INVERTIBILITY AND INPUT-TO-STATE STABILITY OF SWITCHED SYSTEMS AND APPLICATIONS IN ADAPTIVE CONTROL

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2007

Urbana, Illinois

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ABSTRACT

This dissertation aims to study basic properties–invertibility and stability–of switched systems and their applications in control.

We formulate the invertibility problem for switched linear systems, which concerns finding conditions on a switched system so that one can uniquely recover the switching signal and the input from an output and an initial state. In solving the invertibility problem, we introduce the concept of singular pairs and we provide a necessary and sufficient solution for invertibility of switched linear systems, which says that every subsystem should be invertible and there is no singular pair. We propose a switching inversion algorithm for switched systems to find inputs and switching signals that generates a given output starting at a given initial state.

Another result of the dissertation addresses stability of switched nonlinear systems. Unlike switched linear systems where there always exists a constant switching gain among the Lyapunov functions of the subsystems, the existence of such constant gains is not guaranteed for switched nonlinear systems in general. We provide conditions on how slow the switching signals should be in order to guarantee input-to-state stability (or asymptotic stability) of a switched system if all the subsystems are input-to-state stable (or asymptotically stable, respectively), both when a constant switching gain exists and when it does not. The slowly switching conditions are characterized via switching profiles, dwell-time switching, and average dwell-time switching.

As control applications of switched systems, we apply our stability results for switched systems to the problem of adaptively controlling uncertain nonlinear plants and linear time-varying plants. For uncertain nonlinear plants with bounded noise and disturbances, we show that using supervisory control, all the closed-loop signals can be kept bounded for arbitrary initial conditions when the controllers provide the

ISS property with respect to the estimation errors. We also show that supervisory control is capable of stabilizing uncertain linear plants with large parameter variation in the presence of unmodeled dynamics and bounded noise and disturbances, provided that the unmodeled dynamics are small enough and the parameters vary slowly enough as described by switching profiles.

To mom, dad, and my other half

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my adviser, Professor Daniel Liberzon. I have been very fortunate to have an adviser who completely understands and supports me throughout my graduate study. Daniel gave me freedom to explore on my own, and at the same time, guidance and advice when I needed them. Sometimes the research went slowly, and Daniel's patience and support helped me overcome crises and finish the dissertation. I am also grateful to Daniel for holding me to a high research standard, never compromising thoroughness in favor of a quick publication. More than just being an academic adviser, Daniel gave me valuable advice in career and job situations (and also social situations). I am also thankful to Daniel for encouraging me to improve my writing and communication skills in English (which is my second language).

I also thank Professor Tamer Başar, Professor Rayadurgam Srikant, and Professor Christoforos Hadjicostis for serving on my thesis committee and giving comments and constructive criticisms on my research and on the dissertation. I especially thank Professor Hadjicostis who was on sabbatical at the time of the final defense but still participated via phone conference despite his local time being evening.

I thank my friends and fellows in the Control Group: Debasish Chatterjee, Shreyas Sundaram, Peter Hokayem, Aneel Tanwani, and Yoav Sharon, for giving me feedback on my research and sharing research ideas, joy, and fun.

For their various forms of support during my graduate study, I also thank the following former and current staff in the Control Group : Frances Bridges, Rebecca Lonberger, Ronda Rigdon, John Bambenek, and Dan Block. I also thank the staff in the Publications Office in the department of Electrical and Computer Engineering (ECE) for helping correct typos in the draft of the dissertation.

The financial support provided by my adviser through a research assistantship via grants from the National Science Foundation (NSF) and the Defense Advanced Research Projects Agency (DARPA) and the financial support provided by the ECE department through a teaching assistantship are gratefully acknowledged.

Last but not least, I thank my parents for all the love and support they have provided me through the years, and Thu Pham, my other half, for all her love, care, understanding, and never-failing support.

