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IFAC INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL
AACC American Automatic Control Council

How to Park a Car Blindfolded

Guilherme Scabin Vicinansa
Daniel Liberzon

1. Introduction

- Consider the following driftless control system

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t), \qquad (1)$$
$$x(0) = x_0,$$

subject to the constraints

1. System (1) satisfy the Lie Algebra Rank Condition (LARC) everywhere;
2. A global upper bound $\mathcal{D}$ on the non-holonomy degree is known;
3. The vector functions g_i, for $i = \{1, \cdots, m\}$ are smooth, i.e. $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

- Consider the *coder map at time n*

$$\gamma_1 : x(t_1) \mapsto q_1,$$
$$\gamma_n : (q_1, \cdots, q_{n-1}, x(t_1), \cdots, x(t_n)) \mapsto q_n,$$

where $q_n \in \mathcal{C}_n$, with $\mathcal{C}_n$ a finite alphabet, for all $n \in \mathbb{N}$.

- Consider the *controller map at time n*

$$\delta_1 : (q_1, i) \mapsto (u_{[t_1,t_2]}, k)$$
$$\delta_n : ((q_1, \cdots, q_n), i) \mapsto (u_{[t_n,t_{n+1}]}, k),$$

where $k \in \mathbb{N}$ and $i \in \mathbb{N}$, and $u_{[t_1,t_2]} \in (U^m)^{[t_n,t_{n+1}]}$ is the set of functions from $[t_n, t_{n+1}]$ to U^m.

- Let the *average data-rate* be given by

$$b := \limsup_{j \to \infty} \frac{1}{t_j} \sum_{i=1}^{j} \log(\#\mathcal{C}_i).$$

In this presentation $\mathcal{C}_i = \{-1, 1\}, \forall i \in \mathbb{N}$, and $U = [-1, 1]$.

- Our problem is: **Can we find a constructive algorithm that gives us coder and controller maps such that system (1) is globally asymptotically stabilized with minimum average data-rate?**

- The answer is **yes** and we can do it with **average data-rate zero**.

2. The Driver

- System (1) can be rewritten in its integral form as

$$x(t_n) = x(t_{n-1}) + \sum_{i=1}^{m} \int_{t_{n-1}}^{t_n} g_i(x(\tau))u_i(\tau)d\tau, \qquad (2)$$
$$x(0) = x_0.$$

Define the parameter $\alpha_n := \left|\left|u_{[t_{n-1},t_n]}\right|\right|_\infty, \forall n \in \mathbb{N}$. Also, let $x_n := x(t_n), \forall n \in \mathbb{N}$. Furthermore, define $v_n := \sum_{i=1}^m \int_{t_{n-1}}^{t_n} \frac{g_i(x(\tau))u_i(\tau)}{\alpha_n} d\tau$. Note that the function $\frac{u_{i_{[t_{n-1},t_n]}}}{\alpha_n}$ has its image on $[-1, 1]$ by definition of α_n, but it is an arbitrary piecewise constant function, otherwise.

- Therefore the following equation holds

$$x_n = x_{n-1} + \alpha_n v_n, \qquad (3)$$

- Consider $V : \mathbb{R}^d \to \mathbb{R}$ a convex, radially unbounded, $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$, function with Lipschitz derivative around the origin.

- The idea is to choose a control law $u_{[t_{n-1},t_n]}$ such that v_n can be made into a decreasing direction for the cost function V departing from x_{n-1}. This is the idea behind the compass search method [A. R. Conn, K. Scheinberg, and L. N. Vicente, 2009].

- If we choose a nondecreasing direction, we can go back, due to the *strong reversibility* property of driftless systems.

We need to be careful about two things:

- The step size α_n needs to be small

- If the decrease in the function value between two consecutive iterations is not large enough we might get stuck.

- To solve the second problem: introuce a function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the properties: (1) non-decreasing, (2) continuous, (3) $\lim_{t \downarrow 0} \frac{\rho(t)}{t^{\mathcal{D}}}$.

- Declare the iteration successful only if $V(\phi(t_n, x_{n-1}, u_{[t_{n-1},t_n]})) - V(x_{n-1}) + \rho(\alpha_n)) < 0$

- To generate the directions $v_i \in \mathcal{V}$, we pick control laws that generate approximations to all Lie brackets up to order $\mathcal{D}$, i.e. $v_n = \alpha^{p-1} X_p + \frac{o(\alpha^p)}{\alpha}$.

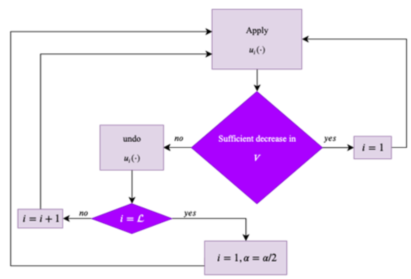

Figure 1: Block diagram with the control logic

3. Parking the Car

- Consider the equations of the Dubin's car

$$\dot{x}_1 = u_1 \cos(\theta)$$
$$\dot{x}_2 = u_1 \sin(\theta) \qquad (4)$$
$$\dot{\theta} = u_2$$

where states x_1 and x_2 represent the $x - y$ coordinates of a unicycle, while θ is the angle.

