NORM-CONTROLLABILITY, or How a Nonlinear System Responds to Large Inputs

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TALK OUTLINE

- Motivating remarks
- Definitions
- Main technical result
- Examples
- Conclusions

CONTROLLABILITY: LINEAR vs. NONLINEAR

Point-to-point controllability

Linear systems: $\dot{x} = Ax + Bu$

- can check controllability via matrix rank conditions
- Kalman contr. decomp. $\longrightarrow$ controllable modes

Nonlinear control-affine systems: $\dot{x} = f(x) + g(x)u$

similar results exist but more difficult to apply

General nonlinear systems: $\dot{x} = f(x, u)$

controllability is not well understood

NORM-CONTROLLABILITY: BASIC IDEA

$$\dot{x} = f(x,u)$$

Instead of point-to-point controllability:

- look at the norm $|x|$
- ask if can get large $|x|$ by applying large $|u|$ for long time

This means that x can be steered far away at least in some directions (output map $y = h(x)$ will define directions)

Possible application contexts:

- process control: does more reagent yield more product ?
- economics: does increasing advertising lead to higher sales ?

NORM-CONTROLLABILITY vs. ISS

$$\dot{x} = f(x, u)$$

Input-to-state stability (ISS)
[Sontag '89]:

small / bounded u $\Rightarrow$ small / bounded x

ISS gain:
upper bound on x for all u

Norm-controllability (NC):

large u $\Rightarrow$ large x

NC gain: lower bound on x for worst-case u

Lyapunov characterization
[Sontag–Wang '95]:

$$|x| \geq \rho(|u|) \Rightarrow \dot{V} < 0$$

Lyapunov sufficient condition:

$$|x| \leq \rho(|u|) \Rightarrow \dot{V} > 0$$

(to be made precise later)

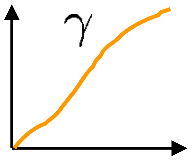
NORM-CONTROLLABILITY vs. NORM-OBSERVABILITY

For dual notion of observability, [Hespanha–L–Angeli–Sontag '05] followed a conceptually similar path

$$\dot{x} = f(x), \quad y = h(x)$$

Instead of reconstructing x precisely from measurements of y, norm-observability asks to obtain an upper bound on $|x|$ from knowledge of norm of y:

$$|x(0)| \leq \gamma\left(\|y\|_{[0,\tau]}\right) \qquad \text{where } \gamma \in \mathcal{K}_\infty$$



Precise duality relation – if any – between norm-controllability and norm-observability remains to be understood

FORMAL DEFINITION

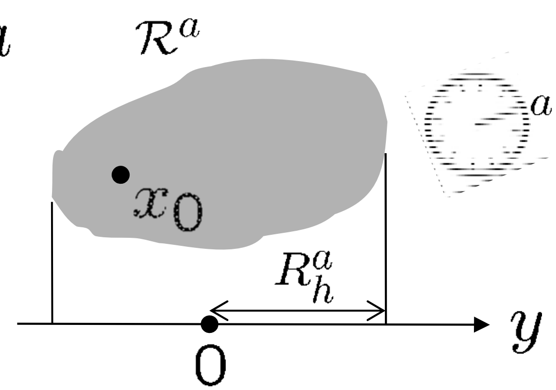
$$\dot{x} = f(x, u), \quad x(0) = x_0$$

$$y = h(x) \ \text{(e.g., } h(x) = x\text{)}$$

$$\mathcal{U}_{a,b} := \{u(\cdot) : |u(t)| \le b \ \forall t \in [0, a]\}$$

$\mathcal{R}^a\{x_0, \mathcal{U}_{a,b}\} :=$ reachable set at $t = a$ from $x(0) = x_0$ using $u(\cdot) \in \mathcal{U}_{a,b}$

$R_h^a(x_0, \mathcal{U}_{a,b}) :=$ radius of smallest ball around $y = 0$ containing $h(\mathcal{R}^a)$



Definition: System is norm-controllable (NC) if

$$R_h^a(x_0, \mathcal{U}_{a,b}) \ge \gamma(a, b) \quad \forall a, b > 0$$

where γ is nondecreasing in a and class $\mathcal{K}_\infty$ in b

(related notion: excitability index [Fradkov])

LYAPUNOV-LIKE SUFFICIENT CONDITION

Theorem: $\dot{x} = f(x,u), \;\; y = h(x)$

is NC if $\exists\, V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

- $\alpha_1(|\omega(x)|) \leq V(x) \leq \alpha_2(|\omega(x)|)$
  $\nu(|\omega(x)|) \leq |h(x)|$ (e.g., $\omega(x) = h(x)$)
  where $\omega : \mathbb{R}^n \to \mathbb{R}^\ell$ continuous, $\alpha_1, \alpha_2, \nu \in \mathcal{K}_\infty$

