COMMUTATORS, ROBUSTNESS, and STABILITY of SWITCHED LINEAR SYSTEMS

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SWITCHED SYSTEMS

Switched system:

$$\dot{x} = f_\sigma(x)$$

- $\dot{x} = f_p(x), \;\; p \in \mathcal{P}$ is a family of systems
- $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal

Switching can be:

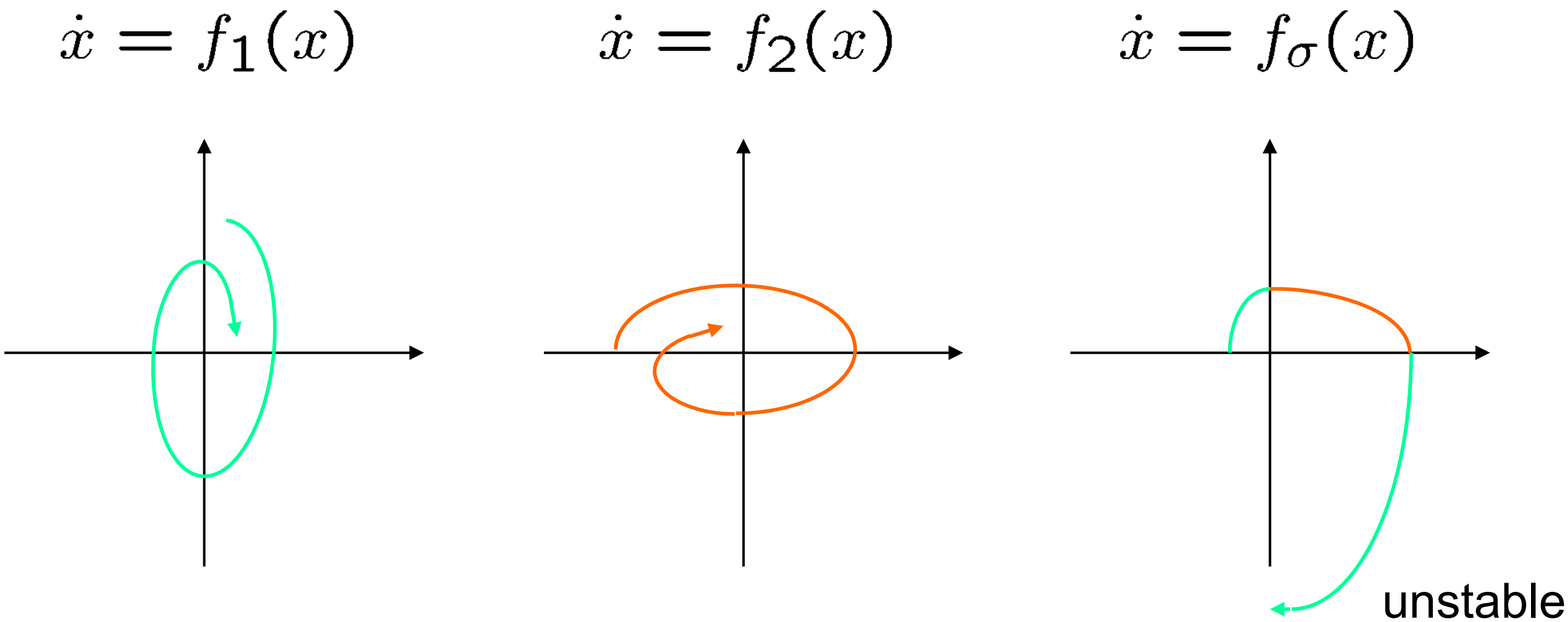
- State-dependent or time-dependent
- Autonomous or controlled

Details of discrete behavior are “abstracted away”

Discrete dynamics $\longrightarrow$ classes of switching signals

Properties of the continuous state x: stability

STABILITY ISSUE



Asymptotic stability of each subsystem is

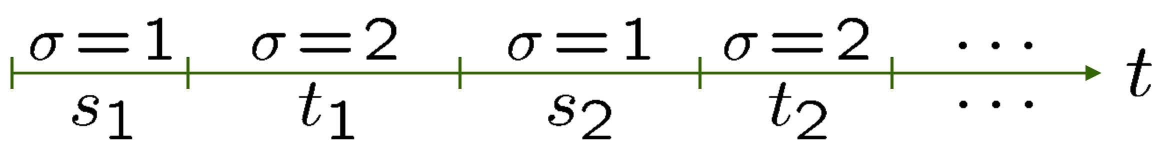
not sufficient for stability under arbitrary switching

COMMUTING STABLE MATRICES => GUES

Switched linear system: $f_p(x) = A_p x$

$\mathcal{P} = \{1, 2\},\ A_1 A_2 = A_2 A_1$

(commuting Hurwitz matrices)



$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2 (t_k + \ldots + t_1)} e^{A_1 (s_k + \ldots + s_1)} x(0) \to 0$$

For > 2 subsystems – similarly

COMMUTING STABLE MATRICES => GUES

Alternative proof:

$\exists$ quadratic common Lyapunov function

[Narendra–Balakrishnan ’94]

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$

$$\vdots$$

$$P_m A_m + A_m^T P_m = -P_{m-1}$$

$x^T P_m x$ is a common Lyapunov function

SUMMARY of KNOWN RESULTS

Lie algebra $\{A_p : p \in \mathcal{P}\}_{\text{LA}}$ w.r.t. $[A_1, A_2] = A_1 A_2 - A_2 A_1$

Stability is preserved under arbitrary switching for:

- commuting subsystems: $[A_p, A_q] = 0 \ \forall p, q \in \mathcal{P}$

  $\bigcap$
- nilpotent Lie algebras (suff. high-order Lie brackets are 0)

  $\bigcap$ e.g. $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$
- solvable Lie algebras (triangular up to coord. transf.)

