. Also, let $\alpha \geq 0$ be the prescribed exponential decay rate for the estimation error.

Step 0:

In this step, we define an estimate $\hat{x}(0)$ for $\xi(x,0) = x$.

- Define
$$B^0 \coloneqq \left\{ \sum_{i=1}^{d} \gamma_i v_i^0 : \underline{\kappa}_i^0 \leq \gamma_i < \overline{\kappa}_i^0 \right\}, \tag{2.13}$$
where $\underline{\kappa}_i^0$ and $\overline{\kappa}_i^0$ are such that B^0 is the smallest set of such type that contains the initial set K.

- Write $\xi(x,0) = \sum_{i=1}^{d} \beta_i^0 v_i^0$. Then, the symbol related to the quantized value of $\xi(x,0)$ is given by $q^0 = (q_1^0, \ldots, q_d^0)$, constructed as follows. Define $\mathcal{C}_i^0 \coloneqq \left\{1, \ldots, \left\lceil d\frac{\overline{\kappa}_i^0 - \underline{\kappa}_i^0}{\epsilon} \right\rceil \right\}$. We define q_i^0, for every $i \in \{1, \ldots, d\}$, as the $k \in \mathcal{C}_i^0$ such that
$$\beta_i^0 \in \left[\underline{\kappa}_i^0 + \frac{\epsilon}{d}(k-1), \underline{\kappa}_i^0 + \frac{\epsilon}{d}k\right) \tag{2.14}$$
holds true.

- Denote
$$\hat{\beta}_i^0 \coloneqq \underline{\kappa}_i^0 + \frac{\epsilon}{d}\left(q_i^0 - 1/2\right). \tag{2.15}$$
Our estimate for the state at the moment $t = 0$ is
$$\hat{x}(0) \coloneqq \sum_{i=1}^{d} \left(\underline{\kappa}_i^0 + \frac{\epsilon}{d}\left(q_i^0 - 1/2\right)\right) v_i^0.$$

We could describe this step 0 in words as follows. B^0 is divided into cubic boxes with sides of length ϵ/d; q_i^0 encodes the position of the box in the i-th dimension that contains x; and $\hat{x}(0)$ is the center of this box.

Step 1:

In this step, we define an estimate $\hat{x}(t)$ for $\xi(x,t)$ with $1 \leq t \leq \ell$.

[10]Notice that these λ_i's are not the same as the Lyapunov exponents with multiplicity since the v_i^j's are not a normal basis for the Oseledets' filtration in principle.

Notice that we generated a box

$$\bar{B}^0 := \left\{ \sum_{k=1}^{d} \mu_k v_k^0 : \underline{\kappa}_k^0 + \frac{\epsilon}{d}(q_k^0 - 1) \leq \mu_k < \underline{\kappa}_k^0 + \frac{\epsilon}{d} q_k^0 \right\} \tag{2.16}$$

at the end of Step 0 and that $x \in \bar{B}^0$. Now, in this step, we generate the smallest box aligned with the new basis $\{v_i^1\}_{i=1}^d$ that contains $\bar{B}^0$. This box takes the form

$$\tilde{B}^1 := \left\{ \sum_{i=1}^{d} \gamma_i v_i^1 : \underline{\kappa}_i^1 \leq \gamma_i < \overline{\kappa}_i^1 \right\}.$$

To compute the bounds $\underline{\kappa}_i^1$ and $\overline{\kappa}_i^1$, let $y = \sum_{k=1}^d \mu_k v_k^0$ be an arbitrary point in $\bar{B}^0$. Thus, its coordinate relative to each v_i^1 is $\gamma_i = \langle \sum_{k=1}^d \mu_k v_k^0, v_i^1 \rangle = \sum_{k=1}^d \mu_k \langle v_k^0, v_i^1 \rangle$.

Hence, to find the smallest such box, we need to take

$$\begin{aligned} \underline{\kappa}_i^1 := \min \Big\{ & \sum_{k=1}^{d} \mu_k \langle v_k^0, v_i^1 \rangle : \\ & \underline{\kappa}_k^0 + \frac{\epsilon}{d} \big(q_k^0 - 1\big) \leq \mu_k \leq \underline{\kappa}_k^0 + \frac{\epsilon}{d} q_k^0, \quad k = 1, \ldots, d \Big\}, \end{aligned} \tag{2.17}$$

for every $i \in \{1, \ldots, d\}$. Notice that this is a linear programming problem. Therefore, the solution will occur at the boundary. Moreover, this set of inequalities forms a box, and we only need to check its vertices to find the optimal value. The upper bounds $\overline{\kappa}_i^1$ are defined similarly but with max instead of min. Finally, we define the box

$$B^1 := \left\{ \sum_{i=1}^{d} \gamma_i \Phi_1 v_i^1 : \underline{\kappa}_i^1 \leq \gamma_i < \overline{\kappa}_i^1 \right\} \tag{2.18}$$

by flowing the box $\tilde{B}^1$ forward by Φ_1. We can write the procedure of this step in the following itemized way.

- Define $B^1 := \left\{ \sum_{i=1}^d \gamma_i \Phi_1 v_i^1 : \underline{\kappa}_i^1 \leq \gamma_i < \overline{\kappa}_i^1 \right\}$, where $\underline{\kappa}_i^1$ is obtained as described above, and $\overline{\kappa}_i^1$ is obtained in an analogous fashion by changing min by max.
- Write $\xi(x, 1) = \sum_{i=1}^d \beta_i^1 \Phi_1 v_i^1$. Then, the symbol related to the quan-

tized value of $\xi(x,1)$ is given by $q^1 = (q_1^1, \ldots, q_d^1)$. Define $\mathcal{C}_i^1 := \Big\{1, \ldots, \Big\lceil d\Gamma_i^1 e^{T_p\alpha\ell}\frac{\overline{\kappa}_i^1-\underline{\kappa}_i^1}{\epsilon}\Big\rceil\Big\}$. We define q_i^1, for every $i \in \{1,\ldots,d\}$, as the $k \in \mathcal{C}_i^1$ such that

$$\beta_i^1 \in \left[\underline{\kappa}_i^1 + \frac{\epsilon}{d}\frac{e^{-T_p\alpha\ell}}{\Gamma_i^1}(k-1), \underline{\kappa}_i^1 + \frac{\epsilon}{d}\frac{e^{-T_p\alpha\ell}}{\Gamma_i^1}k\right) \tag{2.19}$$

holds true.

- Denote by

$$\hat{\beta}_i^1 := \underline{\kappa}_i^1 + \frac{\epsilon}{d}\frac{e^{-T_p\alpha\ell}}{\Gamma_i^1}(q_i^1 - 1/2). \tag{2.20}$$

Our estimate for the state at the moments $1 \leq t \leq \ell$ is

$$\hat{x}(t) := \sum_{i=1}^{d} \hat{\beta}_i^1 \Phi_t v_i^1.$$

Step j+1:

In this step, we define an estimate $\hat{x}(t)$ for $\xi(x,t)$ with $j\ell + 1 \leq t \leq (j+1)\ell$. Notice that we generated a box

$$\bar{B}^j := \Big\{\sum_{k=1}^{d} \mu_k v_k^j : \underline{\kappa}_k^j + \frac{\epsilon}{d}\frac{e^{-T_p\alpha j\ell}}{\Gamma_k^j}(q_k^j - 1) \leq \quad \mu_k < \underline{\kappa}_k^j + \frac{\epsilon}{d}\frac{e^{-T_p\alpha j\ell}}{\Gamma_k^j}q_k^j\Big\} \tag{2.21}$$

at the end of Step j and that $x \in \bar{B}^j$. Now, in this step, we generate the smallest box aligned with the new basis $\{v_i^{j+1}\}_{i=1}^d$ that contains $\bar{B}^j$. We define this smallest box as

$$\tilde{B}^{j+1} := \left\{\sum_{i=1}^{d} \gamma_i v_i^{j+1} : \underline{\kappa}_i^{j+1} \leq \gamma_i < \overline{\kappa}_i^{j+1}\right\},$$

and obtain $\underline{\kappa}_i^{j+1}$ and $\overline{\kappa}_i^{j+1}$ in an analogous manner as we obtained $\underline{\kappa}_i q$ and $\overline{\kappa}_i^1$ in Step 1. Finally, we define the box B^{j+1} as the box obtained after flowing $\tilde{B}^{j+1}$ forward by $\Phi_{j\ell+1}$. We describe the procedure in the following itemized way.

- Define
$$B^{j+1} := \left\{ \sum_{i=1}^{d} \gamma_i \Phi_{j\ell+1} v_i^{j+1} : \underline{\kappa}_i^{j+1} \le \gamma_i < \overline{\kappa}_i^{j+1} \right\}, \tag{2.22}$$
where
$$\underline{\kappa}_i^{j+1} := \min \left\{ \sum_{k=1}^{d} \mu_k \langle v_k^j, v_i^{j+1} \rangle : \underline{\kappa}_k^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_k^j} \left(q_k^j - 1 \right) \le \right. \tag{2.23}$$
$$\left. \mu_k \le \underline{\kappa}_k^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_k^j} q_k^j, k = 1, \ldots, d \right\},$$
and $\overline{\kappa}_i^{j+1}$ is obtained in an analogous fashion by changing min by max.

- Write $\xi(x, j\ell + 1) = \sum_{i=1}^{d} \beta_i^{j+1} \Phi_{j\ell+1} v_i^{j+1}$. Then, the symbol related to the quantized value of $\xi(x, j\ell+1)$ is given by $q^{j+1} = \left(q_1^{j+1}, \ldots, q_d^{j+1}\right)$. Let
$$\mathcal{C}_i^{j+1} := \left\{ 1, \ldots, \left\lceil d e^{T_p \alpha (j+1) \ell} \Gamma_i^{j+1} \frac{\overline{\kappa}_i^{j+1} - \underline{\kappa}_i^{j+1}}{\epsilon} \right\rceil \right\}.$$
We define q_i^{j+1} as the $k \in \mathcal{C}_i^{j+1}$ such that
$$\beta_i^{j+1} \in \left[\underline{\kappa}_i^{j+1} + \frac{\epsilon}{d} \frac{e^{-T_p \alpha (j+1) \ell}}{\Gamma_i^{j+1}} (k-1), \underline{\kappa}_i^{j+1} + \frac{\epsilon}{d} \frac{e^{-T_p \alpha (j+1) \ell}}{\Gamma_i^{j+1}} k \right) \tag{2.24}$$
holds true.

- Denote by
$$\hat{\beta}_i^{j+1} := \underline{\kappa}_i^{j+1} + \frac{\epsilon}{d} \frac{e^{-T_p \alpha (j+1) \ell}}{\Gamma_i^{j+1}} (q_i^{j+1} - 1/2). \tag{2.25}$$
Then, our state estimate for the time instants $j\ell + 1 \le t \le (j+1)\ell$ is
$$\hat{x}(t) := \sum_{i=1}^{d} \hat{\beta}_i^{j+1} \Phi_t v_i^{j+1}.$$

Theorem 2.3.1 proves that Algorithm 2.3.1 generates a coding scheme that permits us to reconstruct a state estimate with an exponentially decaying error with a prescribed rate of decay. That theorem also gives us an upper bound on the average data-rate used by our coding scheme.

Theorem 2.3.1. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices that comes from the exact discretization of the system (2.1) with sampling time $T_p > 0$. Then, the algorithm from Section 2.3.1 gives a sequence of estimates $(\hat{x}(t))_{t\in\mathbb{Z}_{\geq 0}}$ such that $||\hat{x}(t) - \xi(x,t)|| \leq \frac{\epsilon}{2}e^{-T_p\alpha t}$. Further, the average data-rate of the algorithm from Section 2.3.1 is given by $b = \limsup_{j\to\infty} \frac{1}{T_p t\ell}\sum_{j=0}^{t} \log(\#\mathcal{C}^j)$, with $\mathcal{C}^j \coloneqq \prod_{i=1}^d \mathcal{C}_i^j$ and $\#\mathcal{C}^j \coloneqq \prod_{i=1}^d \#\mathcal{C}_i^j$, where

$$\#\mathcal{C}_i^{j+1} \leq \left\lceil e^{T_p\alpha\ell}\frac{\Gamma_i^{j+1}}{\Gamma_i^j}\sum_{k=1}^{d}\left|\langle v_k^j, v_i^{j+1}\rangle\right|\right\rceil$$

for $j \in \mathbb{Z}_{\geq 0}$ and

$$\#\mathcal{C}_i^0 \leq \left\lceil d\frac{\operatorname{diam}(B^0)}{\epsilon}\right\rceil.$$

Proof. **Step 0**:
Recall that $\left|\hat{\beta}_i^0 - \beta_i^0\right| \leq \epsilon/2d$ by equations (2.14) and (2.15). Then,

$$||\hat{x}(0) - \xi(x,0)|| = \left\|\sum_{i=1}^{d}\left(\hat{\beta}_i^0 - \beta_i^0\right)v_i^0\right\| \leq \frac{\epsilon}{2}$$

and $\#\mathcal{C}_i^0 = \left\lceil d\frac{\overline{\kappa}_i^0 - \underline{\kappa}_i^0}{\epsilon}\right\rceil \leq \left\lceil d\frac{\operatorname{diam}(B^0)}{\epsilon}\right\rceil$. Finally, notice that $x \in \bar{B}^0$ by equations (2.16) and (2.15).

Step 1:
We need to show that

$$\Phi_1\left(\bar{B}^0\right) = \left\{\sum_{i=1}^{d}\gamma_i\Phi_1 v_i^0 : \underline{\kappa}_i^0 + \frac{\epsilon}{d}(q_i^0 - 1) \leq \gamma_i < \underline{\kappa}_i^0 + \frac{\epsilon}{d}q_i^0\right\} \subset B^1.$$

Take $y \in \bar{B}^0$ and write it as $y = \sum_{k=1}^d y_k v_k^0$ and recall that $\underline{\kappa}_k^0 + \frac{\epsilon}{d}(q_k^0 - 1) \leq y_k \leq \underline{\kappa}_k^0 + \frac{\epsilon}{d}q_k^0$ for $k \in \{1,\ldots,d\}$ by equation (2.16). Now, rewriting $y = \sum_{i=1}^d\left(\sum_{k=1}^d y_k\langle v_k^0, v_i^1\rangle\right)v_i^1$, we can check that $\underline{\kappa}_i^1 \leq \left(\sum_{k=1}^d y_k\langle v_k^0, v_i^1\rangle\right) \leq \overline{\kappa}_i^1$ by the definitions of $\underline{\kappa}_i^1$ and $\overline{\kappa}_i^1$. Thus, $\Phi_1 y \in B^1$ by equation (2.18). Since $y \in \bar{B}^0$ was arbitrary, we have that $\Phi_1\left(\bar{B}^0\right) \subset B^1$.
Now, we need to find an estimate for $\#\mathcal{C}_i^1$. First, let $(\underline{\gamma}_1^1,\ldots,\underline{\gamma}_d^1)$ be any argument of the minimum corresponding to the minimization used to define $\underline{\kappa}_i^1$, and let $(\overline{\gamma}_1^1,\ldots,\overline{\gamma}_d^1)$ be any argument of the maximum corresponding to the maximization used to define $\overline{\kappa}_i^1$. Next, notice that $|\overline{\kappa}_i^1 - \underline{\kappa}_i^1| = \left|\sum_{k=1}^d\left(\overline{\gamma}_k^1 - \underline{\gamma}_k^1\right)\langle v_k^0, v_i^1\rangle\right| \leq \frac{\epsilon}{d}\sum_{k=1}^d|\langle v_k^0, v_i^1\rangle|$, because $\left|\overline{\gamma}_k^1 - \underline{\gamma}_k^1\right| \leq \epsilon/d$ by

the fact that[11] $\underline{\kappa}_i^0 + \frac{\epsilon}{d}(q_i^0 - 1) \le \gamma_i < \underline{\kappa}_i^0 + \frac{\epsilon}{d}$ for every $i \in \{1, \dots, d\}$. Thus, we get the upper bound $\#\mathcal{C}_i^1 \le \left\lceil \Gamma_i^1 e^{T_p \alpha \ell} \sum_{k=1}^d |\langle v_k^0, v_i^1 \rangle| \right\rceil$.
Further, by equations (2.19) and (2.20), we have that $\left| \hat{\beta}_i^1 - \beta_i^1 \right| \le \frac{\epsilon}{2d} \frac{e^{-T_p \alpha \ell}}{\Gamma_i^1}$. Then, for $1 \le t \le \ell$ we have that

$$||\hat{x}(t) - \xi(x,t)|| = \left|\left| \sum_{i=1}^d \left(\hat{\beta}_i^1 - \beta_i^1 \right) \Phi_t v_i^1 \right|\right| \le \frac{\epsilon}{2d} e^{-T_p \alpha \ell} \left|\left| \sum_{i=1}^d \frac{\Phi_t v_i^1}{\Gamma_i^1} \right|\right| \le \frac{\epsilon}{2} e^{-T_p \alpha t},$$

where the last inequality comes from the facts that $\left|\left| \frac{\Phi_t v_i^1}{\Gamma_i^1} \right|\right|$
≤ 1 and $1 \le t \le \ell$. Finally, notice that $x \in \bar{B}^1$ because $\sum_{i=1}^d \beta_i^1 v_i^1 \in \bar{B}^1$ by the fact that[12] $\Phi_1 \bar{B}^1 \subset B^1$ and equation (2.18).
Step j+1:
By our induction hypothesis, we have that $x \in \bar{B}^j$. We need to show that

$$\Phi_{j\ell+1} \left(\bar{B}^j \right) = \Big\{ \sum_{i=1}^d \gamma_i \Phi_{j\ell+1} v_i^j : \underline{\kappa}_i^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_i^j} (q_i^j - 1) \le \gamma_i < \underline{\kappa}_i^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_i^j} q_i^j \Big\} \subset B^{j+1}.$$

Take $y \in \bar{B}^j$ and write it as $y = \sum_{k=1}^d y_k v_k^j$ and recall that $\underline{\kappa}_k^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_i^j} (q_k^j - 1) \le y_k \le \underline{\kappa}_k^j + \frac{\epsilon}{d} \frac{e^{-T_p \alpha j \ell}}{\Gamma_i^j} q_k^j$ for $k \in \{1, \dots, d\}$ by equation (2.21). Now, rewriting $y = \sum_{i=1}^d \left(\sum_{k=1}^d y_k \langle v_k^j, v_i^{j+1} \rangle \right) v_i^{j+1}$, we can check that $\underline{\kappa}_i^{j+1} \le \left(\sum_{k=1}^d y_k \langle v_k^j, v_i^{j+1} \rangle \right) \le \overline{\kappa}_i^{j+1}$ by the definitions of $\underline{\kappa}_i^{j+1}$ and $\overline{\kappa}_i^{j+1}$. Thus, $\Phi_{j\ell+1} y \in B^{j+1}$ by equation (2.22). Since $y \in \bar{B}^j$ was arbitrary, we have that $\Phi_{j\ell+1} \left(\bar{B}^j \right) \subset B^{j+1}$.
Now, we need to find an estimate for $\#\mathcal{C}_i^{j+1}$. First, let $(\underline{\gamma}_1^{j+1}, \dots, \underline{\gamma}_d^{j+1})$ be any argument of the minimum corresponding to the minimization used to define $\underline{\kappa}_i^{j+1}$, and let $(\overline{\gamma}_1^{j+1}, \dots, \overline{\gamma}_d^{j+1})$ be any argument of the maximum corresponding to the maximization used to define $\overline{\kappa}_i^{j+1}$. Next, notice that

$$\left| \overline{\kappa}_i^{j+1} - \underline{\kappa}_i^{j+1} \right| = \left| \sum_{k=1}^d \left(\overline{\gamma}_k^{j+1} - \underline{\gamma}_k^{j+1} \right) \langle v_k^j, v_i^{j+1} \rangle \right| \le \frac{\epsilon}{d} \frac{e^{-T_p \alpha \ell}}{\Gamma_i^j} \sum_{k=1}^d \left| \langle v_k^j, v_i^{j+1} \rangle \right|,$$

[11]See equation (2.17) and the discussion below.
[12]To see this, look at equation (2.18) and compare with equation (2.21) with $j = 1$.

because $\left|\overline{\gamma}_k^{j+1} - \underline{\gamma}_k^{j+1}\right| \leq \frac{\epsilon}{d}\frac{e^{-T_p\alpha j\ell}}{\Gamma_i^j}$ by the fact that[13] $\underline{\kappa}_i^j + \frac{\epsilon}{d}\frac{e^{-T_p\alpha j\ell}}{\Gamma_i^j}(q_i^j - 1) \leq \gamma_i < \underline{\kappa}_i^j + \frac{\epsilon}{d}\frac{e^{-T_p\alpha j\ell}}{\Gamma_i^j}q_i^j$. Thus, we arrive at the bound

$$\#\mathcal{C}_i^{j+1} \leq \left\lceil e^{T_p\alpha\ell}\frac{\Gamma_i^{j+1}}{\Gamma_i^j}\sum_{k=1}^{d}\left|\langle v_k^j, v_i^{j+1}\rangle\right|\right\rceil.$$

Further, by equations (2.24) and (2.25), we have the inequality

$$\left|\hat{\beta}_i^{j+1} - \beta_i^{j+1}\right| \leq \frac{\epsilon}{2d}\frac{e^{-T_p\alpha(j+1)\ell}}{\Gamma_i^{j+1}}.$$

Then, for $j\ell + 1 \leq t \leq (j+1)\ell$ we have that

$$\begin{aligned} ||\hat{x}(t) - \xi(x,t)|| &= \left|\left|\sum_{i=1}^{d}\left(\hat{\beta}_i^{j+1} - \beta_i^{j+1}\right)\Phi_t v_i^{j+1}\right|\right| \\ &\leq \frac{\epsilon}{2d}e^{-T_p\alpha(j+1)\ell}\left|\left|\sum_{i=1}^{d}\frac{\Phi_t v_i^{j+1}}{\Gamma_i^{j+1}}\right|\right| \leq \frac{\epsilon}{2}e^{-T_p\alpha t}, \end{aligned}$$

where the last inequality comes from the facts that[14] $\left|\left|\frac{\Phi_t v_i^{j+1}}{\Gamma_i^{j+1}}\right|\right| \leq 1$ and $j\ell+1 \leq t \leq (j+1)\ell$. Finally, notice that $x \in \bar{B}^{j+1}$ because $\sum_{i=1}^{d}\beta_i^{j+1}v_i^{j+1} \in \bar{B}^{j+1}$ by the fact that $\Phi_{j\ell+1}\bar{B}^{j+1} \subset B^{j+1}$ and equation (2.22). □

The previous theorem gives us the following corollary.

Corollary 2.3.2. Let $\delta > 0$, $\alpha \geq 0$, and $\ell \in \mathbb{N}$. If $\mathcal{V}_j = \mathcal{V}$ for all $j \in \mathbb{Z}_{\geq 0}$, where $\mathcal{V}$ is a normal basis for the Oseledets' filtration, then

$$b \leq \frac{1}{T_p\ell}\sum_{i=1}^{d}\log\left\lceil e^{T_p(\lambda_i+\alpha)\ell}\right\rceil$$

if the system is known to be regular and

$$b \leq \frac{1}{T_p\ell}\sum_{i=1}^{d}\log\left\lceil e^{T_p(\lambda_i+\alpha+\delta)\ell}\right\rceil,$$

otherwise. Furthermore, b can be made as close as desired to $h_{\text{est}}(\alpha, K)$ by choosing ℓ large enough in case the system is known to be regular, or b can

[13] See equation (2.23) and the discussion below.

[14] This is implied by the defining equations (2.11) and (2.12).

be made as close as desired to $\sum_{i=1}^{d}\max\{\lambda_i+\alpha+\delta,0\}$, otherwise.

Proof. If $\mathcal{V}=\{v_1,\ldots,v_d\}$ is a normal basis for the Oseledets' filtration of a tempered matrix sequence $(A_j)_{j\in\mathbb{N}}$ and $\mathcal{V}_j=\mathcal{V}$, i.e. $v_i^j=v_i$ for $j\in\mathbb{Z}_{\geq 0}$ and every $i\in\{1,\ldots,d\}$, then $\sum_{k=1}^{d}\left|\langle v_k^j, v_i^{j+1}\rangle\right|=1$, and $\lambda_i=\limsup_{j\to\infty}\frac{1}{j}\log\left(\left\|\Phi_j v_i^j\right\|\right)=\limsup_{j\to\infty}\frac{1}{j}\log\left(\|\Phi_j v_i\|\right)$, i.e., λ_i's will be the Lyapunov exponents with multiplicity. We know that for every $\eta>0$, there exists $N\in\mathbb{N}$ such that $\forall j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$ and all $i\in\{1,\ldots,d\}$, we have that $\|\Phi_t v_i\|\leq e^{T_p(\lambda_i+\eta)t}\leq e^{T_p(\lambda_i+\delta+\eta)t}$ for all $t\geq N$ and this δ is the same as the one used in the definition of Γ_i^j in the algorithm from Section 2.3.1. Further, we know that for $\eta>0$ sufficiently small, $\lambda_i+\delta+\eta<0$ for all $\lambda_i+\delta<0$ with $i\in\{1,\ldots,d\}$. Therefore, for $j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$ we have that $\max_{\{0,\ldots,\ell-1\}}\{\|\Phi_{j\ell-k}v_i\|\}\leq\max\left\{e^{T_p(\lambda_i+\delta+\eta)j\ell}, e^{T_p(\lambda_i+\delta+\eta)((j-1)\ell+1)}\right\}$.
Hence, as a consequence of our previous discussion and equation (2.12), if $\lambda_i+\delta<0$, then we have that $\Gamma_i^j=e^{T_p(\lambda_i+\delta)((j-1)\ell+1)}$ $\forall j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$ and all $i\in\{1,\ldots,d\}$, otherwise we have that $\Gamma_i^j=e^{T_p(\lambda_i+\delta)j\ell}$ $\forall j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$ and all $i\in\{1,\ldots,d\}$. Note that for $\lambda_i+\delta\geq 0$, we have $e^{T_p(\lambda_i+\delta-\eta)j\ell}\leq\Gamma_i^j\leq e^{T_p(\lambda_i+\delta+\eta)j\ell}$ and that $e^{T_p(\lambda_i+\delta-\eta)((j-1)\ell+1)}\leq\Gamma_i^j\leq e^{T_p(\lambda_i+\delta+\eta)((j-1)\ell+1)}$ if $\lambda_i+\delta<0$. Therefore, we have that $\frac{\Gamma_i^{j+1}}{\Gamma_i^j}\leq e^{T_p(\lambda_i+\delta+2\eta)\ell}$ independently of the sign of $\lambda_i+\delta$. Thus, by Theorem 2.3.1, we have that $\#\mathcal{C}_i^{j+1}\leq\left\lceil e^{T_p(\lambda_i+\alpha+\delta+2\eta)\ell}\right\rceil$, $\forall j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$ and every $i\in\{1,\ldots,d\}$. We conclude, by showing that the first $\left\lceil\frac{N-1}{\ell}+1\right\rceil+1$ terms of the sum in the definition of b go to zero and that $\#\mathcal{C}^j\leq\prod_{i=1}^{d}\left\lceil e^{T_p(\lambda_i+\alpha+\delta+2\eta)\ell}\right\rceil$ for all $j\geq\left\lceil\frac{N-1}{\ell}+1\right\rceil$, that[15] $b\leq\frac{1}{T_p\ell}\sum_{i=1}^{d}\log\left\lceil e^{T_p(\lambda_i+\alpha+\delta+2\eta)\ell}\right\rceil$. Also, because η can be arbitrarily small, we have that $b\leq\frac{1}{T_p\ell}\sum_{i=1}^{d}\log\left\lceil e^{T_p(\lambda_i+\alpha+\delta)\ell}\right\rceil$. Finally, by choosing ℓ large enough, b can get as close to $\sum_{i=1}^{d}\max\{\lambda_i+\alpha+\delta,0\}$ as desired.
Following analogous steps, we can prove a similar result for the case when the system is known to be regular. To see this, note that, under the regularity assumption, for every $\eta>0$ there exists $N\in\mathbb{N}$ such that $e^{T_p(\lambda_i-\eta)t}\leq\|\Phi_t v_i\|\leq e^{T_p(\lambda_i+\eta)t}$ for all $t\geq N$. Then, we notice that for $\lambda_i\geq 0$, we have $e^{T_p(\lambda_i-\eta)j\ell}\leq\Gamma_i^j\leq e^{T_p(\lambda_i+\eta)j\ell}$ and that $e^{T_p(\lambda_i-\eta)((j-1)\ell+1)}\leq\Gamma_i^j\leq e^{T_p(\lambda_i+\eta)((j-1)\ell+1)}$ if $\lambda_i<0$. Next, we get the inequality $\frac{\Gamma_i^{j+1}}{\Gamma_i^j}\leq e^{T_p(\lambda_i+2\eta)\ell}$ independently of the sign of λ_i. Now, we replace this inequality in our previous argument to get that $b\leq\frac{1}{T_p\ell}\sum_{i=1}^{d}\log\left\lceil e^{T_p(\lambda_i+\alpha)\ell}\right\rceil$, and by choosing ℓ large enough, b can get as

[15] These steps are similar to those used in the proof of the entropy's upper bound in Theorem 2.2.6.

close to $\sum_{i=1}^{d} \max\{\lambda_i + \alpha, 0\}$ as desired. These results are summarized in the next Corollary 2.3.2. $\square$

Remark 2.3.3. We note that Algorithm 2.3.1 reconstructs the state at the end of the interval $j\ell + 1 \leq t \leq (j+1)\ell$ for $j \in \mathbb{Z}_{\geq 0}$. By that, we mean that we must wait until time $(j+1)\ell$ to build our estimate. We could, analogously, build an estimate at the beginning of the interval by making a simple modification: choose an arbitrary $\bar{\delta} > 0$ and redefine, for all $i \in \{1, \ldots, d\}$ and all $j \in \mathbb{Z}_{\geq 0}$, Γ_i^j as $\Gamma_i^{\prime j} := \|\Phi_{j\ell} v_i^j\| e^{T_p \bar{\delta}(j+1)\ell}$, if the system is known to be regular or $\Gamma_i^{\prime j} := \max\{\|\Phi_{j\ell} v_i^j\|, e^{T_p(\lambda_i+\delta)j\ell}, e^{T_p(\lambda_i+\delta)((j-1)\ell+1)} e^{T_p\bar{\delta}(j+1)\ell}\}$, otherwise. This latter modification works because the property of temperedness of the sequence implies that there exists some $N \in \mathbb{N}$ such that we have $\|\Phi_t v_i^t\| \leq e^{T_p \bar{\delta}(t-j\ell)} \|\Phi_{j\ell} v_i^j\| \leq e^{T_p \bar{\delta}(j+1)\ell} \|\Phi_{j\ell} v_i^j\|$ for all $t \geq N$. This latter fact, by its turn, tells us that $\frac{\|\Phi_t v_i^t\|}{\Gamma_i^{\prime j}} \leq 1$, which is all that is needed for the proof of Theorem 2.3.1 to hold. We also note that the quantity Γ_i^j only appears in the fraction $\frac{\Gamma_i^{j+1}}{\Gamma_i^j}$ that we use to compute our data-rate estimate. Thus, the data-rate analysis presented in the proof of Corollary 2.3.2 holds with the minor change that $\frac{\Gamma_i^{\prime j+1}}{\Gamma_i^{\prime j}} \leq e^{T_p(\lambda_i + 2\eta + \bar{\delta})\ell}$ for all $i \in \{1, \ldots, d\}$ and for all $j \geq N$, where $N \in \mathbb{N}$. Since $\bar{\delta} > 0$ is arbitrary, our claim in Corollary 2.3.2 remains unchanged.

2.3.2 Finding $(\mathcal{V}_j)_{j \in \mathbb{Z}_{\geq 0}}$ Online

In general, knowing a family of bases $(\mathcal{V}_j)_{j \in \mathbb{Z}_{\geq 0}}$ that makes our algorithm operate with an average data-rate close to the estimation entropy, e.g., a constant family equal to a normal bases for the Oseledets' filtration, is impossible. That happens because to compute the Lyapunov exponent, described in Definition 2.2.4, we must calculate a limit superior, which requires us to know the entire sequence $(A_n)_{n \in \mathbb{N}}$ from the beginning. Analogously, we cannot know the Oseledets' filtration beforehand. We hope that Examples 2.2.1 and 2.2.3 should elucidate these latter points.

So, naturally, we ask ourselves if there is a way to construct the family of bases online. The answer is affirmative when the system is regular. We prove this fact in the next theorem, first presented in [20].

Theorem 2.3.4. Assume that $(A_n)_{n \in \mathbb{N}}$ is regular. Let $Q_j := \left(\Phi_j^\top \Phi_j\right)^{\frac{1}{2j}}$ for $j \in \mathbb{Z}_{\geq 0}$ and let its eigenvalues be $e^{\rho_i(j)}$, where $i \in \{1, \ldots, d\}$ and

$e^{\rho_1(j)} \leq \cdots \leq e^{\rho_d(j)}$. Also, let $\mathcal{V}_j = \left\{v_1^j, \ldots, v_d^j\right\}$ be an orthonormal basis that diagonalizes Q_j, with an order induced by the order on their corresponding eigenvalues $e^{\rho_i(j)}$. Then the average data-rate of the algorithm from Section 2.3.1 is upper bounded by $\sum_{i=1}^d \max\left\{\alpha + \lambda_i + \frac{1}{T_p \ell}, 0\right\}$, if the Lyapunov exponents are simple, or $\sum_{i=1}^d \max\left\{\alpha + \lambda_i + \frac{\log\left(\sqrt{d}\right)+1}{T_p \ell}, 0\right\}$, otherwise.

Proof. Our goal is to find an upper bound for $\#\mathcal{C}_i^j$ for j large enough. For that purpose, we will use the upper bound obtained in Theorem 2.3.1. So, we need to find upper bounds or expressions for $\sum_{k=1}^d \left|\langle v_k^j v_i^{j+1}\rangle\right|$ and $\frac{\Gamma_i^{j+1}}{\Gamma_i^j}$.
First, we show that $\lambda_i = \limsup_{j\to\infty} \frac{1}{j} \log\left\|\Phi_j v_i^j\right\|$, which appear in the definition of the algorithm from Section 2.3.1 for $i \in \{1, \ldots, d\}$, are the Lyapunov exponents with multiplicity, and that they are given by $\lambda_i = \lim_{j\to\infty} \rho_i(j)$. To see that, notice that $\left\|Q_j v_i^j\right\| = e^{\rho_i(j)}$ and that

$$\begin{aligned}
\lambda_i &= \limsup_{j\to\infty} \frac{1}{j} \log\left\|\Phi_j v_i^j\right\| \\
&= \limsup_{j\to\infty} \frac{1}{j} \log\left((v_i^j)^\top \Phi_j^\top \Phi_j v_i^j\right)^{1/2} \\
&= \limsup_{j\to\infty} \frac{1}{j} \log\left((v_i^j)^\top Q_j^{2j} v_i^j\right)^{1/2} \\
&= \limsup_{j\to\infty} \rho_i(j)
\end{aligned}$$

where the second equality comes from the fact that the Euclidean norm and the infinity norm are equivalent. Also, the last equality comes from the fact that any basis that diagonalizes Q_j also diagonalizes Q_j^{2j}.
As a consequence of regularity, by the third bullet of Lemma 2.2.5, Q_j has a limit. Therefore, its eigenvalues $e^{\rho_i(j)}$ have a limit as well. Hence, we conclude that $\lambda_i = \lim_{j\to\infty} \rho_i(j)$ because the limit on the right exists.
Second, denote the limit of Q_j by $Q \coloneqq \lim_{j\to\infty} Q_j$. Because Lyapunov exponents are simple, there exists $N_0 \in \mathbb{N}$ such that for all $j \geq N_0$ the eigenvalues of Q_j are simple as well. Now, a symmetric matrix with simple eigenvalues has a unique, up to a change of signs and subject to the order indicated in the theorem statement, orthonormal basis that diagonalizes it. This implies that for any $\eta_1 > 0$, there exists $N_1 \in \mathbb{N}$ such that $\sum_{k=1}^d \left|\langle v_k^j, v_i^{j+1}\rangle\right| \leq 1 + \eta_1$ for all $j \geq N_1$ and $i \in \{1, \ldots, d\}$. To see this, denote by $\{v_1, \ldots, v_d\}$ a basis that diagonalizes Q. Now, we can change the signs of $\left\{v_1^j, \ldots, v_d^j\right\}$ if

necessary, so that v_i^j converges to v_i, and notice that changing the sign does not change the absolute value of the inner products mentioned above. Because these are orthonormal bases, there exists $N_1 \in \mathbb{N}$ such that, for every $i \in \{1, \dots, d\}$, we have $|\langle v_k^j, v_i^{j+1} \rangle| \leq \eta_1/d$ if $k \neq i$ and $|\langle v_k^j, v_i^{j+1} \rangle| \leq 1 + \eta_1/d$ if $k = i$, and we proved this claim. Notice, however, that the inequalities $\sum_{k=1}^{d} \left| \langle v_k^j, v_i^{j+1} \rangle \right| \leq \sqrt{d}$ for every $i \in \{1, \dots, d\}$ always hold, even without simplicity.

Third, again because of regularity, for $\eta_2 > 0$ such that $\lambda_i + \eta_2 < 0$ for all $\lambda_i < 0$, but otherwise arbitrary[16], there exists $N_2 \in \mathbb{N}$ such that for all $j \geq N_2$ and all $i \in \{1, \dots, d\}$ we have that $\lambda_i - \eta_2 \leq \rho_i(j) \leq \lambda_i + \eta_2$. Thus, $\Gamma_i^j \coloneqq \max_{k \in \{0, \dots, \ell-1\}} \left\| \Phi_{j\ell - k} v_i^j \right\| = \max_{k \in \{0, \dots, \ell-1\}} \left\| e^{\rho_i (j\ell - k)} \right\|$. Then, we arrive at the inequalities $e^{T_p(\lambda_i - \eta_2) j\ell} \leq \Gamma_i^j \leq e^{T_p(\lambda_i + \eta_2) j\ell}$, if $\lambda_i \geq 0$, and $e^{T_p(\lambda_i - \eta_2)((j-1)\ell + 1)} \leq \Gamma_i^j \leq e^{T_p(\lambda_i + \eta_2)((j-1)\ell + 1)}$, if $\lambda_i < 0$. Then, $\frac{\Gamma_i^{j+1}}{\Gamma_i^j} \leq e^{T_p(\lambda_i + 2\eta_2)\ell}$ for $j \geq N_2$ and $i \in \{1, \dots, d\}$.

Now, recall the definition of average data-rate

$$b = \limsup_{t \to \infty} \frac{1}{T_p t \ell} \sum_{j=0}^{t} \sum_{i=1}^{d} \log\left(\#\mathcal{C}_i^j\right).$$

Denote $N \coloneqq \max\{N_1, N_2\}$. So, for $j \geq N$ we have that

$$\#\mathcal{C}_i^j \leq \left\lceil e^{T_p(\alpha + \lambda_i + 2\eta_2)\ell} (1 + \eta_1) \right\rceil.$$

Further, define $M = \sum_{j=0}^{N-1} \sum_{i=1}^{d} \log\left(\#\mathcal{C}_j^i\right)$. We can upper-bound the average data-rate by $b \leq \limsup_{t \to \infty} \frac{1}{T_p t \ell} \Big(M + \sum_{k=N}^{t} \sum_{i=1}^{d} \log(g_i) \Big)$, where $g_i = \left\lceil e^{T_p(\alpha + \lambda_i + 2\eta_2)\ell} (1 + \eta_1) \right\rceil$.

Notice that $\log(\lceil x \rceil) \leq \max\{\log(x) + 1, 0\}$. To see that, we study two cases. If $x \geq 1$, then $2x \geq x + 1$ and $\log(2x) = \log(2) + \log(x) = 1 + \log(x) \geq \log(x + 1) \geq \log(\lceil x \rceil)$. If $x < 1$, then $\log(\lceil x \rceil) = 0$. Therefore, we can derive the upper bound

$$\log\left(\left\lceil e^{T_p(\alpha + \lambda_i + 2\eta_2)\ell} (1 + \eta_1) \right\rceil\right) \leq \max\{T_p(\alpha + \lambda_i + 2\eta_2)\ell(1 + \eta_1) + 1, 0\}.$$

[16] Notice that η_2 can be chosen to be as small as desired.