- $\exists\, \rho, \chi \in \mathcal{K}_\infty$: $\forall\, b > 0$, $\forall\, x$ s.t. $|\omega(x)| \leq \rho(b)$
  $\exists\, u\,, |u| \leq b$ that gives $V'(x; f(x,u)) \geq \chi(b)$
  where $V'(x;h) := \liminf\limits_{t \searrow 0,\, \bar{h} \to h} \dfrac{V(x+t\bar{h}) - V(x)}{t}$

See paper for extension using higher-order derivatives

IDEA of CONTROL CONSTRUCTION

For simplicity take $h(x) = x$ and $\omega(x) = x$. Fix $a, b > 0$.

- $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$
- when $|\omega(x)| \le \rho(b)$, $\exists\, u$ with $|u| \le b$
  s.t. $V'(x; f(x,u)) \ge \chi(b)$ (u is “good” for x)

1) Given x_0, pick a “good” value u_0

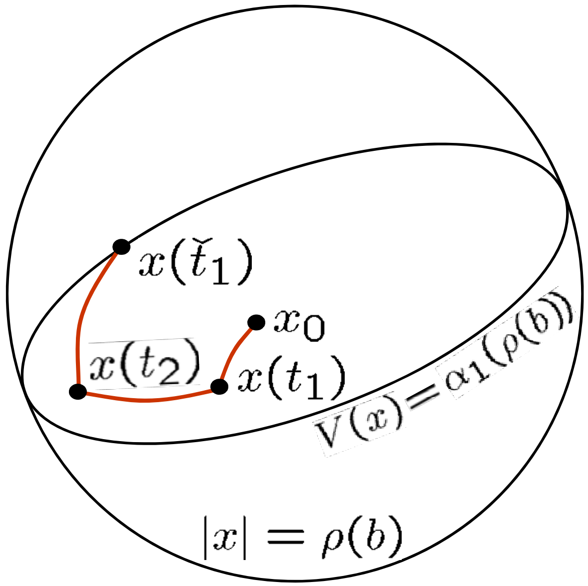
For $u \equiv u_0$ $\exists$ time $t_1 > 0$ s.t.

$V(x(t)) \ge V(x_0) + (1 - \varepsilon) t \chi(b)$

$\forall t \in [0, t_1]$, where ε is arb. small

Repeat for $x(t_1), x(t_2), \ldots$ until

time $\check{t}_1$ s.t. $V(x(\check{t}_1)) = \alpha_1(\rho(b))$



IDEA of CONTROL CONSTRUCTION

For simplicity take $h(x) = x$ and $\omega(x) = x$. Fix $a, b > 0$.

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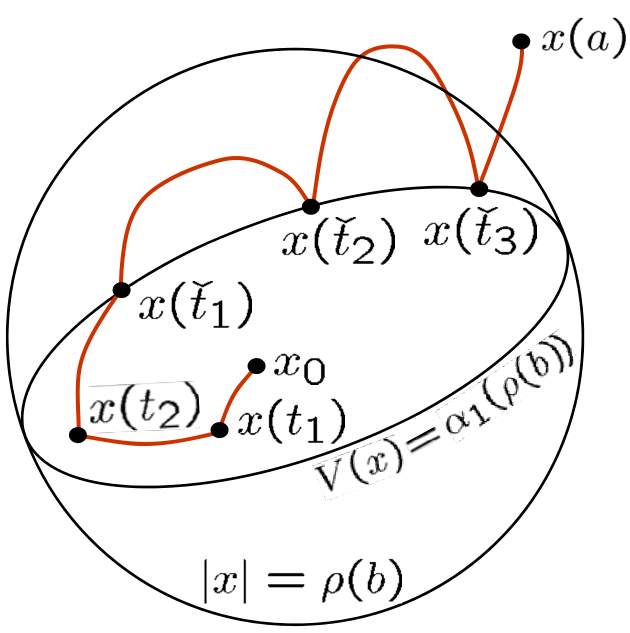
- when $|\omega(x)| \le \rho(b)$, $\exists\, u$ with $|u| \le b$
  s.t. $V'(x; f(x,u)) \ge \chi(b)$ (u is "good" for x)

2) Pick a "good" $\check{u}_1$ for $x(\check{t}_1)$

For $u \equiv \check{u}_1$ $\exists$ time $\check{t}_2 > \check{t}_1$ s.t.

