NORM-CONTROLLABILITY, or
How a Nonlinear System Responds to Large Inputs

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TALK OUTLINE

• Motivating remarks
• Definitions
• Main technical result
• Examples
• Conclusions
CONTROLLABILITY: LINEAR vs. NONLINEAR

Point-to-point controllability

**Linear systems:** \( \dot{x} = Ax + Bu \)
- can check controllability via matrix rank conditions
- Kalman contr. decomp. \( \rightarrow \) controllable modes

**Nonlinear control-affine systems:** \( \dot{x} = f(x) + g(x)u \)
similar results exist but more difficult to apply

**General nonlinear systems:** \( \dot{x} = f(x, u) \)
controllability is not well understood
NORM-CONTROLLABILITY: BASIC IDEA

\[ \dot{x} = f(x, u) \]

Instead of point-to-point controllability:

- look at the norm \(|x|\)
- ask if can get large \(|x|\) by applying large \(|u|\) for long time

This means that \(x\) can be steered far away at least in some directions (output map \(y = h(x)\) will define directions)

Possible application contexts:

- process control: does more reagent yield more product?
- economics: does increasing advertising lead to higher sales?
**NORM-CONTROLLABILITY vs. ISS**

\[ \dot{x} = f(x, u) \]

**Input-to-state stability (ISS)**
[Sontag ’89]:
small / bounded \( u \) \( \Rightarrow \)
small / bounded \( x \)
ISS gain:
upper bound on \( x \) for all \( u \)

**Norm-controllability (NC):**
large \( u \) \( \Rightarrow \) large \( x \)
NC gain: lower bound on \( x \) for worst-case \( u \)

**Lyapunov characterization**
[Sontag–Wang ’95]:
\[ |x| \geq \rho(|u|) \Rightarrow \dot{V} < 0 \]

Lyapunov sufficient condition:
\[ |x| \leq \rho(|u|) \Rightarrow \dot{V} > 0 \]
(to be made precise later)
NORM-CONTROLLABILITY vs. NORM-OBSERVABILITY

For dual notion of observability, [Hespanha–L–Angeli–Sontag ’05] followed a conceptually similar path

\[ \dot{x} = f(x), \quad y = h(x) \]

Instead of reconstructing \( x \) precisely from measurements of \( y \), norm-observability asks to obtain an upper bound on \( |x| \) from knowledge of norm of \( y \):

\[ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \quad \text{where} \quad \gamma \in \mathcal{K}_\infty \]

Precise duality relation – if any – between norm-controllability and norm-observability remains to be understood
FORMAL DEFINITION

\[ \dot{x} = f(x, u), \quad x(0) = x_0 \]
\[ y = h(x) \quad (\text{e.g., } h(x) = x) \]

\[ \mathcal{U}_{a,b} := \{ u(\cdot) : |u(t)| \leq b \ \forall t \in [0, a] \} \]

\[ \mathcal{R}^a\{x_0, \mathcal{U}_{a,b}\} := \text{reachable set at } t = a \]
from \( x(0) = x_0 \) using \( u(\cdot) \in \mathcal{U}_{a,b} \)

\[ R_h^a(x_0, \mathcal{U}_{a,b}) := \text{radius of smallest ball around } y = 0 \text{ containing } h(\mathcal{R}^a) \]

**Definition:** System is norm-controllable (NC) if

\[ R_h^a(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b) \quad \forall a, b > 0 \]

where \( \gamma \) is nondecreasing in \( a \) and class \( \mathcal{K}_\infty \) in \( b \)

(related notion: excitability index [Fradkov])
LYAPUNOV-LIKE SUFFICIENT CONDITION