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CHAPTER 1

INTRODUCTION

1.1 Switched Systems

Many complex systems exhibit switching behaviors. These switching behaviors may come from physical changes in the systems, such as an aircraft during different thrust modes [1], a walking robot during leg impact and leg swing modes [2], or different formations of a group of vehicles [3]. Switching behaviors can also arise as results of control design, such as in gain scheduling control [4] or supervisory adaptive control [5]. In numerous situations, dynamical systems with switching behaviors can be described by switched systems. Formally, a *switched system* comprises:

1. A *family of dynamical subsystems* parameterized over an *index set* $\mathcal{P}$. Denote the family of the subsystems by $\{\Gamma_p,\ p \in \mathcal{P}\}$, where Γ_p is the system with index p. The subsystem dynamics are described by ordinary differential equations as for nonswitched systems.

2. A *switching signal* $\sigma : \mathcal{D} \to \mathcal{P}$ that indicates the active subsystem at every time instant, where $D \subseteq [0, \infty)$ is a time domain of the switched system. Switching signals are piecewise right-continuous functions that take constant values in between every two consecutive discontinuities. The discontinuities of σ are called *switching times* or *switches*. We assume that there is a finite number of switches in every finite time interval. The phenomenon of infinitely many switches in a finite interval is called Zeno behavior (see, for example, [6]). We explicitly rule out Zeno behavior as it may cause finite escape time even if all the subsystems are forward complete (i.e., solutions are defined over $[0, \infty)$ for all initial states). Zeno behavior also translates to hardware components moving at very high frequency and thus is not desirable in practice.

The following are standard assumptions on switched systems:

1. We assume that all the subsystems live in the same state space, which is also the state space of the switched system. While it is theoretically feasible to obtain more general results in the case the state spaces of the individual subsystems are different, virtually all of the switched systems encountered in practice have subsystems of the same state dimension. We will remark on the case of the subsystems having different state spaces when it is applicable.

2. We assume that there are no state jumps at switching times unless it is clearly stated otherwise. It is possible to extend the results in this thesis to include the case of state jumps at switching times such that if τ is a switching time, $x(\tau) = m_{\sigma(\tau),\sigma(\tau^-)}(x(\tau^-))$ where $m_{p,q} : \mathbb{R}^n \to \mathbb{R}^n$, $p, q \in \mathcal{P}$ is a *reset map* for the switched system's state when the active subsystem changes from subsystem with index p to subsystem with index q (here $\mathbb{R}^n$ denotes the state space of the switched system). We will remark on more general cases of nonidentity reset maps when it is applicable.

Remark 1.1 *The switched system model described in this dissertation places emphasis on the sequence of active modes over the time and neglects the actual switching mechanism (which may depend on the states as well as the inputs of the subsystems). While we may lose certain detail, the sequence of active modes over the time is often sufficient to draw conclusions about system properties (such as asymptotic stability or input-to-state stability) of the switch system (see Chapter 3). However, it is noted that in certain situations, more information about the switching signal (such as the switching mechanism) may become useful [7].*

1.1.1 Autonomous switched systems

For autonomous switched systems, the subsystem dynamics do not depend on external signals. The subsystems can be written as

$$\Gamma_p : \begin{cases} \dot{x} = f_p(x), \\ y = h_p(x), \end{cases} \quad p \in \mathcal{P}, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^r$, f_p are locally Lipschitz, and h_p are continuous for all $p \in \mathcal{P}$. The switched system with the subsystems (1.1) is written as

$$\Gamma_\sigma : \begin{cases} \dot{x} = f_\sigma(x), \\ y = h_\sigma(x). \end{cases} \tag{1.2}$$

For every switching signal σ, $f_\sigma(x)$ is piecewise right-continuous in t and locally Lipschitz in x. Therefore, for every initial state $x(t_0) = x_0$, the nonlinear equation $\dot{x} = f_\sigma(x)$ has a unique solution over some interval $[t_0, T_{max})$ (see, for example, [8, Theorem 3.1]). In particular, the solution can be written as

$$x(t) = x_0 + \int_{t_0}^{t} f_{\sigma(\tau)}(x(\tau))d\tau. \tag{1.3}$$

The function (1.3) satisfies the differential equation $\dot{x} = f_\sigma(x)$ everywhere except at switching times. At a switching time τ, $\lim_{t\to\tau^+} \frac{dx}{dt} = f_{\sigma(\tau)}(x(\tau))$ and $\lim_{t\to\tau^-} \frac{dx}{dt} = \lim_{t\to\tau^-} f_{\sigma(\tau^-)}(x(\tau))$. In essence, solutions of (1.2) can be viewed as concatenations of solutions of the active subsystems along the time. If every subsystem is forward complete, then the switched system is also forward complete (i.e., solutions are defined on $[0, \infty)$) in view of the fact that switching signals do not have Zeno behavior.