Thus,

$$b \leq \limsup_{t\to\infty} \frac{1}{T_p t\ell}\Big(M + (t-N)\sum_{i=1}^{d} \max\big\{T_p(\alpha+\lambda_i+2\eta_2)\ell + \\ + \log(1+\eta_1) + 1, 0\big\}\Big)$$

and since M and N are constants, we conclude that

$$b \leq \sum_{i=1}^{d} \max\left\{\alpha+\lambda_i+2\eta_2+\frac{\log(1+\eta_1)}{T_p\ell}+\frac{1}{T_p\ell}, 0\right\}.$$

Since $\eta_1 > 0$ and $\eta_2 > 0$ can be chosen to be arbitrarily small, we have that $b \leq \sum_{i=1}^{d} \max\left\{\alpha+\lambda_i+\frac{1}{T_p\ell}, 0\right\}$.
Finally, if we drop the simplicity assumption, we could replace $\log(1+\eta_1)$ by $\log\left(\sqrt{d}\right)$ and obtain $b \leq \sum_{i=1}^{d} \max\left\{\alpha+\lambda_i+\frac{\log(\sqrt{d})+1}{T_p\ell}, 0\right\}$, and, therefore, in both cases, by choosing ℓ sufficiently large, the upper bound on b can be made arbitrarily close to the estimation entropy $h_{\text{est}}(\alpha, K)$ as given by the last statement of Theorem 2.2.6. □

Remark 2.3.5. Some of the results still hold even without regularity and simplicity. Note that $\sum_{k=1}^{d}\left|\langle v_k^j, v_i^{j+1}\rangle\right| \leq \sqrt{d}$ always holds for every $i \in \{1,\ldots,d\}$. Also, removing the regularity assumption, it is true that for every $\eta_2 > 0$, there exists $N \in \mathbb{N}$ such that $\frac{\Gamma_i^{j+1}}{\Gamma_i^j} \leq e^{T_p(\lambda_i+\delta+2\eta_2)\ell}$ for all for $j \geq N$, where $\delta > 0$ is the same that appears in the definition of Γ_i^j in algorithm 2.3.1. Further, these inequalities lead us to the conclusion that $\#\mathcal{C}_i^j \leq \left\lceil e^{T_p(\alpha+\lambda_i+\delta+2\eta_2)\ell}\sqrt{d}\right\rceil$ for $j \geq N$ and $i \in \{1,\ldots,d\}$. We can use this to upper-bound $\#\mathcal{C}_i^j$ and, following the same steps as in the previous proof, we conclude that $b \leq \sum_{i=1}^{d} \max\left\{(\alpha+\lambda_i+\delta)+\frac{\log(\sqrt{d})+1}{T_p\ell}, 0\right\}$. Finally, note that these λ_i's aren't the Lyapunov exponents with multiplicity. These λ_i's are the *upper growth rates* of the singular values of Q_j as j goes to infinity (see, e.g., Chapter 6 of [36]). It is well-known that these λ_i's are smaller than or equal to the Lyapunov exponents when we don't have regularity. For that reason, this algorithm might work at an average data-rate smaller than the entropy's upper bound obtained in Theorem 2.2.6. Understanding this gap is the topic of future research.

We note that, without the regularity assumption, we need to have a priori knowledge either of the λ_i's, or an upper bound to them. Both hypotheses are unreasonable if we want to have a causal algorithm since the λ_i's depend on the entire sequence $(A_n)_{n\in\mathbb{N}}$. We also remark that the simplicity of the Lyapunov exponents is a generic property, and we expect that most systems will have it (see, e.g., Chapter 8 of [29]).

2.4 Sufficient Conditions for Regularity

In this section, we show that many pratical systems satisfy the Lyapunov regularity condition. We first prove that sampling continuous-time regular systems gives us a discrete-time regular system as well. After that, we deal with probabilistic switched systems, i.e., systems for which the switching signal is a random process in some sense. An interesting subclass is that of ergodic Markov Jump Linear Systems (MJLS) (see, e.g., [40, 41]).

2.4.1 Sampled Continuous-time Regular Systems

We can define regularity of continuous-time systems analogously to the discrete-time case. To do that, however, we need to adapt some other auxiliary notions. We define the the Lyapunov exponent of system (2.1) as $\lambda^c(v) := \limsup_{t\to\infty} \frac{1}{t} \log ||\Phi(t,0)v||$, where $\Phi(t,0)$ is the state transition matrix of system (2.1) (see, e.g., Chapter 3 of [36]). We further define the Oseledets' filtration and Lyapunov exponents with multiplicities, λ_i^c with $i \in \{1,\ldots,d\}$, analogously as the discrete-time Definition 2.2.6 by simply changing the definition of Lyapunov exponent used. Now, we are ready to define regularity for continuous-time systems. System (2.1) is regular if $\lim_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{tr}(\mathcal{A}_{\sigma(\tau)})d\tau = \sum_{i=1}^d \lambda_i^c$.

Another instrumental concept we need to adapt to the continuous-time case is tempredness. We say that system (2.1) is tempered if

$$\lim_{t\to\infty} \frac{1}{t}\int_t^{t+1} \left\|\mathcal{A}_{\sigma(\tau)}\right\| d\tau = 0.$$

We note that there exists an analogous result to that of Lemma 2.2.5 is true for the continuous-time case (see, e.g., Chapter 4 of [36]). More specifically, we use in this subsection a consequence of the analogue of the second bullet in Lemma 2.2.5, i.e., for tempered and regular systems it holds that $\lambda_i^c = \lim_{t\to\infty} \frac{\log(\|\Phi(t,0)v_i\|)}{t}$ where $\{v_1, \ldots, v_d\}$ is a normal basis for the Oseledets' filtration.

Our next proposition, first presented in [20], proves that if we sample a continuous-time linear systems that is regular and tempered, then its corresponding discrete-time system preserves those properties.

Proposition 2.4.1. Consider a continuous-time switched linear system as in equation (2.1). Define $x_n := x(T_p n)$ and $A_n := \Phi(nT_p, (n-1)T_p)$ for $n \in \mathbb{N}$, where $\Phi(t, 0)$ is the fundamental matrix of (2.1), and T_p is the sampling time. If the continuous-time system is tempered and regular, then the sequence $(A_n)_{n\in\mathbb{N}}$ is tempered and regular.

Proof. First, note that since $\lambda^c(v) = \lim_{t\to\infty} \frac{1}{t} \log(\|\Phi(t,0)v\|)$, we can take a subsequence $t_j = T_p j$ and conclude that $\lambda^c(v) = \lim_{j\to\infty} \frac{1}{T_p j} \log(\|\Phi_j v\|) = \frac{\lambda(v)}{T_p}$. Thus, $\lambda_i^c = \frac{\lambda_i}{T_p}$. Notice that, by Liouville's formula, we have that $\log\Big(|\det(\Phi(t,0))|\Big) = \int_0^t \operatorname{tr}(\mathcal{A}_{\sigma(\tau)})d\tau$. Finally, we conclude that

$$\begin{aligned}\sum_{i=1}^d \lambda_i &= T_p \sum_{i=1}^d \lambda_i^c \\ &= T_p \lim_{t\to\infty} \frac{\log\Big(|\det(\Phi(t,0))|\Big)}{t} \\ &= \lim_{j\to\infty} \frac{\log\Big(|\det(\Phi(T_p j,0))|\Big)}{j}.\end{aligned}$$

Therefore, the sampled system is regular. Now, for temperedness, notice that $\|A_n\| \leq e^{\int_{(n-1)T_p}^{nT_p} \|\mathcal{A}_{\sigma(\tau)}\| d\tau}$ by the Bellman-Grönwall lemma (see, e.g., Chapter 2 of [42]). Taking the logarithm on both sides we get that $\frac{\log(\|A_n\|)}{nT_p} \leq \frac{1}{nT_p}\int_{(n-1)T_p}^{nT_p} \left\|\mathcal{A}_{\sigma(\tau)}\right\| d\tau$ and after a change of variables and using the fact that temperedness implies $\lim_{n\to\infty} \frac{1}{n}\int_{n-1}^{n} \left\|\mathcal{A}_{\sigma(\tau)}\right\| d\tau = 0$ we get that

$$\limsup_{n\to\infty} \frac{\log(\|A_n\|)}{n} \leq 0.$$

For the lower bound, note that we can apply Bellman-Grönwall lemma to conclude that $\|A_n\| \leq e^{\int_{nT_p}^{(n-1)T_p} \|\mathcal{A}_{\sigma(\tau)}\| d\tau}$ and get that $\limsup_{n\to\infty} \frac{\log\left(\left\|A_n^{-1}\right\|\right)}{n} \leq 0$. Finally, recall that $\|A_n^{-1}\| \geq \|A_n\|^{-1}$, which implies that $\liminf_{n\to\infty} \frac{\log(\|A_n\|)}{n} \geq 0$, and we conclude that $\lim_{n\to\infty} \frac{\log(\|A_n\|)}{n} = 0$. ∎

2.4.2 Randomly Switched Systems

As mentioned at the beginning of this section, we focus our attention on discrete-time randomly switched systems. Informally, in this case, our switching signal is a realization of a random process, which takes values over all possible signals. We formalize that idea by introducing the concept of linear cocycle later in this subsection. Our interest is to find conditions that ensure that the realizations of such a process are regular with probability one. To help us make these ideas more concrete, we present the next intuitive example: let $\{B_1, \dots, B_m\}$ with B_i an invertible $d \times d$ matrix for $i \in \{1, \dots, m\}$ be the set of modes. We assume that at each instant $k \in \mathbb{N}$, the probability that the mode B_i is active at a time is p_i for each $i \in [m]$. Repeating this process, we get a sequence $(B_{i_n})_{n\in\mathbb{N}}$. Kolmogorov's extension theorem (see, e.g., Chapter III of [43]) tells us that we can assign probabilities to sets in the space $\{(A_n)_{n\in\mathbb{N}} : A_n \in \{B_1, \dots, B_m\}\}$. Thus, it seems natural to ask: what is the probability of the set of regular sequences? The remainder of this subsection is devoted to addressing this issue. We start by proving some definitions.

Definition 2.4.1 (Linear Cocycle [29])**.** Let $(M, \mathscr{B}, \mu)$ be a probability space, $f : M \to M$ be a measure-preserving map. Let[17] $L : M \to \mathrm{GL}(\mathbb{R}, d)$. The *linear cocycle* defined by L over f is the transformation $F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d$ with $F(x, v) = (f(x), L(x)v)$. It follows that $F^n(x, v) = (f^n(x), L(f^n(x)) \cdots L(x)v)$ for every $n \geq 1$. Moreover, if f is invertible, then so is F, with inverse $F^{-1}(x, v) = (f^{-1}(x), (L(f^{-1}(x)))^{-1}v)$.

At first glance, it might seem counter-intuitive why we should define linear cocycles to study switched systems. Nonetheless, there is a natural way to model a linear switched system as a linear cocycle. This model allows us to use powerful tools from dynamical systems to study the switched system properties. To do that, we need to make a few definitions. For simplicity, denote the set of modes as $\mathbf{B} \coloneqq \{B_1, \dots, B_m\} \subset \mathrm{GL}\,(d, \mathbb{R})$. Then, define $M \coloneqq \mathbf{B}^{\mathbb{N}}$, i.e., our sample space is the set of all possible sequences of modes. Choose $f : M \to M$ to be the shift, i.e., $f\left((A_n)_{n\in\mathbb{N}}\right) = (A_{n+1})_{n\in\mathbb{N}}$. Finally, let $L : M \to \mathrm{GL}(d, \mathbb{R})$ be the projection to the first coordinate, i.e., $L\left((A_n)_{n\in\mathbb{N}}\right) = A_1$. We see that $F^n((A_j)_{j\in\mathbb{N}}, v) = ((A_{j+n})_{j\in\mathbb{N}}, A_n \cdots A_1 v)$

[17] Recall that $\mathrm{GL}(\mathbb{R}, d)$ is the set of $d \times d$ invertible matrices.

for any sequence $(A_j)_{j\in\mathbb{N}} \in M$. Since $A_n \cdots A_1 v = \Phi_n v$ is the solution to equation (2.2) with[18] $\tilde{A}_k = A_n$ and initial condition $v \in \mathbb{R}^d$, it seems natural to expect that such a linear cocycle should give us information about the switched system. Now, our goal is to study properties of linear cocycles. Indeed, Theorem 2.4.2 gives us that the set of regular realizations has probability one.

Before we proceed, we recall some classical definitions: a *cylinder of rank* k is a set of the form $\big[(A_n)_{n\in\mathbb{N}} : A_1 = B_{i_1}, \ldots, A_k = B_{i_k}\big]$, where $i_j \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$. Also, we define $\mathscr{B}$ as the smallest σ-algebra that contains the cylinder sets of all ranks (see, e.g., Section 2 of [44]).

Theorem 2.4.2 (Oseledets [29,36,38])**.** Let $(M, \mathscr{B}, \mu)$ be a probability space, $f : M \to M$ be a measure-preserving map. Let $L : M \to \mathrm{GL}(\mathbb{R}, d)$ be such that[19] $\log^+ ||L|| \in L^1(\mu)$. Also consider the linear cocycle defined by L over f. Further, denote $\Phi_n(x) = L(f^n(x)) \cdots L(x)$.

Then, for μ-almost every $x \in M$, there are $k(x)$ positive integers, $\lambda_k(x) > \cdots > \lambda_1(x)$, and a filtration $\{0\} = E^1_x \subsetneq \cdots \subsetneq E^k_x = \mathbb{R}^d$ such that $\forall i = 1, \ldots, k(x)$:

- $k(f(x)) = k(x)$ and $\lambda_i(f(x)) = \lambda_i(x)$ and $L(x)(E^i_x) = E^i_{f(x)}$;
- $\lim_{n\to\infty} \frac{1}{n} \log ||\Phi_n(x)v|| = \lambda_i(x)$ for all $v \in E^{i+1}_x \setminus E^i_x$, with $E^1_x = \{0\}$,
- The $\lim_{n\to\infty} \big(\Phi_n^\top(x)\Phi_n(x)\big)^{\frac{1}{2n}}$ exists.

Furthermore, if f is ergodic, the multiplicities $k(x)$ of the Lyapunov exponents $\lambda_i(x)$ are constant and, consequently, so are the dimension of the subspaces E^i_x. Also, in the ergodic case $\lambda_i(x) = \lambda_i$ is constant a.e..

Since the third bullets of Theorem 2.4.2 and Lemma 2.2.5 are the same, we see that the set C, which we call the set of *regular realizations*, is a set of regular sequences. The fact that C is measurable and has $\mu(C) = 1$ tells us that, with probability one, the realization of our random process will be regular. This latter fact is true for any shift-invariant probability measure over $\mathscr{B}$. Note, however, that different measures give us distinct sets C.

The simplest case we can analyze with the previous theorem is that of periodically switched systems, which we do in the next corollary. We remark that this and the next corollary were presented first in [20].

[18] See Section 2.2 for a discussion about the indices.
[19] Here we use the notation $\log^+(x) = \max\big\{\log(x), 0\big\}$.

Corollary 2.4.3 (Periodically Switched Systems)**.** Let $(A_n)_{n\in\mathbb{N}} \subset \mathbf{B}^{\mathbb{N}}$ be such that $A_{n+T} = A_n$ for some $T \in \mathbb{N}$ and every $n \in \mathbb{N}$. Then, this sequence is regular.

Proof. Let $\mathcal{N} \in \mathscr{B}$, $x = (A_n)_{n\in\mathbb{N}}$, and $f(x) = (A_{n+1})_{n\in\mathbb{N}}$. Define the measure $\mu(\mathcal{N}) = \frac{1}{T}\sum_{i=0}^{T-1}\delta_{f^i(x)}(\mathcal{N})$, where δ_x is a Dirac measure, i.e. $\delta_x(\mathcal{N}) = 1$ if $x \in \mathcal{N}$ and $\delta_x(\mathcal{N}) = 0$, otherwise. This measure is trivially forward invariant under the shift and, because $\|A_n\| < \infty$, we have that $\log^+\|L\| \in L^1(\mu)$. Therefore, we can apply Oseledets' Theorem and conclude that there exists $C \in \mathscr{B}$ with $\mu(C) = 1$ such that all of its realizations are regular. Notice that $K := \cup_{i\geq 0} f^i(x) = \cup_{i=0}^{T-1} f^i(x) \in \mathscr{B}$ and that $\mu(K) = 1$ by construction. Finally, notice that $C \cap K = K$. To see this, notice that K is a finite set, and μ gives the same measure for each point of K, more specifically $\mu(f^i(x)) = \frac{1}{n}$ for $i \in \{0,\ldots,T-1\}$. Hence, if $\#C \cap K < \#K$, we would have that $1 = \mu(C \cap K) \leq \mu(K) - \frac{1}{n}$, which is a contradiction. Therefore, the sequence $(A_n)_{n\in\mathbb{N}}$ is regular. Also, notice that the Lyapunov exponents with multiplicity are constant on K. □

A more interesting class of systems with several practical applications is that of Markov Jump Linear Systems. Before we discuss that case, we must recall some definitions regarding discrete-time Markov Chains (see, e.g., Chapter 1 of [45]). Let $P = (p_{ij})$ be the $m \times m$ transition probability matrix of a discrete-time Markov chain. A *stationary distribution* of such chain $\pi^* = (\pi_1, \ldots, \pi_m)$, is defined as a solution of $\pi^{*\top} = P\pi^{*\top}$, where $\sum_{i=1}^m \pi_i^* = 1$ and $\pi_i^* \geq 0$ for all $i \in \{1, \ldots, m\}$. Recall that if a Markov chain is irreducible and positively recurrent, it has a unique stationary distribution. Now we can define a measure on the cylinder sets by choosing a vector $\pi^0 = (\pi_1^0, \ldots, \pi_m^0)$, with $\pi_i^0 \geq 0$ for $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^m \pi_i^0 = 1$, and using our transition probability matrix P. To do that, we define the value of the measure μ on cylinders of rank k for each $k \in \mathbb{N}$ in the following manner:

$$\mu(\mathcal{N}_k) = \pi_{i_1}^0 p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{k-1} i_k},$$

where $\mathcal{N}_k = \left[(A_n)_{n\in\mathbb{N}} \in \mathbf{B}^{\mathbb{N}} : A_1 = B_{i_1}, \ldots, A_k = B_{i_k}\right]$ is an arbitrary cylinder of rank k and k is an arbitrary natural number. As mentioned earlier in this subsection, Kolmogorov extension theorem tells us that the measure of the cylinders defines the measure in the entire sample space (see, e.g., Section 24 of [44], or Chapter III of [43]). We call such measure the *probability measure induced by* π^0 *and* P.

It is worth noticing that the measure of the cylinder $\mathcal{N}_k$ equals the probability of seeing the event $(B_{i_1}, \ldots, B_{i_k})$ given an initial distribution π^0 on the modes if the chain is irreducible and positive recurrent. We can informally rephrase the last sentences as follows: the measure of a cylinder is the probability of seeing a sequence given an initial distribution. Finally, we remark that we can choose $\pi^0 = \pi^*$, the unique stationary distribution of the irreducible and positively recurrent chain, which tells us that the probability of being in mode i is constant for all times since $\pi^{*\top} = P\pi^{*\top}$.

Corollary 2.4.4 (Markov Jump Linear Systems)**.** Let $P = (p_{ij})$ be the $m \times m$ transition matrix of an irreducible and aperiodic discrete-time discrete-state Markov chain, that represents the switching of the modes $B_i \in \mathbf{B}$. Let $\pi^{*\top} \in \mathbb{R}^d$ be the Markov chain's stationary distribution. Let $\mu^* : \mathscr{B} \to [0,1]$ be the probability measure induced by π^* and P. Then, the set of regular realizations with respect to μ^* has full probability.

Proof. Let $\mathcal{N}_k = \left\{(A_{n+1})_{n\in\mathbb{N}} \in \mathbf{B}^{\mathbb{N}} : A_1 = B_{i_1}, \ldots, A_k = B_{i_k}\right\}$ be a cylinder of rank k and let $f((A_n)_{n\in\mathbb{N}}) = (A_{n+1})_{n\in\mathbb{N}}$ be the shift. Notice that the probability measure induced by π^* is ergodic under the shift f, see e.g Chapter 1 of [46] or Section 24 of [44]. Because $\#\{B_1, \ldots, B_m\} = m$, we have that $\log^+ \|L\| \in L^1(\mu^*)$. Therefore, we can apply Oseledets' Theorem and get that the set of regular realizations C of a Markov Jump Linear System with an irreducible and aperiodic probability transition matrix has probability 1 under μ. Furthermore, because of the ergodicity, the Lyapunov exponents with multiplicity are constant, i.e. have the same value for any realization, in the set C. □

Remark 2.4.5. Assume that our initial distribution π^0 on the modes is arbitrary, i.e., the distribution might differ from π^*. Since the distribution $\pi^0 P^n$ converges to the stationary distribution π^* as n goes to infinity, we expect the previous result to still hold. Our goal is to prove that the measure μ^n induced by $\pi^n = \pi^0 P^n$ and P converges to the stationary measure μ^* induced by π^* and P on the set of regular realizations C. We prove that μ^n converges to μ^* in the total variation distance, i.e., in the distance defined by $\|\mu^n - \mu^*\| = \sup_{\mathcal{B}\in\mathscr{B}} \|\mu^n(\mathcal{B}) - \mu^*(\mathcal{B})\|$. Noticing that for each cylinder of the form $\mathcal{N}_k = \left[(A_n)_{n\in\mathbb{N}} \in \mathbf{B}^{\mathbb{N}} : A_1 = B_{i_1}, \ldots, A_k = B_{i_k} \right]$ of rank k, we have

$$\|\mu_n(\mathcal{N}_k) - \mu^*(\mathcal{N}_k)\| = \left\|\pi^n_{i_1} - \pi^*_{i_1}\right\| p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \leq \left\|\pi^n_{i_1} - \pi^*_{i_1}\right\|.$$

Therefore, as n goes to infinity, μ^n converges to μ^* on the cylinders and, consequently, on any measurable set. Thus, the fact that $\lim_{n\to\infty} \|\mu_n - \mu^*\| = 0$. In particular, $\mu^n(C) \to 1$ leads to the conclusion that, with probability 1, our realizations will be regular.

Corollary 2.4.4 answers the question we posed at the beginning of this subsection, i.e., the set of regular sequences $(B_{i_n})_{n\in\mathbb{N}}$ has probability 1. Now, we revisit Example 2.1.1 and analyze the average data-rate needed for Algorithm 2.3.1 to work in that case.

Example 2.4.6 (Example 2.1.1 revisited.)**.** Corollary 2.4.4 tells us that, with probability one, the realizations of the system presented in Example 2.1.1 are regular. Thus, we conclude that the upper bound (2.10), in Example 2.2.7, is an equality. Explicitly, with probability one, the estimation entropy of the system is given by

$$h_{\text{est}}(\alpha, K) = \max\left\{\frac{1}{2}\log(0.99) + \alpha, 0\right\} + \max\{\alpha, 0\} \text{ nats/sample}$$

or, equivalently,

$$h_{\text{est}}(\alpha, K) = \log_2(e)\Big(\max\left\{\frac{1}{2}\log(0.99) + \alpha, 0\right\} + \max\{\alpha, 0\}\Big) \text{ bits/sample.}$$

Hence, we can use the algorithm from Section 2.3 with the choice of bases from Subsection 2.3.2 for a randomly chosen realization of our system. This is what we do in the following simulation. The parameters chosen were $\alpha = 0.05$, $\epsilon = 0.01$, and the time horizon for our simulation was 140 time units. Further, $K = [0.5, 1.5] \times [1.5, 2.5]$, $x(0) = (1.3, 2.207)^\top$. Notice that, for this α, we get $h_{\text{est}}(0.05, K) \approx 0.137$ bits/sample.
In Figure 2.1, we see the simulation results for our algorithm using different block lengths. We depict the result corresponding to the block length $\ell = 1$ in blue, to $\ell = 3$ in red, and to $\ell = 5$ in yellow. We can see that the error is upper bounded by the purple curve $\epsilon e^{-\alpha t}/2$ for all values of ℓ. Further, the empirical average data-rate, i.e., $\frac{1}{t\ell}\sum_{j=1}^{t} \log\left(\mathcal{C}_i^j\right)$, is portrayed in Figure

2.2, which shows that the data-rate decreases with the block length, as the theory we developed previously predicted. Note, however, that the average data-rate obtained from simulation is much higher than the upper bound derived in Theorem 2.3.4. The reason why that is the case is related to the fact that the results from Theorem 2.3.4 are only asymptotic. For now, we conjecture that this might be related to the rate of convergence of the subadditive ergodic theorem for this class of problems. However, this is a topic for future research. We remark that these figures appeared in [20].

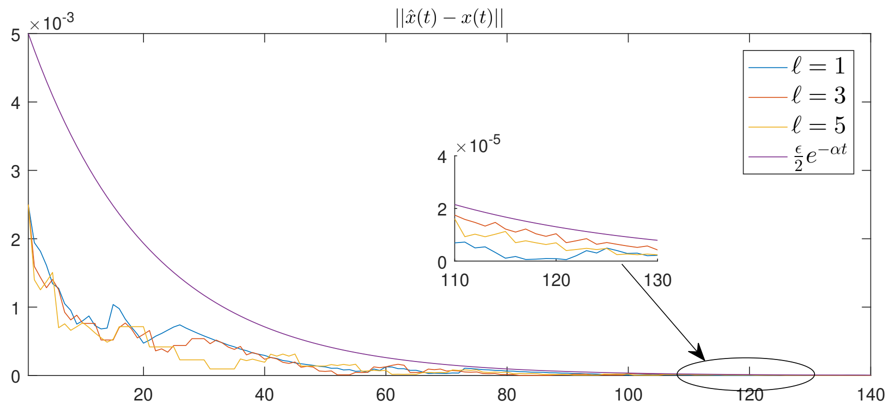

Figure 2.1: Evolution of error for several block lengths.

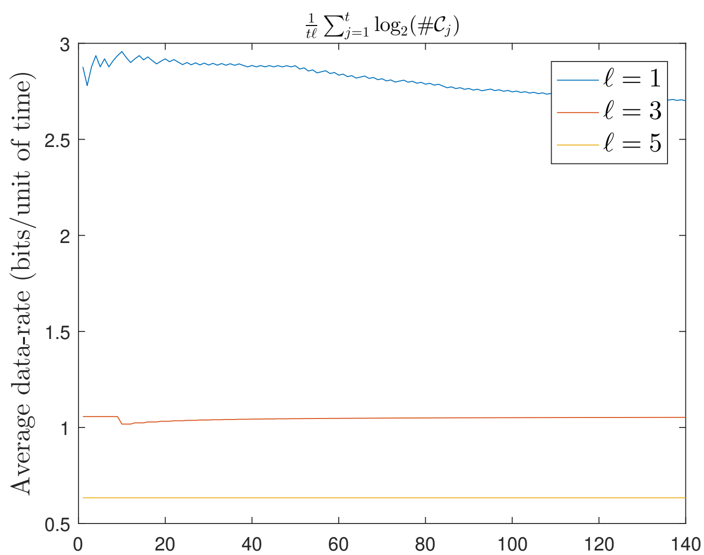

Figure 2.2: Evolution of the empirical average data-rate for several block lengths.

2.5 Conclusion

In this chapter, we studied how the concepts of Lyapunov exponents relate to the estimation entropy of a switched linear system. Also, we discussed how the geometric concept of Oseledets' filtration is associated with those notions. Further, we addressed the problem of finding a quantization scheme that operates close to the minimum average data-rate for regular switched linear systems. Furthermore, we showed how to adapt our algorithm to work close to the optimal data rate, even if the underlying system is not regular. Additionally, we showed that regular switches occur in several practical conditions including periodic switching and almost all switches that come from Markov Jump Linear Systems. Finally, we presented simulation results.

CHAPTER 3

CONTROLLABILITY FOR LINEAR TIME-VARYING SYSTEMS WITH A FINITE DATA-RATE

3.1 Chapter Overview

In the present chapter, we study controllability of linear time-varying systems that operates with finite data-rate. The motivation behind this study is that many practical systems today use computers or other digital circuits in their controller implementation. Digital circuits, by their turn, operate with sampled and quantized data. Moreover, since those circuits only have a finite number of possible output values for any given clock cycle, they must work with finite data-rate. We saw earlier in Chapter 2 that the data-rate available to our system limits what estimation problems we can solve. Similarly, the data-rate we can use limits what control problems we can solve. Interestingly, finding fundamental limitations in control systems has been a prolific endeavor in providing new insights that helped develop new controller design techniques [47]. For example, Kalman introduced the concept of controllability in the paper [48] to answer what plant dynamics' intrinsic properties impede us from designing controllers with determined properties for it. In that same paper, he showed how to construct a controller for a controllable plant that sends the system's state to zero as fast as possible, extending the work [49].

In light of this discussion, we ask a natural question: what new constraints arise from the fact that our controller must operate with a finite data-rate? The so-called data-rate theorems [4], which provide the minimum data-rates for stabilizing plants, give part of the answer. Indeed, the control over communication networks community devoted much of its attention to studying such theorems [8,9,12] since communication channels restrict the data-rate of the control laws used. Nonetheless, these theorems are not the only restrictions to finite data-rate control. Indeed, in this chapter, we prove that, in general, a finite data-rate controller can only make the system's state norm decay exponentially at the fastest. This fact shows us that the usual concept of controllability, as defined in the reference [48], is unfit for studying the problem of making the state go to the origin as quickly as possible when data-rate constraints are present. Thus, this motivates us to introduce a new controllability notion suited to this case. We do so with the help of concepts from the paper [11]. In that article, the author introduced a concept of stabilization with a finite data-rate, which, loosely speaking, is the ability to drive the state of a system to zero with a prescribed exponential rate of decay. In our work, we strengthen that notion to allow for arbitrary exponential rates of decay. This latter concept is compatible with the idea of being able to drive the state to zero as fast as possible, as we argue later.

We take this opportunity to note that the literature on conditions for stabilization with quantized control of linear time-invariant systems is extensive, e.g., the references [5, 11, 50]. Also, there exists a corresponding literature for linear time-varying (LTV) systems focused on switched linear systems [24, 51, 52]. However, most results in this literature deal with sufficient conditions for stabilizing switched linear systems, but the same result for general LTV systems is lacking. Furthermore, even in the switched case, necessary conditions for stabilization with quantized controls are missing. In view of this, another goal of this chapter is to present a necessary condition and a sufficient condition for controllability with quantized controls and finite data-rate for LTV systems. With this, we hope to lessen the gap on the literature we mentioned above. We also note that the results from this chapter can be found in a slightly different form in the work [53].

The structure of the present chapter is as follows: first, in Section 3.1, we introduce the motivation and notations. Next, in Section 3.2, we describe the problem and needed concepts. Further, we introduce the concept of controllability with finite data-rate and discuss why this concept is natural. Then, in Section 3.3, we state some necessary results, recall the concept of complete controllability, and define persistent complete controllability. After that, in Subsection 3.3.1, we prove that persistent complete controllability and another condition, the exponential energy-growth condition, are sufficient for an LTV system to be controllable in the sense we defined. Furthermore, in Subsection 3.3.2, we prove that complete controllability is a necessary condition for an LTV system to be controllable with finite data-rate. Finally, in Section 3.4, we conclude the chapter and present some future research directions.

Notations: We denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of the positive (nonnegative) integers. We denote by $\mathbb{R}$ ($\mathbb{R}_{>a}$) the set of real numbers (larger than $a \in \mathbb{R}$). Given $n \in \mathbb{Z}_{>0}$, we denote $[n] \coloneqq \{1, \ldots, n\}$. Given a set S, we denote by $\#S$ its cardinality. Let $\mathcal{M}^d$ be the set of $d \times d$ real matrices. We denote the transpose of an element $A \in \mathcal{M}^d$ by A'. For every $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote by $|x| \coloneqq \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$ the Euclidean norm. Also, if A is a $d \times d$ real matrix we denote by $\|A\| \coloneqq \max\{|Ax| : |x| = 1, x \in \mathbb{R}^d\}$ the induced norm. For a matrix $A \in \mathcal{M}^d$, we denote by $\mathcal{N}(A)$ its null space. We denote by $L^{\infty}_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$ the set of all integrable locally essentially bounded functions from $[t_0, \infty)$ to $\mathbb{R}^m$ where $t_0 \in \mathbb{R}_{\geq 0}$ and $m \in \mathbb{Z}_{>0}$, i.e., the set of integrable functions $u(\cdot)$ such that for every compact set $L \subset [t_0, \infty)$, we have that $u(L) \subset \mathbb{R}^m$ is bounded. Also, we denote by $L^2([a, b], \mathbb{R}^m)$ the set of square-integrable functions on the interval $[a, b] \subset \mathbb{R}$ with image on $\mathbb{R}^m$. Let $u : A \to B$ and let $C \subset A$, then we denote by $u_{|C} : C \to B$ the restriction of the function u to the subset C of the domain A. Finally, we denote by $B(x, r) \subset \mathbb{R}^d$ the open ball of radius $r \in \mathbb{R}_{>0}$ and center $x \in \mathbb{R}^d$.

3.2 Preliminaries

In this section, we motivate the study of controllability of linear time-varying systems with finite data-rate. We first state some necessary definitions. Next, we provide a definition of controllability that makes sense when our controller operates with a finite data-rate. Finally, we motivate the study of our controllability notion through an example.

Our primary goal is to study the controllability with quantized controls and finite data-rate of systems described by equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3.1}$$

where the initial state is given by $x(t_0) = x_0 \in K \subset \mathbb{R}^d$ with K compact with nonempty interior, the initial time is given by $t_0 \in \mathbb{R}_{\geq 0}$, time is such that $t \in \mathbb{R}_{\geq t_0}$, $A(t)$ is a $d \times d$ real matrix, $B(t)$ is a $d \times m$ real matrix, and

$u(t) \in \mathbb{R}^m$. Also, we assume that the functions $A(\cdot)$ and $B(\cdot)$ are bounded[1] and piecewise-continuous on $\mathbb{R}_{\geq t_0}$. Further, we define by $\Phi(t,\tau)$ for $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the *state-transition matrix* associated with the unforced response of system (3.1). Furthermore, we assume that $u(\cdot) \in L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^m)$.

Now, our objective is to define controllability with finite data-rate. Our next definition borrows concepts and definitions from the article [11]. We name some sets and properties not named in [11] to improve readability in later discussions. However, these concepts were first introduced in [11].

Definition 3.2.1. We say that system (3.1) satisfies the *exponential decay condition* with rate $\mu \in \mathbb{R}_{>0}$, with $M \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$ if for each $x_0 \in K \subset \mathbb{R}^d$ there exists $u(\cdot) \in L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^m)$ such that

$$|x(t)| \leq \big(M|x_0| + \epsilon\big)e^{-\mu(t-t_0)} \tag{3.2}$$

for all $t \in \mathbb{R}_{\geq t_0}$. For given $\mu \in \mathbb{R}_{>0}$, $M \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and $K \subset \mathbb{R}^d$ as above, we call the set $\mathscr{R}(\epsilon,M,K,\mu) \subset L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^m)$ a *stabilizing control set* of system (3.1) if for every $x_0 \in K$, there exists a control function $u(\cdot) \in \mathscr{R}(\epsilon,M,K,\mu)$ such that (3.2) holds. Furthermore, we denote by

$$\mathscr{R}_T(\epsilon,M,K,\mu) := \{u_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0,T],\mathbb{R}^m) : u(\cdot) \in \mathscr{R}(\epsilon,M,K,\mu)\}$$

a *set of restrictions of stabilizing controls*, where $T > t_0$ is arbitrary. Moreover, we define the *data-rate* associated with system (3.1) in the following manner. First, given a stabilizing control set $\mathscr{R}(\epsilon,M,K,\mu)$, we define the quantity

$$b(\mathscr{R}(\epsilon,M,K,\mu)) := \limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{R}_T(\epsilon,M,K,\mu)).$$

Next, we define the data-rate as[2]

$$\begin{aligned} b(M,\mu) := \lim_{\epsilon\to 0}\inf\{b(\mathscr{R}(\epsilon,M,K,\mu)) : \mathscr{R}(\epsilon,M,K,\mu) \\ \text{is a stabilizing control set of (3.1)}\}. \end{aligned}$$

Finally, we say that system (3.1) can be *stabilized with finite data-rate* with

[1] That means that $A(\mathbb{R}_{\geq t_0})$ and $B(\mathbb{R}_{\geq t_0})$ are bounded subsets of $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively.

[2] Note that $b(M,\mu)$ also depends on the set of initial conditions K. We drop that dependence to make the notation simpler.

$M \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathbb{R}_{\geq 0}$ if $b(M, \mu) < \infty$.

We analyze this definition thoroughly, including the role of ϵ, in the next Chapter. To continue our discussion, we recall the usual definition of controllability for LTV systems. See, e.g., Chapter 9 of [54].

Definition 3.2.2. We say that system (3.1) is *controllable in the usual sense* on $[t_0, T]$, where $T \geq t_0$, if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ there exists a function $u : [t_0, T] \to \mathbb{R}^m$ such that $x(T) = 0$.

Now, we are ready to define controllability with finite data-rate.

Definition 3.2.3. We say that system (3.1) is *controllable with finite data-rate* if for every $\mu \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{R}_{\geq 0}$ such that system (3.1) is stabilizable with finite data-rate $b(M, \mu) < \infty$.

We remark that the previous definition was first stated in the author's paper [53]. We further note that this definition differs from the one given in reference [11] for stabilization with finite data-rate, in the sense that, in our case, $\mu \in \mathbb{R}_{>0}$ is arbitrary. The reader might wonder why we need a new definition of controllability for the case where the data-rate is finite. We answer this question in Section 4.2 from Chapter 4 once we have more tools.

Before we continue our discussion, we recall the definition of controllability Gramian.

Definition 3.2.4 (Chapter 6 of [55])**.** Consider the system given by Equation (3.1). We define the *controllability Gramian* from t_0 to t of system (3.1) as $W(t, t_0) \coloneqq \int_{t_0}^{t} \Phi(t, \tau)B(\tau)B'(\tau)\Phi'(t, \tau)d\tau$.

We naturally ask if the usual controllability condition for LTV systems, based on the invertibility of the controllability Gramian, implies that system (3.1) is controllable with finite data-rate. The following Example 3.2.1 shows that the answer is negative.

Example 3.2.1. Consider the LTV system (3.1) in the specific case when $A(t) = I_d$ and $B(t) = (1, 0)$ for $0 < t < 1$, and $A(t) = I_d$ and $B(t) = (0, 1)$ for $t \geq 1$. Further, we assume that the initial time is $t_0 = 0$. We readily see that the Gramian $W(2, 0)$ is invertible, implying that system (3.1) is controllable in the usual sense. However, our results in Section 3.3 show that this system is not controllable with finite data-rate.

The previous example motivates the development of our theory. In the next section, we provide necessary and sufficient conditions for system (3.1) to be controllable with finite data-rate.

3.3 Controllability with Finite Data-Rate

In this section, we present this chapter's main contribution. We state and prove a sufficient and a necessary condition for LTV systems to be controllable with finite data-rate. To do that, we first introduce some new definitions and technical results.

Our first goal is to present definitions that allow us to characterize controllability with finite data-rate. Definition 3.3.1 plays an instrumental role in our theory. We use it to present our main results, namely Theorems 3.3.5 and 3.3.6. We note that it is easier to check if a system satisfies the following conditions than to check if a system is controllable with a finite data-rate directly.

Definition 3.3.1. We say that system (3.1) is *completely controllable* if there exists an increasing sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ with $s_0 = t_0$ and $s_n \to \infty$ such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$. If the sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ also satisfies[3] $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, then we say that system (3.1) is *persistently completely controllable*.

Remark 3.3.1. We take this opportunity to make some remarks. We note that Kalman defined complete controllability in the paper [56] differently from the way we did it in Definition 3.3.1. We prove the equivalence of both definitions in the Appendix. We mention, however, that the concept of persistent complete controllability is new, and the author first stated it in [53]. We further notice that there are necessary conditions and sufficient conditions for the complete controllability of LTV systems in the literature. For instance, the article [57] provides some conditions[4] for an LTV system to be completely controllable when the matrices A(t) and B(t) are differentiable

[3] This is equivalent to the statement: there exists $M \in \mathbb{R}_{>0}$ such that $\frac{s_{n+1}}{s_n} \leq M$ for all $n \in \mathbb{Z}_{\geq 0}$.

[4] We note that complete controllability and complete controllability on an interval are distinct notions.

functions of time. Finally, the quantity $s_{n+1} - s_n$ does not need to be bounded in either statement from Definition 3.3.1.

Now, we state some technical results. The proofs of all of the lemmas are in the Appendix. We start stating Lemma 3.3.2, which will be useful in the proof of Theorem 3.3.5.

Lemma 3.3.2. Let system (3.1) be persistently completely controllable. Then, there exists a sequence $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$, that $\limsup_{n \to \infty} \frac{s_{n+1}}{s_n} < \infty$, and that $\limsup_{n \to \infty} \frac{n}{s_n} < \infty$.

Before we proceed, we introduce some notation: let $\lambda^t := \sup\{\frac{1}{s} \log(\|\Phi(s, t_0)\|) : t \geq s \geq t_0\}$, $\xi := \sup\{\|A(t)\| : t \geq t_0\}$, and $\bar{\lambda} := \limsup_{t \to \infty} \lambda^t$. Now, we state Lemma 3.3.3, which collects some known results about the state transition matrix (see, e.g., Chapter 4 of [54]).