$V(x(t)) \ge \alpha_1(\rho(b))$ $\forall t \in [\check{t}_1, \check{t}_2]$

Repeat for $x(\check{t}_2), x(\check{t}_3), \ldots$
until we reach $t = a$



IDEA of CONTROL CONSTRUCTION

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$

$u(\cdot) \in \mathcal{U}_{a,b}$ piecewise constant

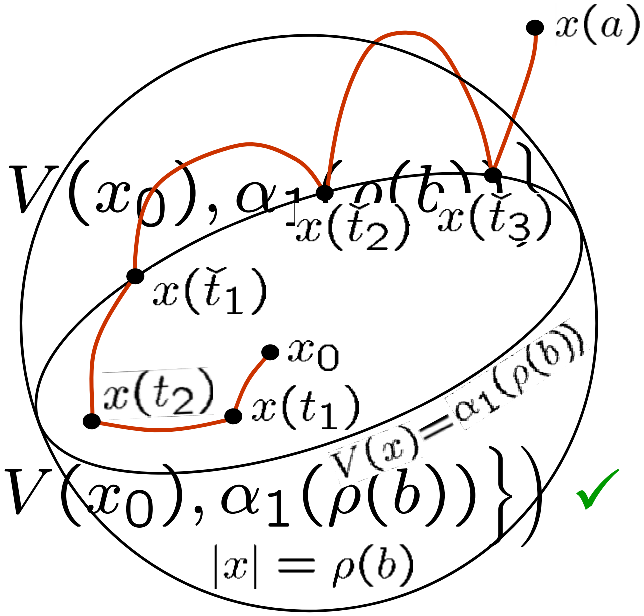
Can prove:

t_i's and $\check{t}_i$'s don't accumulate

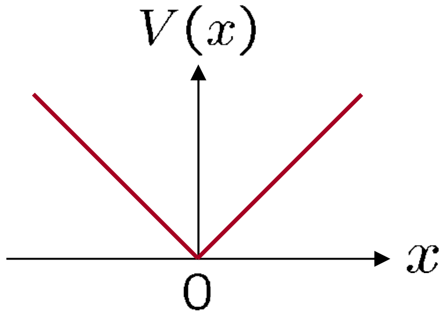
$$V(x(a)) \ge \min\Big\{(1-\varepsilon)a\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}$$

$$\Downarrow$$

$$R^a(x_0, \mathcal{U}_{a,b}) \ge \alpha_2^{-1}\Big(\min\Big\{a\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big) \checkmark$$



EXAMPLES



1) $\dot{x} = -x^3 + u, \quad y = x$

Take $V(x) = |x|$

For $x \neq 0, \quad \dot{V} = -|x|^3 + \mathrm{sgn}(x)u$

$$= -|x|^3 + \theta\mathrm{sgn}(x)u + (1-\theta)\mathrm{sgn}(x)u$$

where $\theta \in (0,1)$ is arbitrary

For each $b > 0$, "good" u are $|u| = b$ and $xu \geq 0$:

$\dot{V} \geq (1-\theta)b =: \chi(b)$ when $|x| \leq (\theta b)^{1/3} =: \rho(b)$

For $x = 0, \quad V'(0;u) = |u|$

so any u with $|u| = b$ is "good": $V'(0;u) = b \geq \chi(b)$

V non-smooth at 0 can be increased from 0 at desired speed

System is NC with $\gamma(a,b) = \min\left\{(1-\theta)ab + |x_0|, \sqrt[3]{\theta b}\right\}$ ✓