1.1.2 Switched systems with inputs

For switched systems with inputs, the subsystems can be written as

$$\Gamma_p : \begin{cases} \dot{x} = f_p(x, u), \\ y = h_p(x, u), \end{cases} \qquad p \in \mathcal{P}, \tag{1.4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^\ell$, $y \in \mathbb{R}^r$, f_p are locally Lipschitz, and h_p are continuous for all $p \in \mathcal{P}$. The switched system with the subsystems (1.4) is then written as

$$\Gamma_\sigma : \begin{cases} \dot{x} = f_\sigma(x, u), \\ y = h_\sigma(x, u). \end{cases} \tag{1.5}$$

While inputs in (1.4) are functions, inputs in (1.5) are actually not functions in the usual sense: an input u of the switched system (1.5) is a concatenation of the input segments of the active subsystems, and these segments can have different dimensions. To describe inputs and outputs of switched systems in general, we need to use the concept of *hybrid functions* and *compatible switching signals and inputs.*

Definition 1.1 *A function $f : \mathcal{D} \to \{\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_k}\}$ is a* (piecewise right-continuous) hybrid function *if*

(i) *for all $t \in \mathcal{D}$, the function value $f(t) \in \mathbb{R}^{m_i}$ for some $i, 1 \leqslant i \leqslant k$, and $\lim_{s\uparrow t} f(s)$ and $\lim_{s\downarrow t} f(s)$ exist;*

(ii) *f has a finite number of discontinuities in every finite interval, where t is a discontinuity of f if $\lim_{s\uparrow t} f(s) \neq \lim_{s\downarrow t} f(s)$;*

(iii) *f is continuous in between every two consecutive discontinuities and f is right continuous at discontinuities, i.e., $f(t) = \lim_{s\downarrow t} f(s)$.*

Denote by $\mathcal{H}^{\mathrm{pc},m_1,\ldots,m_k}$ the set of piecewise right-continuous hybrid functions on $\{\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_k}\}$ (the order of $m_1, \ldots, m_k$ is irrelevant). When the dimensions $m_1, \ldots, m_k$ are not important, we will write $\mathcal{H}^{\mathrm{pc}}$ instead.

Definition 1.2 *Consider the switched system* (1.5) *where the input dimensions of the subsystems are $m_p, p \in \mathcal{P}$. A hybrid function u is* compatible *with a switching signal σ if the domains of σ and u are the same, and further,* $\dim(u(t)) = m_{\sigma(t)}$ *for all t in the domain.*

The concept of hybrid functions and compatible switching signals and input functions allows us to describe solutions of the switched system (1.5) rigorously. For the switched system (1.5), denote by $m_{\mathcal{P}}$ the set of input dimensions. Denote by $\mathcal{H}^{\mathrm{pc},m_{\mathcal{P}}}$ the corresponding set of hybrid functions. This set $\mathcal{H}^{\mathrm{pc},m_{\mathcal{P}}}$ is the set of all admissible inputs of the switched system. For every initial state x_0, a switching signal σ, and a compatible hybrid function $u \in \mathcal{H}^{\mathrm{pc},m_{\mathcal{P}}}$, a solution of (1.5) exists and is unique, which follows from the existence and uniqueness of solutions for each individual subsystem. The solution can be written as

$$x(t) = x_0 + \int_{t_0}^{t} f_{\sigma(s)}(x(s), u(s))ds. \tag{1.6}$$

Equation (1.6) satisfies the differential equation $\dot{x} = f_\sigma(x, u)$ everywhere except at switching times. If τ is a switching time, then $\lim_{t\to\tau^+} \frac{dx}{dt} = f_{\sigma(\tau)}(x(\tau), u(\tau))$ and $\lim_{t\to\tau^-} \frac{dx}{dt} = \lim_{t\to\tau^-} f_{\sigma(\tau^-)}(x(\tau), u(\tau^-))$. Similarly to autonomous switched systems, solutions of switched systems with inputs can be viewed as concatenations of solutions of the active subsystems along the time.