Lemma 3.3.3. Consider Equation (3.1) and let $\xi < \infty$. Then, $e^{-\xi(t-t_0)} \leq |\Phi(t, t_0)v| \leq e^{\xi(t-t_0)}$ for all $t \geq t_0$ and all $v \in \mathbb{R}^d$ with $|v| = 1$. In particular, it is also true that $\|\Phi(t, t_0)\| \leq e^{\xi(t-t_0)}$.

Since $\xi < \infty$, Lemma 3.3.3 tells us that both $\bar{\lambda}$ and λ^t are finite. Our next lemma gives a bound for $\|W^{-1}(s_n, s_{n+1})\|$ as n goes to infinity. We use this fact to prove Theorem 3.3.5.

Lemma 3.3.4. For every sequence $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ with $s_n \nearrow \infty$, the Gramian $W(s_{n+1}, s_n)$ associated with system (3.1) satisfies

$$\|W(s_{n+1}, s_n)\| \leq \sup\{\|B(t)\|^2 : t \geq t_0\} \frac{e^{2\xi(s_{n+1} - s_n)} - 1}{2\xi}.$$

Lemma 3.3.4 shows that the norm of the Gramian can only grow exponentially fast with n when $A(\cdot)$ and $B(\cdot)$ are bounded matrices.

Definition 3.3.2. Let $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ be an increasing sequence such that

$$\limsup_{n \to \infty} s_n = \infty.$$

Then, we say that system (3.1) satisfies the *exponential energy-growth condition* if there exists $\theta \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{R}_{>0}$ such that $\|W^{-1}(s_{n+1}, s_n)\| \leq N e^{\theta s_{n+1}}$.

The reader might be asking what is the rationale behind this property's name. To understand the idea behind it, we need to remember a result related to the minimum energy control of LTV systems on time intervals of the form $[s_n, s_{n+1}]$. We recall the classical result (see, e.g., Theorem 1 in Chapter 22 from [42]) that the minimum cost for any control that drives the state $x(s_n)$ at time s_n to the origin at time s_{n+1} in the $L^2([s_n, s_{n+1}], \mathbb{R}^m)$ sense, is given by $x'(s_n)W^{-1}(s_n, s_{n+1})x(s_n)$. Therefore, the exponential energy-growth condition tells us that the energy needed to drive a given state to zero over time intervals of the form $[s_n, s_{n+1}]$ cannot grow faster than an exponential as n grows to infinity. We are finally ready to state and prove our necessary and sufficient conditions for system (3.1) to be controllable with finite data-rate.

3.3.1 Sufficient Condition

In this subsection, we state and prove Theorem 3.3.5. This result is our sufficient condition for system (3.1) to be controllable with finite data-rate. We note that this theorem gives us a characterization of controllability with finite data-rate.

Theorem 3.3.5. System (3.1) is controllable with finite data-rate if it is persistently completely controllable and satisfies the exponential energy-growth condition.

Proof. Let $\{e_1, \cdots, e_d\} \subset \mathbb{R}^d$ be the canonical basis of $\mathbb{R}^d$. Pick an arbitrary $\tilde{\epsilon} \in \mathbb{R}_{>0}$ and an arbitrary $\mu \in \mathbb{R}_{>0}$. Also, let $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence that satisfies the conditions given in Definition 3.3.1 for system (3.1) to be persistently completely controllable. By Lemma 3.3.2, without loss of generality, we assume that $\limsup_{n\to\infty} \frac{n}{s_n} = Q < \infty$. Further, denote by $\alpha := 4\xi + \theta + \mu$ for simplicity. Finally, let $C = e^{\alpha(s_1 - t_0)}$, $\epsilon = \frac{\sqrt{d}(2C+1)N\sup\{\|B(t)\|^2 : t\geq t_0\}}{2\xi}\tilde{\epsilon}$, and $M = \frac{\sqrt{d}CN\sup\{\|B(t)\|^2 : t\geq t_0\}}{\xi}$.

Our proof can be divided into four parts: first, we construct a set of controls $\mathscr{U}(\epsilon, M, K, \mu)$, where each control corresponds to an initial condition in K. Second, we prove by induction that for every initial condition $x \in K$, there exists a control in $\mathscr{U}(\epsilon, M, K, \mu)$ such that $|x(s_n)| \leq C\big(|x(t_0)| + \tilde{\epsilon}\big)e^{-\alpha(s_{n+1} - t_0)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Third, we prove for any $n \in \mathbb{Z}_{\geq 0}$ and any $t \in [s_n, s_{n+1})$ we have a bound $|x(t)| \leq \big(M|x(t_0)| + \epsilon\big)e^{-\mu(t-t_0)}$, i.e., we show that $\mathscr{U}(\epsilon, M, K, \mu)$

is a stabilizing control set. Finally, we show that the data-rate $b(M,\mu)$ is finite for every possible $\mu \in \mathbb{R}_{>0}$ and our choice of $M \in \mathbb{R}_{>0}$ by proving an upper bound for $b(\mathscr{U}(\epsilon, M, K, \mu)) = \limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{U}_T(\epsilon, M, K, \mu))$ that is constant for every $\epsilon \in \mathbb{R}_{>0}$.

Part 1*:* Consider the following recursive definitions:

For $n \geq 0$ and for each $x \in K$, we define.

- For $n = 0$, define the constant function

$$\underline{\kappa}_i^0(x) := \min\{\langle x, e_i\rangle : x \in K\}$$

  and

$$\overline{\kappa}_i^0(x) := \max\{\langle x, e_i\rangle : x \in K\}$$

  for every $i \in [d]$. For $n \geq 1$, define the piecewise-constant functions

$$\underline{\kappa}_i^n(x) := \underline{\kappa}_i^{n-1}(x) + \Gamma_i^{n-1}(q_i^{n-1}(x) - 1)$$

  and

$$\overline{\kappa}_i^n(x) := \underline{\kappa}_i^{n-1}(x) + \Gamma_i^{n-1} q_i^{n-1}(x)$$

  for every $i \in [d]$;

- Define the constant $\Gamma_i^n := \frac{\tilde{\epsilon}}{d} e^{-(\lambda^{s_{n+1}}+\alpha)s_{n+1}}$ and the positive integer $\mathcal{C}_i^n := \left\{1, \ldots, \left\lceil \frac{\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x)}{\Gamma_i^n} \right\rceil \right\}$ for each $i \in \{1, \ldots, d\}$ and each $n \in \mathbb{Z}_{>0}$. Note that, by the defining equations of $\underline{\kappa}_i^n(x)$ and $\overline{\kappa}_i^n(x)$, $\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x) = \Gamma_i^{n-1}$. Thus, $\frac{\overline{\kappa}_i^n(x) - \underline{\kappa}_i^n(x)}{\Gamma_i^n} = e^{(\lambda^{s_{n+1}}+\alpha)s_{n+1} - (\lambda^{s_n}+\alpha)s_n}$ for every $i \in [d]$, every $x \in K$, and every $n \in \mathbb{Z}_{\geq 0}$.

- Define the quantized value of the $i-$th projection of the initial state into the vector space $\operatorname{span}\{e_i\}$ at time s_n by

$$q_i^n(x) := \{l \in \mathcal{C}_i^n : \underline{\kappa}_i^n(x) + \Gamma_i^n(l-1) \leq \langle x, e_i\rangle < \underline{\kappa}_i^n(x) + \Gamma_i^n l\}$$

  for each $i \in [d]$;

- Define the quantized value of the $i-$th projection of the initial state into the vector space $\operatorname{span}\{e_i\}$ at time s_n by

$$\hat{\beta}_i^n(x) := \underline{\kappa}_i^n(x) + \Gamma_i^n(q_i^n(x) - 1/2)$$

for each $i \in [d]$;

- Define the i-th projection of the initial state into the vector space $\text{span}\{e_i\}$ at time s_n by

$$\beta_i^n(x) \coloneqq \langle x, e_i \rangle;$$

- With the notation $\sum_{i=1}^{b} c_i = 0$ for any $b \in \mathbb{Z}$ such that $b < 1$. Then, define the quantity[5]

$$\hat{x}(s_n) \coloneqq \sum_{i=1}^{d} \hat{\beta}_i^n(x)\Phi(s_n, s_0)e_i + \\ + \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} \Phi(s_n, s)B(s)u(q^0(x), \cdots, q^k(x), s)ds;$$

- Define the control law in the interval $[s_n, s_{n+1})$ corresponding to the initial state x by

$$u(q^0(x), \cdots, q^n(x), t) \coloneqq -B'(t)\Phi'(s_{n+1}, t)W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$

for $t \in [s_n, s_{n+1})$ where $q^n(x) \coloneqq (q_1^n(x), \cdots, q_d^n(x))$. Further define $v(x,t) \coloneqq u(q^0(x), \cdots, q^{n-1}(x), t)$, where n is the smallest integer such that $t < s_n$. Finally, define by $\mathscr{U}(\epsilon, M, K, \mu)$ the set of all such $v(x, \cdot)$. Also, denote by $\mathscr{U}_T(\epsilon, M, K, \mu)$ the set of restrictions of controls in $\mathscr{U}(\epsilon, M, K, \mu)$ from time t_0 to T. More explicitly $\mathscr{U}_T(\epsilon, M, K, \mu) \coloneqq \{v_{|[t_0,T)}(x, \cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m) : v(x, \cdot) \in \mathscr{U}(\epsilon, M, K, \mu)\}$.

- *Part 2:*

Step 0: Trivially, we have that $|x(t_0)| \leq |x(t_0)| + \tilde{\epsilon} = C(|x(t_0)| + \tilde{\epsilon})e^{-\alpha(s_1 - t_0)}$ and we proved the base case, i.e., $|x(s_n)| \leq C(|x(t_0)| + \tilde{\epsilon})e^{-\alpha(s_{n+1} - t_0)}$ for $C \in \mathbb{R}_{>1}$ and for $n = 0$.

Step $n+1$: Recall that for each $x \in K$ and for $t \in [s_n, s_{n+1})$ the control law we defined in the first part is given by

$$u(q^0(x), \cdots, q^n(x), t) = -B'(t)\Phi'(s_{n+1}, t)W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$

[5] This can be seen as an state estimate at time s_n.

where

$$\hat{x}(s_n) = \sum_{i=1}^{d} \hat{\beta}_i^{s_{n+1}}(x)\Phi(s_n, s_0)e_i + \\ + \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} \Phi(s_n, s)B(s)u(q^0(x), \cdots, q^{k-1}(x), s)ds.$$

Now, writing down the variation of parameters formula at time s_{n+1} we get that

$$x(s_{n+1}) = \Phi(s_{n+1}, s_n)x(s_n) - \int_{s_n}^{s_{n+1}} \Phi(s_{n+1}, \tau)B(\tau)B'(\tau)\Phi'(s_{n+1}, \tau)d\tau \times \\ \times W^{-1}(s_{n+1}, s_n)\Phi(s_{n+1}, s_n)\hat{x}(s_n)$$

from which we conclude that

$$x(s_{n+1}) = \Phi(s_{n+1}, s_n)\big(x(s_n) - \hat{x}(s_n)\big) = \sum_{i=1}^{d}(\beta_i^n(x) - \hat{\beta}_i^n(x))\Phi(s_{n+1}, s_0)e_i.$$

Then, by taking the norm on both sides and applying the triangle inequality, we conclude that

$$|x(s_{n+1})| \leq \sum_{i=1}^{d} |\beta_i^n(x) - \hat{\beta}_i^n(x)||\Phi(s_{n+1}, s_0)e_i|.$$

Now, by the definition of λ^t,[6] we get that $|\Phi(s_{n+1}, s_0)e_i| \leq e^{\lambda^{s_{n+1}} s_{n+1}}$ for all $i \in [d]$. Further, by recalling the expression of Γ_i^n and by the definitions of $\hat{\beta}_i^n$, β_i^n and $q_i^n(x)$, we conclude that $|\beta_i^n(x) - \hat{\beta}_i^n(x)| \leq \frac{\tilde{\epsilon}}{d} e^{-(\lambda^{s_{n+1}} + \alpha)s_{n+1}}$. Hence, we get that

$$|x(s_{n+1})| \leq \sum_{i=1}^{d} \frac{\tilde{\epsilon}}{d} e^{-\alpha s_{n+1}} = \tilde{\epsilon} e^{-\alpha s_{n+1}}.$$

Therefore, $|x(s_{n+1})| \leq \tilde{\epsilon} e^{-\alpha s_{n+1}} \leq C\big(|x(t_0)| + \tilde{\epsilon}\big)e^{-\alpha(s_{n+1} - t_0)}$ and we proved the case for step $n+1$.

- *Part* 3:

Now, pick any $n \in \mathbb{Z}_{\geq 0}$ and any $t \in [s_n, s_{n+1})$. Note that the variation of

[6] Recall that $\lambda^{s_{n+1}} = \sup\{\frac{1}{t}\log(\|\Phi(t, t_0)\|) : s_{n+1} \geq t \geq t_0\}$.

parameters formula gives us that

$$x(t) = \Phi(t,s_n)x(s_n) - \int_{s_n}^{t} \Phi(t,s)B(s)B'(s)\Phi(s_{n+1},s)ds W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)\hat{x}(s_n).$$

Notice that

$$\int_{s_n}^{t} \Phi(t,s)B(s)B'(s)\Phi'(s_{n+1},s)ds = \Phi(t,s_{n+1})\int_{s_n}^{t} \Phi'(s_{n+1},s)B(s)B'(s)\Phi(s_{n+1},s)ds.$$

Next, let

$$\Omega(t,s_{n+1},s_n) \coloneqq \int_{s_n}^{t} \Phi(s_{n+1},s)B(s)B'(s)\Phi(s_{n+1},s)ds$$

and let

$$\Theta(t,s_{n+1},s_n) \coloneqq \int_{t}^{s_{n+1}} \Phi(s_{n+1},s)B(s)B'(s)\Phi(s_{n+1},s)ds.$$

Further, note that[7] $\Omega(t,s_{n+1},s_n) \succcurlyeq 0$, $\Theta(t,s_{n+1},s_n) \succcurlyeq 0$, and $W(s_{n+1},s_n) \succ 0$. Also, the definitions imply that $W(s_{n+1},s_n) = \Omega(t,s_{n+1},s_n)+\Theta(t,s_{n+1},s_n)$ for any $t \in [s_n,s_{n+1})$. The two latter facts imply that $\|\Omega(t,s_{n+1},s_n)\| \leq \sqrt{d}\|W(s_{n+1},s_n)\|$ and $\|\Theta(t,s_{n+1},s_n)\| \leq \sqrt{d}\|W(s_{n+1},s_n)\|$.
Recall the semigroup property for the transition matrix, i.e.,
$\Phi(t,z) = \Phi(t,r)\Phi(r,z)$ for any $z \geq t_0$, $t \geq t_0$ and any $z \geq t_0$. So, we get

$$\begin{aligned} x(t) &= \Phi(t,s_n)x(s_n) - \\ &\quad \Phi(t,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)\hat{x}(s_n) = \\ &\quad \Phi(t,s_n)\Big(x(s_n) - \Phi(s_n,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)\hat{x}(s)\Big). \end{aligned}$$

By rewritting $\hat{x}(s_n) = \hat{x}(s_n) - x(s_n) + x(s_n)$ and using the fact that

[7] Since $\Phi(s_{n+1},s)B(s)B'(s)\Phi'(s_{n+1},s) \succcurlyeq 0$ for all $s \in [s_n,s_{n+1})$.

$\Phi(t,r)\Phi(r,t) = I_d$ for every $t \geq t_0$ and every $r \geq t_0$, we get

$$x(t) = \Phi(t,s_n)\Big(\Phi(s_n,s_{n+1})(I - \Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n))\Phi(s_{n+1},s_n)x(s)$$
$$- \Phi(s_n,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)(\hat{x}(s_n) - x(s_n))\Big) =$$
$$\Phi(t,s_{n+1})\Theta(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)x(s)$$
$$- \Phi(t,s_{n+1})\Omega(t,s_{n+1},s_n)W^{-1}(s_{n+1},s_n)\Phi(s_{n+1},s_n)(\hat{x}(s_n) - x(s_n)).$$

Taking the norm on both sides and using the triangle inequality yields

$$|x(t)| \leq \|\Phi(t,s_{n+1})\|\|\Theta(t,s_{n+1},s_n)\|\|W^{-1}(s_{n+1},s_n)\|\times$$
$$\times\|\Phi(s_{n+1},s_n)\||x(s)| + \|\Phi(s_n,s_{n+1})\|\|\Omega(t,s_{n+1},s_n)\|\times$$
$$\times\|W^{-1}(s_{n+1},s_n)\|\|\Phi(s_{n+1},s_n)\||\hat{x}(s_n) - x(s_n)|.$$

We invoke Lemma 3.3.4 and notice that it implies that

$$\|W(s_{n+1},s_n)\| \leq \frac{\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi}e^{2\xi s_{n+1}}.$$

Then, we combine that with the fact that

$$\max\{\|\Omega(t,s_{n+1},s_n)\|, \|\Theta(t,s_{n+1},s_n)\|\} \leq \sqrt{d}\|W(s_n,s_{n+1})\|,$$

to conclude that

$$\max\{\|\Omega(t,s_{n+1},s_n)\|,\|\Theta(t,s_{n+1},s_n)\|\} \leq$$
$$\frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi}e^{(2\xi+\theta)(s_{n+1})}.$$

By the exponential energy-growth condition, we know that there exist $\theta \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{R}_{>0}$ such that $\|W^{-1}(s_{n_1},s_n)\| \leq Ne^{\theta s_{n+1}}$. So, we have that

$$|x(t)| \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi}e^{(2\xi+\theta)(s_{n+1})}\|\Phi(s_{n+1},s_n)\|\times$$
$$\times\Big(\|\Phi(t,s_{n+1})\||x(s)| + \|\Phi(s_n,s_{n+1})\||\hat{x}(s_n) - x(s_n)|\Big).$$

By Lemma 3.3.3, for any $t \in [s_n,s_{n+1})$, we get $\|\Phi(t,s_n)\| \leq e^{\xi(t-t_0)}$, which implies that

$$|x(t)| \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)s_{n+1}}\big(|x(s_n)| + |\hat{x}(s_n)|\big) \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)(s_{n+1}-t_0)}\big(|x(s_n)| + |\hat{x}(s_n)|\big),$$

where the last inequality follows from the fact that $t_0 \geq 0$. Note that $|x(s_n) - \hat{x}(s_n)| \leq \sum_{i=1}^{d} \|\Phi(s_n, s_0)e_i\| |\beta_i^n(x) - \hat{\beta}_i^n(x)| \leq \tilde{\epsilon}e^{-\alpha(s_{n+1}-t_0)}$ by the defining equations of β_i^n, $\hat{\beta}_i^n$, and $\hat{x}(s_n)$ presented in part 1 of the proof, from which we conclude that

$$|\hat{x}(s_n)| \leq \tilde{\epsilon}e^{-\alpha s_{n+1}} + |x(s_n)|.$$

So, we can write

$$|x(t)| \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)(s_{n+1}-t_0)}\big(2|x(s_n)| + \tilde{\epsilon}e^{-\alpha(s_{n+1}-t_0)}\big).$$

Thus, by the conclusion of the proof of part 2, we get

$$|x(t)| \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi} e^{(4\xi+\theta)(s_{n+1}-t_0)}\big(2C(|x(t_0)| + \tilde{\epsilon}) + \tilde{\epsilon}\big)e^{-\alpha(s_{n+1}-t_0)} \leq \frac{\sqrt{d}N\sup\{\|B(t)\|^2 : t \geq t_0\}}{2\xi}\big(2C|x(t_0)| + (2C+1)\tilde{\epsilon}\big)e^{-\mu(s_{n+1}-t_0)}.$$

Since $\alpha = (4\xi + \theta + \mu)$. Finally, recall that $\epsilon = \frac{\sqrt{d}(2C+1)N\sup\{\|B(t)\|^2 : t\geq t_0\}}{2\xi}\tilde{\epsilon}$ and $M = \frac{\sqrt{d}CN\sup\{\|B(t)\|^2 : t \geq t_0\}}{\xi}$. Hence, we conclude that

$$|x(t)| \leq (M|x(t_0)| + \epsilon)e^{-\mu(s_{n+1}-t_0)} \leq (M|x(t_0)| + \epsilon)e^{-\mu(t-t_0)}$$

for all $t \geq t_0$. Therefore, we proved that $\mathscr{U}_T(\epsilon, M, K, \mu)$ is a stabilizing control set, concluding the proof of part 3.

- *Part 4*: Note that there is a bijection between the elements of $\prod_{j=0}^{n}\prod_{i=1}^{d}\mathcal{C}_i^j$ and those of $\mathscr{U}_T(\epsilon, M, K, \mu)$ by the definition of $v(x,t)$. So, $\#\mathscr{U}_T(\epsilon, M, K, \mu) = \prod_{j=0}^{n}\prod_{i=1}^{d}\#\mathcal{C}_i^j$. Also, by the same equations, we have that $\#\mathscr{U}_T(\epsilon, M, K, \mu)$ is constant for $T \in [s_n, s_{n+1})$ for each $n \in \mathbb{Z}_{\geq 0}$. Thus,

$$\frac{1}{T}\log\big(\#\mathscr{U}_T(\epsilon, M, K, \mu)\big) \leq \frac{1}{s_n}\log\big(\#\mathscr{U}_T(\epsilon, M, K, \mu)\big)$$

for $T \in [s_n, s_{n+1})$. Also, note that

$$\#\mathcal{C}_i^n = \left\lceil e^{(\lambda^{s_{n+1}}+\alpha)s_{n+1}-(\lambda^{s_n}+\alpha)s_n} \right\rceil$$

for every $i \in [d]$ and $n \in \mathbb{Z}_{\geq 1}$. Therefore,

$$\log\left(\prod_{j=1}^{n}\prod_{i=1}^{d}\mathcal{C}_i^j\right) \leq d\Big((\lambda^{s_{n+1}}+\alpha)s_{n+1} - (\lambda^{s_1}+\alpha)s_1 + n\Big),$$

where the inequality comes from the facts that $\log(\lceil e^y \rceil) \leq y+1$ for $y \in \mathbb{R}_{\geq 1}$ and from the property of telescoping series. Combining our previous results, we arrive at $\frac{1}{T}\log(\#\mathscr{U}_T(\epsilon, M, K, \mu)) \leq \frac{d}{s_n}\Big((\lambda^{s_{n+1}}+\alpha)s_{n+1} - (\lambda^{s_1} + \alpha)s_1 + n\Big) + \frac{\sum_{i=1}^{d}\log\left(\#\mathcal{C}_i^0\right)}{s_n}$. Taking the limit superior on the left hand side with T going to infinity implies that we are taking the limit superior on the right hand sided with n going to infinity because $n = \inf\{l \in \mathbb{Z}_{\geq 0} : s_l \leq T \text{ and } s_{l+1} > T\}$. Hence, we get $\limsup_{T\to\infty}\frac{1}{T}\log(\#\mathscr{U}_T(\epsilon, M, K, \mu)) \leq \limsup_{n\to\infty}\frac{d(\lambda^{s_{n+1}}+\alpha)s_{n+1}}{s_n} + \frac{n}{s_n} \leq d(\bar{\lambda}+\alpha)R + Q$. The first inequality follows from the fact that $\sum_{i=1}^{d}\log(\#\mathcal{C}_i^0)$ and $(\lambda^{s_1}+\alpha)s_1$ are finite. The last inequality follows because $\limsup_{n\to\infty}\frac{n}{s_n} = Q$ and because given two sequences of positive numbers $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ and $(b_n)_{\mathbb{Z}_{\geq 0}}$, then $\limsup_{n\to\infty} a_n b_n \leq \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n$ and we have that

$$\limsup_{n\to\infty} \lambda^{s_{n+1}} = \bar{\lambda}$$

and $\limsup_{n\to\infty}\frac{s_{n+1}}{s_n} = R$ by persistent complete controllability. Since our bound does not depend on ϵ, we have that the previous inequality gives an upper bound for $b(M, \mu)$. In this way, we proved that

$$b(M,\mu) < \lim_{\epsilon\to 0} b(\mathscr{U}(\epsilon, M, K, \mu)) < d(\bar{\lambda}+\alpha)R + Q < \infty$$

for every $\mu \in \mathbb{R}_{>0}$ and our chosen M. Thus, we conclude the proof of the theorem.

In the next chapter, we will see that controllable LTI systems are controllable with a finite data-rate. In the following subsection, we state our necessary condition for controllability with finite data-rate. Then, we use it to show why the system from Example 3.2.1 is not controllable with finite data-rate.

3.3.2 Necessary Condition

In this subsection, we state a necessary condition for system (3.1) to be controllable with finite data-rate. We prove this theorem in the Appendix A.6.

Theorem 3.3.6. System (3.1) is controllable with finite data-rate only if it is completely controllable.

Remark 3.3.7. We now analyze the gap between the hypotheses of the necessary condition and those of the sufficient condition. More explicitly, the sufficient condition requires the system to satisfy the exponential energy-growth rate and the persistency of complete controllability in addition to the necessary condition's assumptions. We examine the role of each of those two conditions separately: first, we note that the exponential energy-growth rate is only used in part three of the proof of Theorem 3.3.5. Its purposed is to bound the growth of the state between times s_n and s_{n+1} for $n \in \mathbb{Z}_{\geq 0}$. We do not know at this time if this condition is necessary or if it is a consequence of our choice of stabilizing control set $\mathscr{U}(\epsilon, M, K, \mu)$ in the proof of Theorem 3.3.5. Assuming that our system satisfies the exponential energy-growth rate is reasonable, however. This latter fact is true because, in practice, we require the control energy to remain bounded, a stronger assumption. Second, we use the persistency of the complete controllability in the last part of the proof of Theorem 3.3.5 to bound the data-rate. Nonetheless, at the moment, it is not clear if we can remove it from the statement of Theorem 3.3.5.

We now have tools to understand why Example 3.2.1 cannot be controllable with finite data-rate. Given an arbitrary increasing sequence (s_n) with $\lim_{n\to\infty} s_n = \infty$, we know that there exists $n_0 \in \mathbb{Z}_{\geq 0}$ such that $s_n > 1$ for all $n \geq n_0$. Thus, the matrices $W(s_{n+1}, s_n)$ are not invertible for each $n \geq n_0$. This argument shows that that system is not controllable with finite data-rate.

3.4 Conclusion

In this chapter, we discussed the problem of controlling LTV systems when our controller operates with a finite data-rate. We motivated our study by showing that systems that are controllable in the usual sense might not be controllable when our controller needs to operate with a finite the data-rate. Then, we introduced a definition of controllability with finite data-rate that extends the usual notion of arbitrary pole placement, characteristic of LTV systems that are controllable in the usual sense. After that, we introduced additional concepts to help us characterize when our system is controllable with a finite data-rate. Namely, we introduced the notions of persistent complete controllability and exponential energy-growth rate. Finally, we gave a necessary and a sufficient condition for an LTV system to be controllable when the data-rate is finite in terms of the latter mentioned notions.

CHAPTER 4

CONTROLLABILITY AND CONTROLLER DESIGN FOR SWITCHED LINEAR SYSTEMS WITH A FINITE DATA-RATE

4.1 Chapter Overview

In this Chapter, we continue the discussion we started in Chapter 3 about controllability with a finite data-rate of linear time-varying systems by specializing to switched linear systems [58]. For this case, we provide some sufficient conditions for our system to be controllable with a finite data-rate. The first of them is just a special case of the sufficient condition provided in the previous chapter. The second of them has a constructive proof and provides an algorithm for designing a controller. Additionally, this controller can handle packet losses naturally. We illustrate this condition and algorithm with a practical example. We consider a switched linear system with controllable modes that we control over a communication network. However, we assume that that communication channel is under a Denial-of-Service (DoS) attack, i.e., the transmitted symbols, which carry information about the quantized measurements of the plant state, may not always arrive at the sampling times because of an attack. For this example, our task is to design a switching signal and a controller. Further, we present a necessary condition for our controllability notion to hold, although there is a gap between this and the sufficient condition we provide. This latter fact is the topic of future research.

We take this opportunity to connect this chapter with Chapter 3, where we discussed controllability with a finite data-rate for linear time-varying (LTV) systems. In that chapter, we presented a necessary condition and a sufficient condition for a general LTV system to be controllable with a finite data-rate. There, we gave a different sufficient condition for controllability that works for more general LTV systems than the class of switched linear ones. In the present chapter, we use the switched linear structure to derive a different sufficient condition, which helps us associate controllability with a finite data-rate of the switched system with the controllable subspaces of each mode and the switching signal. Also, the switched structure appears in our controller design technique. Further, we also consider the possibility of packet losses, which we did not in our previous chapter. This latter fact allows us to discuss some applications, such as the DoS attacks mentioned above. Furthermore, we give a more in-depth explanation to concepts that we only explained briefly in Chapter 3.

The structure of this chapter is as follows: First, in Section 4.2, we pose our problem and describe why we need a new controllability notion. We also introduce the practically motivated example we want to study. Still in Section 4.2, we define controllability with a finite data-rate. Next, in Section 4.3, we state our necessary condition for controllability with a finite data-rate. Also in that section, we introduce some new concepts needed for our sufficient condition, illustrate them with figures and examples, and we state our sufficient condition. We finish Section 4.3 with a disscussion on the gap between the necessary condition and the sufficient one. After that, in Section 4.4, we describe our control algorithm and prove its correctness. Then, we use that same algorithm to prove the sufficient condition. In Section 3.4, we conclude and present future research directions.

Notation: We denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of the positive (nonnegative) integers. We denote by $\mathbb{R}$ the set of real numbers. We denote by $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) the set of positive (nonnegative) real numbers. Given $m \in \mathbb{Z}_{>0}$, we define the set $[m] := \{1, \ldots, m\}$ and $[m]_0 := [m] \cup \{0\}$. Given two integers a and b, we denote by (a mod b) the remainder of the division of a by b. Given $m \in \mathbb{Z}_{>0}$, we define the set $[m] := \{1, \ldots, m\}$. Given a set S, we denote by $\#S$ its cardinality. For a set $S \subset \mathbb{Z}_{\geq 0}$, we denote by S^c its complement in the set $\mathbb{Z}_{\geq 0}$. Let $S \subset \mathbb{R}^{d_x}$ and $\varepsilon \in \mathbb{R}_{\geq 0}$, we define the *$\varepsilon$-collar* of S by[1] $S^\varepsilon := \{x \in \mathbb{R}^{d_x} : \inf\{|x - y| : y \in S\} \leq \varepsilon\}$. Let $d_x \in \mathbb{Z}_{>0}$ and $d_u \in \mathbb{Z}_{>0}$, we denote by $\mathcal{M}^{d_x \times d_u}$ the set of $d_x \times d_u$ real matrices. Let $d_x \in \mathbb{Z}_{>0}$, then we denote by I_{d_x} the $d_x \times d_x$ identity matrix. We denote the transpose of a matrix $A \in \mathcal{M}^{d_x \times d_u}$ by $A' \in \mathcal{M}^{d_u \times d_x}$. Given a pair of matrices (A, B) with $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_u}$, we denote by $\langle A|B \rangle$ their controllable subspace. Given $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_x}$ two symmetric positive semi-definite matrices, we write that

[1]Although, when ε is a positive integer, the notation S^ε might cause confusion with the Cartesian product of S, we make it clear in the text when we are using a collar.

$A \geq B$ ($A > B$) if $A - B$ is positive semidefinite (definite). For $k \in [d_x]$ and $A \in \mathcal{M}^{k\times k}$, we define the matrix measure of A as $\mu(A) \coloneqq \lim_{\delta\downarrow 0} \frac{\|I_k+\delta A\|-1}{\delta}$. We denote by $B(x, r) \subset \mathbb{R}^{d_x}$ ($B[x, r] \subset \mathbb{R}^{d_x}$) the open (closed) ball of radius $r \in \mathbb{R}_{>0}$ and center $x \in \mathbb{R}^{d_x}$. For a matrix $A \in \mathcal{M}^{d_x\times d_x}$, we denote by $\mathcal{N}(A)$ its null space. If A is a $d_x \times d_x$ real matrix and $|\cdot|$ is a vector norm[2] in $\mathbb{R}^{d_x}$, we denote by $\|A\| \coloneqq \max\{|Ax| : |x| = 1, x \in \mathbb{R}^{d_x}\}$ the norm induced by that vector norm. For a set $S \subset \mathbb{R}^{d_x}$, we define its maximum distance from the origin as $\text{dist}(S) \coloneqq \sup\{|x| : x \in S\}$. We denote by $\log(a)$ the natural logarithm of $a \in \mathbb{R}_{>0}$. We denote by $L^{\infty}_{\text{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ the set of all Lebesgue integrable (see, e.g., Chapter 2 of [59]) locally essentially bounded functions from $[t_0, \infty)$ to $\mathbb{R}^{d_u}$ where $t_0 \in \mathbb{R}_{\geq 0}$ and $d_u \in \mathbb{Z}_{>0}$. Finally, given a function $u : I \subset \mathbb{R} \to \mathbb{R}^{d_u}$ and a set $J \subset I$, we denote by $u_{|J}(\cdot)$ the restriction of the function $u(\cdot)$ to the subset J.

4.2 Motivation

We start this section by describing the class of systems we study in most of the paper. Then, we explain why the usual notion of controllability is unfit to deal with systems that have quantization and data-rate constraints. Further, we present a new notion of controllability with finite data-rate, which addresses the issues that appear in our setting. Next, we introduce some definitions and preliminary results. Finally, we provide a partial characterization of our controllability notion.

4.2.1 The Model

In this work, we study the controllability with a finite data-rate of switched linear systems, i.e., systems described by equation

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{4.1}$$

where the current time is $t \in [t_0, \infty)$, the initial time is $t_0 \in \mathbb{R}_{\geq 0}$, the initial state is $x(t_0) = x_0$ and it belongs to a compact set with nonempty interior $K \subset \mathbb{R}^{d_x}$, $m \in \mathbb{Z}_{>0}$ is the number of modes, $\sigma : [t_0, \infty) \to [m]$ is the switching

[2] If not stated otherwise, we assume that $|\cdot|$ is the Euclidean norm.

signal, $u : [t_0, \infty) \to \mathbb{R}^{d_u}$ is the control function, and $A_p \in \mathcal{M}^{d_x \times d_x}$ and $B_p \in \mathcal{M}^{d_x \times d_u}$ are the matrices of each mode $p \in [m]$. We also assume that $u(\cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ and that $\sigma(\cdot)$ is a càdlàg function[3]. We denote by[4] t_n the n-th discontinuity point of $\sigma(\cdot)$ and we call such points the *switching times*. Finally, we define by $\Phi_\sigma(t, \tau)$ for $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the *state-transition matrix* associated with the autonomous part of system (4.1), i.e., $\Phi_\sigma(t, \tau)$ is the unique solution to the differential equation $\dot{\Phi}_\sigma(t, \tau) = A_{\sigma(t)}\Phi_\sigma(t, \tau)$ with $\Phi_\sigma(\tau, \tau) = I_{d_x}$.

A *control law* is a set $\mathscr{U}(K)$ of functions $u(x, \cdot)$ indexed by initial conditions $x \in K \subset \mathbb{R}^{d_x}$, i.e., each initial state $x \in K$ corresponds to a unique control $u(x, \cdot) \in \mathscr{U}(K)$. Denote by $\mathscr{U}_T(K) \coloneqq \{v_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0, T], \mathbb{R}^{d_u}) : v(\cdot) \in \mathscr{U}(K)\}$ the set of restrictions of functions from our control law. We define the *data-rate of the control law* $\mathscr{U}(K)$ as

$$b(\mathscr{U}(K)) \coloneqq \limsup_{T\to\infty} \frac{1}{T} \log(\#\mathscr{U}_T(K))$$

and we say that the control law $\mathscr{U}(K)$ *operates with a finite data-rate* if it satisfies $b(\mathscr{U}(K)) < \infty$. We can, alternatively, look at the control law as a function with two parameters $u(\cdot, \cdot)$, where the first parameter is the initial state and the second is the time. However, looking at the control law as a set allows us to define the data-rate more naturally.

4.2.2 The Need for a New Controllability Notion

In this subsection, we explain why the usual notion of controllability of LTV systems is not suitable when we consider control systems that use quantization and that operate with finite data-rate.

To do that, we start by recalling the usual controllability notion (see, e.g., Chapter 9 of [54].) for LTV systems.

Definition 4.2.1. We say that system (4.1) is *controllable in the usual sense* on $[t_0, T]$, where $T \geq t_0$, if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^{d_x}$ there exists a function $u : [t_0, T] \to \mathbb{R}^{d_u}$ such that $x(T) = 0$.

[3] A function that is right-continuous and has a left limit everywhere.

[4] We consider t_0 the 0-th discontinuity point of $\sigma(\cdot)$ to keep the notation simple.

To see why this notion is unfit when we work with finite data-rate, we consider the following simple Example 4.2.1.

Example 4.2.1. Let $\dot{x}(t) = u(t)$ where $t \in \mathbb{R}$, $x_0 \in K \subset \mathbb{R}$ with K compact with a nonempty interior and $u(t) \in \mathbb{R}^{d_u}$. We can easily solve this equation to get that $x(T) = x_0 + \int_{t_0}^{T} u(\tau)d\tau$. Note that, if $u(t) \in \mathbb{R}^{d_u}$, this system is controllable in the usual sense on the interval $[t_0, T]$. If we impose that this control function comes from a control law that operates with finite data-rate, we have that the set of possible controls $u_{[t_0,T]}(\cdot)$ on any interval of time $t \in [t_0, T]$ has a finite cardinality. Therefore, the integral $\int_{t_0}^{T} u(\tau)d\tau$ attains at most finitely many values, but x_0 belongs to the set K, which has infinitely many points. Hence, it is not possible to make $x(T) = 0$ for an arbitrary initial condition in K.

The goal of the previous example is to make the straightforward observation that we cannot have $x(T) = 0$ for an arbitrary initial condition in K, which supports the claim that the usual controllability notion is unfit for the case where we have finite data-rate. Thus, we must define a new notion of controllability in this setting. One way of doing so is to think of controllability as the property of being able to drive the state as fast as possible to the origin. The following Proposition 4.2.2 shows that, in general, the fastest mode of decay for the norm of the state of system (4.1) using finite data-rate is exponentially fast. Indeed, a stronger claim is true for a much larger class of systems. We provide a proof of this proposition in Appendix A.5.

Proposition 4.2.2. Let the set of possible initial states $K \subset \mathbb{R}^{d_x}$ have a nonempty interior, let $m \in \mathbb{Z}_{>0}$ be the number of modes, and let $t_0 \in \mathbb{R}_{\geq 0}$ be the initial time. Consider the switched nonlinear time-varying dynamics given by

$$\dot{x}(t) = f(t, \sigma(t), u(x_0, t), x(t)), \tag{4.2}$$

where $x(t_0) = x_0 \in K$ is a initial state, $u(x_0, \cdot) \in \mathscr{U}(K)$ is the control function that corresponds to the initial state x_0, $\mathscr{U}(K)$ is a control law that operates with a finite data-rate, $\sigma : [t_0, \infty) \to [m]$ is a càdlàg switching signal, and $f : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$. Also, define $\mathcal{R}_u \coloneqq \overline{\{u(x, t) \in \mathbb{R}^{d_u} : (x, t) \in P \text{ and } u(x, \cdot) \in \mathscr{U}(K)\}}$, where[5] $P \coloneqq \{(x, t) \in K \times [t_0, \infty) : |u(x, t)| < \infty \text{ for } u(x, \cdot) \in \mathscr{U}(K)\}$. We assume that:

[5]Note that $(K \times [0, \infty)) \setminus P$ has measure zero since $u(\cdot, \cdot)$ is locally essentially bounded.

- Equation (4.2) has a unique forward-complete[6] Caratheodory solution for each initial state $x_0 \in K$ and the initial time t_0. We denote by[7] $\xi(t, t_0, x_0)$ the Caratheodory solution of (4.2) at time t when the initial time is t_0 and the initial state is x_0.

- There exists a compact set $B_x \subset \mathbb{R}^{d_x}$ such that[8] $\{|\xi(t, t_0, x_0)| : x_0 \in K, t \in [t_0, \infty)\} \subset B_x$.

- $f(\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable in its fourth argument. Define the Jacobian of $f(\cdot,\cdot,\cdot,\cdot)$ in its fourth argument as $f_x : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ where $(f_x(\cdot,\cdot,\cdot,\cdot))_{(i,j)} := \frac{\partial f_i}{\partial x_j}(\cdot,\cdot,\cdot,\cdot)$ for each pair $(i, j) \in [d_x]^2$. We assume that $f_x(\cdot,\cdot,\cdot,\cdot)$ is a continuous function. Further, the quantity $\underline{a} := \operatorname{ess\,sup}\{\|f_x(p_1, p_2, p_3, p_4)\| : p_1 \in [t_0, \infty), p_2 \in [m], p_3 \in \mathcal{R}_u, p_4 \in B_x\}$ is finite.