Theorem: \( \dot{x} = f(x, u), \ y = h(x) \)

is NC if \( \exists V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) satisfying

\[
\begin{align*}
\alpha_1(|\omega(x)|) & \leq V(x) \leq \alpha_2(|\omega(x)|) \\
\nu(|\omega(x)|) & \leq |h(x)| \quad \text{(e.g., } \omega(x) = h(x))
\end{align*}
\]

where \( \omega : \mathbb{R}^n \rightarrow \mathbb{R}^l \) continuous, \( \alpha_1, \alpha_2, \nu \in \mathcal{K}_\infty \)

\[
\begin{align*}
\exists \rho, \chi \in \mathcal{K}_\infty : \forall b > 0, \ \forall x \ \text{s.t.} \ |\omega(x)| & \leq \rho(b) \\
\exists u, |u| & \leq b \text{ that gives } V'(x; f(x, u)) \geq \chi(b)
\end{align*}
\]

where \( V'(x; h) := \liminf_{t \searrow 0, \bar{h} \rightarrow h} \frac{V(x + t\bar{h}) - V(x)}{t} \)

See paper for extension using higher-order derivatives
**IDEA of CONTROL CONSTRUCTION**

For simplicity take \( h(x) = x \) and \( \omega(x) = x \). Fix \( a, b > 0 \).

- \( \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \)
- when \( |\omega(x)| \leq \rho(b) \), \( \exists u \) with \( |u| \leq b \)
  
  s.t. \( V'(x; f(x, u)) \geq \chi(b) \) \hspace{1cm} (\( u \) is “good” for \( x \))

1) Given \( x_0 \), pick a “good” value \( u_0 \)

For \( u \equiv u_0 \) \( \exists \) time \( t_1 > 0 \) s.t.

\[ V(x(t_1)) \geq V(x_0) + (1 - \varepsilon)t\chi(b) \]

\( \forall t \in [0, t_1] \), where \( \varepsilon \) is arb. small

Repeat for \( x(t_1), x(t_2), \ldots \) until time \( \tilde{t}_1 \) s.t. \( V(x(\tilde{t}_1)) = \alpha_1(\rho(b)) \)
IDEA of CONTROL CONSTRUCTION

For simplicity take $h(x) = x$ and $\omega(x) = x$. Fix $a, b > 0$.

- $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$
- when $|\omega(x)| \leq \rho(b)$, $\exists u$ with $|u| \leq b$
  s.t. $V'(x; f(x, u)) \geq \chi(b)$  ($u$ is “good” for $x$)

2) Pick a “good” $\tilde{u}_1$ for $x(\tilde{t}_1)$

For $u \equiv \tilde{u}_1 \exists$ time $\tilde{t}_2 > \tilde{t}_1$ s.t.

$V(x(t)) \geq \alpha_1(\rho(b)) \quad \forall t \in [\tilde{t}_1, \tilde{t}_2]$  

Repeat for $x(\tilde{t}_2), x(\tilde{t}_3), \ldots$

until we reach $t = a$.
IDEA of CONTROL CONSTRUCTION

\[ \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \]

\[ u(\cdot) \in \mathcal{U}_{a,b} \text{ piecewise constant} \]

Can prove:

\[ t_i \text{'s and } \tilde{t}_i \text{'s don't accumulate} \]

\[ V(x(a)) \geq \min \left\{ (1 - \varepsilon)a\chi(b) + V(x_0), \alpha_1(\rho(b)) \right\} \]

\[ R^a(x_0, \mathcal{U}_{a,b}) \geq \alpha_2^{-1} \left( \min \left\{ a\chi(b) + V(x_0), \alpha_1(\rho(b)) \right\} \right) \]
EXAMPLES

1) \( \dot{x} = -x^3 + u, \quad y = x \)

Take \( V(x) = |x| \)

For \( x \neq 0 \), \( \dot{V} = -|x|^3 + \text{sgn}(x)u \)

\[ = -|x|^3 + \theta \text{sgn}(x)u + (1 - \theta) \text{sgn}(x)u \]

where \( \theta \in (0, 1) \) is arbitrary

For each \( b > 0 \), “good” \( u \) are \( |u| = \Theta b \) and \( xu \geq 0 \):

\[ \dot{V} \geq (1 - \theta)b =: \chi(b) \quad \text{when} \quad |x| \leq (\theta b)^{1/3} =: \rho(b) \]

For \( x = 0 \), \( V'(0; u) = |u| \)

so any \( u \) with \( |u| = \Theta b \) is “good”: \( V'(0; u) = b \geq \chi(b) \)