1.2 Examples of Switched Systems

We provide here some representative examples of switched systems encountered in practice. The first two examples are switched systems resulting from controller design even though the plants are nonswitched in nature. The last three examples are examples of plants which are switched in nature.

1. **Gain scheduling**

   Gain scheduling is a common control technique used in practice to handle plants' nonlinearity (see, e.g., the survey paper [4]). Suppose that one wants to stabilize or regulate a nonlinear plant $\dot{x} = f(x,u), y = h(x)$. One obtains a linear parameter-varying (LPV) model for the plant, either by rewriting nonlinear terms as time-varying parameters (quasi-LPV method) or by linearizing the plant around a family of operating points $\{\hat{x}_p, p \in \mathcal{P}\}$ (linearization scheduling method). For each linear plant $\dot{x} = A_p x + B_p u$, $y = C_p x$ with a fixed parameter $p \in \mathcal{P}$, one designs a linear controller $\mathbf{C}_p : \dot{x}_c = \tilde{A}_p x_C + \tilde{B}_p x, u_p = \tilde{C}_p x_C + \tilde{D}_p x$ that meets the control objective. One constructs a variable θ (which is called *scheduling variable*) that can be measured directly, and devises a map M : $\text{Range}(\theta) \to \mathcal{P}$ that maps behavior of θ to specific controllers. Then the closed-loop control system can be approximated by the switched system $\dot{x}_{CL} = \bar{A}_\sigma x_{CL}$, where $x_{CL} = (x, x_C)$, $\bar{A}_p = \begin{bmatrix} A_p + B_p\tilde{D}_p & B_p\tilde{C}_p \\ \tilde{B}_p & \tilde{A}_p \end{bmatrix}$, and $\sigma(t) := M(\theta(t))$ is the switching signal that returns the index of the active controller at every time based on the value of θ.

2. **Switching supervisory control**

   Switching supervisory control (see, e.g., [9, Chapter 6] and the references therein; we will also cover supervisory control in detail in Chapter 4) is an adaptive control scheme that employs switching among multiple controllers (in contrast to other continuously tuning adaptive schemes). Consider a parameterized uncertain plant $\dot{x} = f(p^*, x, u)$, $y = h(p^*, x)$ where p^* is the unknown constant parameter. The range Ω of p^* is partitioned into a finite number of subsets $\Omega_i, i \in \mathcal{P}$ such that $\bigcup_{i\in\mathcal{P}} \Omega_i = \Omega$, where $\mathcal{P}$ is the index set. For each subset Ω_i, one selects a nominal value $p_i \in \Omega_i$. One then designs a family of controllers, one for each plant with $p = p_i, i \in \mathcal{P}$. One also designs a supervisor which produces a switching signal $\sigma : [0, \infty) \to \mathcal{P}$ that indicates the active controller at every time. Denote $\dot{x}_p = \tilde{f}(p, x_p, y), u_p = h(p, x_p, y)$ the dynamics of the controller with index p. It follows that the closed-loop control system is the switched system $\dot{x}_{CL} = g_\sigma(x_{CL})$, where $x_{CL} = (x, x_\sigma)$ and $g_p(x_{CL}) = \begin{pmatrix} f(p^*, x, h(p, x_p, x)) \\ \tilde{f}(p, x_p, x) \end{pmatrix}, p \in \mathcal{P}$.