Denote by $\operatorname{dist}(t, t_0, K) := \sup\{|\xi(t, t_0, x_0)| : x_0 \in K\}$ the maximum distance from a point in the reachable set of (4.2) at time $t \in [t_0, \infty)$ and the origin of $\mathbb{R}^{d_x}$ when the initial condition belongs to K. Then, we have that

$$\liminf_{t\to\infty} \frac{\log(\operatorname{dist}(t, t_0, K))}{t} > -\infty.$$

In particular, if $f(t, \sigma(t), u(x_0, t), x(t)) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(x_0, t)$, $\mathcal{R}_u$ is a bounded subset of $\mathbb{R}^{d_u}$, and the second bullet above is true, then this result holds.

Thus, it seems natural to relax the usual controllability notion by asking the norm of the state to converge to zero with an arbitrary exponential rate of decay instead of asking the state to equal zero in finite time. To formally state our controllability notion, we use the following Definition 4.2.2, which is an adaptation from the definitions given in [11] about stabilization with finite data-rate. To improve readability, we name some sets and properties that were not named in [11]. We remark that the following definitions appeared in Chapter 3. We rewrite them here for the reader's convenience.

[6]This means that the solution is defined for all $t \in [t_0, \infty)$. See, e.g., Section 1.5 from [60] for sufficient conditions on $f(\cdot,\cdot,\cdot,\cdot)$ for this assumption to hold.

[7]Note that the control is defined by the initial state.

[8]Informally, we are asking the control law to keep the state bounded uniformly over all possible initial states.

Definition 4.2.2. We say that system (4.1) satisfies the *exponential decay condition* with rate $\alpha \in \mathbb{R}_{\geq 0}$, with $M \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$ if for each $x_0 \in K \subset \mathbb{R}^{d_x}$ there exists $u(\cdot) \in L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^{d_u})$ such that the corresponding solution satisfies

$$|x(t)| \leq \big(M|x_0| + \epsilon\big)e^{-\alpha(t-t_0)} \tag{4.3}$$

for all $t \in \mathbb{R}_{\geq t_0}$. For given $\alpha \in \mathbb{R}_{\geq 0}$, $M \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and $K \subset \mathbb{R}^{d_x}$ as above, we call a set[9] $\mathscr{R}(\epsilon, M, K, \alpha) \subset L^\infty_{\text{loc}}([t_0,\infty),\mathbb{R}^{d_u})$ a *stabilizing control set* of system (4.1) if for every $x_0 \in K$, there exists a control function $u(\cdot) \in \mathscr{R}(\epsilon, M, K, \alpha)$ such that (4.3) holds for the corresponding solution. Furthermore, we denote by

$$\begin{aligned}\mathscr{R}_T(\epsilon, M, K, \alpha) := \{u_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0,T],\mathbb{R}^{d_u}) :\\ u(\cdot) \in \mathscr{R}(\epsilon, M, K, \alpha)\}\end{aligned} \tag{4.4}$$

a *set of restrictions of stabilizing controls*, where $T > t_0$ is arbitrary. We define the *data-rate* associated with system (4.1) in the following manner. First, given a stabilizing control set $\mathscr{R}(\epsilon, M, K, \alpha)$, we define the *data-rate of the stabilizing control set* $\mathscr{R}(\epsilon, M, K, \alpha)$ as[10]

$$b(\mathscr{R}(\epsilon, M, K, \alpha)) := \limsup_{T\to\infty} \frac{1}{T}\log(\#\mathscr{R}_T(\epsilon, M, K, \alpha)).$$

Next, we define the *data-rate of system* (4.1) as[11]

$$\begin{aligned}b(M,\alpha) := \lim_{\epsilon\to 0}\big(\inf\{b(\mathscr{R}(\epsilon, M, K, \alpha)) : \mathscr{R}(\epsilon, M, K, \alpha)\\ \text{is a stabilizing control set of (4.1)}\}\big).\end{aligned} \tag{4.5}$$

Finally, we say that system (4.1) can be *stabilized with finite data-rate* with $M \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$ if $b(M,\alpha) < \infty$.

The reader might wonder if we can remove the ϵ term from inside inequality (4.3) and still get a reasonable notion of stabilizability with finite data-rate. The answer is negative and is proved in Proposition 2.2 of [11] where the

[9] We note that this set can be infinite in general.

[10] The corresponding quantity in [11] uses the limit inferior instead of limit superior. Because of that, if the quantity given in [11] is infinite, ours is also infinite.

[11] Note that $b(M,\alpha)$ also depends on the set of initial conditions K. We drop that dependence to make the notation simpler.

author showed that, for any pair $(\alpha, M) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, LTI systems with poles with a nonnegative real part cannot satisfy (4.3) with $\epsilon = 0$ and have $b(M, \alpha) < \infty$. Also, we take this opportunity to note that the limit on the right-hand side of Equation (4.5) exists. That happens because the infimum on the right-hand side of that equality is a monotonically decreasing function of ϵ. Consequently, that limit can be replaced by the supremum over $\epsilon \in \mathbb{R}_{>0}$. We also note that $\mathscr{R}(\epsilon, M, K, \alpha)$ is a control law[12] that operates with the data-rate $b(\mathscr{R}(\epsilon, M, K, \alpha))$. Now, we are ready to define controllability with a finite data-rate, which is one of the contributions of this paper.

Definition 4.2.3. We say that system (4.1) is *controllable with a finite data-rate* if for every $\alpha \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{R}_{\geq 0}$ such that system (4.1) can be stabilized with finite data-rate $b(M, \alpha) < \infty$.

In light of our discussion, Definition 4.2.3 captures the property of the norm of the state converging to zero as fast as possible in our setting. We believe that it is a natural candidate for extending the concept of controllability to switched linear systems with finite data-rate. It is important to remark that the previous definition is new and it differs from the definition of stabilization with finite data-rate, originally given in [11], in the sense that it captures the possibility of stabilization with an arbitrary convergence rate $\alpha \in \mathbb{R}_{\geq 0}$, while in [11] α was taken to be a fixed parameter.

4.2.3 Motivating Example

In this subsection, we present an example to motivate the study of controllability with a finite data-rate concept. We answer the questions we pose in this subsection later in the paper once we have enough tools. Before we continue, we need to introduce a few concepts that we will use throughout the remainder of this document.

First, define the sequence of *sampling times* $(\tau_n)_{n \in \mathbb{Z}_{\geq 0}} \subset [t_0, \infty)$ by

$$\tau_n := t_0 + nT_p, \tag{4.6}$$

where $T_p \in \mathbb{R}_{>0}$ is the *sampling period*. When $(t_n)_{n \in \mathbb{Z}_{\geq 0}} \subset (\tau_n)_{n \in \mathbb{Z}_{\geq 0}}$, we say that the switching happens *synchronously* with the sampling. Now, let

[12]See Subsection 4.2.1

$\mathcal{S} := \{n \in \mathbb{Z}_{\geq 0} : \sigma(\tau_n) \neq \sigma(t) \text{ for some } t \in [\tau_n, \tau_{n+1})\}$, i.e., $n \in \mathcal{S}$ if a switching occurs in the interior of the time interval $[\tau_n, \tau_{n+1})$. Note that $\mathcal{S} = \emptyset$ only if the switchings happen synchronously with the samplings, or if there are no switchings.

Now we are ready to present our motivating example. Our example deals with a class of cyberphysical attacks called DoS attacks. We refer to [61] for a discussion on their practical relevance.

Example 4.2.3. *(DoS attack)* Assume that we want to control a switched linear system described by Equation (4.1) over a communication network, and suppose that the modes (A_j, B_j) are controllable for each $j \in [m]$. We also assume that the encoder sends a quantization symbol q_n corresponding to the state $x(\tau_n)$ at time $\tau_n \in \mathbb{R}_{>0}$ for $n \in \mathbb{Z}_{\geq 0}$. The channel may transmit the symbol or erase it (when an attack occurs), i.e., the decoder may receive either q_n or an empty symbol (in which case we say that we lost a packet). We further suppose that the encoder knows when an attack occurs. We ask if we can drive the state of this system to zero exponentially fast with an arbitrarily prescribed exponential decay rate. We give a sufficient condition for this problem to have a solution in Section 4.3 and, for that case, we present a controller design technique in Section 4.4.

We take this opportunity to informally introduce some sets, related to packet losses, that are instrumental in our later discussion. Let $\mathcal{L} \subset \mathbb{Z}_{\geq 0}$ be the set of sampling time indices such that no packet loss occurs at time τ_n. Then, we define $\mathcal{R} := \mathcal{L} \cup \mathcal{S}^c$ as the set of sampling time indices' n such that no switching occurs in the interior of the time interval $[\tau_n, \tau_{n+1})$ nor a packet loss occurs at τ_n. The *generalized set of losses associated to system* (4.1) is $\mathcal{R}^c$. Notice that if there are no packet losses and the switchings are synchronous, we have that $\mathcal{R}^c = \emptyset$, explaining the name of the set. We define both sets $\mathcal{L}$ and $\mathcal{R}$ formally in Subsection 4.3.6.

4.3 Controllability with a Finite Data-Rate

In this section, we state a necessary condition and sufficient condition for a switched linear system to be controllable with a finite data-rate. We recall some classical concepts related to controllability to state our necessary condition. Next, we introduce some new controllability concepts, which require some discussion and examples, to state our sufficient condition. Then, we introduce concepts related to quantized control and the coder-decoder/controller scheme, which we use to prove our sufficient condition in a constructive manner. Finally, we briefly discuss the gap between the necessary condition and the sufficient condition, and we provide examples of applications. In particular, we answer the questions posed in Example 4.2.3 as an application.

4.3.1 The Necessary Condition

We start this subsection by recalling the notion of complete controllability, first stated in [56].

Definition 4.3.1. We say that system (4.1) is *completely controllable* if, for each $\bar{t} \in [t_0, \infty)$, there exists $t_1 \in (\bar{t}, \infty)$ such that (4.1) is controllable in the usual sense[13] on the time interval $[\bar{t}, t_1]$.

We have two remarks about this definition. First, some authors, such as [62] in Chapter 4, use the term "complete controllability" to refer to usual controllability on a given time interval. The difference is that Definition 4.3.1 requires system (4.1) to be controllable over infinitely many intervals, while the definition given in [62] requires the system to be controllable on a single time interval. Second, the next definition is helpful.

Definition 4.3.2. *The controllability Gramian of system* (4.1) *is given by*

$$W(t,s) := \int_s^t \Phi_\sigma(t,\tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(t,\tau) d\tau \tag{4.7}$$

for any $t \in \mathbb{R}_{>0}$ *and* $s \in \mathbb{R}_{>0}$.

Then, it is a well-known fact, see, e.g., [48], that complete controlabillity is equivalent to the statement: for every $\bar{t} \in \mathbb{R}_{\geq 0}$ there exists some $t_1 > \bar{t}$ such that $W(t_1, \bar{t})$ is invertible. This result gives us an operational way to check if a system is completely controllable. Now, we are ready to state our necessary condition in Theorem 4.3.1. We provide a proof of it in Appendix A.6.

[13] See Definiton 4.2.1.

Theorem 4.3.1. System (4.1) is controllable with finite data-rate only if it is completely controllable.

This statement is interesting because it gives a simple condition that guarantees that, if not satisfied, we can rule out the possibility of our system being controllable with a finite data-rate. This theorem appears in Chapter 3 of this thesis, stated for a more general class of LTV systems. In fact, in Example 3.2.1 from Chapter 3, we present a system that does not satisfy the necessary condition, with the goal of illustrating the usage of our previous theorem. The reason why we restate this theorem is because we want to remind the reader about it before we argue in Section 4.4.3 that this condition is close to our sufficient condition stated in Theorem 4.3.2, which we present in the next subsection.

4.3.2 The Sufficient Condition: the case without packet losses

In this subsection, we study the sufficient condition when no packet losses occur. We deal with packet losses in Subsection 4.3.4. To state the sufficient condition, we must first recall a classical controllability notion for LTV systems.

Definition 4.3.3. We say that system (4.1) is *uniformly completely controllable* (UCC) if there exist $T \in \mathbb{R}_{>0}$ and some $\underline{w} \in \mathbb{R}_{>0}$ such that the controllability Gramian satisfies $\underline{w} I_{d_x} \leq W(t+T,t)$ for all $t \in \mathbb{R}_{>0}$, where the inequality here denotes the partial order relation on symmetric positive definite matrices.

We remark that this concept was introduced by Kalman in works [48] and [56] using different conditions from the one we stated. It was [63] who proved that, if $A_{\sigma(\cdot)}$ and $B_{\sigma(\cdot)}$ are uniformly bounded for all times, then the condition we present in Definition 4.3.3 is equivalent to UCC. Now, we are ready to state our sufficient condition:

Theorem 4.3.2. System (4.1) is controllable with a finite data-rate if it is UCC.

This result is a consequence of Theorem 3.3.5 from Chapter 3 of this dissertation. It happens that being UCC is a stronger condition than the condition presented in that chapter. Thus, an LTV UCC system is controllable with a finite data-rate. We prove this latter fact in Appendix A.7.

The previous result applies to any LTV system[14], and it requires us to prove that our system is UCC, which might be difficult in general. However, assuming that our system is given by the switched linear dynamics (4.1), we can prove results that involve the controllable subspaces of the modes and some properties of the switching signal.

4.3.3 Applications

To extend our theory to deal with a more general case that considers packet losses, we discuss two relatively simple examples of practical relevance that we can analyze with what we have discussed. First, we assume that the switchings are synchronous and derive a relatively simple condition for UCC. We illustrate that case with an example where the modes are unstabilizable, but the switched system is UCC. After that, we remove the assumption on synchronicity and state a result that gives us another condition for UCC when the modes are controllable. In particular, when system (4.1) satisfies an average dwell-time condition (ADT), we arrive at an elementary relation involving the chatter bound, the average dwell-time, and the sampling period that guarantee UCC.

To state our condition for uniform complete controllability when the switchings are synchronous, we must introduce a new controllability definition. We briefly recall that $\langle A|B\rangle$ denotes the controllable subspace of the pair (A, B).

Definition 4.3.4. Let $\ell \in \mathbb{Z}_{>0}$ be a discrete time-horizon and let $\mathcal{S} = \emptyset$. For each $k \in \mathbb{Z}_{\geq 0}$, let $n = n(k) := \lfloor \frac{k}{\ell} \rfloor$. Define $\mathcal{V}_k := \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. We say that system (4.1) is $\ell-$*uniformly completely controllable* if

$$\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^{d_x} \tag{4.8}$$

for each[15] $k \in \mathbb{Z}_{\geq 0}$.

[14]Any system with $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, where the function $u(\cdot)$ is integrable and locally essentially bounded an the matrix functions $A(\cdot)$ and $B(\cdot)$ are locally integrable and bounded.

[15]Note that for each $l \in \mathbb{Z}_{\geq 0}$ there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $l = \lfloor \frac{k}{\ell} \rfloor$. Thus, $n(k)$

To help the reader better understand the idea behind Definition 4.3.4, we first discuss its relationship with classical controllability notions. Notice that Equation (4.8) is the same as the condition for complete controllability on the interval $[\tau_{\ell n}, \tau_{\ell(n+1)}]$ given in Chapter 4 of [62] for some fixed $n \in \mathbb{Z}_{\geq 0}$[16]. In fact, more is true. Since Equation (4.8) holds for each $n \in \mathbb{Z}_{\geq 0}$, a stronger controllability property must hold. The following lemma shows that Definition 4.3.4 and UCC are equivalent when the switchings are synchronous. Therefore, in the synchronous case, the existence of $\ell \in \mathbb{Z}_{\geq 0}$ such that our system is ℓ-uniformly completely controllable is sufficient for our system to be controllable with a finite data-rate.

Lemma 4.3.3. Let $\mathcal{S} = \emptyset$. Then, there exists some $\ell \in \mathbb{Z}_{>0}$ such that system (4.1) is $\ell-$uniformly completely controllable if, and only if, system (4.1) is UCC.

The following example should help us illustrate how we can apply Lemma 4.3.3 to show a nontrivial result.

Example 4.3.4. Let $\ell \in \mathbb{Z}_{>0}$, let $m = 2$, and let $t_0 = 0$. Let $\{e_1, e_2\} \subset \mathbb{R}^2$ be the canonical basis. Assume that, for each $n \in \mathbb{Z}_{\geq 0}$, there exists at least one integer $k_i(n)$ such that $\ell n \leq k_i(n) < \ell(n+1)$ and that $\sigma(k_i(n)) = i$ for $i \in [2]$. Also, let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $B_1 = e_1$, and $B_2 = e_2$. Note that each individual mode is unstabilizable. A simple calculation shows that $\langle A_i | B_i \rangle = \operatorname{span}\{e_i\}$ for $i \in [2]$. Also, since the matrix A_i is diagonal for each $i \in [2]$, we have that $\Phi_\sigma(t,s)$ is diagonal for each $t \in \mathbb{R}_{\geq 0}$ and $s \in [t, \infty)$. This latter fact implies that $\Phi_\sigma^{-1}(t,s)\langle A_i | B_i \rangle = \langle A_i | B_i \rangle = \operatorname{span}\{e_i\}$ for each $i \in [2]$, all $t \in \mathbb{R}_{\geq 0}$, and all $s \in [t, \infty)$. In particular, for each $n \in \mathbb{Z}_{\geq 0}$ and each $i \in [2]$, we have that $\mathcal{V}_{k_i(n)} = \operatorname{span}\{e_i\}$. Thus, we conclude that $\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j \supset \mathcal{V}_{k_1(n)} + \mathcal{V}_{k_2(n)} = \mathbb{R}^2$, which implies that our system is ℓ-uniformly completely controllable. Thus, by Lemma 4.3.3, our system is controllable with a finite data-rate.

The previous example used the fact that the switchings are synchronous to conclude that the switched system is controllable with a finite data-rate, even though the individual modes are unstabilizable. This result, however, relies

is a surjective function.

[16]We also notice that there exists an analogous characterization for the concept of complete observability, given in [64].

on synchronicity. At this point, the reader might wonder if there are simple conditions that ensure that the conditions from Theorem 4.3.2 hold when we do not require the switchings to be synchronous. The next proposition answers this questions affirmatively.

Proposition 4.3.5. Let $\ell \in \mathbb{Z}_{\geq 0}$. If, for each index $n \in \mathbb{Z}_{\geq 0}$, there exists some index $k(n) \in \mathbb{Z}_{\geq 0}$ such that $\ell n \leq k(n) < \ell(n+1)$, that $k(n) \notin \mathcal{S}$, and that $\langle A_{\sigma(\tau_{k(n)})} | B_{\sigma(\tau_{k(n)})} \rangle = \mathbb{R}^{d_x}$, then system (4.1) is UCC.

Informally, the last proposition is saying the following: if each interval of the form $[\tau_{\ell n}, \tau_{\ell(n+1)})$, where $n \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$ is given, has a sampling subinterval without a switching in its interior and a controllable mode is active on that subinterval, then the conditions of Theorem 4.3.2 hold. This latter condition is verified, for instance, when we have a "safe" mode, which we visit at least once in each time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$, i.e., we vist the controllable mode "frequently enough".

Interestingly, Proposition 4.3.5 has an immediate corollary of practical interest. First, we recall the definition of average dwell-time.

Definition 4.3.5 (Average Dwell-Time [65])**.** We say that system (4.1) satisfies an *average dwell-time condition* [65] if there exists a *chatter bound* $N_0 \in \mathbb{Z}_{\geq 0}$ and an *average dwell-time* $\tau_D \in \mathbb{R}_{>0}$ such that the number of switches $N_\sigma(t, \tau)$ on any time interval of the form $[\tau, t) \subset [t_0, \infty)$ satisfies $N_\sigma(t, \tau) \leq N_0 + \frac{t-\tau}{\tau_D}$.

The next result gives us a simple relation between the sampling period, the chatter bound, and the dwell-time of our switching signal that ensures that system (4.1) is controllable with a finite data-rate. We prove this corollary in the Appendix A.10.

Corollary 4.3.6. Assume that system (4.1) satisfies the ADT condition with average dwell-time $\tau_D \in \mathbb{R}_{>0}$ and chatter bound $N_0 \in \mathbb{Z}_{\geq 0}$. Further, assume that system (4.1) modes' are controllable. If $\frac{\tau_D}{N_0+2} \geq T_p$, then the system is controllable with a finite data-rate.

4.3.4 The Sufficient Condition: the general case

In the previous two subsections, we described a sufficient condition for a switched linear system to be controllable with a finite data-rate when no packet losses are present. Also, we showed how we can apply that result to cases of practical interest. In this subsection, we finally address the case where packet losses can occur.

To state the sufficient condition, we must make a small change to Definition 4.3.4 to accommodate packet losses, which will also be useful when we discuss our controller design technique. We briefly recall that S^c denotes the complement for a set $S \subset \mathbb{Z}_{\geq 0}$.

Definition 4.3.6. Let $\ell \in \mathbb{Z}_{>0}$ be a discrete time-horizon and let $\mathcal{R} \subset \mathbb{Z}_{\geq 0}$ be the set of generalized losses[17]. For each $k \in \mathbb{Z}_{\geq 0}$, let $n = n(k) := \lfloor \frac{k}{\ell} \rfloor$. Define $\mathcal{V}_k := \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$, if $k \in \mathcal{R}$, and $\mathcal{V}_k := \{0\}$, otherwise. We say that system (4.1) is *$\ell-$uniformly completely controllable with losses in $\mathcal{R}^c$* if

$$\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^{d_x} \tag{4.9}$$

for each[18] $k \in \mathbb{Z}_{\geq 0}$.

We take this opportunity to make a few comments about Definition 4.3.6. Note that ℓ-uniform controllability, introduced in Definition 4.3.4, is the same as ℓ-uniform controllability with losses in the empty set. In this sense, Definition 4.3.6 generalizes Definition 4.3.4. Also, recall from Subsection 4.2.3 that the set $\mathcal{R}$ is the set of sampling time indices' where no switching occurs on the interior of the time interval $[\tau_k, \tau_{k+1})$ and that no packet losses happen at times τ_k for each $k \in \mathcal{R}$.

Even though Definition 4.3.6 plays a major role in our theory, our sufficiency theorem requires one more condition, which only appears because we work with a continuous-time system with data-rate constraints for its control. To precisely state that additional condition we need yet another technical definition, which we present below.

Definition 4.3.7. *(compatible projection matrix)* Let $\ell \in \mathbb{Z}_{>0}$ be a discrete time-horizon and let $\mathcal{R}^c \subset \mathbb{Z}_{\geq 0}$ be the generalized set of losses of system (4.1). For each $k \in \mathbb{Z}_{\geq 0}$, let $n = n(k) := \lfloor \frac{k}{\ell} \rfloor$. We define a *compatible projection matrix P_k* in the following way.

[17]See, Subsection 4.2.3.

[18]Note that for each $l \in \mathbb{Z}_{\geq 0}$ there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $l = \lfloor \frac{k}{\ell} \rfloor$. Thus, $n(k)$ is a surjective function.

- If $k \in \mathcal{R}$: First, choose a subspace $\bar{\mathcal{W}}_k \subset \mathbb{R}^{d_x}$ such that[19] $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle \oplus \bar{\mathcal{W}}_k = \Phi(\tau_k, \tau_{\ell n}) \sum_{j=k}^{\ell(n+1)-1} \mathcal{V}_j$. Since $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle \cap \bar{\mathcal{W}}_k = \{0\}$, we have that $\bar{\mathcal{W}}_k \subset \Phi(\tau_k, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j$. Next, choose another subspace $\tilde{\mathcal{W}}_k \subset \mathbb{R}^{d_x}$ such that $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle \oplus \bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k = \mathbb{R}^{d_x}$. Then, define $P_k \in \mathcal{M}^{d_x \times d_x}$ as the oblique projection[20] matrix over $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$ that is parallel to $\bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k$, i.e., $P_k x = x$ if $x \in \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$ and $P_k x = 0$ if $x \in \bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k$. Since $\langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle \oplus \bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k = \mathbb{R}^{d_x}$, this construction completely defines the projection matrix P_k.

- If $k \in \mathcal{R}^c$: define $P_k = 0 \in \mathcal{M}^{d_x \times d_x}$.

We first note that P_k is an oblique projection matrix for each $k \in \mathbb{Z}_{\geq 0}$, i.e., $P_k^2 = P_k$. Also, we notice that the specific construction above requires some choices of vector subspaces, namely $\bar{\mathcal{W}}_k$ and $\tilde{\mathcal{W}}_k$, that are up to the designer as long as they comply with the conditions mentioned. Also, because of this freedom, the construction in Definition 4.3.7 is technical and difficult to understand at first. In the next subsection, we present Example 4.3.10, where we walk the reader through each detail of this construction to help clarify it. We are ready to enunciate the assumptions needed to state Theorem 4.3.7.

Assumption 4.3.1. There exists $\ell \in \mathbb{Z}_{>0}$ such that system (4.1) is ℓ-uniformly completely controllable with losses in its generalized set of losses $\mathcal{R}^c$.

Assumption 4.3.2. There exist a real number $g \in \mathbb{R}_{\geq 0}$ and a sequence $(P_k)_{k \in \mathbb{Z}_{\geq 0}} \subset \mathcal{M}^{d_x \times d_x}$ of compatible projection matrices such that $\|P_k\| \leq (1+g)$ for all $k \in \mathbb{Z}_{\geq 0}$.

We now have all the definitions needed to state the sufficient condition for controllability with a finite data-rate.

Theorem 4.3.7. System (4.1) is controllable with a finite data-rate if Assumptions 4.3.1 and 4.3.2 hold.

We present the proof of this Theorem in Section 4.4. The proof relies on Proposition 4.2.2, which states that our control Algorithm 4.4.1 ensures that the norm of the solution of (4.1), under the action of the control law

[19]Recall that for two vector subspaces $V \subset \mathbb{R}^{d_x}$ and $W \subset \mathbb{R}^{d_x}$, when we write the direct sum $V \oplus W$, we implicitly have that $V \cap W = \{0\}$.

[20]This means that it may not be an orthogonal projection in general.

the algorithm constructs, satisfies the exponential decay condition 3.2 with a prescribed exponential rate of decay, that is an algorithm parameter. Then, we show in Subsection 4.4.4 how to use that proposition to prove Theorem 4.3.7. Because of this proof structure, we claim that our proof is constructive since we can use Algorithm 4.4.1 to design controllers in practice.

4.3.5 Understanding the General Sufficient Condition

In Subsection 4.3.3, we discussed how to interpret our results when the switching signal is synchronous. Now, we explain the ideas behind the previous subsection's definitions, which deal with switchings that might be asynchronous and packet losses can be present.

We start by explaining what happens when the set of generalized losses is nonempty and why it matters. The idea behind Definition 4.3.6 is as follows: on each interval of the form $[\tau_{\ell n}, \tau_{\ell(n+1)})$, for an arbitrary $n \in \mathbb{Z}_{\geq 0}$, if we are only allowed to use the control $u(t) = 0$ for each $t \in [\tau_k, \tau_{k+1}) \subset [\tau_{\ell n}, \tau_{\ell(n+1)})$ and each $k \in \mathcal{R}^c \cap \{\ell n, \ldots, \ell(n+1)-1\}$, our system would still be completely controllable if (4.9) was true. Thus, if we have a communication channel between the plant and the controller and the packets carrying information about the system's state[21] at times τ_k, for $k \in \mathcal{R}^c$, were lost, we would still be able to send the state of the system to zero under the assumption that our system is $\ell-$uniform completely controllable with losses in $\mathcal{R}^c$. We state that claim more formally in Lemma 4.3.8, which we prove in Appendix A.11. We can see Definition 4.3.6 as an operational version of the property described in that lemma's statement. Further, we note that this property is not related to data-rate constraints, but it is a characteristic of system (4.1) and the set $\mathcal{R}$.

Lemma 4.3.8. Let $\mathcal{R}^c \subset \mathbb{Z}_{\geq 0}$ be the generalized set of losses of system (4.1) and let $\ell \in \mathbb{Z}_{>0}$. System (4.1) is *$\ell-$uniformly completely controllable with losses in* $\mathcal{R}^c$ if, and only if, for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in \mathbb{R}^{d_x}$, there exists a control $u_n(\cdot) \in L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$ with the following property: If $x(\tau_{\ell n}) = x$, then we have $x(\tau_{\ell(n+1)}) = 0$ and $u_n(t) = 0$ for each $t \in [\tau_k, \tau_{k+1})$ and each $k \in \mathcal{R}^c$.

[21] Here we are assuming that we can transmit perfect information through the channel, i.e., data-rate constraints play no role in this part of the discussion.

We already discussed how the property presented in Assumption 4.3.1 is related to UCC when $\mathcal{R}^c = \emptyset$ and gave an interpretation for it in Lemma 4.3.8. Assumption 4.3.2, by its turn, has a geometric interpretation, which can be understood rigorously by studying the equality presented in the following Lemma 4.3.9. We provide a proof of it in Appendix A.12 for completeness. Note, however, that this result is standard and its proof can be found in Theorem 3.1 from [66]. It is also worth mentioning that a similar quantity, namely, the minimum gap between subspaces, appears in the control systems literature, see, e.g., [67].

Lemma 4.3.9. Let $V \subset \mathbb{R}^{d_x}$ and $W \subset \mathbb{R}^{d_x}$ be such that $V \oplus W = \mathbb{R}^{d_x}$. If $\{0\} \neq V \subset \mathbb{R}^{d_x}$ and $\{0\} \neq W \subset \mathbb{R}^{d_x}$, we denote the *cosine between those two spaces* by $\cos(V, W) := \max\{|\langle v, w\rangle| : v \in V, w \in W, |v| = 1, |w| = 1\}$. If either $V = \{0\}$ or $W = \{0\}$, then we define $\cos(V, W) := 0$. Let $P \in \mathcal{M}^{d\times d}$ be projection matrix such that $Px = x$ when $x \in V$, and $Px = 0$ when $x \in W$. Then, $\|P\| = (1 - \cos^2(V, W))^{-1/2}$, if $V \neq \{0\}$, and $\|P\| = 0$, otherwise.

Thus, we see that Assumption 4.3.2 implies that the cosine between the controllable subspace of the active mode at time τ_k, i.e., $\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$, and its complement space, $\bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k$, is uniformly far from 1 for all $k \in \mathcal{R}$. Note that the cosine between two subspaces will only equal 1 if one is contained in the other. Thus, these inequalities ensure that $\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$ and its complement space $\bar{\mathcal{W}}_k \oplus \tilde{\mathcal{W}}_k$ will not collapse into each other as k goes to infinity.

We finally present Example 4.3.10, which should help the reader understand the construction described in Definition 4.3.7 regarding compatible projection matrices. Also, we explain the interpretation of Assumption 4.3.2 in terms of angles between subspaces using the result from Lemma 4.3.9.

Example 4.3.10. Let $\ell = 5$, let the sampling times be $\tau_n = 2.5n$ for each $n \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{R}^c = \{n \in \mathbb{Z}_{\geq 0} : n \bmod \ell \in \{1, 3, 4\}\}$ be the generalized set of losses. We see that the definition of $\mathcal{R}^c$ is compatible with the switching signal $\sigma(\cdot)$ we defined. Indeed, we see that $\mathcal{S} = \{n \in \mathbb{Z}_{\geq 0} : n \bmod \ell \in \{1, 3\}\}$. Thus, we can also infer that the losses occur at each sampling time τ_n such that $n \bmod \ell = 4$. We consider system (4.1) with two modes described as follows: mode 1 is such that $A_1 = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$ and $B_1 = (1, 1)'$, and mode 2 is such that $A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B_2 = (1, 0)'$. Let $t_{2n+1} = 3.5 + 12.5n$ and $t_{2n+2} = 8 + 12.5n$ for each $n \in \mathbb{Z}_{\geq 0}$ be switching times. We assume that the switching signal $\sigma : \mathbb{R}_{\geq 0} \to [2]$ is such that[22] $\sigma(t) = 1$ for $t \in [t_{2n+1}, t_{2n+2})$ and $\sigma(t) = 2$ for $t \in [\tau_{\ell n}, t_{2n+1}) \cup [t_{2n+2}, \tau_{\ell(n+1)})$ for each $n \in \mathbb{Z}_{\geq 0}$. Note that $\tau_{\ell n+2} \in [t_{2n+1}, t_{2n+2})$.

To continue our construction, we must first compute the spaces $\mathcal{V}_k$ for each $k \in \mathbb{Z}_{\geq 0}$. In what follows, we assume that the claims hold for any $n \in \mathbb{Z}_{\geq 0}$. When $k = \ell n$, we know that $\sigma(\tau_{\ell n}) = 2$. This implies that $\mathcal{V}_{\ell n} = \Phi_\sigma^{-1}(\tau_{\ell n}, \tau_{\ell n})\langle A_2|B_2\rangle = \langle A_2|B_2\rangle = \text{span}\{(1, 0)'\}$. When $k = \ell n + 2$, we also know that $\sigma(\tau_{\ell n+2}) = 1$. This implies that $\mathcal{V}_{\ell n+2} = \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1|B_1\rangle = e^{-3.5A_2}e^{-1.5A_1}\langle A_1|B_1\rangle = e^{-3.5A_2}\langle A_1|B_1\rangle = e^{-3.5A_2}\text{span}\{(1, 1)'\} = \text{span}\{(e^{-3.5}, 1)'\}$. We note that, when $k = \ell n+1$, or $k = \ell n+3$, or $k = \ell n+4$, we have that $k \in \mathcal{R}^c$. Thus, we have that $\mathcal{V}_{\ell n+1} = \mathcal{V}_{\ell n+3} = \mathcal{V}_{\ell n+4} = \{0\}$. Further, notice that $\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j = \mathbb{R}^2$, which implies that our system is $\ell-$uniform completely controllable with losses in $\mathcal{R}^c$. Next, we construct the oblique projection matrices. We note that we have $P_{\ell n+1} = P_{\ell n+3} = P_{\ell n+4} = 0$ by the fact that all such indices belong to the generalized set of losses $\mathcal{R}^c$. To deal with the cases when $k = \ell n$ and $k = \ell n+2$, we must first construct the spaces $\bar{\mathcal{W}}_k$ and $\tilde{\mathcal{W}}_k$. When $k = \ell n$, we choose $\bar{\mathcal{W}}_{\ell n}$ to be an arbitrary subspace of $\mathbb{R}^2$ such that $\langle A_{\sigma(\tau_{\ell n})}|B_{\sigma(\tau_{\ell n})}\rangle \oplus \bar{\mathcal{W}}_{\ell n} = \Phi_\sigma^{-1}(\tau_{\ell n}, \tau_{\ell n}) \sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j = \mathbb{R}^2$. We can choose, for instance, $\bar{\mathcal{W}}_{\ell n} = \text{span}\{(1, 2)'\}$. In this case, $\tilde{\mathcal{W}}_{\ell n} = \{0\}$, since $\tilde{\mathcal{W}}_{\ell n} \cap (\langle A_{\sigma(\ell n)}|B_{\sigma(\ell n)}\rangle \oplus \bar{\mathcal{W}}_{\ell n}) = \{0\}$ by the definition of direct sum. With these choices, we get that $P_{\ell n} = \begin{pmatrix} 1 & -0.5 \\ 0 & 0 \end{pmatrix}$. When $k = \ell n + 2$, we choose $\bar{\mathcal{W}}_{\ell n+2}$ to be an arbitrary subspace of $\mathbb{R}^2$ such that $\langle A_{\sigma(\tau_{\ell n+2})}|B_{\sigma(\tau_{\ell n+2})}\rangle \oplus \bar{\mathcal{W}}_{\ell n+2} = \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n}) \sum_{j=\ell n+2}^{\ell(n+1)-1} \mathcal{V}_j = \langle A_1|B_1\rangle = \text{span}\{(1, 1)'\}$. Hence, $\bar{\mathcal{W}}_{\ell n=2} = \{0\}$, and there are no further choices in this case. Finally, we choose $\tilde{\mathcal{W}}_{\ell n+2}$ as some subspace of $\mathbb{R}^2$ such that $\langle A_1|B_1\rangle \oplus \bar{\mathcal{W}}_{\ell n+2} \oplus \tilde{\mathcal{W}}_{\ell n+2} = \mathbb{R}^2$. We can choose $\tilde{\mathcal{W}}_{\ell n+2} = \text{span}\{(1, 0)'\}$ to get the projection matrices $P_{\ell n+2} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

The purpose of Figure 4.1 is to help us understand the concepts introduced so far. For each sampling time τ_k with $k \in \{\ell n, \dots, \ell(n+1)\}$ we draw a copy of the state space $\mathbb{R}^2$. In the copy at time τ_k, we draw the controllable

[22]We chose a periodic switching signal for simplicity. In this case, the period is 12.5.

subspace $\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$ of mode $\sigma(\tau_k)$, i.e., the controllable subspace of the mode that is active at the sampling time τ_k. If $n \in \mathcal{R}$, we use the color black for the sampling time and the controllable subspace, as is the case of $k = \ell n$, $k = \ell n + 2$, and $k = \ell n + 5$. If $k \in \mathcal{R}^c$, as in the case of $k = \ell n + 1$, $k = \ell n + 3$, and $k = \ell n + 4$, we use the color orange. Also, we use the color red to represent the switching times t_{2n+1} and t_{2n+2}.

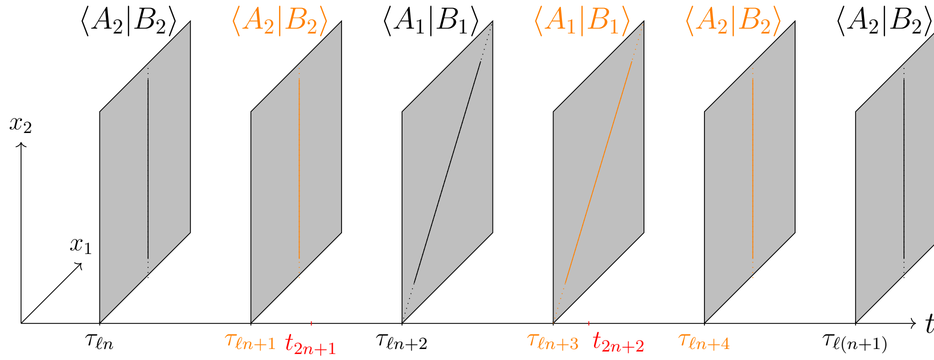

Figure 4.1: Subspaces and sampling times.

There are a few important things to notice in Figure 4.1. First, on the time interval $[\tau_{\ell n}, \tau_{\ell n+1})$, system (4.1) behaves as the LTI system $\dot{x}(t) = A_2 x(t) + B_2 u(t)$ and the controllable subspace is one-dimensional. This implies that we cannot send the state to zero at time $\tau_{\ell n+1}$. Second, we assume that $u(t) = 0$ for $t \in [\tau_{\ell n+1}, \tau_{\ell n+2}) \cup [\tau_{\ell n+3}, \tau_{\ell n+4}) \cup [\tau_{\ell n+4}, \tau_{\ell n+5})$ because those sampling time indices belong to the generalized set of losses $\mathcal{R}^c$. Third, note that our system behaves as the LTI system $\dot{x}(t) = A_1 x(t) + B_1 u(t)$ on the time interval $[\tau_{\ell n+2}, \tau_{\ell n+3})$. Consequently, if $x(\tau_{\ell n}) \neq 0$ and if we impose our previous assumption on $u(t)$, we can only have $x(\tau_{\ell n+5}) = 0$ when $x(\tau_{\ell n+2}) \in \langle A_1|B_1\rangle$. To do that, we must decompose state $x(\tau_{\ell n})$ into a component that belongs to $\mathcal{V}_{\ell n} = \langle A_2|B_2\rangle$, which we can send to zero on the time interval $[\tau_{\ell n}, \tau_{\ell n+1}]$, and another which we can send to zero on the time interval $[\tau_{\ell n+2}, \tau_{\ell n+3}]$. This second component must lie on the vector subspace $\mathcal{V}_{\ell n+2} = \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1|B_1\rangle$ since this vector space is mapped into the vector space $\langle A_1|B_1\rangle$ at time $\tau_{\ell n+2}$. To perform this decomposition, we use the oblique projection matrices. More explicitly, we can write $x(\tau_{\ell n}) = P_{\ell n} x(\tau_{\ell n}) + (I_2 - P_{\ell n}) x(\tau_{\ell n})$, where the first term belongs to $\langle A_2|B_2\rangle$ and the second term belong to the nullspace of[23] $P_{\ell n}$. Note that $\bar{\mathcal{W}}_{\ell n} \oplus \tilde{\mathcal{W}}_{\ell n}$ is the

[23] This follows from $P(I_d - P)x = (P - P^2)x = 0$ for any projection matrix P and any

nullspace of $P_{\ell n}$ by definition 4.3.7. Our previous discussion tells us that $\bar{\mathcal{W}}_{\ell n} \oplus \tilde{\mathcal{W}}_{\ell n} = \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1|B_1\rangle$. This implies that, at time $\tau_{\ell n+2}$, the component $(I_2 - P_{\ell n})x(\tau_{\ell n})$ will belong to the subspace $\Phi_\sigma(\tau_{\ell n+2}, \tau_{\ell n})(\bar{\mathcal{W}}_{\ell n} \oplus \tilde{\mathcal{W}}_{\ell n}) = \langle A_1|B_1\rangle$. These concepts are depicted in Figure 4.2 where we show the state at time $\tau_{\ell n}$ and the state at time $\tau_{\ell n+2}$ as red arrows. As we can see, the state at time $\tau_{\ell n+2}$ lies in $\langle A_1|B_1\rangle$.