  $\bigcap$
- solvable + compact (purely imaginary eigenvalues)

Quadratic common Lyapunov function exists in all these cases

No further extension based on Lie algebra only

[Narendra–Balakrishnan, Gurvits, Kutepov, L–Hespanha–Morse, Agrachev–L]

REMARKS on LIE-ALGEBRAIC CRITERIA

- Checkable conditions

- In terms of the original data

- Independent of representation

- Not robust to small perturbations

In any neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices generating the entire Lie algebra $gl(n, \mathbb{R})$

How to capture closeness to a “nice” Lie algebra?

ALMOST COMMUTING vs. NEAR COMMUTING

Halmos's problem (1976): is a pair of (self-adjoint) matrices which almost commute always close to a pair of commuting matrices?

- Pearcy–Shields (1978): positive result, one matrix must be self-adjoint
- Exel–Loring (1978): counterexample, uses unitary matrices of growing dimension
- Lin (1997): positive result for self-adjoint matrices with bounded norm
- Kachkovskiy–Safarov (2014): quantitative bound for Lin's theorem

- Łojasiewicz (1978): general theorem relating the value of a function to the distance to its zero set
- Ji–Kollar–Shiffman (1992): more specific Łojasiewicz bounds for polynomials or maxima of finitely many polynomials
- Hastings–Loring (2010): Łojasiewicz-type bounds for self-adjoint and unitary matrices with bounded norm

ŁOJASIEWICZ INEQUALITY

$f : \mathbb{R}^n \to \mathbb{R}$ real analytic function

$Z := \{x \in \mathbb{R}^n : f(x) = 0\}$ – zero set of f

Then for every $K \subset \mathbb{R}^n$ compact, $\exists\, C, \alpha > 0$ s.t.

$$C|f(x)| \geq \mathrm{dist}(x, Z)^\alpha \quad \forall\, x \in K$$

Meaning: if $|f(x)| \leq \varepsilon$ then $\mathrm{dist}(x, Z) \leq \delta := (C\varepsilon)^{1/\alpha}$

When f is a maximum of finitely many polynomials, explicit bounds on the exponent α can be obtained

ŁOJASIEWICZ APPLIED TO OUR SET–UP

$\{A_1, A_2, \ldots, A_N\}$ – finite set of Hurwitz matrices

Suppose $\|[A_i, A_j]\|_F \leq \varepsilon \quad \forall\, i, j$

where $\|A\|_F := \sqrt{\mathrm{tr} AA^T}$ is Frobenius norm

Let $f(A_1, A_2, \ldots, A_N) := \max_{i,j} \|[A_i, A_j]\|_F^2$

This is max of $N(N-1)/2$ polynomials of degree 4 in coefficients of A_i (or can use sum to get a single polynomial)

Let δ be distance from $(A_1, \ldots, A_N)$ to nearest N-tuple of pairwise commuting matrices $(B_1, \ldots, B_N)$

(Frobenius norm of difference between stacked matrices)

Łojasiewicz inequality gives $Cf(A_1, A_2, \ldots, A_N) \geq \delta^\alpha$

i.e., $A_i = B_i + \Delta_i$ with $\|\Delta_i\|_F \leq (C\varepsilon^2)^{1/\alpha}$

Higher-order commutators (near nilpotent/solvable) – similar

PERTURBATION ANALYSIS

$A_i = B_i + \Delta_i, \;\; \|\Delta_i\|_F \le (C\varepsilon^2)^{1/\alpha}$

B_i commute (or generate nilpotent or solvable Lie algebra)

$\Rightarrow \exists$ common Lyapunov function $V(x) = x^T P x$:

$$PB_i + B_i^T P = -Q_i < 0 \quad \forall\, i$$

For original matrices: $PA_i + A_i^T P = -Q_i + P\Delta_i + \Delta_i^T P$

which is still negative definite if $\|\Delta_i\|_F < \frac{\lambda_{\min}(Q_i)}{2\lambda_{\max}(P)}$

$\Rightarrow$ stability of $\dot{x} = A_\sigma x$ is ensured if $\boxed{(C\varepsilon^2)^{1/\alpha} < \frac{\lambda_{\min}(Q_i)}{2\lambda_{\max}(P)}}$

For commuting and solvable cases, specific known constructions of P can be used to estimate the right-hand side [Baryshnikov–L, CDC ’13]

C, α depend on N (# of matrices) and on compact set where they live

$\deg f = 4 \Rightarrow \alpha$ can be explicitly estimated

estimating C is a bit more difficult [Ji–Kollar–Shiffman]

CONCLUSIONS

Results:

- Qualitative: switched linear system remains stable $\forall\sigma$ if commutation relations are only approximately satisfied
- Asymptotic: admissible commutator size goes to 0 at rate of stability margin $\lambda_{\mathrm{min}}(Q_i)/(2\lambda_{\mathrm{max}}(P))$ raised to power $\alpha/2$
- Quantitative: explicit estimates are possible in principle, but checking LMI feasibility is probably more numerically efficient

Open questions:

- Relation to approach in [Agrachev–Baryshnikov–L, SCL, 2012]
- Compact but infinite matrix families (trade-off between N and ε)
- Switched nonlinear systems