3. **Aircraft during different flying modes**

Aircraft that have multiple flying modes can be modeled as switched systems [10]. In conventional take-off and landing (CTOL), an aircraft moves in parallel to the horizonal axis while in vertical take-off and landing (VTOL), an aircraft moves up or down without significant change in the horizonal position (such as the McDonnell Douglas YAV-8B Harrier). Some aircraft can operate in both CTOL and VTOL modes, and changing between modes can be done via a transition mode (TRANSITION). Each mode can be described by an ordinary differential equation. Below is the example taken from [10] that describes the dynamics of an aircraft in CTOL, VTOL, and TRANSITION modes, respectively:

$$\Gamma_1 : M \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = R(\theta) \left(R^T(\alpha) \begin{pmatrix} -D \\ L \end{pmatrix} + \begin{pmatrix} T \\ -\epsilon u \end{pmatrix} \right) - \begin{pmatrix} 0 \\ Mg \end{pmatrix},$$

$$\Gamma_2 : M \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = R(\theta) \begin{pmatrix} 0 \\ T - \epsilon u \end{pmatrix} - \begin{pmatrix} 0 \\ Mg \end{pmatrix},$$

$$\Gamma_3 : M \begin{pmatrix} \ddot{x} \\ \ddot{z} \end{pmatrix} = R(\theta) \left(R^T(\alpha) \begin{pmatrix} -D \\ L \end{pmatrix} + \begin{pmatrix} T \cos\delta \\ T \sin\delta - \epsilon u \end{pmatrix} \right) - \begin{pmatrix} 0 \\ Mg \end{pmatrix},$$

where x, z are coordinate variables, L is the lift, D is the drag, M is the mass, θ is the pitch angle, α is the angle of attack, $R(\theta)$ and $R(\alpha)$ are rotation matrices, δ is the nozzle angle, ϵ, g are constants, and T and u are control inputs. Switching between TRANSITION and CTOL modes occurs when $(\delta, \dot{\delta}) = (0, 0)$ and switching between TRANSITION and VTOL modes occurs when $(\delta, \dot{\delta}) = (\pi/2, 0)$. The switched system that describes the aircraft dynamics in this context is Γ_σ where $\sigma : [0, \infty) \to \{1, 2, 3\}$ is a switching signal that indicates the mode the aircraft is in at every instant of time. In [10], the authors used switched systems to analyze trajectory tracking of an aircraft through different flight modes.

4. **Multiagent systems**

Switched systems can arise from switching among different topologies of a network of agents [3, 11, 12]. A topology of a multiagent network is a graph G whose nodes represent agents and edges between nodes denote communications between the corresponding agents. For a node i, denote by $\mathcal{N}_i$ the set of neighbors of node i where neighbors of a node i are all other nodes that have an edge with node i. Denote by $\dot{x}_i = f_i(x_i, u_i)$ the dynamics of agent i where x_i is the state and u_i is the control input. In cooperative multiagent systems, the agents's control law can use not only its own state but also information from neighboring agents and the control law is of the form $u_i = g_i(x_i, \{x_j, j \in \mathcal{N}_i\})$. The collective dynamics of the network of agents are obtained by putting together all the agent dynamics and take the form $\dot{x} = F_G(x)$, where $x = (x_1, \dots, x_n)$ and F_G is a function depending on the structure of the graph G. For example (see [3]), with the agent dynamics $\dot{x}_i = u_i$ and the cooperative control law $u_i = \sum_{j \in \mathcal{N}_i}(x_j - x_i)$ (known as *consensus protocol*), the collective network dynamics are $\dot{x} = -L(G)x$, where $L(G) = [l_{ij}]$, $l_{ij} = \begin{cases} -1 & \text{if } j \in \mathcal{N}_i, j \neq i, \\ |\mathcal{N}_i| & i = j \end{cases}$ is the graph Laplacian of G.

Switching comes into the picture if we allow the topology G to vary over time, $G : [0, \infty) \to \{G_p, p \in \mathcal{P}\}$. We then have a network with switching topology. Networks with switching topology can be modeled as switched systems where the subsystems are the network dynamics with fixed topologies and the switching signal, in this case, indicates the active topology at every time. In particular, the switched system describing multiagent networks with switching topology is $\dot{x} = F_\sigma(x)$, where $\dot{x} = F_p(x)$ is the collective dynamics of the network with topology G_p, $p \in \mathcal{P}$, and $\sigma(t) := p : G(t) = G_p$.