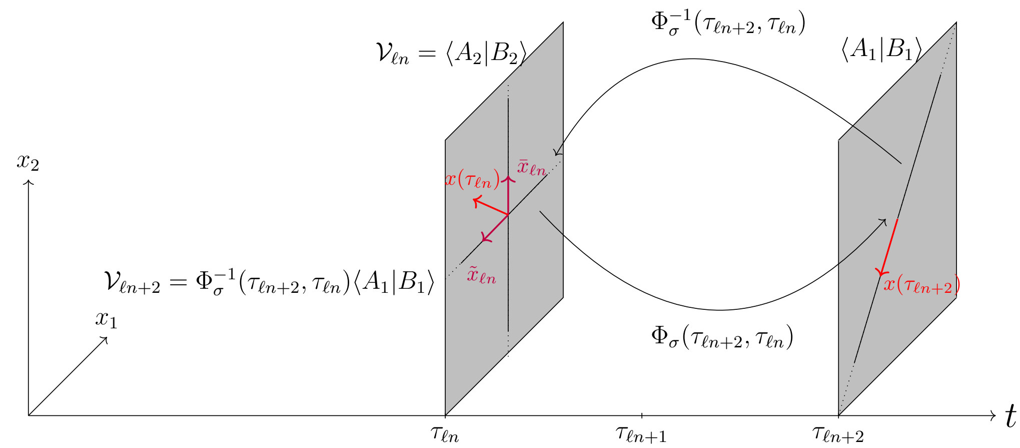

Figure 4.2: Vector spaces flown backwards.

More explicitly, in Figure 4.2 we see the previously mentioned decomposition of the state $x(\tau_{\ell n})$ at the time $\tau_{\ell n}$ into components that lie on $\langle A_2|B_2\rangle$ and $\Phi_\sigma(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1|B_1\rangle$, i.e., we can write $x(\tau_{\ell n}) = \bar{x}_{\ell n} + \tilde{x}_{\ell n}$, where $\bar{x}_{\ell n} = P_{\ell n}x(\tau_{\ell n})$ and $\tilde{x}_{\ell n} = (I_d - P_{\ell n})x(\tau_{\ell n})$. Note that the component $\tilde{x}_{\ell n} \in \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1|B_1\rangle$ and that $\bar{x}_{\ell n} \in \langle A_2|B_2\rangle$.

Finally, we can now interpret Assumption 4.3.2 using the notion of angle between subspaces. We note that, for each $n \in \mathbb{Z}_{\geq 0}$, we have that $\|P_{\ell n+1}\| = \|P_{\ell n+3}\| = \|P_{\ell n+4}\| = 0$. Also, simple calculations show us that $\|P_{\ell n}\| = \frac{\sqrt{5}}{2}$ and $\|P_{\ell n+2}\| = \sqrt{2}$. Thus, if we pick $g = \max\{\frac{\sqrt{5}}{2} - 1, \sqrt{2} - 1\}$, we arrive at the conclusion that our system satisfies Assumption 4.3.2. It is worth mentioning, that these norms depend on the choices of vector spaces $\bar{\mathcal{W}}_k$ and $\tilde{\mathcal{W}}_k$ for $k \in \mathcal{R}$, as mentioned earlier. Indeed, for instance, we could have chosen $\tilde{\mathcal{W}}_{\ell n+2} = \text{span}\{(1, 1 - 1/n)'\}$ for each $n \in \mathbb{Z}_{\geq 0}$. This would still give us that $\langle A_1|B_1\rangle \oplus \bar{\mathcal{W}}_{\ell n+2} \oplus \tilde{\mathcal{W}}_{\ell n+2} = \mathbb{R}^2$. However, for this choice, a compatible projection matrix would be $P'_{\ell n+2} = \begin{pmatrix} 1-n & n \\ 1-n & n \end{pmatrix}$. Thus, no $g \in \mathbb{R}_{\geq 0}$ exists

vector $x \in \mathbb{R}^{d_x}$.

such that $\|P'_{\ell n+2}\| < (1+g)$ since $\|P'_{\ell n+2}\|$ goes to infinity as n goes to infinity. This happens because the cosine between $\langle A_{\sigma(\tau_{\ell n+2})} | B_{\sigma(\tau_{\ell n+2})}\rangle = \langle A_1 | B_1\rangle$ and $\operatorname{span}\{(1,1)'\}$ goes to 1 as n grows. It is important to notice here that the existence of this sequence of matrices is not in contradiction with Assumption 4.3.2 since the assumption only asks for the existence of a some sequence $(P_k)_{k\geq 0} \subset \mathcal{M}^{d\times d}$ of compatible projection matrices with the property that $\|P_k\| \leq (1+g)$.

4.3.6 Quantized Control Concepts

We start this subsection by defining the concept of coder-decoder/controller scheme. These definitions are used in the description of Algorithm 4.4.1 and the proof of Theorem 4.3.7. Also, we finally address the questions asked in Example 4.2.3 about DoS attacks. Furthermore, we also formally define the set $\mathcal{L}$, which we used to define the set $\mathcal{R}$ in Subsection 4.2.3.

Definition 4.3.8 (coder-decoder/controller scheme)**.**
A *coder-decoder/controller scheme* is defined by the following quantities:

- A sequence of *sampling times* $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}} \subset \mathbb{R}_{\geq 0}$ where $\tau_0 = t_0$;
- A sequence of finite cardinality sets $(\mathcal{C}^n)_{n\in\mathbb{Z}_{\geq 0}}$ where each element of the sequence is called a *coder alphabet*;
- A sequence of finite cardinality sets $(\hat{\mathcal{C}}^n)_{n\in\mathbb{Z}_{\geq 0}}$ where each element of the sequence is called a *decoder alphabet*;
- A sequence of *coder functions* $(\gamma^n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $\gamma^0 : \mathbb{R}^{d_x} \to \mathcal{C}^0$ and $\gamma^n : \mathcal{C}^0 \times \cdots \times \mathcal{C}^{n-1} \times \mathbb{R}^{d_x} \to \mathcal{C}^n$ for each $n \in \mathbb{Z}_{>0}$;
- A sequence of *channel functions* $(\mathbf{c}^n)_{n\in\mathbb{Z}_{\geq 0}}$ such that[24] $\mathbf{c}^n : \mathcal{C}^n \to \hat{\mathcal{C}}^n$;
- A sequence of *decoder functions* $(\delta^n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $\delta^n : \hat{\mathcal{C}}^0 \times \cdots \times \hat{\mathcal{C}}^n \to \mathbb{R}^{d_x}$ for each $n \in \mathbb{Z}_{\geq 0}$;
- A *controller* is a sequence of *controller functions*

$$(u_n)_{n\in\mathbb{Z}_{\geq 0}} \subset L^\infty_{\text{loc}}([\tau_n, \tau_{n+1}) \times \mathbb{R}^{d_x}, \mathbb{R}^{d_u}).$$

[24] We could consider more general channel models, e.g., channels with memory. Nonetheless, we will not do this in this document.

Additionally, we make the following definitions. The elements of a coder or decoder alphabet are called *symbols*. Further, we assume that $\hat{\mathcal{C}}^n = \mathcal{C}^n \cup \{\emptyset\}$ where the special symbol $\emptyset$ is called the *empty symbol*. The remaining symbols of a decoder alphabet are called the *valid symbols*. Furthermore, we define the set $\mathcal{L} := \{n \in \mathbb{Z}_{\geq 0} : c_n(q) = q \text{ for all } q \in \mathcal{C}^n\}$, i.e., the set of indices n such that the corresponding channel function is the identity function. If $\mathcal{L} = \mathbb{Z}_{\geq 0}$ we say that our communication channel is *lossless*. If $n \in \mathcal{L}^c$, we say that we have a *packet loss* at the sampling time τ_n. Finally, we define the *received average data-rate of the coder-decoder scheme*[25] as $b_{\mathrm{rx}} := \limsup_{n\to\infty} \frac{1}{\tau_n - \tau_0} \sum_{k\in\mathcal{L}\cap[n]_0} \log\big(\#\mathcal{C}^k\big)$.

A few remarks are in order. First, this definition was inspired by the definition of coder-controller pair first presented in [68] to study the control of stochastic linear systems under data-rate constraints. We also refer to the survey paper [4] for a discussion about such schemes. The main difference between the next definition and that of [68] is that we split our scheme into coder, decoder, and controller instead of just coder and controller. Also, we introduce the channel functions to deal with the issue of packet losses. Second, in the information theory literature, see, e.g., the first three chapters of [69], the channel function is treated as a random function in the sense that, for each $n \in \mathbb{Z}_{\geq 0}$ and each $q \in \mathcal{C}^n$, the value $\mathbf{c}^n(q)$ is random. We do not treat this case here. Third, we note that any control system that measures the state using a digital sensor or analog-to-digital converter, can be described using the previous definitions by suitably choosing the coder, channel, and decoder functions. To do that, let $\mathbb{q} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d_x} \to D \subset \mathbb{R}^{d_x}$ be the *quantizer*, where D is a set of finite cardinality of possible sensor outputs. Let the sampling times be $\tau_k = T_c$, where $T_c \in \mathbb{R}_{\geq 0}$ is the sensor's *clock frequency*, for each $k \in \mathbb{Z}_{\geq 0}$. We assume that the quantizer is a piecewise constant function on its first argument in each interval of the form $[\tau_k, \tau_{k+1})$ for each $k \in \mathbb{Z}_{\geq 0}$. In addition, let the alphabets be $\mathcal{C}^n = \hat{\mathcal{C}}^n = D$ for each $n \in \mathbb{Z}_{\geq 0}$. We choose the functions from the coder-decoder/controller scheme as follows: let[26] $\gamma^n(*, \ldots, *, x(\tau_n)) = \mathbb{q}(\tau_n, x(\tau_n))$, $\mathbf{c}^n(q) = q$, and $\delta^n(*, \ldots, *, q) = q$, for each $q \in \hat{\mathcal{C}}^n$ and each $n \in \mathbb{Z}_{\geq 0}$. Furthermore, if we apply a controller $u(\cdot, \cdot) \in L^\infty_{\mathrm{loc}}([t_0, \infty) \times \mathbb{R}^{d_x}, \mathbb{R}^{d_u})$, then, we can define a sequence of controller

[25]We note that the data-rate does not depend on the controller explicitly.

[26]The symbol $*$ as an argument for a function means that the function is constant in that argument.

functions $(u_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $u_n(t,\cdot) = u(t,\cdot)$ for $t \in [\tau_n, \tau_{n+1})$ for each $n \in \mathbb{Z}_{\geq 0}$. This construction tells us that we can associate a notion of data-rate, given by b_{rx}, to any controller that uses digital sensors or analog-to-digital converters in a natural way.

Now, we are ready to illustrate the concepts developed thus far by answering the questions posed in the motivating Example 4.2.3.

Example 4.3.11 (Example 4.2.3 continued)**.** For simplicity, we assume that the switchings are synchronous. Thus, as we discussed earlier in this subsection, we only need to check Assumption 4.3.1 for Theorem 4.3.7 to hold. Now, assume that there exists $\ell \in \mathbb{Z}_{>0}$ such that in each time interval of the form $[\tau_{\ell n}, \tau_{\ell(n+1)})$ we have at least one subinterval $[\tau_k, \tau_{k+1})$ with $k \in \mathcal{R}$. Since all modes are controllable, we have that Equation (4.9) holds for each $n \in \mathbb{Z}_{\geq 0}$. Hence, our system is $\ell-$uniformly completely controllable and Theorem 4.3.7 holds. This implies that we can send the state of our switched linear system to zero even in the presence of a DoS attack that satisfies this condition. Note that, we are imposing a constraint on the attack duration and on the frequency at which such attacks occur. See, e.g., [61] for a discussion on why imposing such constraints is reasonable.

4.3.7 The Gap Between the Conditions

Now, we discuss the gap between the necessary condition given by Theorem 3.3.6 and the sufficient condition when no packet losses are present. We argue that, although those conditions are "close", the sufficient condition is not necessary.

We see that the difference between the assumptions of Theorem 3.3.6 and 4.3.2 is just uniformity. It seems natural to ask if the sufficient condition is actually necessary. The answer is negative as Example 4.3.12 shows. Before we formally state that example, we take this opportunity to recall some concepts and results presented in [53]. We recall that system (4.1) is called *persistently completely controllable* if there exists an increasing sequence of times $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [t_0, \infty)$ such that $s_0 = t_0$, that $\lim_{n\to\infty} s_n = \infty$, that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, and that $W(s_{n+1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. We also recall that system (4.1) satisfies the *exponential energy-growth condition* if there exist constants $N \in \mathbb{R}_{>0}$ and

$\theta \in \mathbb{R}_{\geq 0}$ such that $\|W^{-1}(s_{n+1}, s_n)\| \leq Ne^{\theta s_{n+1}}$ for each $n \in \mathbb{Z}_{\geq 0}$. The latter condition is related to the minimum control energy needed to drive the state $x(s_n)$ at time s_n to zero at time s_{n+1} for each $n \in \mathbb{Z}_{\geq 0}$. We refer to [53] for a discussion on this latter point. Now, Theorem 3.1 from [53] says that if an LTV system is persistently completely controllable and satisfies the exponential energy-growth condition, then it is controllable with a finite data-rate. We use this result in our next example to show that UCC is not a necessary condition.

Example 4.3.12. Let $t_0 = 2$ and consider the equation $\dot{x}(t) = b_{\sigma(t)}u(t)$ with $b_{\sigma(t)} = 1$, when $t \in \cup_{n\geq 1}[2^n, 2^n + 1)$, and $b_{\sigma(t)} = 0$, otherwise. We claim that this system is controllable with a finite data-rate but it is not UCC. We start by choosing a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [2, \infty)$ such that $s_n = 2^{n+1}$ for $n \in \mathbb{Z}_{\geq 0}$. Naturally, $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ is an increasing sequence that grows to infinity. Also, we have that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = 2$. Further, for each $n \in \mathbb{Z}_{\geq 0}$, on the interval $[2^{n+1}, 2^{n+2})$, we have that $b_{\sigma(t)} = 1$ only on the time subinterval $[2^{n+1}, 2^{n+1}+1)$ and $b_{\sigma(t)} = 0$ for the remainder of the total interval. Therefore, we get that $W(s_{n+1}, s_n) = \int_{s_n}^{s_{n+1}} b^2_{\sigma(\tau)} d\tau = \int_{2^{n+1}}^{2^{n+1}+1} 1 d\tau = 1$ for each $n \in \mathbb{Z}_{\geq 0}$, i.e., $W(s_{n+1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. Finally, we can easily see that $|W^{-1}(s_{n+1}, s_n)| = 1$ for every $n \in \mathbb{Z}_{\geq 0}$, which implies that our system satisfies the exponential energy-growth condition with $N = 1$ and $\theta = 0$. Thus, our system satisfies all the conditions for Theorem 3.1 from [53] to hold. We therefore conclude that this system is controllable with a finite data-rate. Nonetheless, this system is not UCC. To see that, note that for every $T \in \mathbb{R}_{>0}$ there exists some $n \in \mathbb{Z}_{\geq 0}$ so that $W(s_n + 1 + T, s_n + 1) = 0$. Indeed, this follows from the fact that $b_{\sigma(t)} = 0$ for all $t \in [s_n + 1, s_n + 1 + T)$ if $T < 2^{n+1} - 1$ since $s_n + 1 + T < 2^{n+2}$. This proves the claim.

As we can see, in Chapter 3, we work with a more general class of systems, i.e., general LTV systems, and we obtain a sufficient condition that works in more general settings than the one we present in Theorem 4.3.2. Hence, one might ask why we need a new chapter to study a more restrictive class of systems and even obtain more restrictive results. We note two main differences between those results, which should justify the relevance of the results from the present chapter. First, the sufficient condition from Chapter 3 has no clear geometric interpretation. In the switched case, however, we can attribute a geometric meaning related to the controllable subspaces of the modes to our sufficient conditions. We note that we can check Assumption 4.3.1 using Equation (4.9), and we recall that we have already discussed the geometrical interpretation of Assumption 4.3.2. Thus, the conditions from Theorem 4.3.2 are geometric, whereas Theorem 3.3.5 has no natural geometric interpretation. We should not expect a geometric characterization of this latter property for general LTV systems since these systems do not keep nonzero subspaces of $\mathbb{R}^{d_x}$ invariant over any time interval. A switched linear system, however, behaves as an LTI system between two consecutive switchings, thus allowing us to analyze controllability properties using each mode's controllable subspace. Indeed, this geometric characterization allowed us find easy conditions for controllability with a finite data-rate, as we present in the Subsection 4.3.3. Second, in the present chapter, we allow for packet losses in Theorem 4.3.7, which is not a case we consider in Chapter 3. Thus, we cannot draw a fair comparison between those two cases when losses are present.

In the next section, we design a control law that works with a finite data-rate. Also in that section, we prove properties of such control law and we use such controller to prove Theorem 4.3.7.

4.4 Controller Design

In this section, we present an algorithm that, under Assumptions 4.3.1 and 4.3.2, designs a control law that will be used in proving Theorem 4.3.7. First, we present a high-level description of how our algorithm works and, to make the concepts clear, we illustrate that idea with a figure. Then, we formally describe the algorithm. After that, we prove its correctness in Proposition 4.4.4. Finally, we present the proof of Theorem 4.3.7.

4.4.1 Algorithm's Idea

Here, we describe the high-level idea of our control algorithm. We do that inductively and with the help of Figures 4.4 and 4.3. We use Figure 4.3 to describe what happens between two consecutive sampling times and we use Figure 4.4 to explain the working mechanism behind our algorithm. Before we begin, we need to introduce some additional notation. Let $x(t) :=$

$\phi(t, t_0, x_0, u(\cdot))$, where $u(\cdot)$ is the control given by our algorithm.[27] Next, let $q^0 := \gamma^0(x(\tau_0))$ and let $q^k := \gamma^k(q^0, \ldots, q^{k-1}, x(\tau_k))$ for each $k \in \mathbb{Z}_{>0}$. Analogously, we define $\hat{q}^0 := \mathbf{c}^0(q^0)$ and $\hat{q}^k := \mathbf{c}^k(q^k)$ for each $k \in \mathbb{Z}_{>0}$. Then, we define the k-th *state estimate* as $\hat{x}(\tau_k) := \delta^k(\hat{q}^0, \ldots, \hat{q}^k)$ and the k-th *estimation error* $\mathbf{e}(t) := \delta^k(\mathbf{c}^0(q^0), \ldots, \mathbf{c}^k(q^k)) - x(t)$ for each $t \in [\tau_k, \tau_{k+1})$ and each $k \in \mathbb{Z}_{\geq 0}$. Further, let $\ell \in \mathbb{Z}_{>0}$ be such that system (4.1) satisfies the assumptions from Theorem 4.3.7, let $k \in \mathbb{Z}_{\geq 0}$, and let $n = n(k) = \lfloor \frac{k}{\ell} \rfloor$. Furthermore, let $\alpha \in \mathbb{R}_{\geq 0}$ be a prescribed exponential rate of decay and let $\epsilon \in \mathbb{R}_{>0}$. Now we have all quantities we need to describe the algorithm's idea.

At time τ_k, for an arbitrary $k \in \mathbb{Z}_{\geq 0}$, we assume that our system's state $x(\tau_k)$ belong to a compact set $B^k \subset \mathbb{R}^{d_x}$, which is an overapproximation to the reachable set such that $0 \in B^k$. The set B^k, by its turn, is the union of several disjoints subsets called the *quantization regions*. Each quantization region corresponds to a symbol in the coding alphabet $\mathcal{C}^k$. We denote by $\bar{B}^k$ the quantization region that contains $x(\tau_k)$ at time τ_k and its corresponding symbol is q^k. Informally, we say that we measure the state $x(\tau_k)$ and the encoder encodes that measurement as the symbol q^k. The symbol q^k is transmitted over the communication channel, and the symbol $\hat{q}^k$ arrives at the decoder[28] at time τ_k. Then, $\hat{q}^k$ is converted by the decoder into a state estimate $\hat{x}(\tau_k)$. Next, we implement a control law $u_k(\cdot, P_k\hat{x}(\tau_k))$ that drives the compatible projection of our state estimate, i.e., $P_k\hat{x}(\tau_k)$, to zero over the time interval $[\tau_k, \tau_{k+1})$. We repeat this procedure for all $k \in \mathbb{Z}_{\geq 0}$.

On the left of Figure 4.3, we see the set B^k subdivided into squares. Each one of the squares represents a quantization region. The state $x(\tau_k)$ at time τ_k is represented by a red dot, and the state estimate $\hat{x}(\tau_k)$ is represented by a blue dot. The state estimate is always at the center of a quantization region. The origin is represented by the black dot in the center of B^k. When we apply the control law $u_k(t, P_k\hat{x}(\tau_k))$, our system follows the red trajectory and lands on the set B^{k+1} at time τ_{k+1}, i.e., $x(\tau_{k+1}) \in B^{k+1}$, which we can see represented on the right side of Figure 4.3. Then, we repeat the process for $k + 1$.

[27] To be precise, we only need the restriction of $u(\cdot)$ to the time interval $[t_0, t)$. We make this remark to emphasize that our algorithm is causal.

[28] We assume that there is no communication delay.

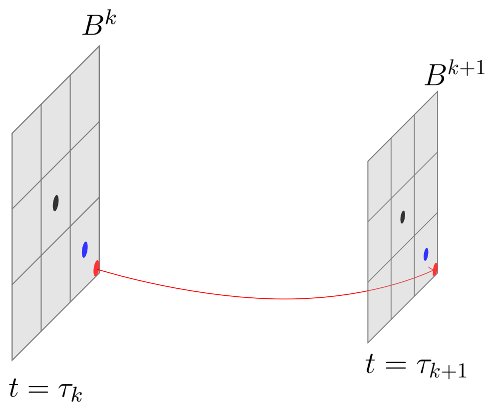

Figure 4.3: Quantization region and state trajectory.

On Figure 4.4, once again we consider the system we described in Example 4.3.10. We analyze the time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$ for some $n \in \mathbb{Z}_{\geq 0}$. As before, we see several copies of the state space $\mathbb{R}^2$ at each sampling time. As in Figure 4.3, we represent the origin as a black dot and the state at time τ_k by the red dot for each $k \in \{\ell n, \ldots, \ell(n+1)\}$. At time $\tau_{\ell n}$, we see a dashed region, which represents $B^{\ell n}$. At time $\tau_{\ell n+1}$, however, the dashed region depicts a subset of $B^{\ell n+1}$. To explain what set that dashed represents, we first need to understand what the control is doing. We start by assuming that there is no estimation error, i.e., $x(\tau_k) = \hat{x}(\tau_k)$ for $k \in \mathbb{Z}_{\geq 0}$, and we drop that assumption later in our discussion. Our goal is to drive $x(\tau_{\ell n})$ to zero. Nevertheless, we know that we cannot do that on the time interval $[\tau_{\ell n}, \tau_{\ell n+1})$ since the controllable subspace of the active mode on that interval is $\langle A_2 | B_2 \rangle = \text{span}\{(1,0)'\}$, which is a proper subspace of $\mathbb{R}^2$. We can, nonetheless, drive the compatible projection $P_{\ell n} x(\tau_{\ell n})$ to zero, which is what the control $u_{\ell n}(\cdot, P_{\ell n} x(\tau_{\ell n}))$ does. This leads us to the conclusion that $x(\tau_{\ell n+1}) \in \Phi_\sigma(\tau_{\ell n+1}, \tau_{\ell n})(\sum_{j=\ell n+1}^{\ell(n+1)-1} \mathcal{V}_j) = \Phi_\sigma(\tau_{\ell n+1}, \tau_{\ell n})(\Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n})\langle A_1 | B_1 \rangle) = \Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n+1})\langle A_1 | B_1 \rangle$, which is a one-dimensional subspace of the state space $\mathbb{R}^2$. Now, due to quantization, the estimation error is nonzero in general. Because of that, we have that $x(\tau_{\ell n+1})$ belong to the $\varepsilon_{\ell n+1}-$collar of the set $\Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n+1})\langle A_1 | B_1 \rangle)$ for some[29] $\varepsilon_{\ell n+1} \in \mathbb{R}_{>0}$. In other words, we drive the state to an approximation of the subspace $\Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n+1})\langle A_1 | B_1 \rangle$, i.e., $x(\tau_{\ell n+1}) \in$ $(\Phi_\sigma^{-1}(\tau_{\ell n+2}, \tau_{\ell n+1})\langle A_1 | B_1 \rangle))^{\varepsilon_{\ell n+1}}$.We see this latter fact in Figure 4.4 represented by the fact that the dashed region at time $\tau_{\ell n+1}$ shrinks along the

[29]We give a precise value for each ε_k in the proof of Proposition 4.4.4, for each $k \in \mathbb{Z}_{\geq 0}$.

direction of the subspace $\langle A_2|B_2\rangle$. More precisely, the dashed region at time $\tau_{\ell n+1}$ represents the set $B^{\ell n+1}\cap(\Phi_\sigma^{-1}(\tau_{\ell n+2},\tau_{\ell n+1})\langle A_1|B_1\rangle)^{\varepsilon_{\ell n+1}}$. The same reasoning holds for the other time intervals $[\tau_k,\tau_{k+1})$ for $k\in\{\ell n+2,\dots,\ell n+4\}$. We finally notice that, if the diameter of $B^{\ell(n+1)}$ decreases exponentially with n, we can ensure that the sate decays exponentially fast. We explain, in the next subsection 4.4.2, how to choose our algorithm parameters so that we achieve an exponential decays with a prescribed exponential decay rate.

Remark 4.4.1. It is worth mentioning that the idea of driving the center of a quantization region to zero is not new. In [5], the author did something similar for discrete-time linear time-invariant systems. In the proof of his Proposition 2.2, we see that he designed a control sequence that steers the center of a quantization region to its corner. The goal was to drive the state to a neighborhood of the origin and keep it there. In some sense, our algorithm builds on that idea to design a controller that operates with a finite data-rate for switched linear systems. However, we also need to emphasize some differences here. The fact that we work in continuous time complicates the analysis. Indeed, when we deal with sampled continuous-time systems, the inter-sample growth of the state norm plays a prominent role in the controller design [70, 71]. In fact, if system (4.1) was a discrete-time linear switched system, we could simplify the proof of Proposition 4.4.4 by removing the analysis of the inter-sample behavior, which we use to prove Theorem 4.3.7. It is important to note that Assumption 4.3.2 is only used in the inter-sample behavior analysis. Thus, the discrete-time case is simpler to analyze.

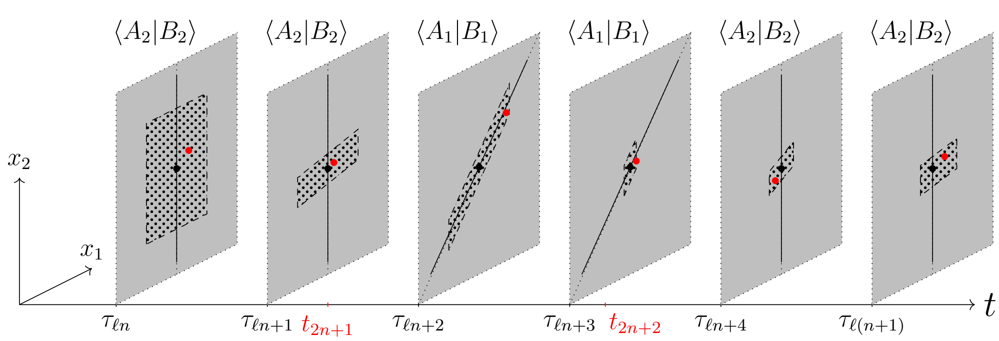

Figure 4.4: Algorithm's Idea.

Before we resume our discussion, we need the following technical lemma, which we prove in Appendix A.13.

Lemma 4.4.2. Let $k \in \mathcal{R}$. If $x(\tau_k) \in \langle A_{\sigma(\tau_k)} | B_{\sigma(\tau_k)} \rangle$, then there exists some $u_k(\cdot,\cdot) \in L^\infty_{\text{loc}}([\tau_k, \tau_{k+1}) \times \mathbb{R}^{d_x}, \mathbb{R}^{d_u})$ such that $x(\tau_{k+1}) = 0$ and such that

$$u_k(t,x) = F_{\sigma(\tau_k)} x \tag{4.10}$$

for each $t \in [\tau_k, \tau_{k+1})$ and some $F_{\sigma(\tau_k)} \in \mathcal{M}^{d \times m}$ that only depends on the active mode at time τ_k.

Now, we are ready to formally present our control algorithm.

4.4.2 Algorithm's Formal Description

Algorithm 4.4.1. We denote the canonical basis in $\mathbb{R}^{d_x}$, where the matrices A_i and B_i are defined for every $i \in [m]$, by $\mathcal{V}_0 = \{e_1, \ldots, e_{d_x}\}$. For convenience, we denote $x_k := x(\tau_k)$ for $n \in \mathbb{Z}_{\geq 0}$. Prescribe a parameter $\epsilon \in \mathbb{R}_{>0}$ and an exponential decay rate $\alpha \in \mathbb{R}_{\geq 0}$. Also, we can precompute the quantities: $\eta := \max\{\|A_i\| : i \in [m]\}$, $e^{\eta T_p}$, and $b := \max\{\|B_i\| : i \in [m]\}$. Further, for each mode $i \in [m]$, we can precompute: the matrix $e^{A_i T_p}$ and the matrix measure $\mu(A_i)$. Furthermore, we can define and compute $J_i := e^{\eta T_p}$ for each $i \in [m]$. Finally, choose $\epsilon \in \mathbb{R}_{>0}$ and a prescribed exponential decay rate $\alpha \in \mathbb{R}_{\geq 0}$.

i) **Coder Side**:

Online at time τ_0:

We describe the coder function at time τ_0 ($\gamma^0 : K \to \mathcal{C}^0$):

- Define the overapproximation of the initial set at time τ_0 as

$$B^0 := K; \tag{4.11}$$

- Define the constants[30] $\underline{\kappa}_i^0 := \min\{\langle x, e_i \rangle : x \in K\}$ and $\overline{\kappa}_i^0 := \max\{\langle x, e_i \rangle : x \in K\}$ for every $i \in [d_x]$;
- Define the quantity $\Gamma_i^0 := \frac{\epsilon}{d_x}$ and a set of positive integers $\mathcal{C}_i^0 := \left\{1, \ldots, \lceil \frac{\overline{\kappa}_i^0 - \underline{\kappa}_i^0}{\Gamma_i^0} \rceil \right\}$ for every $i \in [d_x]$. Further, we define the set of

[30] Note that, if K is a box, i.e., a set of the form $\{x \in \mathbb{R}^{d_x} : x = \sum_{i=1}^{d_x} c_i e_i, c_i \in [a_i, b_i]\}$ where $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ with $b_i > a_i$ for each $i \in [d_x]$, then this optimization problem becomes one of checking the values of that inner product at the vertices of that box.

quantization symbols at time τ_0 to be

$$\mathcal{C}^0 := \mathcal{C}_1^0 \times \cdots \times \mathcal{C}_{d_x}^0; \tag{4.12}$$

- Define the i-th quantization symbol at time τ_0, i.e. the quantization symbol related to the projection of the initial state into the vector space span$\{e_i\}$ by

$$q_i^0(x_0) := \{p \in \mathcal{C}_i^0 : \underline{\kappa}_i^0 + \Gamma_i^0(p-1) \leq \langle x_0, e_i\rangle < \underline{\kappa}_i^0 + \Gamma_i^0 p\} \tag{4.13}$$

for each $i \in [d_x]$. Further, if $0 \in \mathcal{R}$, we define the quantization region that contains $x(\tau_0)$ at time τ_0 as

$$\begin{aligned}\bar{B}^0 := \{x \in \mathbb{R}^{d_x} : \underline{\kappa}_i^0 + \Gamma_i^0(q_i^0(x_0) - 1) \leq \langle x, e_i\rangle \\ < \underline{\kappa}_i^0 + \Gamma_i^0 q_i^0(x_0) \text{ for each } i \in [d_x]\},\end{aligned} \tag{4.14}$$

and $\bar{B}^0 := B^0$, otherwise;

- Define the coder function $\gamma^0 : K \to \mathcal{C}^0$ at step zero as

$$\gamma^0(x_0) = (q_1^0(x_0), \ldots, q_{d_x}^0(x_0)). \tag{4.15}$$

Online at time τ_k with $k \in \mathbb{Z}_{>0}$:

We describe the coder function at time τ_k ($\gamma^k : B^k \times \mathcal{C}^0 \times \cdots \times \mathcal{C}^{k-1} \to \mathcal{C}^k$):

- If $n \in \mathcal{R}$, then we define the overapproximation to the reachable set at time τ_k as

$$\begin{aligned}B^k := \{x \in \mathbb{R}^{d_x} : x = e^{A_{\sigma(\tau_k)} T_p} y + \\ + \int_0^{T_p} e^{A_{\sigma(\tau_k)}(T_p - \tau)} B_{\sigma(\tau_k)} u_k(\tau, \hat{y}_{k-1}) d\tau, y \in \bar{B}^{k-1}\}.\end{aligned} \tag{4.16}$$

If $n \notin \mathcal{R}$, then we define the overapproximation to the reachable set at time τ_k as

$$\begin{aligned}B^k := \{x \in \mathbb{R}^{d_x} : |x - \hat{x}(\tau_{k-1})| \leq J_{\sigma(\tau_k)} |\hat{x}(\tau_{k-1})| + \\ + e^{\eta T_p} \sqrt{d_x} \max\{\Gamma_i^{k-1} : i \in [d_x]\}\};\end{aligned} \tag{4.17}$$

- Define the constants

$$\underline{\kappa}_i^k := \min\{\langle y, e_i\rangle : y \in B^k\} \tag{4.18}$$

$$\overline{\kappa}_i^k := \max\{\langle x, e_i\rangle : x \in B^k\} \tag{4.19}$$

for every $i \in \{1, \dots, d\}$;

- Define the constant $\Gamma_i^k := e^{-\alpha T_p}\Gamma_i^{k-1}$. Also, define a set of positive integers $\mathcal{C}_i^k := \left\{1, \dots, \lceil \frac{\overline{\kappa}_i^k - \underline{\kappa}_i^k}{\Gamma_i^k} \rceil \right\}$ for each $i \in [d_x]$ and each $n \in \mathbb{Z}_{\geq 1}$. Further, we define the set of quantization symbols at time τ_k to be

$$\mathcal{C}^k := \mathcal{C}_1^k \times \cdots \times \mathcal{C}_d^k; \tag{4.20}$$

- Define the quantized value of the $i-$th projection of the state at time τ_k into the vector space span$\{e_i\}$ by

$$q_i^k(x_k) := \{p \in \mathcal{C}_i^k : \underline{\kappa}_i^k + \Gamma_i^k(p-1) \leq \langle x_k, e_i\rangle < \underline{\kappa}_i^k + \Gamma_i^k p\} \tag{4.21}$$

for each $i \in [d_x]$. Further, if $k \in \mathcal{R}$, we define the quantization region that contains $x(\tau_k)$ at time τ_k as

$$\begin{aligned} \bar{B}^k := \{x \in \mathbb{R}^{d_x} : \underline{\kappa}_i^k + \Gamma_i^k(q_i^k(x_k) - 1) \leq \langle x, e_i\rangle \\ < \underline{\kappa}_i^k + \Gamma_i^k q_i^k(x_k) \text{ for each } i \in [d_x]\}, \end{aligned} \tag{4.22}$$

and $\bar{B}^k := B^k$, otherwise;

- Define the coder function $\gamma^k : B^k \times \mathcal{C}^0 \times \cdots \times \mathcal{C}^{k-1} \to \mathcal{C}^k$ at step n as

$$\gamma^k(x_k) := (q_1^k(x_k), \dots, q_{d_x}^k(x_k)). \tag{4.23}$$

ii) **Decoder/Controller Side**:

Online at time τ_0:

We describe the decoder function at time τ_0 ($\delta^0 : \mathcal{C}^0 \to B^0$):

- Compute the same constants $\underline{\kappa}_i^0$ and $\overline{\kappa}_i^0$ for every $i \in [d_x]$ as in the coder side. Also, compute the quantization region $\bar{B}^0$;
- If $k \in \mathcal{R}$, define the quantized value of the $i-$th projection of the

initial state into the vector space span$\{e_i\}$ at time τ_0 by

$$\hat{\beta}_i^0(x_0) := \underline{\kappa}_i^0 + \Gamma_i^0\Big(q_i^0(x_0) - 1/2\Big) \tag{4.24}$$

for each $i \in [d_x]$, i.e., the midpoint of the interval[31] $\underline{\kappa}_i^0 + \Gamma_i^0(q_i^0(x_0) - 1) \leq \langle x, e_i \rangle < \underline{\kappa}_i^0 + \Gamma_i^0 q_i^0(x_0)$.

If $k \notin \mathcal{R}$, define $\hat{\beta}_i^0(x_0) := 0$ for each $i \in [d_x]$;

- Define the state estimate at time τ_0 as

$$\hat{x}(\tau_0) := \sum_{i=1}^{d_x} \hat{\beta}_i^0(x_0) e_i. \tag{4.25}$$

Furthermore, we define the decoder at time τ_0 as $\delta^0(q^0) := \hat{x}(\tau_0)$;

We describe the controller function ($u_0 : B^0 \times [\tau_0, \tau_1) \to \mathbb{R}^{d_u}$):

- Define the control law on the time interval $[\tau_0, \tau_1)$ as

$$u_0(\hat{x}(\tau_0), t) = F_{\sigma(\tau_0)}(t - \tau_0)\hat{x}(\tau_0). \tag{4.26}$$

Online at time τ_k with $n \in \mathbb{Z}_{>0}$:

We describe the decoder at time τ_k ($\delta^k : \mathcal{C}^0 \times \cdots \times \mathcal{C}^k \to B^k$):

- Define the quantization regions on the decoder side as in the coder side, i.e., using Equations(4.18) and (4.19). Also, compute $\bar{B}^k$;
- If $k \in \mathcal{R}$, define the value of the projection of the state at time τ_k into the vector space span$\{e_i\}$ by[32]

$$\hat{\beta}_i^k(x_k) := \underline{\kappa}_i^k + \Gamma_i^k(q_i^k(x_k) - 1/2) \tag{4.27}$$

for each $i \in [d_x]$.

If $k \notin \mathcal{R}$, define $\hat{\beta}_i^k(x_k) := 0$;

- Define the state estimate at time τ_k as

$$\hat{x}(\tau_k) := \sum_{i=1}^{d_x} \hat{\beta}_i^k(x_k) e_i. \tag{4.28}$$

[31] Note that $|\langle x_0, e_i \rangle - \hat{\beta}_i^0(x)| \leq \Gamma_i^0/2$.

[32] Note that $|\langle x_k, e_i \rangle - \hat{\beta}_i^k(x_k)| \leq \Gamma_i^k/2$.

Furthermore, we define the decoded value of x_k as
$\delta^k(\gamma^0(x_0), \dots, \gamma^k(x_k)) \coloneqq \hat{x}(\tau_k)$;

We describe the controller function ($u_k : B^k \times [\tau_k, \tau_{k+1}) \to \mathbb{R}^{d_u}$):

- Define the projection of the state estimate $\hat{x}(\tau_k)$ at time τ_k as

$$\hat{y}(\tau_k) \coloneqq P_k \hat{x}(\tau_k). \tag{4.29}$$

- Define the control law for t on the time interval $[\tau_k, \tau_{k+1})$ as

$$u_k(\hat{x}(\tau_k), t) = F_{\sigma(\tau_k)}(t - \tau_k)\hat{y}(\tau_k). \tag{4.30}$$

4.4.3 Controller Analysis

In this subsection, we prove Proposition 4.4.4. This lengthy result will be used in the proof of Theorem 4.3.7 in the next subsection. Also, this result can be seen as the proof of correctness of our Algorithm 4.4.1. We start by stating the simple technical Lemma 4.4.3, whose proof can be found in Appendix A.14.

Lemma 4.4.3. Let $(y_k)_{k\in\mathbb{Z}_{\geq 0}} \subset \mathbb{R}_{\geq 0}$, $a \in \mathbb{R}_{>0}$, and $b \in \mathbb{R}_{\geq 0}$. If $y_{k+1} \leq a y_k + b$ for each $k \in \mathbb{Z}_{\geq 0}$, then we have that $y_k \leq a^k y_0 + b\frac{a^k - 1}{a-1}$ holds for each $k \in \mathbb{Z}_{\geq 0}$.