\( V \) non-smooth at 0 can be increased from 0 at desired speed

System is NC with \( \gamma(a, b) = \min \left\{ (1 - \theta)ab + |x_0|, \frac{3}{\sqrt[3]{\theta}}b \right\} \)
EXAMPLES

2) \( \dot{x}_1 = (1 + \sin(x_2u))|u| - x_1 \)
\[ \dot{x}_2 = x_1 - 0.2x_2 \]
\[ y = x_1 \]
Assume \( x_1(0) \geq 0 \Rightarrow x_1(t) \geq 0 \ \forall t \)
Take \( V(x) = |x_1| \)
For \( x_1 \neq 0 \), \( \dot{V} = \dot{x}_1 \)
“Good” inputs: \( |u| = b, \ \sin(x_2u) \geq 0 \) \Rightarrow \( \dot{V} \geq b - x_1 \geq (1 - \theta)b =: \chi(b) \)
when \( x_1 \leq \theta b =: \rho(b), \ \theta \in (0, 1) \)
For \( x_1 = 0 \), \( V'(0; u) = \dot{x}_1 \) – same as above
System is NC with \( \gamma(a, b) = \min\{ (1-\theta)ab + |x_1(0)|, \theta b \} \) ✓
EXAMPLES

3) Isothermal continuous stirred tank reactor with irreversible 2nd-order reaction from reagent A to product B

\[
\begin{align*}
\dot{x}_1 &= -cx_1 - kx_1^2 + cu \\
\dot{x}_2 &= kx_1^2 - cx_2 \\
y &= qx_2
\end{align*}
\]

\(x_1, x_2 = \) concentrations of species A and B
\(u = \) concentration of reagent A in inlet stream
\(y = \) amount of product B per time unit
\(q = \) flow rate, \(k = \) reaction rate
\(c = q/V, \ V = \) reactor volume

This system is NC, but showing this is not straightforward:

- Lyapunov sufficient condition only applies when initial conditions satisfy \(x_2(0) \leq (k/c)x_1^2(0)\) – meaning that enough of reagent A is present to increase the amount of product B
- Relative degree = 2 \(\Rightarrow\) need to work with \(V^{(2)}(x; f(x, u))\)
- Several regions in state space need to be analyzed separately
LINEAR SYSTEMS

\[ \dot{x} = Ax + Bu \quad S := \text{span}(B, AB, \ldots, A^{n-1}B) \]

If \( \ell^\top \) is a real left eigenvector of \( A \) not orthogonal to \( S \) then system is NC w.r.t. \( h(x) = \ell^\top x \)

\[
(V(x) = |\ell^\top x| \Rightarrow \dot{V} = \text{sgn}(\ell^\top x)\ell^\top \dot{x} = \lambda |\ell^\top x| + \text{sgn}(\ell^\top x)(\ell^\top Bu)\]

Corollary: If \((A, B)\) is controllable and \( A \) has \( \geq 1 \) real eigenvalues, then \( \dot{x} = Ax + Bu, \ y = x \) is NC

Let \( \text{eig}(A) \) be real. Then \((A, B)\) is controllable if and only if system is NC w.r.t. \( h(x) = \ell^\top x \) \( \forall \) left eigenvectors \( \ell^\top \) of \( A \)

More generally, consider Kalman controllability decomposition

\[
\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \tilde{B}u, \quad \dot{x}_2 = A_{22}x_2
\]

If \( \tilde{\ell}^\top \) is a real left eigenvector of \( A_{11} \), then system is NC w.r.t. \( h(x) = (\tilde{\ell}^\top 0)x \) from initial conditions \( x_0 \in S \)
CONCLUSIONS

Contributions:
• Introduced a new notion of norm-controllability for general nonlinear systems
• Developed a Lyapunov-like sufficient condition for it ("anti-ISS Lyapunov function")
• Established relations with usual controllability for linear systems

Future work:
• Identify classes of systems that are norm-controllable
• Study alternative (weaker) norm-controllability notions
• Develop necessary conditions for norm-controllability
• Treat other application-related examples