EXAMPLES

2) $\dot{x}_1 = (1 + \sin(x_2 u))|u| - x_1$

$$\dot{x}_2 = x_1 - 0.2x_2$$

$$y = x_1$$

Assume $x_1(0) \geq 0 \Rightarrow x_1(t) \geq 0 \ \forall\, t$

Take $V(x) = |x_1|$

For $x_1 \neq 0$, $\dot{V} = \dot{x}_1$

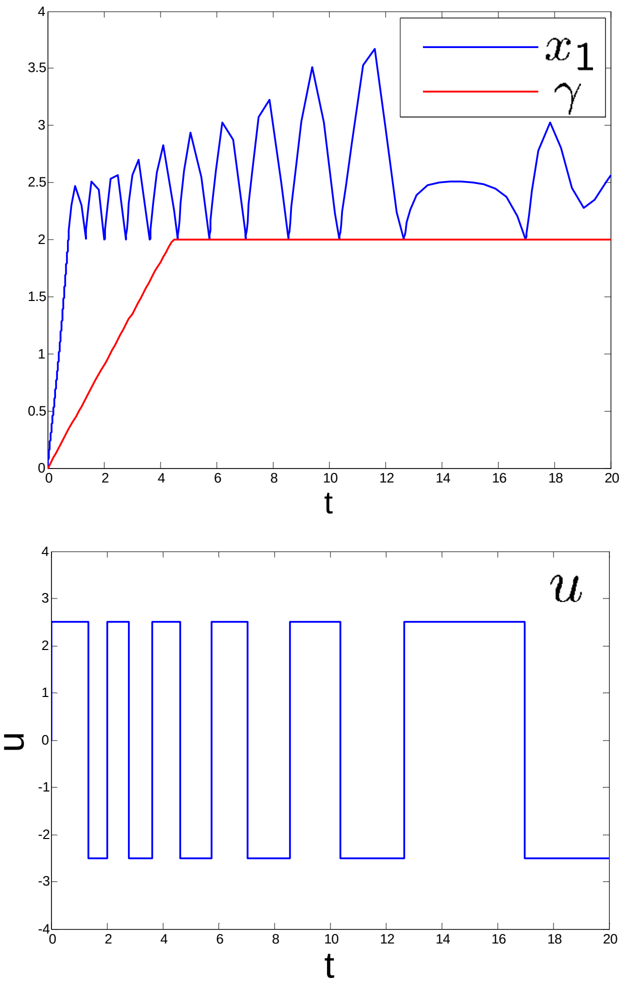
"Good" inputs: $|u| = b$, $\sin(x_2 u) \geq 0$

$\dot{V} \geq b - x_1 \geq (1-\theta)b =: \chi(b)$

when $x_1 \leq \theta b =: \rho(b)$, $\theta \in (0,1)$

For $x_1 = 0$, $V'(0; u) = \dot{x}_1$ – same as above

System is NC with $\gamma(a,b) = \min\{(1-\theta)ab + |x_1(0)|, \theta b\}$ ✓



EXAMPLES

3) Isothermal continuous stirred tank reactor with irreversible 2nd-order reaction from reagent A to product B

$$\dot{x}_1 = -cx_1 - kx_1^2 + cu$$
$$\dot{x}_2 = kx_1^2 - cx_2$$
$$y = qx_2$$

$x_1, x_2 =$ concentrations of species A and B
$u =$ concentration of reagent A in inlet stream
$y =$ amount of product B per time unit
$q =$ flow rate, $k =$ reaction rate
$c = q/V$, $V =$ reactor volume

This system is NC, but showing this is not straightforward:

- Lyapunov sufficient condition only applies when initial conditions satisfy $x_2(0) \leq (k/c)x_1^2(0)$ – meaning that enough of reagent A is present to increase the amount of product B
- Relative degree = 2 $\Rightarrow$ need to work with $V^{(2)}(x; f(x,u))$
- Several regions in state space need to be analyzed separately

LINEAR SYSTEMS

$$\dot{x} = Ax + Bu \qquad \mathcal{S} := \text{span}(B, AB, \ldots, A^{n-1}B)$$

If $\ell^\top$ is a real left eigenvector of A not orthogonal to $\mathcal{S}$ then system is NC w.r.t. $h(x) = \ell^\top x$

$$(V(x) = |\ell^\top x| \Rightarrow \dot{V} = \text{sgn}(\ell^\top x)\ell^\top \dot{x} = \lambda|\ell^\top x| + \text{sgn}(\ell^\top x)\ell^\top Bu)$$

$\neq 0$

Corollary: If (A, B) is controllable and A has ≥ 1 real eigenvalues, then $\dot{x} = Ax + Bu$, $y = x$ is NC

Let $\text{eig}(A)$ be real. Then (A, B) is controllable if and only if system is NC w.r.t. $h(x) = \ell^\top x$ $\forall$ left eigenvectors $\ell^\top$ of A

More generally, consider Kalman controllability decomposition

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \tilde{B}u, \qquad \dot{x}_2 = A_{22}x_2$$

If $\tilde{\ell}^\top$ is a real left eigenvector of A_{11}, then system is NC w.r.t. $h(x) = (\tilde{\ell}^\top \; 0)x$ from initial conditions $x_0 \in \mathcal{S}$

CONCLUSIONS

Contributions:

- Introduced a new notion of norm-controllability for general nonlinear systems
- Developed a Lyapunov-like sufficient condition for it ("anti-ISS Lyapunov function")
- Established relations with usual controllability for linear systems

Future work:

- Identify classes of systems that are norm-controllable
- Study alternative (weaker) norm-controllability notions
- Develop necessary conditions for norm-controllability
- Treat other application-related examples