Now, we are ready to state our proposition.

Proposition 4.4.4. Let the Assumptions 4.3.1 and 4.3.2 hold for some $\ell \in \mathbb{Z}_{>0}$ and $g \in R_{\geq 0}$, respectively. Let $(\mathbf{e}(\tau_n))_{n\in\mathbb{Z}_{\geq 0}}$ be the sequence of state estimation errors produced by Algorithm 4.4.1. Then, the coder-decoder/controller scheme described in Algorithm 4.4.1 ensures that there exist constants $M \in \mathbb{R}_{>0}$ and $P \in \mathbb{R}_{>0}$ such that the solution to (4.1) satisfies

$$|x(t)| \leq M|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s > 0\}}(t) + P\|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty, \tag{4.31}$$

where $\bar{n} \coloneqq \lceil \frac{t-\tau_0}{\ell T_p} \rceil$, $x_0 = x(\tau_0)$, and $\|\mathbf{e}_{|[a,b]}\|_\infty \coloneqq \max\{\sup\{|\mathbf{e}(t)| : t \in [\tau_k, \tau_{k+1})\} : k \in \{a, \dots, b-1\}\}$ for $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{Z}_{\geq 0}$, and $b > a$.

Proof. For $k \in \mathbb{Z}_{\geq 0}$, let $n \in \mathbb{Z}_{\geq 0}$ be $n = n(k) \coloneqq \lfloor \frac{k}{\ell} \rfloor$ and note that this implies that $\tau_k \in [\tau_{\ell n}, \tau_{\ell(n+1)})$. Also, let $\varepsilon_{n\ell} \coloneqq 0$ for $n \in \mathbb{Z}_{\geq 0}$ and $\varepsilon_{k+1} \coloneqq \epsilon e^{-\eta T_p}(\varepsilon_k + g|\mathbf{e}(\tau_k)|)$ for each $k \in \mathbb{Z}_{\geq 0}$ such that $k \neq n\ell - 1$ for any $n \in \mathbb{Z}_{\geq 0}$.

We split this proof into four parts.

Part 1: Analysis of the state at times τ_k for $k \in \mathbb{Z}_{\geq 0}$. In this part, our goal is to prove that[33]

$$x(\tau_k) \in \Big(\Phi(\tau_k, \tau_{\ell n}) \sum_{j=k}^{\ell(n+1)-1} \mathcal{V}_j\Big)^{\varepsilon_k} \tag{4.32}$$

for each $k \in \mathbb{Z}_{\geq 0}$. We do that by induction over $k \in \mathbb{Z}_{\geq 0}$. Before we proceed to prove this fact, we note that we can write

$$x(\tau_{k+1}) = \Phi(\tau_{k+1}, \tau_k)x(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} \Phi(\tau_{k+1}, \tau)B_{\sigma(\tau)}u(\tau)d\tau, \tag{4.33}$$

by the the variation of constants formula applied to system (4.1) with initial state $x(\tau_k)$ at the initial time τ_k when we apply the control $u(\cdot)$. Now, we are ready to prove our induction step

Step 0: note that Assumption 4.3.1 implies that $\left(\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j\right)^{\varepsilon_{n\ell}} = \mathbb{R}^d$. So, $x(\tau_{\ell n}) \in \left(\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j\right)^{\varepsilon_{n\ell}}$ for each $n \in \mathbb{Z}_{\geq 0}$, which implies that our hypothesis is correct for $k = \ell n$ for any $n \in \mathbb{Z}_{\geq 0}$. In particular, this proves the base case $x(\tau_0) \in \mathbb{R}^d$. Furthermore, by the same reasoning, we conclude that we only need to study the cases when $k \neq \ell n$ for any $n \in \mathbb{Z}_{\geq 0}$ to complete our proof.

Step $k+1 \in \mathbb{Z}_{\geq 1}$: as we noted in step 0, we can assume that $k \neq \ell n - 1$ for any $n \in \mathbb{Z}_{\geq 0}$. Further, recall that our induction hypothesis gives us that $x(\tau_p) \in \big(\Phi(\tau_k, \tau_{\ell n}) \sum_{j=p}^{\ell(n+1)-1} \mathcal{V}_j\big)^{\varepsilon_p}$ for all integers $p \leq k$.

Now, let $\bar{P}_k \in \mathcal{M}^{d \times d}$ be the orthogonal projection matrix over the subspace $\Phi(\tau_k, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j$. We note that we can write

$$\begin{aligned} x(\tau_k) &= P_k x(\tau_k) + (I_{d_x} - P_k)x(\tau_k) \\ &= P_k x(\tau_k) + \bar{P}_k(I_{d_x} - P_k)x(\tau_k) + (I_{d_x} - \bar{P}_k)(I_{d_x} - P_k)x(\tau_k), \end{aligned}$$

where $P_k x(\tau_k) \in \Phi(\tau_k, \tau_{\ell n})\mathcal{V}_k$ and $\bar{P}_k(I_{d_x} - P_k)x(\tau_k) \in \Phi(\tau_k, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j$ by how we constructed the matrices P_k and $\bar{P}_k$. To make our discussion simpler, we define $y(\tau_k) \coloneqq P_k x(\tau_k)$, $z(\tau_k) \coloneqq \bar{P}_k(I_{d_x} - P_k)x(\tau_k)$, and $w(\tau_k) \coloneqq (I_{d_x} - \bar{P}_k)(I_{d_x} - P_k)x(\tau_k)$. We notice that $|w(\tau_k)| \leq \varepsilon_k$ since our induction hypothesis tells us that $x(\tau_k) \in \big(\Phi(\tau_k, \tau_{\ell n}) \sum_{j=k}^{\ell(n+1)-1} \mathcal{V}_j\big)^{\varepsilon_k}$ and $y(\tau_k) + z(\tau_k) \in$

[33] Recall that A^{ε} is the ε-collar of the set $A \subset \mathbb{R}^{d_x}$.

$\Phi(\tau_k, \tau_{\ell n}) \sum_{j=k}^{\ell(n+1)-1} \mathcal{V}_j$.
Recall that we use Equations (4.26) and (4.30) to define the control sequence $(u_k(\cdot,\cdot))_{k\in\mathbb{Z}_{\geq 0}}$. Also recall that, when $k \in \mathcal{R}$, Lemma 4.4.2 gives us that $-\Phi_\sigma(\tau_{k+1}, \tau_k)y = \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau_k)} u_k(y, \tau) d\tau$ for any $y \in \mathbb{R}^{d_x}$. Now, we break our analysis into two cases:
Case of $k \notin \mathcal{R}$: In this case, $u_k(t) = 0$ for all $t \in [\tau_k, \tau_{k+1})$. Thus, equation (4.33) gives us that

$$x(\tau_{k+1}) = \Phi(\tau_{k+1}, \tau_k)x(\tau_k) = \Phi(\tau_{k+1}, \tau_k)\big(y(\tau_k) + z(\tau_k) + w(\tau_k)\big).$$

Since $\mathcal{V}_k = \{0\}$, we have that $y(\tau_k) = 0$. Further, recall that $z(\tau_k) \in \Phi(\tau_k, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j$. Therefore, we have that $\Phi(\tau_{k+1}, \tau_k)\big(y(\tau_k) + z(\tau_k)\big) \in \big(\Phi(\tau_{k+1}, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j\big)$, since $\Phi(\tau_{k+1}, \tau_k)\Phi(\tau_k, \tau_{\ell n}) = \Phi(\tau_{k+1}, \tau_{\ell n})$. Furthermore, since $|w(\tau_k)| \leq \varepsilon_k$, we have that $|\Phi(\tau_{k+1}, \tau_k)w(\tau_k)| \leq e^{\eta(\tau_{k+1}-\tau_k)}\varepsilon_k \leq e^{\eta T_p}(\varepsilon_k + g|\mathbf{e}(\tau_k)|) = \varepsilon_{k+1}$. This proves the induction step in this case.
Case of $k \in \mathcal{R}$: In this case, equation (4.30) gives us that $u(t) = F_{\sigma(\tau_k)}(t - \tau_k)\hat{y}(\tau_k)$. Further, since there are no switches on the time interval $[\tau_k, \tau_{k+1})$, we have that $\Phi(\tau_{k+1}, \tau) = e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau)}$ and $B_{\sigma(\tau)} = B_{\sigma(\tau_k)}$ for all $\tau \in [\tau_k, \tau_{k+1})$. Combining these facts with equation (4.33), we get that $x(\tau_{k+1}) = e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\big(y(\tau_k) + z(\tau_k) + w(\tau_k)\big) + \int_{\tau_k}^{\tau_{k+1}} e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau)} B_{\sigma(\tau_k)} F_{\sigma(\tau_k)}(\tau - \tau_k) d\tau \hat{y}(\tau_k)$. Equation (4.10) gives us that $\int_{\tau_k}^{\tau_{k+1}} e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau)} B_{\sigma(\tau_k)} F_{\sigma(\tau_k)}(\tau - \tau_k) d\tau \hat{y}(\tau_k) = -e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\hat{y}(\tau_k)$.
Hence, we get that

$$x(\tau_{k+1}) = e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\big(y(\tau_k) - \hat{y}(\tau_k) + z(\tau_k) + w(\tau_k)\big). \tag{4.34}$$

Since $z(\tau_k) \in \bar{\mathcal{W}}_k \subset \Phi(\tau_k, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j$, we conclude that

$$e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)} z(\tau_k) \in \big(\Phi(\tau_{k+1}, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j\big).$$

Also, we note that $|e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\big(y(\tau_k) - \hat{y}(\tau_k)\big)| \leq e^{\eta(\tau_{k+1}-\tau_k)}|y(\tau_k) - \hat{y}(\tau_k)| = e^{\eta(\tau_{k+1}-\tau_k)}|P_k(x(\tau_k) - \hat{x}(\tau_k))| \leq g e^{\eta(\tau_{k+1}-\tau_k)}|\mathbf{e}(\tau_k)|$ where the first inequality follows from the fact that $\|e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\| \leq e^{\eta(\tau_{k+1}-\tau_k)}$ and that $\|\cdot\|$ is an operator norm. The equality follows from equation (4.29). Finally, the last inequality follows from $|P_k| \leq g$, by our uniform complete controllability with losses assumption, and our assumption that $|x(\tau_k) - \hat{x}(\tau_k)| \leq |\mathbf{e}(\tau_k)|$. Next, we proceed as before to get that $|e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)} w(\tau_k)| \leq e^{\eta(\tau_{k+1}-\tau_k)}\varepsilon_k$. Hence, combining these two inequalities, the fact that $\tau_{k+1} - \tau_k \leq T_p$, and using the triangle inequality, we get that $|e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}\big(y(\tau_k) - \hat{y}(\tau_k) + w(\tau_k)\big)| \leq e^{\eta T_p}(\varepsilon_k + d|\mathbf{e}(\tau_k)|) \leq \varepsilon_{k+1}$. Thus, we get that $x(\tau_{k+1}) \in \big(\Phi(\tau_{k+1}, \tau_{\ell n}) \sum_{j=k+1}^{\ell(n+1)-1} \mathcal{V}_j\big)$ concluding the proof of Part 1.

Part 2: Analysis of the state at times $\tau_{n\ell}$ for $n \in Z_{\geq 0}$. In this part, our goal is to prove that

$$|x(\tau_{n\ell})| \leq E \max\{|\mathbf{e}(\tau_{n\ell+j-1})| : j \in [\ell]\}, \tag{4.35}$$

where $E \coloneqq g e^{\eta T_p} \frac{1-e^{\eta T_p \ell}}{1-e^{\eta T_p}}$. First, we claim that $x(\tau_{n\ell-1}) = y(\tau_{n\ell-1}) + w(\tau_{n\ell-1})$, i.e., $z(\tau_{n\ell-1}) = 0$. Indeed, $z(\tau_{n\ell-1}) \in \big(\Phi(\tau_{n\ell}, \tau_{\ell n}) \sum_{j=n\ell}^{\ell(n+1)-1} \mathcal{V}_j\big) = \{0\}$. Next, we divide our analysis into two cases. If $n\ell \notin \mathcal{R}$, we know that $y(\tau_{n\ell-1}) \in \{0\}$. Thus, equation (4.33) gives us that $|x(\tau_{n\ell})| \leq |\Phi(\tau_{\ell n}, \tau_{\ell n-1}) w(\tau_{\ell n-1})| \leq e^{\eta T_p}\varepsilon_{n\ell-1} \leq e^{\eta T_p}(\varepsilon_{n\ell-1} + g \max\{|\mathbf{e}(\tau_{n\ell+j-1})| : j \in [\ell]\})$. If $n\ell \in \mathcal{R}$, we can follow the same steps as in Part 1 and conclude that the state $x(\tau_{\ell n})$ satisfies equation (4.34), i.e., $x(\tau_{\ell n}) = e^{A_{\sigma(\tau_k)}(\tau_{k+1}-\tau_k)}(y(\tau_{\ell n-1}) - \hat{y}(\tau_{\ell n-1}) + w(\tau_{\ell n-1}))$. Next, we can follow the same steps that we used after equation (4.34) in part 1 to conclude that $|x(\tau_{n\ell})| \leq e^{\eta T_p}(\varepsilon_{n\ell-1} + g|\mathbf{e}(\tau_{\ell n-1})|) \leq e^{\eta T_p}(\varepsilon_{n\ell-1} + g \max\{|\mathbf{e}(\tau_{n\ell+j-1})| : j \in [\ell]\})$. Now, we need to find the value of $\varepsilon_{\ell n-1}$; we obtain that value by solving the recursion for $\varepsilon_{k+1} = e^{\eta T_p}\varepsilon_k + e^{\eta T_p} g \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\}$ with $\varepsilon_{\ell n} = 0$ for $k = \ell n - 1$ to obtain that $\varepsilon_{\ell n-1} = g e^{\eta T_p} \frac{1-e^{\eta T_p(\ell-1)}}{1-e^{\eta T_p \ell}} \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\}$. Finally, we arrive at $|x(\tau_{\ell n})| \leq e^{\eta T_p}\big(g e^{\eta T_p} \frac{1-e^{\eta T_p(\ell-1)}}{1-e^{\eta T_p \ell}} \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\} + g \times$ $\times \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\}\big) = g e^{\eta T_p} \frac{1-e^{\eta T_p \ell}}{1-e^{\eta T_p \ell}} \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\}$. This concludes the proof of Part 2.

Part 3: Analysis of the state's inter-sample behavior, i.e., for $t \in [\tau_k, \tau_{k+1})$. In this part, let $k \in \mathbb{Z}_{\geq 0}$ be arbitrary. Our goal is to prove that, for any $t \in [\tau_k, \tau_{k+1})$, we have that

$$|x(t)| \leq C|x(\tau_k)| + D \max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\}, \tag{4.36}$$

where the constants are $C \coloneqq \max\{e^{\eta T_p} + \frac{\|B_{i_j}\| \|F_{i_j}(t-\tau_k)\|_\infty}{\eta} : j \in [m]\}$ and $D \coloneqq \max\{g \frac{\|B_{i_j}\| \|F_{i_j}(t-\tau_k)\|_\infty}{\eta} : j \in [m]\}$. First, we use the variation of constants formula to get that $|x(t)| = \left|\Phi(t, \tau_k)x(\tau_k) + \int_{\tau_k}^{t} \Phi(t, \tau) B_{\sigma(\tau)} u(\tau) d\tau\right| \leq$

$e^{\eta(t-\tau_k)}|x(\tau_k)| + \|B_{\sigma(\tau_k)}\|\|F_{\sigma(\tau_k)}(\tau-\tau_k)\|_\infty \int_{\tau_k}^t e^{\eta(t-\tau)}d\tau|\hat{y}(\tau_k)|$, where the inequality follows from the triangle inequality and the facts that $\|\Phi(t,\tau_k)\| \leq e^{\eta(t-\tau_k)}$ and that

$$|B_{\sigma(\tau)}u(\tau)| \leq \max\{0, |B_{\sigma(\tau_k)}F_{\sigma(\tau_k)}(t-\tau_k)\hat{y}(\tau_k)|\}$$
$$\leq \|B_{\sigma(\tau_k)}\|\|F_{\sigma(\tau_k)}(\tau-\tau_k)\|_\infty|\hat{y}(\tau_k)|.$$

Thus, by upper-bounding the integral by $\frac{1}{\eta}$, we get that $|x(t)| \leq e^{\eta(t-\tau_k)}|x(\tau_k)| + \frac{\|B_{\sigma(\tau_k)}\|\|F_{\sigma(\tau_k)}(\tau-\tau_k)\|_\infty}{\eta}|\hat{y}(\tau_k)|$. Finally, we can write

$$\begin{aligned}|\hat{y}(\tau_k)| &= |P_k\hat{x}(\tau_k)| \\ &\leq d|\hat{x}(\tau_k)| \\ &\leq g\big(|\hat{x}(\tau_k) - x(\tau_k)| + |x(\tau_k)|\big) \\ &\leq g\max\{|\mathbf{e}(\tau_{\ell n+j-1})|) : j \in [\ell]\} + g|x(\tau_k)|.\end{aligned}$$

By using the fact that $e^{\eta(t-\tau_k)} \leq e^{\eta T_p}$ for any $t \in [\tau_k, \tau_{k+1})$ and rearranging the terms, we arrive at inequality (4.36). This concludes this part.

Part 4: Bounds on the norm of the state for time $t \geq t_0$. In this final part, our goal is to prove that the inequality

$$|x(t)| \leq M|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t) + P\|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty, \tag{4.37}$$

where $\bar{n} := \lceil \frac{t-\tau_0}{\ell T_p} \rceil$, $M := C$, and $P := \left(C^\ell E + D\left(\frac{C^\ell - 1}{C-1}\right)\right)$, holds for all $t \geq t_0$. Before we do that, for each $k \in [\ell]$, we define $y_k := \sup\{|x(t)| : t \in [\tau_{\ell(n-1)+k-1}, \tau_{\ell(n-1)+k})\}$. Further, we define $y_0 := |x(\tau_{\ell(n-1)})|$.

We start by analyzing what happens on the interval $[\tau_{\ell(n-1)}, \tau_{\ell n})$ for an arbitrary $n \in \mathbb{Z}_{\geq 1}$. Let $a = C$ and $b = D\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty$. Note that, by taking the supremum over $t \in [\tau_{\ell(n-1)+k-1}, \tau_{\ell(n-1)+k})$ on both sides of inequality (4.36) we get that $y_k = \sup\{|x(t)| : t \in [\tau_{\ell(n-1)+k-1}, \tau_{\ell(n-1)+k})\} \leq C|x(\tau_{\ell(n-1)+k-1})| + D\|\mathbf{e}_{[\ell(n-1),\ell n]}\|_\infty \leq Cy_{k-1} + D\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty = ay_{k-1} + b$ where the first inequality comes from inequality (4.36) and the second inequality follows from the fact that[34]

$$y_{k-1} = \sup\{|x(t)| : t \in [\tau_{\ell(n-1)+k-2}, \tau_{\ell(n-1)+k-1})\}$$

[34]Note that $|x(\tau_{\ell(n-1)+k-1})| \leq \sup\{|x(t)| : t \in [\tau_{\ell(n-1)+k-2}, \tau_{\ell(n-1)+k-1})\}$ since $x(\cdot)$ is a continuous function on the closed interval $[\tau_{\ell(n-1)+k-2}, \tau_{\ell(n-1)+k-1}]$.

if $k \geq 2$ and $y_{k-1} = |x(\tau_{\ell(n-1)})|$ if $k = 1$. Now, we can apply Lemma 4.4.3 and arrive at $y_k \leq a^k y_0 + b\Big(\frac{a^k-1}{a_1-1}\Big)$. Further, since $a \geq 1$, we get that $y_k \leq a^\ell y_0 + b\Big(\frac{a^\ell-1}{a-1}\Big)$ for any $k \in [\ell]$. Hence, for any $t \in [\tau_{\ell(n-1)}, \tau_{\ell n})$, we get $|x(t)| \leq \max\{\sup\{|x(t)| : t \in [\tau_{\ell(n-1)+k-1}, \tau_{\ell(n-1)+k})\} : k \in [\ell]\} \leq C^\ell |x(\tau_{\ell(n-1)})| + D\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty \Big(\frac{C^\ell-1}{C-1}\Big)$. Now, we break our analysis into two cases. First, let $n \geq 2$. In this case, we can use inequality (4.35) to get that $|x(\tau_{\ell(n-1)})| \leq E\|\mathbf{e}_{|[\ell(n-2),\ell(n-1)]}\|_\infty$. Thus, by the inequality above, we get that

$$
\begin{aligned}
|x(t)| &\leq C^\ell E\|\mathbf{e}_{[\ell(n-2),\ell(n-1)]}\|_\infty + D\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty \Big(\frac{C^\ell-1}{C-1}\Big) \\
&\leq \Big(C^\ell E + D\Big(\frac{C^\ell-1}{C-1}\Big)\Big)\|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty,
\end{aligned}
$$

where we have used the fact that $\max\{\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty, \|\mathbf{e}_{|[\ell(n-2),\ell(n-1)]}\|_\infty\} \leq \|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty$, for any $t \in [\tau_{\ell(n-1)}, \tau_{\ell n})$. Second, when $n = 1$, we get that

$$
\begin{aligned}
|x(t)| \leq& C^\ell |x(\tau_0)| + D\|\mathbf{e}_{|[\ell(n-1),\ell n]}\|_\infty \Big(\frac{C^\ell-1}{C-1}\Big) \\
\leq& C^\ell |x(\tau_0)| + \Big(C^\ell E + D\Big(\frac{C^\ell-1}{C-1}\Big)\Big)\|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty.
\end{aligned}
$$

for any $t \in [\tau_0, \tau_\ell)$. Note that we can write $n \in \mathbb{Z}_{\geq 0}$ implicitly as $\lceil \frac{t-\tau_0}{\ell T_p} \rceil$ since $t \in [\tau_{\ell(n-1)}, \tau_{\ell n})$. Finally, combining the previous inequalities, we conclude the proof of the proposition.

□

4.4.4 Data-Rate Analysis

In this subsection, we finally prove Theorem 4.3.7. Also, we note that in the following proof, we indirectly prove that Algorithm 4.4.1 works with a finite data-rate. Further, we discuss the relationship of the following proof with some results from the topological entropy literature in Appendix A.15.

Proof of Theorem 4.3.7. Before we continue, let $J := \max\{J_i : i \in [m]\}$ and $R_0 := \max\{|x| : x \in K\}$. Also, for this proof, we make the two following definitions: Let $i \in [m]$ be a mode, then we define the *perceived asymptotic* *active rate* as

$$\rho_i^p \coloneqq \limsup_{n\to\infty} \frac{1}{n}\sum_{j\in\mathcal{R}}^{n} \chi_{\{i\}}(\sigma(\tau_j)).$$

Further, for simplicity, we define $\omega \coloneqq \ell - \min\{\#\{\ell n,\dots,\ell(n+1)-1\}\cap\mathcal{R} : n\in\mathbb{Z}_{\geq 0}\}$.

Pick $\epsilon \coloneqq \frac{2\bar{\epsilon}e^{-2\alpha T_p}}{P}$ in Algorithm 4.4.1 and pick $\bar{M} \coloneqq Me^{\alpha\ell T_p}$, where $M\in\mathbb{R}_{>0}$ is the same as in Theorem 4.4.4. Further, definitions (4.24) and (4.27) give us that $|\mathbf{e}(\tau_n)| \leq e^{-\alpha(\tau_n-\tau_0)}$ for all $n\in\mathbb{Z}_{\geq 0}$. Consequently, for any $t\in[\tau_0,\infty)$, we have that $\|\mathbf{e}_{|[\ell(\bar{n}-2),\ell\bar{n}]}\|_\infty \leq \frac{\epsilon}{2}e^{-\alpha\max\{\lceil\frac{t-\tau_0}{\ell T_p}\rceil-2,0\}\ell T_p} \leq \frac{\epsilon}{2}e^{-\alpha\max\{t-\tau_0-2\ell T_p,0\}}$, where $\bar{n}=\lceil\frac{t-\tau_0}{\ell T_p}\rceil$. Choosing $n\in\mathbb{Z}_{\geq 0}$ and combining these previous results with the bound (4.31), we get that

$$\begin{aligned}
|x(t)| &\leq M|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell-s>0\}}(t) + P\|\mathbf{e}_{|[\ell(n-2),\ell n]}\|_\infty\\
&\leq M|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell-s>0\}}(t) + \frac{\epsilon P}{2}e^{-\alpha\max\{t-\tau_0-2\ell T_p,0\}}\\
&= \bar{M}e^{-\alpha\ell T_p}|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell-s>0\}}(t) + \bar{\epsilon}e^{-\alpha\max\{t-\tau_0,2\ell T_p\}}\\
&\leq \bar{M}e^{-\alpha\ell T_p}|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell-s>0\}}(t) + \bar{\epsilon}e^{-\alpha(t-\tau_0)}\\
&\leq (\bar{M}|x_0|+\bar{\epsilon})e^{-\alpha(t-t_0)}
\end{aligned}$$

for all $t\in[\tau_0,\tau_n)$. Here, we used the fact that $\tau_0=t_0$ on the last inequality. The third inequality follows from $\max\{t-\tau_0-2\ell T_p,0\}\geq t-\tau_0$ for all $t\geq\tau_0$. The final inequality follows from the facts that $e^{-\alpha\ell T_p}\leq e^{-\alpha(t-\tau_0)}$ for $t\in[\tau_0,\tau_\ell)$ and that $0=\bar{M}|x_0|\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell-s>0\}}(t) < \bar{M}|x_0|e^{-\alpha(t-t_0)}$ for all $t\geq\tau_\ell$.

Thus, we just proved that for any given $\alpha\in\mathbb{R}_{\geq 0}$, $\bar{M}\in\mathbb{R}_{>0}$, $\bar{\epsilon}\in\mathbb{R}_{>0}$, and any compact set with nonempty interior $K\subset\mathbb{R}^{d_x}$, the set of control laws generated by Algorithm 4.4.1 forms a stabilizing control set[35] $\mathscr{U}(\bar{\epsilon},\bar{M},K,\alpha)$ for system (4.1). From our discussion, we know that the cardinality of the set $\mathscr{U}_T(\bar{\epsilon},\bar{M},K,\alpha)$ is upper-bounded by $\prod_{j=0}^{\lceil\frac{T-t_0}{T_p}\rceil}\#\mathcal{C}^j$. Now, our goal is to provide upper bounds for $\#\mathcal{C}^n$ for each $n\in\mathbb{Z}_{\geq 0}$.

For $n=0$, we have that $\#\mathcal{C}^0\leq\lceil\frac{dR_0}{\epsilon}\rceil$. For $n\in\mathbb{Z}_{\geq 1}$, we divide the analysis into two cases. The first case happens when $n\in\mathcal{R}$ and $n\neq 0$. In this case, $\overline{\kappa}_i^n-\underline{\kappa}_i^n\leq e^{(\alpha+\mu(A_i))T_p}\max\{\Gamma_i^n : i\in[d_x]\}$. Since, Γ_i^n is constant in $i\in[d_x]$, we get that $\#\mathcal{C}_i^n\leq\lceil e^{(\alpha+\mu(A_i))T_p}\rceil$. Thus, we conclude that $\#\mathcal{C}^n\leq\lceil e^{(\alpha+\mu(A_i))T_p}\rceil^d$.

[35] Compare with definition (4.2.2).

The second case happens when $n \notin \mathcal{R}$. In that case, we have that

$$\overline{\kappa}_i^n - \underline{\kappa}_i^n \leq 2(J_{i_n}|\hat{x}(\tau_{n-1})| + e^{\eta T_p} \max\{\Gamma_i^{n-1} : i \in [d_x]\}). \tag{4.38}$$

Using the triangle inequality, we get that $|\hat{x}(\tau_{n-1})| \leq |x(\tau_{n-1})| + |\hat{x}(\tau_{n-1}) - x(\tau_{n-1})| \leq \bar{M}R_0\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t) + \frac{\epsilon P}{2}e^{-\alpha\max\{t-\tau_0,2\ell T_p\}} + \max\{\Gamma_i^{n-1} : i \in [d_x]\}$, where the second inequality comes from (4.31) and the fact that $|x_0| \leq R_0$. Now, we can substitute that on (4.38) to get that

$$\begin{aligned}\overline{\kappa}_i^n - \underline{\kappa}_i^n \leq 2(J_{i_n}(\bar{M}R_0\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t) + \frac{\epsilon P}{2}e^{-\alpha\max\{t-\tau_0-2\ell T_p,0\}} + \\ \max\{\Gamma_i^{n-1} : i \in [d_x]\}) + e^{\eta T_p}\max\{\Gamma_i^{n-1} : i \in [d_x]\}).\end{aligned} \tag{4.39}$$

Dividing both sides by $\Gamma_i^n = e^{-\alpha T_p}\Gamma_i^{n-1} = \frac{\epsilon}{d}e^{-\alpha n T_p}$ and recalling that Γ_i^{n-1} is constant for $i \in [d_x]$, we get that

$$\begin{aligned}\frac{\overline{\kappa}_i^n - \underline{\kappa}_i^n}{\Gamma_i^n} \leq 2(J_{i_n}(\frac{d\bar{M}R_0 e^{\alpha n T_p}\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t)}{\epsilon} + \\ + \frac{dP}{2}e^{-\alpha\max\{-nT_p+(t-\tau_0)-2\ell T_p,-nT_p\}} + e^{\alpha T_p}) + e^{(a+\alpha)T_p}).\end{aligned} \tag{4.40}$$

Note that $e^{\alpha n T_p}\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t) \leq e^{\alpha\ell T_p}$ and that $-\max\{-nT_p + (t-\tau_0) - 2\ell T_p, -nT_p\} = \min\{\lceil\frac{t-\tau_0}{T_p}\rceil T_p - (t-\tau_0) + 2\ell T_p, \lceil\frac{t-\tau_0}{T_p}\rceil T_p\}$. Using the fact that $\lceil\frac{t-\tau_0}{T_p}\rceil T_p - (t-\tau_0) \leq T_p$, we get that $-\max\{-nT_p + (t-\tau_0) + 2\ell T_p, -nT_p\} \leq \min\{(2\ell+1)T_p, nT_p\} \leq (2\ell+1)T_p$ for all $n \in \mathbb{Z}_{\geq 0}$. Therefore,

$$\begin{aligned}\frac{\overline{\kappa}_i^n - \underline{\kappa}_i^n}{\Gamma_i^n} \leq 2(J_{i_n}(\frac{d\bar{M}R_0 e^{\alpha\ell T_p}\chi_{\{s\in\mathbb{R}_{\geq 0}:\tau_\ell - s>0\}}(t)}{\epsilon} + \\ + \frac{dP}{2}e^{\alpha(2\ell+1)T_p} + e^{\alpha T_p}) + e^{(\eta+\alpha)T_p}).\end{aligned} \tag{4.41}$$

Hence, using the fact that $J = \max\{J_i : i \in [m]\}$ and recalling that, by Equation (4.20), we have that $\#\mathcal{C}^n = \left(\prod_{i=1}^d \#\mathcal{C}_i^n\right) = \left(\prod_{i=1}^d \lceil\frac{\overline{\kappa}_i^n - \underline{\kappa}_i^n}{\Gamma_i^n}\rceil\right)$. Therefore, we conclude that $\#\mathcal{C}^n \leq \lceil\frac{2Jd\bar{M}R_0e^{\alpha\ell T_p}}{\epsilon} + Jde^{\alpha(2\ell+1)T_p} + 2Je^{\alpha T_p} + 2e^{(\alpha+\eta)T_p}\rceil^d$, for $n < \ell$ and $n \notin \mathcal{R}$, and that $\#\mathcal{C}^n \leq \lceil Jde^{\alpha(2\ell+1)T_p} + 2Je^{\alpha T_p} + 2e^{(\alpha+\eta)T_p}\rceil^d$, for $n \geq \ell$ and $n \notin \mathcal{R}$. For $n \in \mathbb{Z}_{\geq 1}$, let $q(n) := \lfloor n/\ell \rfloor$ be the integer part of the division of n by ℓ and let $r(n) := n - q(n)\ell$ be the remainder of that same division. Further, given a sequence $(c_n)_{n\in\mathbb{Z}_{\geq 0}}$, for $a \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}_{\geq 0}$

with $a > b$ we define $\sum_{j=a}^{b} c_j = 0$. Additionally, define[36] $k := \lceil \frac{T}{T_p} \rceil$. Now, we can write that

$$
\begin{aligned}
& b(\mathscr{U}(\bar{\epsilon}, \bar{M}, K, \alpha)) \\
& \leq \limsup_{T\to\infty} \frac{1}{T} \sum_{j=0}^{\lceil \frac{T}{T_p} \rceil} \log\bigl(\#\mathcal{C}^j\bigr) \leq \limsup_{k\to\infty} \frac{1}{kT_p} \sum_{j=0}^{k} \log\bigl(\#\mathcal{C}^j\bigr) \\
& = \limsup_{k\to\infty} \frac{1}{kT_p} \Bigl(\sum_{j=0}^{\ell-1} \log\bigl(\#\mathcal{C}^j\bigr) + \sum_{j=\ell}^{\ell q(k)} \log\bigl(\#\mathcal{C}^j\bigr) + \sum_{j=\ell q(k)+1}^{\ell q(k)+r(k)} \log\bigl(\#\mathcal{C}^j\bigr) \Bigr) \\
& \leq \limsup_{n\to\infty} \frac{1}{kT_p} \Bigl(\sum_{j=0}^{\ell-1} \log\bigl(\#\mathcal{C}^j\bigr) + \omega(q(k)-1) \times \\
& \times (d \log\bigl(\lceil Jde^{\alpha(2\ell+1)T_p} + 2Je^{\alpha T_p} + 2e^{(\alpha+\eta)T_p} \rceil\bigr)) + \sum_{\substack{j=\ell \\ j\in\mathcal{R}}}^{\ell q(k)} \log\bigl(\#\mathcal{C}^j\bigr) + \\
& + \sum_{j=\ell q(k)+1}^{\ell q(k)+r(k)} \log\bigl(\#\mathcal{C}^j\bigr) \Bigr),
\end{aligned} \tag{4.42}
$$

where the first inequality follows from our coder-decoder/controller pair description[37] and the second inequality follows from the fact that $\lceil \frac{T}{T_p} \rceil = k$ and that $kT_p \leq T$. The third inequality follows from the fact that there are at most $\omega(q(k)-1)$ values of $j \notin \mathcal{R}$ on the interval $\{\ell, \ldots, \ell q(k)\}$ by Assumption 4.3.1. Next, note that the first and last terms on right-hand side of (4.42) are bounded since those sums are finite. Therefore, considering the division by $\frac{1}{kT_p}$, their limit superior equals zero. The ratio $\frac{\omega(q(k)-1)}{kT_p}$ has a limit superior that equals $\frac{\omega}{\ell T_p}$ since $k = q(k)\ell + r(k)$ with $0 \leq r(k) < \ell$. Also, notice that $\log(\#C^j) \leq \sum_{i\in[m]} \chi_{\{i\}}(\sigma(j)) d \log\bigl(\lceil e^{(\mu(A_i)+\alpha)T_p} \rceil\bigr)$ for any $j \in \mathcal{R}$ since $\sigma(j) \in [m]$ for each $j \in \mathbb{Z}_{\geq 0}$ and because of the upper bound we derived

[36] k depends on T, but we drop that dependency to simplify the notation.
[37] See Equations (4.26) and (4.30) in our Algorithm 4.4.1 description.

for $\#\mathcal{C}$ when $n \in \mathcal{R}$. Further, note that (4.4.4) can be written as

$$\rho_i^p = \limsup_{n\to\infty} \frac{1}{n}\Big(\sum_{\substack{j \in \mathcal{R}}}^{\ell-1} \chi_{\{i\}}(\sigma(j)) + \sum_{\substack{j=\ell \\ j \in \mathcal{R}}}^{\ell q(n)} \chi_{\{i\}}(\sigma(j)) + \sum_{\substack{j=\ell q(n)+1 \\ j \in \mathcal{R}}}^{\ell q(n)+r(n)} \chi_{\{i\}}(\sigma(j))\Big) = \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{j=\ell \\ j \in \mathcal{R}}}^{\ell q(n)} \chi_{\{i\}}(\sigma(j)),$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{kT_p} \sum_{\substack{j=\ell \\ j \in \mathcal{R}}}^{\ell q(k)} \log\big(\#\mathcal{C}^j\big) \\
&\leq \limsup_{k\to\infty} \frac{1}{kT_p} \sum_{\substack{j=\ell \\ j \in \mathcal{R}}}^{\ell q(k)} \sum_{i \in [m]} \chi_{\{i\}}(\sigma(j)) d \log\big(\lceil e^{(\mu(A_i)+\alpha)T_p} \rceil\big) \\
&\leq \frac{1}{T_p} \sum_{i \in [m]} \Big(\limsup_{k\to\infty} \frac{1}{k} \sum_{\substack{j=\ell \\ j \in \mathcal{R}}}^{\ell q(k)} \chi_{\{i\}}(\sigma(j)) \Big) d \log\big(\lceil e^{(\mu(A_i)+\alpha)T_p} \rceil\big) \\
&= \frac{1}{T_p} \sum_{i \in [m]} \rho_i^p d \log\big(\lceil e^{(\mu(A_i)+\alpha)T_p} \rceil\big).
\end{aligned} \tag{4.43}$$

Hence, we can write

$$\begin{aligned}
&b(\mathscr{U}(\bar{\epsilon}, \bar{M}, K, \alpha)) \\
&\leq \frac{\omega}{\ell} \frac{d \log\big(\lceil Jde^{\alpha(2\ell+1)T_p} + 2Je^{\alpha T_p} + 2e^{(\alpha+a)T_p} \rceil\big)}{T_p} + \\
&+ \frac{1}{T_p} \sum_{i=1}^{m} \rho_i^p d \log\big(\lceil e^{(\mu(A_i)+\alpha)T_p} \rceil\big).
\end{aligned}$$

Finally, note that Equation (4.5) gives us that

$$\begin{aligned} b(\bar{M},\alpha) &= \lim_{\bar{\epsilon}\to 0}\inf\{b(\mathscr{R}(\bar{\epsilon},\bar{M},K,\alpha))\} \\ &\leq \lim_{\bar{\epsilon}\to 0} b(\mathscr{U}(\bar{\epsilon},\bar{M},K,\alpha)) \\ &\leq \frac{\omega}{\ell}\frac{d\log\left(\lceil Jde^{\alpha(2\ell+1)T_p} + 2Je^{\alpha T_p} + 2e^{(\alpha+a)T_p}\rceil\right)}{T_p} + \\ &+ \frac{1}{T_p}\sum_{i=1}^{m}\rho_i^p d\log\left(\lceil e^{(\mu(A_i)+\alpha)T_p}\rceil\right). \end{aligned}$$

Note that the right-hand side of the previous inequality is an upper bound for Algorithm 4.4.1 data-rate

$$b_{\mathrm{rx}}(\epsilon,\alpha) = \limsup_{n\to\infty}\frac{1}{nT_p}\sum_{i=1}^{n}\log(\#\mathcal{C}^n),$$

concluding the proof.

□

4.5 Conclusion

We discussed in this chapter why we need a new controllability notion for systems that operate with a finite data-rate. Then, we presented a necessary condition and a sufficient condition for switched linear systems to be controllable with a finite data-rate. Next, we used the switched linear system's structure to get simpler sufficient conditions. The first condition stated in Lemma 4.3.3, uses the controllable subspaces of the modes and a mild assumption on the switching signal to establish controllability with a finite data-rate. The second one, stated in Proposition 4.3.5, required us to activate some controllable mode frequently enough. In particular, when all the modes are controllable, this latter condition boils down to a simple inequality for the sampling frequency that guarantees that a system that satisfies an ADT condition is controllable with a finite data-rate. After that, we studied a sufficient condition for controllability with a finite data-rate when packet losses might be present. We proved this condition by introducing an algorithm, which gives us a controller design technique.