- The cost function is $V(x_1, x_2, \theta) = x_1^2 + x_2^2 + \theta^2$.

$$f_1(x_1, x_2, \theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f_2(x_1, x_2, \theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix},$$

$$[f_1, f_2](x_1, x_2, \theta) = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix},$$

- The controls are the functions $u_1^\alpha(t) = (\alpha, 0)$ for $t \in \left[0, \frac{T_p}{2}\right]$, $u_2^\alpha(t) = (0, \alpha)$ for $t \in \left[0, \frac{T_p}{2}\right]$,

$$u_3^\alpha(t) = \begin{cases} (\alpha, 0) & t \in \left[0, \frac{T_p}{8}\right) \\ (0, \alpha) & t \in \left[\frac{T_p}{8}, \frac{T_p}{4}\right) \\ (-\alpha, 0) & t \in \left[\frac{T_p}{4}, \frac{3T_p}{8}\right) \\ (0, -\alpha) & t \in \left[\frac{3T_p}{8}, \frac{T_p}{2}\right] \end{cases},$$

and $u_{j+3}^\alpha = -u_j$ for $j = 1, 2, 3$.

- The cost value reaches a plateau due to (i) local convergence properties of direct search methods [A. R. Conn, K. Scheinberg, and L. N. Vicente, 2009], and (ii) by the fast convergece of θ to zero.

4. Analysis

- It can be shown that the following assumptions are satisfied if the first set of assumptions hold. From this we conclude GAS.

(i) The level set $L(x_0) = \{x \in \mathbb{R}^d : V(x) \leq V(x_0)\}$ is compact;

(ii) If for some real constant $\alpha > 0$, the step size at iteration k, α_k, is such that $\alpha_k > \alpha$, for all $k \in \mathbb{N}$. Then the algorithm visits only a finite number of points;

(iii) Let $\xi_1, \xi_2 > 0$ be some fixed constants. The positive spanning sets $\mathcal{V}_k$ used in the algorithm are chosen from the set

$$\{\mathcal{V}_k \text{ positively span } \mathbb{R}^d : \text{cm}(\mathcal{V}_k) > \xi_1, \ ||\bar{v}|| \leq \xi_2, \forall \bar{v} \in \mathcal{V}_k\}$$

Recall that the cosine measure of a finite positively spanning set $\mathcal{V} \subset \mathbb{R}^d$ is $\text{cm}(\mathcal{V}) = \min_{w,||w||=1} \max_{v \in \mathcal{V}} \frac{\langle v,w \rangle}{||v||}$.

(iv) The gradient ∇V is Lipschitz continuous in an open set containing $L(x_0)$, with Lipschitz constant ν.

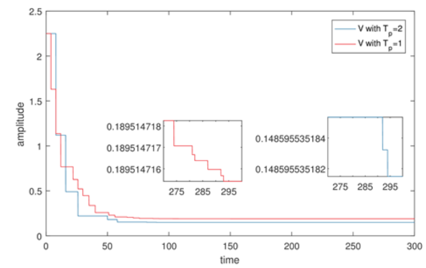

Figure 2: Cost function evolution

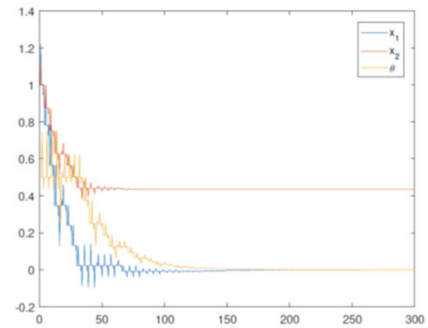

Figure 3: Evolution of the states for $T_p = 1$

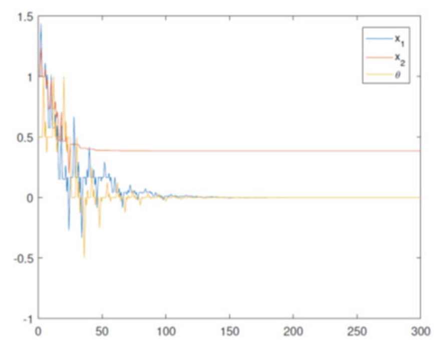

Figure 4: Evolution of the states for $T_p = 2$

- Let $P(x_k)$ be the positively spanning set generated by the method at iteration k

- Denote by κ the minimum of the norm of the nonzero Lie brackets of $\{f_i\}_{i=1}^m$ up to order $\mathcal{D}$ evaluated at the iterate x_k.

- By Theorem 5.1 of the paper

$$||\nabla V(x_k)|| \leq \frac{\nu}{2} \text{cm}(P(x_k))^{-1} \max_{d \in P(x_k)} ||v||\alpha_k + \text{cm}(P(x_k))^{-1} \frac{\rho(\alpha_k)}{\min_{v \in P(x_k)} ||v||\alpha_k},$$

- By assumption (iii) and the fact that $\frac{\rho(\alpha_k)}{\min_{v \in P(x_k)} ||v||\alpha_k} \leq \frac{\rho(\alpha_k)}{\kappa(\alpha_k)^{\mathcal{D}}} \downarrow 0$, for $\alpha_k \downarrow 0$, we conclude that $||\nabla V(x_k)|| \downarrow 0$.

- Therefore, $(x_k)_{k \in \mathbb{N}}$ converges to 0 proving GAS.

5. Conclusion

- We presented a limited information control algorithm that globally asymptoticaly stabilizes a subclass of non-holonomic driftless affine systems.

- We proved that the minimum average data-rate for achieving GAS for this class is 0.

- Future works include extending the constructive method presented here to other classes of control systems and providing data-rate theorems for more general classes of systems.