CHAPTER 5

FUTURE RESEARCH DIRECTIONS

In the future, our goal will be to extend the present results to the nonlinear case. More specifically, we want to extend the notion of controllability with a finite data-rate to systems of the form

$$\dot{x}(t) = f(x(t), u(t)). \tag{5.1}$$

At the moment, it seems intuitive that such a generalization is possible when we consider controllability around a nominal trajectory. More formally, let $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ be the nominal control with the nominal initial condition $\gamma_0 \in \mathbb{R}^d$ and let the corresponding solution $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ of (5.1) be the nominal trajectory, i.e., $\gamma(\cdot)$ is the unique solution of (5.1) when the control is equal to $\nu(\cdot)$ and initial condition $\gamma(0) = \gamma_0$. In practice, we only know that $\gamma_0 \in K \subset \mathbb{R}^d$, where K is a compact set. It is reasonable to assume that we know an estimate $\bar{\gamma}_0 \in K$ for γ_0 so that the error $|\gamma_0 - \bar{\gamma}_0|$ is small in some sense. If system (5.1) is contractive, i.e., its nearby trajectories get closer to each other exponentially fast, our control strategy should work. However, if the nominal system $\dot{y}(t) = \bar{f}(y(t)) \coloneqq f(y(t), \nu(t))$ has positive topological entropy, we should expect that the true trajectory will diverge from the nominal one. Intuitively, our system needs feedback, and how much data we need to transmit from the system to the controller is related to the topological entropy. To address this issue, we should make small corrections to the nominal control $\nu(\cdot)$ as we receive more information from our system's output. We can do that by applying a small additive perturbation, i.e., $u(t) = \nu(t) + \omega(t)$ for some function $\omega(\cdot)$. Denoting the solution of (5.1) with initial condition $\bar{\gamma}_0 \in K$ and control $u(\cdot)$ by $\bar{\gamma}(\cdot)$, we define the error as $e(t) \coloneqq \gamma(t) - \bar{\gamma}(t)$. Using the Taylor expansion, we conclude that $\dot{e}(t) = \dot{x}(t) - \dot{\bar{x}}(t) = f_x(\gamma(t), \nu(t))e(t) + f_u(\gamma(t), \nu(t))\omega(t) + r(t, e(t), \omega(t))$, where $r(t, e(t), \omega(t))$ is the Taylor expansion's remainder, and the functions $f_x(\cdot, \cdot)$

and $f_u(\cdot,\cdot)$ denote the Jacobians of $f(\cdot,\cdot)$ with respect to the first and second variable, respectively. If we ignore the higher-order terms, we get that

$$\dot{e}(t) = f_x(\gamma(t), \nu(t))e(t) + f_u(\gamma(t), \nu(t))\omega(t),$$

which is an LTV system. Thus, we can apply the theory of Chapter 3 and the data-rate for controlling the original system will be related to the data-rate to control the linearized one. Nonetheless, a careful analysis is needed here to deal with the higher-order terms since their contribution can be relevant. Therefore, we will focus on this problem as an interesting future research direction.

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APPENDIX A

A.1 Proof of equivalence between the two definitions of complete controllability

Proof. The definition of complete controllability in [56] can be understood as follows: For every[1] $t \in \mathbb{R}_{\geq t_0}$, there exists $\bar{t} \geq t$, such that $W(\bar{t}, t)$ is positive definite. First, we prove that this definition implies the complete controllability definition given in Definition 4.3.1. We prove this fact by induction. For our base step, pick $s_0 = t_0$. By the defintion of complete controllability from [56], we know that there exists $s_1 > s_0$ such that $W(s_1, s_0)$ is positive definite, which implies that it is invertible. Now we consider the step $n \in \mathbb{Z}_{\geq 1}$. Note that there exists $s_{n+1} > s_n$ such that $W(s_{n+1}, s_n)$ is positive definite. Hence, we proved that there exists an increasing sequence $(s_n)_{n \in \mathbb{Z}_{\geq 0}}$ such that $W(s_{n_1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. Therefore, we proved the first part of the claim.

Now, we assume Definition 4.3.1 and we show that this implies the definition given in [56]. For any $t \in \mathbb{R}_{\geq t_0}$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $t \leq s_n$. Consider $W(s_{n+1}, t)$. Note that

$$W(s_{n+1}, t) = W(s_{n+1}, s_n) + \Phi(s_{n+1}, s_n) W(s_n, t) \Phi'(s_{n+1}, s_n).$$

By hypothesis, we know that $W(s_{n+1}, s_n)$ is positive definite[2] and we know that $\Phi(s_{n+1}, s_n) W(s_n, t) \Phi'(s_{n+1}, s_n)$ is positive semi-definite. Therefore, $W(s_{n+1}, t)$ is positive definite and we proved the claim.

□

[1]Here we are imposing that the initial time is $t_0 \in \mathbb{R}$, which was not required in [56].
[2]This is equivalent to invertibility of the Gramian.

A.2 Proof of Lemma 3.3.2

Proof. Let $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ be such that $W(s_{n+1}, s_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$ and that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = R$. Recursively define $\bar{s}_0 := s_0$ and $\bar{s}_n := \min\{s \in (s_n)_{n\in\mathbb{Z}_{\geq 0}} : s \geq \bar{s}_{n-1} + 1\}$ for every $n \in \mathbb{Z}_{\geq 1}$. First, notice that $W(\bar{s}_{n+1}, \bar{s}_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$ because there exists at least two distinct elements from $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ in the interval $[\bar{s}_n, \bar{s}_{n+1}]$. Next, note that, for every $n \geq 1$, we have that $\bar{s}_n \geq 1$ because $s_0 \geq 0$ and the fact that $\bar{s}_n \geq \bar{s}_1 \geq 1$. Now, for every $n \in \mathbb{Z}_{\geq 0}$ we have that $\bar{s}_n \in (s_n)_{n\in\mathbb{Z}_{\geq 0}}$. Thus, there exists $m_n \in \mathbb{Z}_{\geq 0}$ such that $\bar{s}_n = s_{m_n}$. Write $\frac{s_{m_{n+1}}}{s_{m_n}}$. By the definition of $\bar{s}_{n+1}$, we have that $s_{m_{n+1}-1} < s_{m_n} + 1$. Hence, $\frac{s_{m_{n+1}-1}}{s_{m_n}} < \frac{s_{m_n}+1}{s_{m_n}} \leq 2$, where the last inequality comes from the fact that $s_{m_n} \geq 1$. With this, we conclude that $\limsup_{n\to\infty} \frac{\bar{s}_{n+1}}{\bar{s}_n} = \limsup_{n\to\infty} \frac{s_{m_{n+1}}}{s_{m_n}} \frac{s_{m_{n+1}-1}}{s_{m_{n+1}-1}} \leq$
$\limsup_{n\to\infty} 2\frac{s_{m_{n+1}}}{s_{m_{n+1}-1}} = 2R$. Further, note that $\bar{s}_{i+1} - \bar{s}_i \geq 1$ for every $i \in \mathbb{Z}_{\geq 0}$. Thus, $\bar{s}_n - \bar{s}_0 = \sum_{i=0}^{n} \bar{s}_{i+1} - \bar{s}_i \geq n$, where the first equality comes from the equality for telescoping sums. Hence, $\frac{\bar{s}_{n+1}-\bar{s}_0}{\bar{s}_n} \geq \frac{n}{\bar{s}_n}$. Taking the limit superior when n goes to infinity, we get that $2R \geq \limsup_{n\to\infty} \frac{\bar{s}_{n+1}-\bar{s}_0}{\bar{s}_n} \geq \limsup_{n\to\infty} \frac{n}{\bar{s}_n}$.
Therefore, we proved that given a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ we can build a subsequence $(\bar{s}_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $W(\bar{s}_{n+1}, \bar{s}_n)$ is invertible for every $n \in \mathbb{Z}_{\geq 0}$, that $\limsup_{n\to\infty} \frac{\bar{s}_{n+1}}{\bar{s}_n} < \infty$, and that $\limsup_{n\to\infty} \frac{n}{\bar{s}_n} < \infty$.

□

A.3 Proof of Lemma 3.3.4

Proof. Note that, for every $v \neq 0$ in $\mathbb{R}^d$, we have that

$$\begin{aligned} v'\Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s)v &= |v'\Phi(s_{n+1}, s)B(s)|^2 \\ &\leq \|\Phi(s_{n+1}, s)\|^2\|B(s)\|^2|v|^2. \end{aligned}$$

Also, because $\|\Phi(s_{n+1}, s)\| \leq e^{\xi(s_{n+1}-s)}$ for every $s \in [s_n, s_{n+1})$ by Lemma 3.3.3, we get

$$\Phi(s_{n+1}, s)B(s)B'(s)\Phi(s_{n+1}, s) \preccurlyeq \sup\{\|B(t)\|^2 : t \geq t_0\}e^{2\xi(s_{n+1}-s)}I_d.$$

Now, integrating both sides from s_n to s_{n+1}, we conclude that

$$W(s_{n+1}, s_n) \preccurlyeq \sup\{\|B(t)\|^2 : t \geq t_0\} \frac{e^{2\xi(s_{n+1}-s_n)} - 1}{2\xi} I_d.$$

Finally, taking the norm and noticing that $e^{-2\xi s_n} < 1$, we get

$$\|W(s_{n+1}, s_n)\| \leq \sup\{\|B(t)\|^2 : t \geq t_0\} \frac{e^{2\xi(s_{n+1}-s_n)} - 1}{2\xi}.$$

and we conclude the proof. □

A.4 Proof of Lemma 3.3.3

Proof. Recall that $X(t) = \Phi(t, t_0)$ is the solution to the matrix differential equation

$$\frac{dX(t)}{dt} = A(t)X(t)$$

with $X(t_0) = I_d$ and that $X(\cdot)$ is given by the Peano-Baker series[3]. More explicitly, consider the recursively defined matrices $M_k(t, t_0)$ for $t \geq t_0$ and all $k \in \mathbb{Z}_{\geq 0}$ by

$$M_0(t, t_0) := I_d$$

and

$$M_k(t, t_0) := I_d + \int_{t_0}^{t} A(\tau) M_{k-1}(\tau, t_0) d\tau$$

for $k \in \mathbb{Z}_{>0}$. Now, pick an arbitrary $t_1 > t_0$. It is a well-known fact that $M_k(\cdot, t_0)$ converges uniformly[4] to $X(\cdot) = \Phi(\cdot, t_0)$ on the interval $[t_0, t_1]$.

[3]See, e.g. Chapter 4 of [54] or Chapter 3 of [42].

[4]See, e.g., Theorem 1 of Chapter 3 of [42].

Our goal now is to prove that $\|\Phi(t,t_0)\| \leq e^{\xi(t-t_0)}$ for all $t \in [t_1, t_0]$. We do that by proving that $\|M_k(t,t_0)\| \leq \sum_{i=0}^{k} \xi^i \frac{(t-t_0)^i}{i!}$ holds for every $k \in \mathbb{Z}_{\geq 0}$ using induction. The base case $\|M_0(t,t_0)\| \leq 1$ is trivially true[5]. Now, assume that $\|M_{k-1}(t,t_0)\| \leq \sum_{i=0}^{k-1} \xi^i \frac{(t-t_0)^i}{i!}$ is true. Then,

$$\begin{aligned}\|M_k(t,t_0)\| \leq & 1 + \int_{t_0}^{t} \xi \|M_{k-1}(\tau,t_0) d\tau\| \leq \\ & 1 + \sum_{i=0}^{k-1} \xi^{i+1} \frac{(t-t_0)^{i+1}}{(i+1)!} = \sum_{j=0}^{k} \xi^j \frac{(t-t_0)^j}{j!}\end{aligned}$$

where $j = i+1$ and the inequality holds for all $t \in [t_0, t_1]$. Thus,

$$\|\Phi(t,t_0)\| = \| \lim_{N\to\infty} M_N(t,t_0)\| = \lim_{N\to\infty} \|M_N(t,t_0)\| \leq e^{\xi(t-t_0)}.$$

for all $t \in [t_0, t_1]$. Since $t_1 > t_0$ was arbitrary, $\|\Phi(t,t_0)\| \leq e^{\xi(t-t_0)}$ holds for every $t \geq t_0$. Moreover, by definition of norm, we have that $\|\Phi(t,t_0)\| \geq |\Phi(t,t_0)v|$ for any $v \in \mathbb{R}^d$ with $|v| = 1$. Thus, we get $|\Phi(t,t_0)v| \leq e^{\xi(t-t_0)}$ for all $t \geq t_0$ and all $|v| = 1$, which proves the upper bound.
For the lower bound, let $Z(t) = \Phi'(t_0, t)$. It is a well-known that[6]

$$\frac{dZ(t)}{dt} = -A'(t)Z(t)$$

with $Z(t_0) = I_d$. Thus, we can apply an analogous reasoning to get that $\|\Phi'(t_0,t)\| \leq e^{\xi(t-t_0)}$ since $\xi = \sup\{\| - A'(t)\| : t \geq t_0\}$ as well. Finally, pick any $v \in \mathbb{R}^d$ with $|v| = 1$ and note that

$$1 = |v'v| = |v'I_d v| = |v'\Phi(t_0,t)\Phi(t,t_0)v| \leq |v'\Phi(t_0,t)||\Phi(t,t_0)v|.$$

Now, divide by[7] $|v'\Phi(t_0,t)|$ to get

$$|\Phi(t,t_0)v| \geq |v'\Phi(t_0,t)|^{-1}.$$

Next, note that

[5] We are using the convention that, for $t = t_0$, $(t-t_0)^0 = \lim_{t\to t_0}(t-t_0)^0 = 1$.
[6] See, e.g., Chapter 4 of [54].
[7] $\Phi(\cdot,\cdot)$ is always invertible, so $|v'\Phi(t_0,t)|$ cannot be zero.

$$|\Phi(t,t_0)v| \geq \min\{|\Phi(t,t_0)v| : |v| = 1\} \geq \min\{|v'\Phi(t_0,t)|^{-1} : |v| = 1\}$$
$$= \Big(\max\{|v'\Phi(t_0,t)| : |v| = 1\} \Big)^{-1} = \|\Phi(t_0,t)\|^{-1},$$

where the last equality follows from the definition of norm of a matrix. Finally, recall that $\|\Phi'(t_0,t)\| = \|\Phi(t_0,t)\|$. So, we have, for any $v \in \mathbb{R}^d$ and $|v| = 1$, that

$$|\Phi(t,t_0)v| \geq \|\Phi(t_0,t)\|^{-1} \geq e^{-\xi(t-t_0)}.$$

Therefore, we concluded the proof. □

A.5 Proof of Proposition 4.2.2

To prove this proposition, we first need the following lemma.

Lemma A.5.1. Let $t_0 \in \mathbb{R}_{\geq 0}$, let

$$F(t) = \int_0^1 (1-\tau) f_x(t, \sigma(t), v(t), \tau\xi(t,t_0,x_{k_1}) + (1-\tau)\xi(t,t_0,x_{k_2}))d\tau,$$

where $f_x : [t_0,\infty) \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ is continuous, $\sigma : [t_0,\infty) \to [m]$ is a càdlàg function, $u(x_{k_1},\cdot) : [t_0,\infty) \to \mathbb{R}^{d_u}$ is an integrable locally essentially bounded function that belongs to a control law $\mathcal{U}(K)$, which operates with a finite data-rate. Also, let $x_{k_1} \in K$, let $x_{k_2} \in K$, and let $\xi(t,t_0,x) \in \mathbb{R}^{d_x}$ be the same as in the statement of Proposition 4.2.2 for each $x \in K$ and $t \geq t_0$. Then, $F(\cdot)$ is a locally integrable function.

Proof of Lemma A.5.1. :

Let $D \subset \mathbb{R}$ be a compact set and let $d \in \mathbb{Z}_{>0}$ be arbitrary. We endow $\mathbb{R}$ with the usual Lebesgue σ-algebra $\mathcal{L}(\mathbb{R})$ and $\mathbb{R}^d$ with the product σ-algebra $\mathcal{L}(\mathbb{R}^d) := \otimes_{i=1}^d \mathbb{R}$. Further, we endow any Lebesgue-measurable subset $S \subset \mathbb{R}^d$ with the subset σ-algebra, i.e., the σ-algebra $\mathcal{L}(S, \mathbb{R}^d) := \{X \cap S : X \in \mathcal{L}(\mathbb{R}^d)\}$. Furthermore, we treat any subset of $\mathbb{Z}_{\geq 0}$ as a subset of $\mathbb{R}$. Additionally, we denote by $\lambda_{\mathbb{R}}(\cdot)$ the Lebesgue measure and by $\lambda_{\mathbb{R}^d}(\cdot)$ the d-dimensional Lebesgue measure. We recall that that $\lambda_{\mathbb{R}^d}(\prod_{i=1}^d X_i) = \prod_{i=1}^d \lambda_{\mathbb{R}}(X_i)$ when $X_i \in \mathcal{L}(\mathbb{R})$ for each $i \in [d]$. Finally, we define the Lebesgue measure $\lambda_S(\cdot)$ on a subset $S \subset \mathbb{R}^d$ as the restriction of the function $\lambda_{\mathbb{R}^d}(\cdot)$ to the subspace σ-algebra $\mathcal{L}(S, \mathbb{R}^d)$. We also note that we endow the Cartesian product of two topological spaces with the product topology.

Let $h_1(\cdot) := \sigma_{|D}(\cdot)$, $h_2(\cdot) := u_{|D}(x_{k_1}, \cdot)$, and $h_3(t, \tau) := \tau\xi(t, t_0, x_{k_1}) + (1 - \tau)\xi(t, t_0, x_{k_2})$ for $(t, \tau) \in D \times [0, 1]$. Note that $h_1(\cdot)$ is bounded and measurable[8], $h_2(\cdot)$ is essentially bounded and measurable, and $h_3(\cdot, \cdot)$ is continuous. The latter fact follows from the facts that $\xi(\cdot, t_0, x_{k_1})$ is an absolutely continuous function (see, e.g., Section 1.5 from [60]) and usual properties of sums and products of continuous functions. We claim that the function $G : D \times [0, 1] \to \mathbb{R}^{d_x \times d_x}$ defined as $G(t, \tau) := f_x(t, h_1(t), h_2(t), h_3(t, \tau))$ is a measurable function. Indeed, if we define the function $g : D \times [0, 1] \to \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x}$ as $g(t, \tau) := (t, h_1(t), h_2(t), h_3(t, \tau))$, Proposition 2.4 from [59] tells us that $g(\cdot, \cdot)$ is measurable. Next, note that $G(t, \tau) = (f_x \circ g)(t, \tau)$ for each $(t, \tau) \in D \times [0, 1]$. Since $g(\cdot, \cdot)$ is measurable and $f_x(\cdot, \cdot, \cdot, \cdot)$ is continuous, it follows that $G(\cdot, \cdot)$ is measurable (see, e.g., Section 2.1 from [59]).

[8] Every bounded càdlàg function is Lebesgue measurable since its discontinuity set is countable. See, e.g., Chapter 3 from [72].

We claim that $G(\cdot,\cdot)$ is essentially bounded. To see that, we make some definitions: let $B_i \coloneqq \overline{\{h_i(t) : (D \setminus N)\}}$ for each $i \in [2]$ and note that these sets are compact. This latter claim follows from the fact that $h_i(\cdot)$ is bounded on $D \setminus N$ and the fact that B_i is a closed of an Euclidean space for each $i \in [2]$. Also, let $B_3 \coloneqq \{h_3(t,\tau) : (t,\tau) \in D \times [0,1]\}$, which is also compact because $D \times [0,1]$ is compact and $h_3(\cdot,\cdot)$ is continuous. Since $f_x(\cdot,\cdot,\cdot,\cdot)$ is continuous and $D \times B_1 \times B_2 \times B_3$ is compact (the finite product of compact sets is compact, see, e.g., Theorem 26.7 from [73]), we get that $M_G \coloneqq \sup\{\|f_x(t,p_1,p_2,p_3)\| : (t,p_1,p_2,p_3) \in D \times B_1 \times B_2 \times B_3\} < \infty$. Now, note that

$$\begin{aligned}
\sup\{\|G(t,\tau)\| &: (t,\tau) \in (D \setminus N) \times [0,1]\} \\
&= \sup\{\|f_x(t,h_1(t),h_2(t),h_3(t,\tau))\| : (t,\tau) \in (D \setminus N) \times [0,1]\} \\
&\leq \sup\{\|f_x(t,p_1,p_2,p_3)\| : (t,p_1,p_2,p_3) \in D \times B_1 \times B_2 \times B_3\} \\
&= M_G.
\end{aligned}$$

Thus, $\|G(\cdot,\cdot)\| < \infty$ on $(D \setminus N) \times [0,1]$. All that we need to do now is prove that $N \times [0,1]$ is a null set in $D \times [0,1]$. This follows from the fact that $\lambda_{D\times[0,1]}(N \times [0,1]) = \lambda_D(N)\lambda_{[0,1]}([0,1]) = 0$ since $\lambda_D(N) = 0$, proving the claim. We also note that, since $G(\cdot,\cdot)$ is measurable and essentially bounded, it is integrable.
Finally, note that $F(t) \coloneqq \int_0^1 (1-\tau)G(t,\tau)d\tau$. Since $G(\cdot,\cdot)$ and $1-\tau$ are integrable, we get that $(1-\tau)G(t,\tau)$ is integrable. Then, the Fubini-Tonelli Theorem (see, e.g., Theorem 2.37 from [59]) guarantees that $F(\cdot)$ is integrable as well. Further, note that $F(\cdot)$ is bounded. Indeed, $\|F(t)\| = \|\int_0^1 (1-\tau)G(t,\tau)d\tau\| \leq \int_0^1 \|G(t,\tau)\|d\tau \leq M_G$, proving the lemma.
This concludes the proof of Lemma A.5.1. □

Now, we are ready to prove the Proposition. Since $b(\mathscr{U}(K)) = \limsup_{t\to\infty} \frac{1}{t-t_0}\log(\#\mathscr{U}_t(K))$, we know that, for each $\epsilon \in \mathbb{R}_{>0}$, there exists $s^1 = s^1(\epsilon) \in [t_0,\infty)$ such that

$$\frac{1}{t-t_0}\log(\mathscr{U}_t(K)) < \epsilon + b(\mathscr{U}(K)) \tag{A.1}$$

for all $t \geq s^1$.

Since $K \subset \mathbb{R}^{d_x}$ has nonempty interior, there exists some point $\bar{x} \in K$ and some $\delta = \delta(\bar{x}) \in \mathbb{R}_{>0}$ such that the open ball $\mathbb{B}(\bar{x}, \delta) := \{x \in \mathbb{R}^d : |x| < \delta\}$ is contained in K. Further, since K is closed, the closed ball $\mathbb{B}[\bar{x}, \delta] := \{x \in \mathbb{R}^d : |x| \leq \delta\}$ is also contained in K.

Next, choose an arbitrary $s \in \mathbb{R}_{>0}$ so that $s > s^1 = s^1(\epsilon)$ and define $N_s := \#\mathscr{U}_s(K)$. Also, choose $N_s + 1$ points $S := \{x_1, \ldots, x_{N_s+1}\} \subset \mathbb{B}[\bar{x}, \delta]$ such that $|x_i - x_j| \geq \frac{\delta}{N_s+1}$ for each pair $(i, j) \in [N_s + 1]^2$ with $i \neq j$. Note that we can always choose a set of $N_s + 1$ points with this property. One example is the set $S = \{x \in \mathbb{R}^{d_x} : x = \bar{x} + \frac{\delta}{N_s+1} p e_1 \text{ for } p \in [N_s + 1]\}$, where $e_1 = (1, 0, \ldots, 0)' \in \mathbb{R}^{d_x}$. It is easy to verify that given $x \in S$ and $y \in S$ distinct, we have $|x - y| = \frac{\delta}{N_s+1} q \geq \frac{\delta}{N_s+1}$ for some $q \in [N_s]$. Also, note that any $x \in S$ is such that $|\bar{x} - x| = \frac{\delta}{N_s+1} p \leq \delta$, proving that $S \subset \mathbb{B}[\bar{x}, \delta]$. Now, by the pigeonhole principle, there are at least two indices $k_1 \in [N_s + 1]$ and $k_2 \in [N_s + 1]$ of points in S such that $u(x_{k_1}, t) = u(x_{k_2}, t)$ for all $t \in [t_0, s]$. For simplicity, we define $v(t) := u(x_{k_1}, t)$ for each $t \in [t_0, s]$.
Now, for each fixed $t \in [t_0, s]$, we can apply the Taylor Theorem with remainder in its integral form (see, e.g., Theorem 2.68 from [74]) to get

$$f(t, \sigma(t), v(t), \xi(t, t_0, x_{k_1})) = f(t, \sigma(t), v(t), \xi(t, t_0, x_{k_2})) + F(t)(\xi(t, t_0, x_{k_1}) - \xi(t, t_0, x_{k_2})) \tag{A.2}$$

where $F(t) = \int_0^1 (1 - \tau) f_x(t, \sigma(t), v(t), \tau\xi(t, t_0, x_{k_1}) + (1 - \tau)\xi(t, t_0, x_{k_2})) d\tau$ is the remainder. To continue the proof, let $\mathscr{X}(t) := \xi(t, t_0, x_{k_1}) - \xi(t, t_0, x_{k_2})$ for each $t \in [t_0, s]$. Then, we can write

$$\begin{aligned} \dot{\mathscr{X}}(t) &= \dot{\xi}(t, t_0, x_{k_1}) - \dot{\xi}(t, t_0, x_{k_2}) \\ &= f(t, v(t), \xi(t, t_0, x_{k_1})) - f(t, v(t), \xi(t, t_0, x_{k_2})) \\ &= F(t)\mathscr{X}(t). \end{aligned} \tag{A.3}$$

We can interpret equation (A.3) as a variational equation (see, e.g., [75] Section 4.2.4) for nonlinear time-varying controlled switched systems. We notice three things about equation (A.3). First, we prove in Lemma A.5.1 that $F(\cdot)$ is locally integrable. Second, note that $\|F(\cdot)\|$ is bounded on $[t_0, \infty)$. This follows from

$$\begin{aligned}\|F(t)\| \leq & \int_0^1 (1-\tau)d\tau \times \\ & \times \operatorname{ess\,sup}\{\|f_x(p_1,p_2,p_3,p_4)\| : p_1 \in [t_0,\infty), p_2 \in [m], p_3 \in \mathcal{R}_u, p_4 \in B_x\} \\ & \leq \underline{a}.\end{aligned}$$

Third, the right-hand side of (A.3) is a linear time-varying system. Thus, we know that this system has a unique Caratheodory solution (see, e.g., Section 1.5 from [60]). In fact, more is true: we can write $\mathscr{X}(t) = \Phi(t,t_0)\mathscr{X}(t_0)$ for each $t \in [t_0, s]$, where $\Phi(t,t_0)$ is the uniform limit over compact sets of the Peano-Baker series (see, e.g., Theorem 1 from [76]).

Since $F(\cdot)$ is locally integrable and its norm is uniformly bounded by $\underline{a}$, Lemma 3.3.3 tells us that $|\Phi(t,t_0)\mathscr{X}(t_0)| \geq e^{-\underline{a}(t-t_0)}|\mathscr{X}(t_0)|$ for each $t \in [t_0, \infty)$. Further, recall that, by our previous definitions of $\mathscr{X}(\cdot)$ and the set S, we have $|\mathscr{X}(t_0)| = |x_{k_1} - x_{k_2}| \geq \frac{\delta}{N_s} \geq \frac{\delta}{2N_s}$. By (A.1) and the fact that $s > s^1$, we know that $\frac{1}{s-t_0}\log(N_s) < \epsilon + b(\mathscr{U}(K))$, which implies that $N_s < e^{(s-t_0)(b(\mathscr{U}(K))+\epsilon)}$, which, by its turn, implies that $\frac{1}{N_s} > e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}$. Therefore,

$$|\mathscr{X}(s)| = |\Phi(s,t_0)\mathscr{X}(t_0)| \geq \frac{\delta}{2}e^{-a(s-t_0)}e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}. \tag{A.4}$$

Note that $|\xi(s,t_0,x_{k_1}) - \xi(s,t_0,x_{k_2})| \leq 2\max\{|\xi(s,t_0,x_{k_1})|, |\xi(s,t_0,x_{k_2})|\} \leq 2\sup\{|\xi(s,t_0,x)| : x \in K\} = 2\text{diam}(s,t_0,K)$, where the first inequality follows from the triangle inequality and the second follows from the fact that $x_{k_1} \in K$ and $x_{k_2} \in K$. Hence, combining inequality (A.4) and the above, we get that

$$2\text{diam}(s,t_0,K) \geq \frac{\delta}{2}e^{-\underline{a}(s-t_0)}e^{-(s-t_0)(b(\mathscr{U}(K))+\epsilon)}. \tag{A.5}$$

Finally, we write

$$\frac{1}{s-t_0}\log(\text{diam}(s,t_0,K)) \geq \frac{1}{s-t_0}\log(\delta/4) - \underline{a} - b(\mathscr{U}(K)) + \epsilon, \tag{A.6}$$

for $s > s^1(\epsilon)$ arbitrary. This implies that

$$\liminf_{s\to\infty} \frac{1}{s-t_0} \log(\mathrm{diam}(s,t_0,K)) \geq -\underline{a} - b(\mathscr{U}(K)) + \epsilon > -\infty \tag{A.7}$$

proving the proposition.
To see why the switched linear case, i.e., $f(t,\sigma(t),u(x_0,t),x(t)) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(x_0,t)$, is a particular case, we have to show that the three bullets that appear in the proposition statement hold. To prove that the first bullet holds, note that the functions $A_{\sigma(\cdot)}$ and $B_{\sigma(\cdot)}$ are integrable since $\sigma(\cdot)$ is measurable and there are only finitely many modes. Further, since $\mathcal{R}_u \subset \mathbb{R}^{d_u}$ is bounded, we have that $f(\cdot,\cdot,\cdot,\cdot)$ satisfies conditions for existence and uniqueness for Caratheodory solutions (see, e.g., Section 1.5 from [60]). All that is left for us is to prove the third bullet. Since there are finitely many modes, we know that the quantity $f_x(t,\sigma(t),u(x_0,t),x(t)) = A_{\sigma(t)}$ remains bounded for all times, proving that all assumptions hold for the switched linear case. This concludes the proof of this proposition.

A.6 Proof of Theorem 3.3.6

We prove this theorem by contradiction. Assume that there exists $s \geq t_0$ such that for all $t \geq s$ we have that the Gramian of system (4.1) $W(t,s)$ is not invertible[9], but system (4.1) can be stabilized with finite data-rate for arbitrary $\alpha \in \mathbb{R}_{\geq 0}$ and arbitrary $\epsilon \in \mathbb{R}_{>0}$. Thus, there exists $w(t) \in \mathbb{R}^d$ for every $t \geq t_0$ such that $w(t) \in \mathcal{N}(W(t,s))$ for all $t \geq s$ and that $|w(t)| = 1$ for all $t \geq s$.
First, note that $w'(t)\int_s^t \Phi(t,\tau)B_{\sigma(\tau)}(\tau)u(\tau)d\tau = 0$ for all $u(\cdot) \in L^\infty_{\mathrm{loc}}([t_0,\infty),\mathbb{R}^m)$. To see that, recall that since $w(t) \in \mathcal{N}(W(t,s))$ for all $t \geq s$, we have that $w'(t)W(t,s)w(t) = 0$. That implies that

$$\begin{aligned} w'(t)\int_s^t \Phi(t,\tau)B_{\sigma(\tau)}(\tau)B'_{\sigma(\tau)}(\tau)\Phi(t,\tau)d\tau w(t) \\ = \int_s^t |w'(t)\Phi(t,\tau)B_{\sigma(\tau)}(\tau)|^2 d\tau = 0., \end{aligned}$$

[9] By the remark following Definition 4.3.1, we know that this implies that system (4.1) is not completely controllable.

which implies that $w'(t)\Phi(t,\tau)B_{\sigma(\tau)}(\tau) = 0$ for almost all[10] $\tau \in [s,t]$. This implies the claim

$$w'(t)\int_s^t \Phi(t,\tau)B_{\sigma(\tau)}(\tau)u(\tau)d\tau = 0.$$

Second, we pick[11] $\alpha > a$ and pick some arbitrary $\epsilon \in \mathbb{R}_{>0}$. Since the data rate is finite, we know that there exists a stabilizing control set $\mathscr{R}(\epsilon, M, K, \alpha)$ such that the cardinality of a set of restrictions of stabilizing controls $N_s = \#\mathscr{R}_s(\epsilon, M, K, \alpha)$ is finite. Thus, if we choose N_s+1 distinct initial conditions $x(t_0)$ we have that at least two of them have the same associated control restriction $u_{|[t_0,s]}(t)$ for all $t \in [t_0, s]$. Now, let $\bar{x} \in K$ be some interior point to K. Pick an open ball $B(\bar{x}, r)$ that is contained in the interior of K. Thus, for each $i \in [d]$, we can pick $N_s + 1$ colinear points that lie on a line that is parallel to e_i. More precisely, define $y_{j,i} = \bar{x} + r\Big(\frac{j-1}{N_s+1} - \frac{1}{2}\Big)e_i$ for every $j \in [N_s+1]$ and every $i \in [d]$. Note that all of such points belong to $B(\bar{x}, r)$. Denote by $u_{j,i}(t) \in \mathbb{R}^m$ the control function from the stabilizing control-set corresponding to the initial condition $y_{j,i}$ at time $t \geq t_0$ for each $i \in [d]$ and $j \in [N_s+1]$, and denote by $x_{j,i}(t)$ the corresponding state trajectory at time $t \geq t_0$ for each $i \in [d]$ and $j \in [N_s+1]$. Then, we can use the variation of constants formula to get

$$x_{j,i}(t) = \Phi(t,t_0)y_{j,i} + \int_{t_0}^t \Phi(t,\tau)B_{\sigma(\tau)}(\tau)u_{j,i}(\tau)d\tau$$

for all $t \geq t_0$. Now, by the pigeonhole principle, for each $i \in [d]$, there exists at least two distinct indices $j_i^* \in [N_s+1]$ and $k_i^* \in [N_s+1]$ such that the restriction of their corresponding controls $(u_{j,i})_{[t_0,s]}(t)$ is the same for $t \in [t_0, s]$. Let $z_i = y_{j_i^*,i} - y_{k_i^*,i} = e_i\frac{r(j_i^*-k_i^*)}{N_s+1}$ for each $i \in [d]$ and notice that $\{z_1, \cdots, z_d\}$ form an orthogonal basis[12] for $\mathbb{R}^d$. Further note that $|z_i| \geq \frac{r}{N_s+1}$ since $j_i^* - k_i^*$ is a nonzero integer. Also, let $\phi_i(t) \coloneqq x_{j_i^*,i}(t) - x_{k_i^*,i}(t)$ for every $i \in [d]$ and all $t \geq t_0$. Therefore, again by the variation of parameters formula, we get that

$$\phi_i(t) = \Phi(t,t_0)z_i$$

[10]With respect to the Lebesgue measure on $[s,t]$.

[11]Recall that $a = \sup\{\|A_k\| : k \in [m]\}$.

[12]We have that z_i is parallel to e_i for each $i \in [d]$.

for $t \in [t_0, s]$ and for $i \in [d]$ and

$$\phi_i(t) = \Phi(t, t_0)z_i + \int_{t_0}^{t} \Phi(t, \tau)B_{\sigma(\tau)}(\tau)(u_{j_i^*,i}(\tau) - u_{k_i^*,i}(\tau))d\tau$$

for $t \geq s$ and for $i \in [d]$. Now, for each $i \in [d]$ multiply $\phi_i(t)$ on the left by $w'(t)$ and note that

$$w'(t)\phi_i(t) = w'(t)\Phi(t, t_0)z_i$$

for all $t \geq t_0$ by the fact that $w'(t)\int_s^t \Phi(t, \tau)B_{\sigma(\tau)}(\tau)u(\tau)d\tau = 0$ for all $t \geq s$ and all integrable $u(\cdot)$. Next, for every fixed time $t \geq t_0$, define coefficients $\xi_i(t) \in \mathbb{R}$ for all $i \in [d]$ such that $\sum_{i=1}^d |\xi_i(t)| = 1$ and $\Phi(t, t_0)z(t) \in \text{span}\{w(t)\}$, where $z(t) := \sum_{i=1}^d \xi_i(t)z_i$. First, note that such coefficients always exist since $\{z_1, \cdots, z_d\}$ forms a basis for $\mathbb{R}^d$ and $\Phi(t, t_0)$ is invertible for every $t \geq t_0$. Hence, we can define $\xi_i(t)$ as $c\langle\Phi^{-1}(t, t_0)w(t), z_i\rangle/|z_i|^2$ for[13] $c = \frac{1}{|\sum_i^d \langle\Phi^{-1}(t,t_0)w(t),z_i\rangle/|z_i|^2|}$. Further, note that $|z(t)| = \sum_{i=1}^d |\xi_i(t)||z_i| \geq \frac{r}{N_s+1}$, where the equality follows from the fact that $\{z_1, \ldots, z_d\}$ is an orthogonal basis and the inequality follows since $\sum_{i=1}^d |\xi_i(t)| = 1$ and the fact that $|z_i| \geq \frac{r}{N_s+1}$ for each $i \in [d]$. Let $\phi(t) := \sum_{i=1}^d \xi_i(t)\phi_i(t)$ for every $t \geq t_0$. Thus, for every $t \geq t_0$ we have

$$w'(t)\phi(t) = w'(t)\Phi(t, t_0)z(t)$$

for every $t \geq t_0$. Taking the norm on both sides and using the Cauchy-Schwarz inequality, we see that

$$|w'(t)\phi(t)| = |\Phi(t, t_0)z(t)|$$

[13] Note that $\sum_{i=1}^d |\xi_i(t)| = |c||\sum_i^d \langle\Phi^{-1}(t, t_0)w(t), z_i\rangle/|z_i|^2| = 1$. Also, $z(t) = c\sum_{i=1}^d \langle\Phi^{-1}(t, t_0)w(t), \frac{z_i}{|z_i|}\rangle \frac{z_i}{|z_i|} = c\sum_{i=1}^d \langle\Phi^{-1}(t, t_0)w(t), e_i\rangle e_i = c\Phi^{-1}(t, t_0)w(t)$.

because $|w'(t)\Phi(t,t_0)z(t)| = |\Phi(t,t_0)z(t)|$ since $\Phi(t,t_0)z(t) \in \text{span}\{w(t)\}$ and $|w(t)| = 1$. Now, recall that, by definition of controllability with finite data-rate, for every $\alpha \geq 0$ and every initial condition $x(t_0)$, we have that $|x(t)| \leq \big(M|x(t_0)| + \epsilon\big)e^{-\alpha(t-t_0)}$ for some $M \in \mathbb{R}_{>0}$, some $\epsilon > 0$, and all $t \geq t_0$. In particular, this must hold for our choice of $\alpha > a$ and our arbitrary choice of ϵ. This implies that

$$
\begin{aligned}
|\phi(t)| = |\sum_{i=1}^{d} \xi_i(t)\phi_i(t)| &\leq \sum_{i=1}^{d} \xi_i(t)(|x_{j_i^*,i}(t)| + |x_{k_i^*,i}(t)|) \\
&\leq 2\big(MR_0 + \epsilon\big)e^{-\alpha(t-t_0)}
\end{aligned}
$$

where the first inequality comes from the triangle inequality. The second inequality follows from the facts that $\sum_{i=1}^{d} |\xi_i(t)| = 1$, by construction, that $\max\{|x_{j_i^*,i}(t)|, |x_{k_i^*,i}(t)|\} \leq \big(M|x(t_0)| + \epsilon\big)e^{-\alpha(t-t_0)}$, by controllability with finite data-rate, and that $|x(t_0)| \leq R_0$. Now, by the Cauchy-Schwarz inequality, we have that $|\Phi(t,t_0)z(t)| = |w'(t)\phi(t)| \leq |\phi(t)|$ since $|w(t)| = 1$. Hence, we arrive at

$$2\big(MR_0 + \epsilon\big)e^{-\alpha(t-t_0)} \geq |\Phi(t,t_0)z(t)|.$$

Finally, note that $|\Phi(t,t_0)z(t)| \geq \frac{r}{N_s+1}e^{-a(t-t_0)}$. That implies that $|\Phi(t,t_0)\frac{z(t)}{|z(t)|}| \geq e^{-a(t-t_0)}$. Thus, $|\Phi(t,t_0)z(t)| \geq e^{-a(t-t_0)}|z(t)| \geq \frac{r}{N_s+1}e^{-a(t-t_0)}$, where the last equality comes from the construction of $z(t)$. Since this must hold for each $t \geq t_0$ and we picked $\alpha > a$, we arrived at a contradiction. Therefore, system (4.1) must be completely controllable.

A.7 Proof of Theorem 4.3.2

Proof. We prove that if system (4.1) is UCC, then it satisfies the assumptions from Theorem 3.3.5 from Chapter 3. The assumptions of that theorem are: system (4.1) is persistently completely controllable and it satisfies the exponential energy-growth condition[14]. First, we prove that UCC implies persistent complete controllability. Since systems (4.1) is UCC, there exists $T \in \mathbb{R}_{>0}$ and $\underline{w} \in \mathbb{R}_{>0}$ such that $W(t+T,t) \geq \underline{w} I_{d_x}$ for all $t \in [t_0, \infty)$. Choose $s_n = t_0 + Tn$ for each $n \in \mathbb{Z}_{\geq 0}$. Note that $W(s_{n+1}, s_n)$ is invertible since $s_{n+1} - s_n = T$ for each $n \in \mathbb{Z}_{\geq 0}$. Further, $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$ since $\lim_{n\to\infty} \frac{t_0+(n+1)T_p}{t_0+nTp} = 1$. Thus, we proved the first claim. Next, note that $W(t+T,t) \geq \underline{w} I_{d_x}$ implies that $\|W^{-1}(s_{n+1}, s_n)\| = \max\{|W^{-1}(s_{n+1}, s_n)v| : |v| = 1\} \leq \underline{w}^{-1}$ for each $n \in \mathbb{Z}_{\geq 0}$, proving the second claim. Thus, we proved our theorem. □

A.8 Proof of Lemma 4.3.3

Proof. We need the following classical identity for the controllability Gramian: let $t_2 > t_1 > t_0$ for $t_0 \in \mathbb{R}_{\geq 0}$, then

$$W(t_2, t_0) = \Phi_\sigma(t_2, t_1) W(t_1, t_0) \Phi'_\sigma(t_2, t_1) + W(t_2, t_1). \tag{A.8}$$

Sufficiency: there exist $T \in \mathbb{R}_{>0}$ and $\underline{w} \in \mathbb{R}_{>0}$ such that $W(t+T,t) \geq I_{d_x}\underline{w}$ for each $t \in \mathbb{R}_{\geq 0}$. Now, let $\ell = \lceil \frac{T}{T_p} \rceil$ and fix some arbitrary $t \in \mathbb{R}_{\geq 0}$. Note that $\ell T_p \geq T$ since $\lceil \frac{T}{T_p} \rceil \geq \frac{T}{T_p}$. Let $t_2 = \ell T_p + t$, $t_1 = t + T$, and $t_0 = t$ in equation (A.8) to get that $W(t+\ell T_p, t) = \Phi_\sigma(t+\ell T_p, t+T) W(t+T,t) \Phi'_\sigma(t+\ell T_p, t+T) + W(t+\ell T_p, t+T)$. Since the controllability Gramian is always positive semi-definite, i.e., $W(t+\ell T_p, t+T) \geq 0$, and $W(t+T,t) \geq I_{d_x}\underline{w}$ we conclude that $W(t+\ell T_p, t) \geq \Phi_\sigma(t+\ell T_p, t+T)\Phi'_\sigma(t+\ell T_p, t)\underline{w}$.

[14]See subsection 4.3.7 for the formal statement of these conditions.

Now, recall that the Rayleigh-Ritz Theorem (see, e.g., Theorem 4.2.2 from [77]) gives us that $AA' \geq \underline{\lambda}(AA')I_{d_x}$ for an arbitrary matrix $A \in \mathbb{R}^{d_x \times d_x}$, where $\underline{\lambda}(AA') \in \mathbb{R}_{\geq 0}$ is the minimum eigenvalue of AA'. Further, remember that $\underline{\lambda}(AA')$ equals the square of the smallest singular value of A (see, e.g. Theorem 2.6.3 from [77]), which we denote by $\underline{\varsigma}(A)$. Thus, $AA' \geq \underline{\varsigma}^2(A)I_{d_x}$. Hence, we can write $W(t+\ell T_p, t) \geq \underline{\varsigma}^2(\Phi_\sigma(t+\ell T_p, t+T))\underline{w}I_{d_x}$. Noticing that $\Phi_\sigma(t+\ell T_p, t+T)$ is always invertible, we conclude that $W(t+\ell T_p, t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$.

For each $t \in \mathbb{R}_{\geq 0}$ and each $x_t \in \mathbb{R}^{d_x}$, the condition $W(t+\ell T_p, t) > 0$ implies that there exists a control $u : [t, t+\ell T_p] \to \mathbb{R}^{d_u}$ such that $\phi(t+\ell T_p, t, x_t, u(\cdot)) = 0$ by Proposition 5.2 from [56]. Thus, Lemma 4.3.8 tells us that system (4.1) is ℓ-uniformly completely controllable.

Necessity: there exists $\ell \in \mathbb{Z}_{>0}$ such that $\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^d$ holds for each $k \in \mathbb{Z}_{\geq 0}$, where $n(k) = \lfloor \frac{k}{\ell} \rfloor$. We note that ℓ-uniform complete controllability with losses in $\mathcal{R}^c$ implies, by Lemma 4.3.8, that system (4.1) is controllable in the usual sense on each interval $[\tau_{\ell n}, \tau_{\ell(n+1)}]$ for $n \in \mathbb{Z}_{\geq 0}$. Again by Proposition 5.2 from [56], we know that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) > 0$ for each $n \in \mathbb{Z}_{\geq 0}$. Now, we prove that there exists $\underline{\tilde{w}} \in \mathbb{R}_{>0}$ such that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) > \underline{\tilde{w}} I_{d_x} \tag{A.9}$$

for every $n \in \mathbb{Z}_{\geq 0}$.

First, recall that $\Phi_\sigma(\tau_{k+1}, \tau_k) = e^{A_{\sigma(\tau_k)}T_p}$. Also, define

$$W_p(0, T_p) \coloneqq \int_0^{T_p} e^{A_p(T_p-\tau)} B_p B_p e^{A'_p(T_p-\tau)} d\tau$$

for each $p \in [m]$. Now, we can rewrite $W(\tau_{\ell(n+1)}, \tau_{\ell n})$ in the following manner:

$$\begin{aligned}
W(\tau_{\ell(n+1)}, \tau_{\ell n}) &= \int_{\tau_{\ell n}}^{\ell(n+1)} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{\ell(n+1)}, \tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{k+1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{\ell(n+1)}, \tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \times \\
&\times \int_{\tau_k}^{k+1} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(\tau_{k+1}, \tau) d\tau \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) W_{\sigma(k)}(0, T_p) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} e^{A_{\sigma(\ell(n+1)-1)}T_p} \cdots e^{A_{\sigma(k+1)}T_p} W_{\sigma(k)}(0, T_p) e^{A'_{\sigma(k+1)}T_p} \cdots e^{A'_{\sigma(\ell(n+1)-1)}T_p}.
\end{aligned} \tag{A.10}$$

Note that the last term in equation (A.10) can only attain at most m^ℓ possible values, each one corresponding to a tuple $(\sigma(\tau_{\ell(n+1)-1}), \ldots, \sigma(\tau_{\ell n})) \in [m]^\ell$. This motivates us to define the simpligying notation

$$W_{(p_1,\ldots,p_\ell)} \coloneqq \sum_{k=1}^{\ell} e^{A_{p_\ell}T_p} \cdots e^{A_{p_{k+1}}T_p} W_{\sigma(k)}(0, T_p) e^{A'_{p_{k+1}}T_p} \cdots e^{A'_{p_\ell}T_p}$$

for each tuple $(p_1, \ldots, p_\ell) \in [m]^\ell$. Thus, we can write $W(\tau_{\ell(n+1)-1}, \tau_{\ell n}) = W_{\sigma(\tau_{\ell n}),\ldots,\sigma(\tau_{\ell(n+1)-1})}$ for each $n \in \mathbb{Z}_{\geq 0}$. This latter equality implies that there exist constants $\underline{w}_{(p_1,\ldots,p_\ell)} \in \mathbb{R}_{>0}$ such that $W_{(p_1,\ldots,p_\ell)} > \underline{w}_{(p_1,\ldots,p_\ell)} I_{d_x}$ for each $(p_1, \ldots, p_\ell) \in [m]^\ell$, since $W(\tau_{\ell(n+1)-1}, \tau_{\ell n}) > 0$ for every $n \in \mathbb{Z}_{\geq 0}$. Hence, if we choose $\underline{\tilde{w}} \coloneqq \min\{\underline{w}_{(p_1,\ldots,p_\ell)} : (p_1, \ldots, p_\ell) \in [m]^\ell\}$, we prove our claim that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) > \underline{\tilde{w}} I_{d_x}$ for every $n \in \mathbb{Z}_{\geq 0}$.

Choose $T = 2\ell T_p$ and fix an arbitrary $t \in [t_0, \infty)$. Next, define $q \coloneqq \lceil \frac{t-\tau_0}{T_p \ell} \rceil$. We claim that $t \leq \tau_{\ell q}$ and $\tau_{\ell(q+1)} < T + t$. The first claim follows from $\tau_{\ell q} = \tau_0 + q\ell T_p = \tau_0 + \lceil \frac{t-\tau_0}{T_p \ell} \rceil \ell T_p \geq \tau_0 + (\frac{t-\tau_0}{T_p \ell}) \ell T_p = t$ and the second claim follows from $\tau_{\ell(q+1)} = \tau_0 + (q+1)\ell T_p = \tau_0 + \lceil \frac{t-\tau_0}{T_p \ell} \rceil \ell T_p + \ell T_p < \tau_0 + (\frac{t-\tau_0}{T_p \ell} + 1)\ell T_p + \ell T_p = t + 2\ell T_p = t + T$.

We use (A.8) with the choices $t_2 = t + T$, $t_1 = \tau_{\ell(q+1)}$, $t_0 = t$ to get that

$$\begin{aligned} & W(t+T,t) \\ & = \Phi_\sigma(t+T,\tau_{\ell(q+1)})W(\tau_{\ell(q+1)},t)\Phi'_\sigma(t+T,\tau_{\ell(q+1)}) + W(t+T,\tau_{\ell(q+1)}) \\ & \geq \Phi_\sigma(t+T,\tau_{\ell(q+1)})W(\tau_{\ell(q+1)},t)\Phi'_\sigma(t+T,\tau_{\ell(q+1)}), \end{aligned} \tag{A.11}$$

where the inequality comes from the fact that any Gramian matrix is always positive semi-definite. Similarly, we use (A.8) with the choices $t_2 = \tau_{\ell(q+1)}$, $t_1 = \tau_{\ell q}$, $t_0 = t$ to get that

$$\begin{aligned} W(\tau_{\ell(q+1)},t) &= \Phi_\sigma(\tau_{\ell(q+1)},\tau_{\ell q})W(\tau_{\ell q},t)\Phi'_\sigma(\tau_{\ell(q+1)},t_1) + W(\tau_{\ell(q+1)},\tau_{\ell q}) \\ &\geq W(\tau_{\ell(q+1)},\tau_{\ell q}) \end{aligned} \tag{A.12}$$

where the inequality follows from the fact that any Gramian matrix is positive semi-definite and by Sylvester law of Inertia (see, e.g., Theorem 4.5.8 from [77]), which states that two congruent[15] symmetric matrices have the same number of positive, negative, and zero eigenvalues. Hence, combining both inequalities (A.11) and (A.12), we get that

$$\begin{aligned} W(t+T,t) &\geq \Phi_\sigma(t+T,\tau_{\ell(q+1)})W(\tau_{\ell(q+1)},\tau_{\ell q})\Phi'_\sigma(t+T,\tau_{\ell(q+1)}) \\ &\geq \Phi_\sigma(t+T,\tau_{\ell(q+1)})\Phi'_\sigma(t+T,\tau_{\ell(q+1)})\underline{\tilde{w}}, \end{aligned} \tag{A.13}$$

where the last inequality follows from inequality (A.9).

Recall that $a := \sup\{\|A_{\sigma(t)}\| : t \in [t_0,\infty)\}$ is finite. Thus, we can apply Lemma 3.3.3 to get that $|\Phi_\sigma(t+T,\tau_{\ell(q+1)})v| \geq e^{-a(t+T-\tau_{\ell(q+1)})}$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Now, we note that $T + t - \tau_{\ell(q+1)} \leq \ell T_p$. This follows from the fact that $T+t-\tau_{\ell(q+1)} \leq 2\ell T_p+\tau_{\ell q}-\tau_{\ell(q+1)} = \ell T_p$. Thus, $|\Phi_\sigma(t+T,\tau_{\ell(q+1)})v| \geq e^{-a\ell T_p}$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. Next, note that $e^{-a2\ell T_p} \leq |\Phi_\sigma(t+T,\tau_{\ell(q+1)})v|^2 = v'\Phi'_\sigma(t+T,\tau_{\ell(q+1)})\Phi_\sigma(t+T,\tau_{\ell(q+1)})v$ for each $v \in \mathbb{R}^{d_x}$ with $|v| = 1$. This last remark implies that $e^{-a2\ell T_p}I_{d_x} \leq \Phi'_\sigma(t+T,\tau_{\ell(q+1)})\Phi_\sigma(t+T,\tau_{\ell(q+1)})$ (see, e.g. Section 7.1 from [77]).

[15] The matrices $A \in \mathbb{R}^{d_x\times d_x}$ and $B \in \mathbb{R}^{d_x\times d_x}$ are congruent to each other if there exists an invertible matrix $P \in \mathbb{R}^{d_x\times d_x}$ such that $A = PBP'$. See, e.g., Definition 4.5.4 from [77].

Let $A \in \mathbb{R}^{d_x \times d_x}$. Once again Rayleigh-Ritz Theorem tells us that a number $r \in \mathbb{R}$ satisfies $AA' \geq rI_{d_x}$ if, and only if, r lower-bounds the minimum eigenvalue of AA'. Consequently, the previous inequality is true if, and only if, r lower-bound the square of the minimum singular value of A. Recalling that the singular values of A and A' are the same (see, e.g., Theorem 2.6.3 from [77]), we conclude that $AA' \geq rI_{d_x}$ if, and only if $A'A \geq rI_{d_x}$. Thus, since $e^{-a2\ell T_p} I_{d_x} \leq \Phi'_\sigma(t{+}T, \tau_{\ell(q+1)})\Phi_\sigma(t{+}T, \tau_{\ell(q+1)})$, we have that $e^{-a2\ell T_p} I_{d_x} \leq \Phi_\sigma(t{+}T, \tau_{\ell(q+1)})\Phi'_\sigma(t{+}T, \tau_{\ell(q+1)})$. This latter inequality, together with (A.13), lets us write

$$W(t+T, t) \geq \underline{\tilde{w}} e^{-a2\ell T_p} I_{d_x}.$$

Defining $\underline{w} := \underline{\tilde{w}} e^{-a2\ell T_p}$, we get that $W(t+T, t) \geq \underline{w} I_{d_x}$. Since $t \in [t_0, \infty)$ is arbitrary, we conlcude the proof of the lemma.

□

A.9 Proof of Proposition 4.3.5

Proof. Once again, we need the following classical identity for the controllability Gramian: let $t_2 > t_1 > t_0$ for $t_0 \in \mathbb{R}_{\geq 0}$, then

$$W(t_2, t_0) = \Phi_\sigma(t_2, t_1) W(t_1, t_0) \Phi'_\sigma(t_2, t_1) + W(t_2, t_1). \tag{A.14}$$

First, fix an arbitrary $n \in \mathbb{Z}_{\geq 0}$. Let $t_2 = \tau_{\ell(n+1)}$, $t_1 = \tau_{k(n)}$, and $t_0 = \tau_{\ell n}$. Then, equation (A.14) gives us that

$$\begin{aligned} W(\tau_{\ell(n+1)}, &\tau_{\ell n}) \\ &= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) W(\tau_{k(n)}, \tau_{\ell n}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) + W(\tau_{\ell(n+1)}, \tau_{k(n)}) \\ &\geq W(\tau_{\ell(n+1)}, \tau_{k(n)}), \end{aligned} \tag{A.15}$$

where the last inequality follows from the fact that

$$\Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) W(\tau_{k(n)}, \tau_{\ell n}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)}) \geq 0$$

(see proof of Lemma 4.3.3).

Next, let $t_2 = \tau_{\ell(n+1)}$, $t_1 = \tau_{k(n)+1}$, and $t_0 = \tau_{k(n)}$. Then, equation (A.14) gives us that

$$\begin{aligned} W(\tau_{\ell(n+1)}, \tau_{k(n)}) &= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) W(\tau_{k(n)+1}, \tau_{k(n)}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) + \\ &\quad + W(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \\ &\geq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) W(\tau_{k(n)+1}, \tau_{k(n)}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}). \end{aligned} \tag{A.16}$$

Let $W_p(0,T) := \int_0^{T_p} e^{A_p(T_p-\tau')} B_p B_p' e^{A_p'(T_p-\tau')} d\tau$ for each $p \in [m]$. Also, for each $p \in [m]$ choose $\underline{w}_p \in \mathbb{R}_{>0}$ so that $W_p(0,T) \geq \underline{w}_p I_{d_x}$. Define $\bar{\underline{w}} := \min\{\underline{w}_p : p \in [m]\}$ and note that $\bar{\underline{w}} \in \mathbb{R}_{>0}$. It is easy to see that

$$\begin{aligned} &W(\tau_{k(n)+1}, \tau_{k(n)}) \\ &= \int_{\tau_{k(n)}}^{\tau_{k(n)+1}} e^{A_{\sigma(\tau_{k(n)})}(\tau_{k(n)+1}-\tau)} B_{\sigma(\tau_{k(n)})} B'_{\sigma(\tau_{k(n)})} e^{A'_{\sigma(\tau_{k(n)})}(\tau_{k(n)+1}-\tau)} d\tau \\ &= \int_0^{T_p} e^{A_{\sigma(\tau_{k(n)})}(T_p-\tau')} B_{\sigma(\tau_{k(n)})} B'_{\sigma(\tau_{k(n)})} e^{A'_{\sigma(\tau_{k(n)})}(T_p-\tau')} d\tau \\ &= W_{\sigma(\tau_{k(n)})}(0, T_p) \geq \bar{\underline{w}} I_{d_x}, \end{aligned}$$

where the second equality follows from the change of variables $\tau' = \tau - \tau_{k(n)}$ and the inequality follows from the definition of $\bar{\underline{w}}$. Combining the previous inequality with inequalities (A.15) and (A.16), we get that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \bar{\underline{w}}.$$

Let $a := \sup\{\|A_{\sigma(t)}\| : t \in [t_0, \infty)\}$. Then, following similar steps as in the necessity part of the proof of Lemma 4.3.3, we get that $e^{-2a(\tau_{\ell(n+1)} - \tau_{k(n)+1})} \leq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})$. Since $\tau_{\ell(n+1)} - \tau_{k(n)+1} \leq \ell T_p$, we conclude that $e^{-2a\ell T_p} I_{d_x} \leq \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1}) \Phi'_\sigma(\tau_{\ell(n+1)}, \tau_{k(n)+1})$. Therefore, we know that

$$W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \bar{\underline{w}} e^{-2a\ell T_p} I_{d_x}.$$

Since $n \in \mathbb{Z}_{\geq 0}$ is arbitrary, we proved that $W(\tau_{\ell(n+1)}, \tau_{\ell n}) \geq \bar{\underline{w}} e^{-2a\ell T_p} I_{d_x}$ for all $n \in \mathbb{Z}_{\geq 0}$.

To finish this proof, we take $T = 2\ell T_p$ and let $t \in [t_0, \infty)$ be arbitrary. Further, define $\underline{\tilde{w}} \coloneqq \underline{\bar{w}} e^{-2a\ell T_p}$. Then, we follow the exact same steps as in the proof of Lemma 4.3.3 from the paragraph that starts with "Chose $T = 2\ell T_p$". We get that $W(t+T, t) \geq \underline{\tilde{w}} e^{-2a\ell T_p} I_{d_x} = \underline{\bar{w}} e^{-4a\ell T_p} I_{d_x}$. Choosing $\underline{w} \coloneqq \underline{\tilde{w}} e^{-2a\ell T_p}$, we conclude that system (4.1) is UCC. This concludes the proof of the Proposition.

□

A.10 Proof Corollary 4.3.6

Proof. Let $\frac{\tau_D}{N_0+2} \geq T_p$. Take $\ell \in \mathbb{Z}_{>0}$ to be $\ell \coloneqq N_0 + 2$. Then, the ADT condition tells us that the number of switchings on any interval of length ℓT_p is upper-bounded by the quantity $N_0 + \frac{\ell T_p}{\tau_D} = N_0 + \frac{(N_0+2)T_p}{\tau_D} \leq N_0 + 1$, where the last inequality follows from our initial assumption. Thus, for each $n \in \mathbb{Z}_{\geq 0}$, the time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$ has at most $N_0 + 1$ switchings and contains $N_0 + 2$ sampling intervals. Therefore, by the pigeonhole principle, for each $n \in \mathbb{Z}_{>0}$, we know that there exists some $k(n) \in \mathbb{Z}_{>0}$ such that $k(n) \notin \mathcal{S}$, i.e., there is no switching on the time interval $[\tau_{k(n)}, \tau_{(k(n)+1)})$. Consequently, since all modes are controllable, we have that $\langle A_{\sigma(k(n))} | B_{\sigma(k(n))} \rangle = \mathbb{R}^d$. Hence, all assumption from Proposition 4.3.5 hold, concluding the proof of the corollary.

□

A.11 Proof of Lemma 4.3.8

Proof. We prove the sufficiency and necessity parts separately. We take this opportunity to make a few definitions and prove some auxiliary results. For each $k \in \mathbb{Z}_{\geq 0}$, define the linear operator

$$L_k : L^\infty_{\text{loc}}([\tau_k, \tau_{k+1}), \mathbb{R}^{d_u}) \to \mathbb{R}^{d_x}$$

given by $L_k(u(\cdot)) \coloneqq \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1},\tau)B_{\sigma(\tau)}u(\tau)d\tau$. Now, given two real vector spaces V and W and a linear operator $L : V \to W$, we define the *range* of L to be the subspace $\mathscr{R}(L) \coloneqq \{w \in W : L(v) = w \text{ for some } v \in V\}$. The next result will be instrumental in what follows: the range of $L_k(\cdot)$ equals $\Phi_\sigma(\tau_{k+1},\tau_{\ell n})\mathcal{V}_k$, i.e.,

$$\mathscr{R}(L_k(\cdot)) = \Phi_\sigma(\tau_{k+1},\tau_{\ell n})\mathcal{V}_k. \tag{A.17}$$

Indeed, note that there are no switchings inside the interval $[\tau_k,\tau_{k+1})$, i.e., system (4.1) behaves as an LTI system on this interval. It is a well-known fact that $\mathscr{R}(L_k(\cdot)) = \mathscr{R}(W(\tau_{k+1},\tau_k))$ (see[16], e.g., Lemma 2.1 from [78]). Furthermore, $\mathscr{R}(W(\tau_{k+1},\tau_k)) = \langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$ (see, e.g., Lemma 2.10 from [78]) since $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are constant on the time interval $[\tau_k,\tau_{k+1})$. Thus, $\mathscr{R}(L_k(\cdot)) = \langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle = \Phi_\sigma(\tau_{k+1},\tau_{\ell n})\mathcal{V}_k$ and we conclude the proof of our remark.

To aid the reader, we now prove some well-known facts about the range of linear operators. Let V and W be real vector spaces, let $L : V \to W$ be a linear operator, and let $S \subset V$ be a subset. We define $LS \coloneqq \{w \in W : L(v) = w \text{ for some } v \in S\}$. Let $L_1 : V \to W$ and $L_2 : W \to Z$ be linear operators, where Z is a real vector space. Then, we see that

$$\begin{aligned}\mathscr{R}(L_2L_2) &= \{z \in Z : L_2L_1(v) = z \text{ for some } v \in V\}\\ &= \{z \in Z : L_2(y) = z \text{ for some } y \in \mathscr{R}(L_1)\} = L_2\mathscr{R}(L_1).\end{aligned} \tag{A.18}$$

Finally, let $L_1 : V \to W$ and $L_2 : V \to W$ be linear operators. Then, we see that

$$\begin{aligned}\mathscr{R}(L_1+L_2) &= \{w \in W : L_1(v) + L_2(v) \text{ for some } v \in V\}\\ &\subset \{w \in W : y_1 + y_2 = w \text{ for some } y_1 \in \mathscr{R}L_1 \text{ and some } y_2 \in \mathscr{R}L_2\}\\ &= \mathscr{R}L_1 + \mathscr{R}L_2\end{aligned} \tag{A.19}$$

[16] Although it is not explicitly mentioned, the proof of Lemma 2.1 provided in [78] shows that the control is integrable locally essentially bounded.

Sufficiency: for each $n \in \mathbb{Z}_{\geq 0}$ and each $x \in \mathbb{R}^{d_x}$, there exist a control $u_n(\cdot) \in L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$ with the following property: if $x(\tau_{\ell n}) = x$, then we have $x(\tau_{\ell(n+1)}) = 0$. We start this proof by noticing that the variation of constants formula lets us write that

$$0 = x(\tau_{\ell(n+1)}) = \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n})x + \int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau.$$

We can rewrite the above integral as

$$\begin{aligned}
&\int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) L_k(u_{n|[\tau_k, \tau_{k+1})}(\cdot)). \qquad \text{(A.20)}
\end{aligned}$$

Since $x \in \mathbb{R}^{d_x}$ is arbitrary and $\Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n})$ is invertible, our initial assumption is equivalent to the fact that

$$\mathbb{R}^{d_x} = \mathscr{R}\left(\int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau) B_{\sigma(\tau)} u_n(\tau) d\tau\right),$$

which, by equation (A.20), is equivalent to the following condition

$$\mathbb{R}^{d_x} = \mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) L_k(u_{n|[\tau_k, \tau_{k+1})}(\cdot))\right). \qquad \text{(A.21)}$$

Using the set inclusion (A.19), we can write

$$\begin{aligned}
&\mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) L_k(u_{n|[\tau_k, \tau_{k+1})}(\cdot))\right) \\
&\quad \subset \sum_{k=\ell n}^{\ell(n+1)-1} \mathscr{R}(\Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) L_k(u_{n|[\tau_k, \tau_{k+1})}(\cdot))).
\end{aligned}$$

Next, using (A.18), we conclude that

$$\begin{aligned}
&\mathscr{R}\left(\sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))\right) \\
&\subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))).
\end{aligned}$$

Thus, the previous inclusion together with the equality (A.21), lets us write that

$$\mathbb{R}^{d_x} \subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n}, \tau_{k+1}) \mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))).$$

We multiply both terms in the previous relation by $\Phi_\sigma(\tau_{\ell n}, \tau_{\ell(n+1)})$ on the left to get that

$$\begin{aligned}
\mathbb{R}^{d_x} &\subset \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n}, \tau_{\ell(n+1)}) \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{k+1}) \mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n}, \tau_{k+1}) \mathscr{R}(L_k(u_{n|[\tau_k,\tau_{k+1})}(\cdot))) \\
&= \sum_{k=\ell n}^{\ell(n+1)-1} \Phi_\sigma(\tau_{\ell n}, \tau_{k+1}) \Phi_\sigma(\tau_{k+1}, \tau_{\ell n}) \mathcal{V}_k = \sum_{k=\ell n}^{\ell(n+1)-1} \mathcal{V}_k,
\end{aligned}$$

where the first equality follows from the semi-group property of the state-transition matrix, and the second equality follows from equation (A.17). Finally, we note that the inclusion is actually an equality because the range of the operator on the right is $\mathbb{R}^{d_x}$. Hence,

$$\mathbb{R}^{d_x} = \sum_{k=\ell n}^{\ell(n+1)-1} \mathcal{V}_k.$$

Since this holds for any $n \in \mathbb{Z}_{\geq 0}$, we conclude the proof of the sufficiency part.

Necessity: there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that system (4.1) is ℓ-uniformly completely controllable. So, for each $x \in \mathbb{R}^{d_x}$ and each $n \in \mathbb{Z}_{\geq 0}$, we can write $x = \sum_{k=\ell n}^{\ell(n+1)-1} x_j$, where $x_j \in \mathcal{V}_j$ for each $j \in \{\ell n, \ldots, \ell(n+1) - 1\}$. To organize our ideas, we split the proof of necessity into four parts.

First, we have that $\Phi_\sigma(\tau_{k+1}, \tau_{\ell n})x_k \in \langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. Indeed, we know that $x_k \in \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. Thus, $\Phi_\sigma(\tau_k, \tau_{\ell n})x_k \in \langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. Furthermore, we also have that $\Phi_\sigma(\tau_{k+1}, \tau_k) = e^{A_{\sigma(\tau_k)}T_p}$, which implies that $\Phi_\sigma(\tau_{k+1}, \tau_k)\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle = \langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$ since $\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$ is $A_{\sigma(\tau_k)}$-invariant (see, e.g., Chapter 1 from [79]).

Second, equation (A.17) gives us that $\mathscr{R}(L_k(\cdot)) = \Phi_\sigma(\tau_{k+1}, \tau_{\ell n})\mathcal{V}_k$ for each $k \in \mathbb{Z}_{\geq 0}$. This implies that there exists a $L^\infty_{\text{loc}}([\tau_k, \tau_{k+1}), \mathbb{R}^{d_u})$ function $u_k : [\tau_k, \tau_{k+1}) \to \mathbb{R}^{d_u}$ such that $-\Phi_\sigma(\tau_{k+1}, \tau_{\ell n})x_k = L_k(u_k(\cdot))$.

Third, define the function $u : [\tau_{\ell n}, \tau_{\ell(n+1)}) \to \mathbb{R}^{d_u}$ as $u(t) = u_k(t)$ for $t \in [\tau_k, \tau_{k+1})$ and each $k \in \{\ell n, \dots, \ell(n+1) - 1\}$. We note that $u(\cdot)$ is $L^\infty_{\text{loc}}([\tau_{\ell n}, \tau_{\ell(n+1)}), \mathbb{R}^{d_u})$. This follows from the facts that each $u_k(\cdot)$ is integrable on its own domain and that the set where $u(\cdot)$ is not bounded is a null Lebesgue set. This latter fact follows from the simple observation that $\{t \in [\tau_{\ell n}, \tau_{\ell(n+1)}) : |u(t)| = \infty\} = \cup_{k=\ell n}^{\ell(n+1)-1}\{t \in [\tau_k, \tau_{k+1}) : |u_k(t)| = \infty\}$ and that the finite union of null sets is a null set. Therefore, $u(\cdot)$ is locally essentially bounded. Fourth, let $x(\tau_{\ell n}) = x$. Then, we can use the variation of constants formula to get that

$$
\begin{aligned}
x(\tau_{\ell(n+1)}) &= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \sum_{k=\ell n}^{\ell(n+1)-1} x_k + \int_{\tau_{\ell n}}^{\tau_{\ell(n+1)}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau)B_{\sigma(\tau)}u(\tau)d\tau \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \sum_{k=\ell n}^{\ell(n+1)-1} x_k + \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{\ell(n+1)}, \tau)B_{\sigma(\tau)}u_k(\tau)d\tau \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \left(\sum_{k=\ell n}^{\ell(n+1)-1} x_k + \sum_{k=\ell n}^{\ell(n+1)-1} \int_{\tau_k}^{\tau_{k+1}} \Phi_\sigma(\tau_{k+1}, \tau)B_{\sigma(\tau)}u_k(\tau)d\tau \right) \\
&= \Phi_\sigma(\tau_{\ell(n+1)}, \tau_{\ell n}) \left(\sum_{k=\ell n}^{\ell(n+1)-1} x_k + L_k(u_k(\cdot)) \right) \\
&= 0.
\end{aligned}
$$

Since $n \in \mathbb{Z}_{\geq 0}$ is arbitrary, we conclude the proof.

□

A.12 Proof of Lemma 4.3.9

We prove the more general fact that given subspaces $V \subset \mathbb{R}^d$ and $W \subset \mathbb{R}^d$ such that $V \oplus W = \mathbb{R}^d$, then the Euclidean norm of the projection matrix P_V over V and parallel to W is given by $\|P_V\| = \frac{1}{(1-\cos^2(V,W))^{1/2}}$. If $V = \{0\}$, then $P_V = 0$ and $\|P_V\| = 0 \leq 1$. Thus, we can assume that $V \neq \{0\}$. First, given any vector $x \in \mathbb{R}^d \setminus \{0\}$, we can write it uniquely as $x = \alpha v + \beta w$, where $v \in V$ and $w \in W$ since $V \oplus W = \mathbb{R}^d$. Then, we see that $P_V x = \alpha v$ since P_V is a projection along V. Assume that $\alpha v \neq 0$ and that $\beta w \neq 0$, i.e., $x \notin V$ and $x \notin W$. Now, we see that

$$\begin{aligned}
|P_V x|^2 &= \frac{|\alpha v|^2}{|\alpha v + \beta w|^2} = \frac{|\alpha v|^2}{|\alpha v|^2 + |\beta w|^2 + 2\alpha\beta\langle v, w\rangle} \\
&= \frac{1}{1 + |\frac{\beta w}{|\alpha v|}|^2 + 2\frac{\alpha}{|\alpha|}\beta\langle \frac{v}{|v|}, \frac{w}{|w|}\rangle \frac{|w|}{|\alpha v|}} \\
&= \frac{1}{1 + |\frac{\beta w}{|\alpha v|}|^2 + 2\langle \frac{v}{|v|}, \frac{w}{|w|}\rangle \frac{\alpha}{|\alpha|}\frac{\beta|w|}{|\alpha v|}},
\end{aligned}$$

by simple algebraic steps. Define $y \coloneqq \frac{\alpha}{|\alpha|}\frac{\beta|w|}{|\alpha v|} \in \mathbb{R}$ and note that $y^2 = |\frac{\beta w}{|\alpha v|}|^2$. Thus, we can write

$$|P_V x|^2 = \frac{1}{1 + 2y\langle \frac{v}{|v|}, \frac{w}{|w|}\rangle + y^2}.$$

Note that the right-hand side can be upper-bounded by $\frac{1}{1-\langle \frac{v}{|v|}, \frac{w}{|w|}\rangle^2}$ since the quadratic function at the denominator achieves its minimum for $y^* = -\langle \frac{v}{|v|}, \frac{w}{|w|}\rangle$. Thus, we get that

$$|P_V x|^2 \leq \frac{1}{1 - \langle \frac{v}{|v|}, \frac{w}{|w|}\rangle^2}.$$

Further, the term $1 - \langle \frac{v}{|v|}, \frac{w}{|w|}\rangle^2$ that can be lower-bounded by $1 - \cos^2(V, W)$. Therefore, we have that

$$|P_V x| \leq \frac{1}{(1 - \cos^2(V, W))^{1/2}} \tag{A.22}$$

for any $x \in \mathbb{R}^d \setminus \{0\}$ such that $x \notin V$ and $x \notin W$. If $x \in V$, then $|P_V x| = \frac{|x|}{|x|} = 1$. If $x \in W$, then $P_V x = 0$. Thus, for any $x \in \mathbb{R}^d \setminus \{0\}$, we have that $|P_V x| \leq \max\{0, 1, \frac{1}{(1-\cos^2(V,W))^{1/2}}\} = \frac{1}{(1-\cos^2(V,W))^{1/2}}$, where the last equality follows from the fact that $\cos(V, W) \in [0, 1)$.
Next, note that there exist $v^* \in V$ and $w^* \in W$ such that $|v^*| = |w^*| = 1$ and that $|\langle v^*, w^* \rangle| = \cos(V, W)$. This follows from the fact that the function $f(v, w) \coloneqq |\langle v, w \rangle|$ is continuous and the set $\{(v, w) \in V \times W : |v| = |w| = 1\}$ is compact. Thus, the maximum is achieved by some pair $(v^*, w^*) \in V \times W$ with $|v^*| = |w^*| = 1$. Since $\cos(V, W) = \max\{f(v, w) : (v, w) \in V \times W \text{ and } |v| = |w| = 1\}$, we conclude the proof of this claim. Now, choose $x = v^* - \cos(V, W) w^*$. Therefore, $\alpha = 1$, $\beta = -\cos(V, W)$, and $y = -\cos(V, W)$. Therefore, we get that $\frac{1}{1+2y\langle v^*, w^* \rangle + y^2} = \frac{1}{(1-\cos^2(V,W))^{1/2}}$. Thus, the maximum on the right-hand side of inequality (A.22) is achieved. Hence, we conclude that $\|P_V\| = \frac{1}{(1-\cos^2(V,W))^{1/2}}$.

A.13 Proof of Lemma 4.4.2

First, the fact that the Riccati equation $\dot{P}_{\sigma(\tau_k)}(t - \tau_k) = -A'_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k) - P_{\sigma(\tau_k)}(t - \tau_k) A_{\sigma(\tau_k)} - Q_{\sigma(\tau_k)} + P_{\sigma(\tau_k)} B'_{\sigma(\tau_k)} R^{-1}_{\sigma(\tau_k)} B_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k)$ with final condition $P_{\sigma(\tau_k)}(\tau_{k+1}) = I_d$ has a unique symmetric positive definite solution for all $t \in [\tau_k, \tau_{k+1}]$ is standard (see, e.g., Chapter 6 of [75]). Furthermore, it is also standard that the system $\dot{x}(t) = A_{\sigma(\tau_k)} x(t) + B_{\sigma(\tau_k)} u(t)$ with initial condition $x(\tau_k)$ at initial time τ_k under the feedback control $u(t) = -R^{-1}_{\sigma(\tau_k)} B'_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k) x(t)$ is such that $x(\tau_{k+1}) = 0$. Note that the feedback system, after making the substitution of $u(t)$, becomes $\dot{x}(t) = (A_{\sigma(\tau_k)} - R^{-1}_{\sigma(\tau_k)} B'_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k)) x(t)$ for $t \in [\tau_k, \tau_{k+1}]$.

Second, note that $\phi_{\sigma(\tau_k)}(t - \tau_k)$ is the matrix transfer function for the LTV system $\dot{x}(t) = (A_{\sigma(\tau_k)} - R^{-1}_{\sigma(\tau_k)} B'_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k)) x(t)$ on the time interval $[\tau_k, \tau_{k+1}]$. Thus, note that $\phi_{\sigma(\tau_k)}(t - \tau_k) x(\tau_k) = x(t)$ by this previous fact. Hence, we can write the previous feedback law as $u(t) = -R^{-1}_{\sigma(\tau_k)} B'_{\sigma(\tau_k)} P_{\sigma(\tau_k)}(t - \tau_k) \phi_{\sigma(\tau_k)}(t - \tau_k) x(\tau_k)$, which proves the lemma.

A.14 Proof of Lemma 4.4.3

We first prove the following inequality by induction over $k \in \mathbb{Z}_{\geq 1}$:

$$y_k \leq a^k y_0 + b \sum_{j=0}^{k-1} a^j.$$

Let $k = 1$, then

$$y_1 \leq a y_0 + b.$$

Now, consider $k = n + 1$ for $n \in \mathbb{Z}_{\geq 1}$. Then,

$$\begin{aligned} y_{n+1} \leq a y_n + b &\leq a \left(a^n y_0 + b \sum_{j=0}^{n-1} a^j \right) + b \\ &= a^{n+1} y_0 + b \sum_{j=1}^{n} a^j + b, \end{aligned}$$

where the second inequality comes from our induction hypothesis, proving the claim. Recalling that $\sum_{j=0}^{n} a^j = \frac{a^{n+1}-1}{a-1}$, we conclude the proof of this lemma.

A.15 Relation of Theorem 4.3.7 and the literature on entropy

Remark A.15.1 (Relationship with topological entropy)**.** There exists a connection between the previous upper bound for the data rate used by our coder-decoder/controller and recent results on upper bound for the topological entropy of linear switched systems. To see that, we make a few simplifying assumptions. First, assume that $\omega = 0$, i.e., the switching times happen synchronously with the sampling times, then the perceived asymptotic active rate is given by $\rho_i^p = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \chi_{\{i\}}(\sigma(\tau))d\tau$. The right-hand side of the latter equation is called *asymptotic active rate* of the i-th mode in [26] and is denoted by $\hat{\rho}_i$. Further, we assume that the previous limit superior is actually a limit[17]. Also, if we take the limit as $T_p \to \infty$ in the last inequality for the data-rate given in the proof of Theorem 4.3.7, we see that $\frac{1}{T_p}\log\left(\lceil e^{(\mu(A_i)+\alpha)T_p}\rceil\right) \to \max\{0, \mu(A_i)+\alpha\}$ since[18] $\frac{1}{T}\log\left(e^{xT}\right) \leq \frac{1}{T}\log\left(\lceil e^{xT}\rceil\right) \leq \max\{0, x+\frac{1}{T}\}$. Thus, inequality the inequality for the data-rate given in Theorem 4.3.7 becomes $b(\bar{M}, \alpha) \leq \sum_{i\in[m]} \rho_i d \max\{0, \mu(A_i)+\alpha\}$. Taking $\alpha = 0$, we can compare our result with the upper-bound for the topological entropy of linear switched systems presented in Theorem 1 from [26]. In that paper, the upper bound presented was

$$\limsup_{t\to\infty} \sum_{i\in[m]} \max\{0, d\mu(A_i)\rho_i(t)\} = \sum_{i\in[m]} \max\{0, d\mu(A_i)\hat{\rho}_i\},$$

where we have used the fact that the limit superior is a limit to get the equality. Comparing, we see that the upper bound given in the proof of the Theorem 4.3.7 becomes $\sum_{i\in[m]} \max\{0, d\mu(A_i)\hat{\rho}_i\}$, which is the same as the upper-bound given in [26]. Informally, if the sampling period T_p is large, system (4.1) satisfies Assumption 4.3.1 with ω/ℓ small, and we use $\alpha = 0$, then the data-rate used by our Algorithm 4.4.1 is close to the upper-bound for the topological entropy provided in [26].

[17] In our current analysis, we can drop this assumption by slightly modifying the inequalities (4.43) and we would obtain the same conclusion without this assumption. However, this would make the proof longer and less clear.

[18] Note that $\log\left(\lceil e^{xT}\rceil\right) = 0$ if $xT \leq 0$ and that $\log\left(\lceil e^{xT}\rceil\right) \leq \log\left(e^{xT}\right) + 1$ if $x > 0$.