5. **Network congestion control**

Switched systems can be used to model and analyze communication networks such as in stability analysis of networks with transmission congestion control [13, 14]. A communication network consists of a number of nodes and there are data packets flowing along communication links between nodes. The objective of a transmission congestion control protocol (TCP) is to regulate the data flows so that the network does not collapse under heavy load while meeting other criteria such as fairness or optimality. Consider a node with m in-flows and one out-flow (the dumbbell topology) under TCP Reno. Let w_i be the window sizes of the in-flows. Let RTT be the round-trip time, which is the time from when a packet arrives at the queue till when an acknowledge signal is received after the packet is sent. If there are no drops, TCP increases w_i by $a \geqslant 1$ for every w_i acknowledgements received. Since each packet takes RTT amount of time for acknowledgement, the window size is increased at the rate a/RTT on average, and the window-size dynamics of the i-th flow can be approximated as $\dot{w}_i = \frac{a}{RTT}$ when no drop occurs. When drops occur, the window size is reduced by a factor $\mu \in (0, 1)$ so that $w_i(\tau) = \mu w_i(\tau^-)$. Let $q(t)$ be the queue length at the node at a time t. The round-trip time RTT can be approximated as a sum of the time $T_q(t)$ a packet spent in the queue and the propagation delay T_p, which is the time taken for a packet to travel across the physical length of the link and an acknowledge signal to come back. It is assumed that T_p is a constant. Let B be the bandwidth of the link in packets per second. Then the queuing time $T_q(t) = q(t)/B$. Thus, $RTT = T_p + \frac{q(t)}{B}$. The queue length dynamics can be approximated as follows: if the queue is not full, there are $\frac{\sum_i w_i}{RTT}$ incoming packets per second and there are B outgoing packets per second; if the queue is full and there still are incoming packets, then incoming packets will be dropped and the window sizes are reduced. We then have $\dot{q} = \begin{cases} 0 & \text{if } q = 0, \frac{\sum_i w_i}{RTT} < B \text{ or } q = q_{max}, \frac{\sum_i w_i}{RTT} > B \\ \frac{\sum_i w_i}{RTT} - B & \text{else,} \end{cases}$, where the first case deals with the fact that the queue length can neither be negative nor exceed the maximal length q_{max}.

A network with TCP Reno can be approximated by a switched system Γ_σ whose subsystems Γ_1 and Γ_2 are the dynamics of the network when the queue is not full and when the queue is full, respectively:

$$\Gamma_1 : \begin{cases} \dot{w}_i = \frac{aB}{BT_p + q}, \\ \dot{q} = B\frac{\sum_i w_i}{BT_p + q} - B, \end{cases} \qquad \Gamma_2 : \begin{cases} \dot{w}_i = \frac{aB}{BT_p + q}, \\ \dot{q} = 0, \end{cases} .$$

The switching signal is $\sigma : [0, \infty) \to \{1, 2\}$. Switching from mode 1 to mode 2 occurs when the queue becomes full, i.e., $q > q_{\max}$ and $\frac{\sum_i w_i}{BT_p + q} > 1$. Switching from mode 2 to mode 1 occurs after RTT amount of time from the time a switch from mode 1 to mode 2 happens. In this example, the reset map $m_{1,2}$ is the identity map but $m_{2,1}(w_i) = \mu w_i$.

1.3 Dissertation Contributions

The contributions of the thesis can be categorized into three main areas:

1. We formulate the invertibility problem for switched systems (Section 2.2), which is a new problem that has not been considered before. The invertibility problem can be seen as an extension of the classic invertibility problem for nonswitched systems [15, 16, 17] to switched systems as well as an extension of the mode detection problem for autonomous switched systems [18, 19] to switched systems with unknown inputs.

We provide a necessary and sufficient condition for a switched linear system to be invertible (Section 2.4). In solving the invertibility problem, we introduce the new concept of singular pairs for switched systems with inputs. This concept can be seen as an extension of the concept of indistinguishable pairs for switched systems without inputs in [18] to systems with unknown inputs. We provide a necessary and sufficient algebraic criterion for checking singular pairs. Using the singular pair concept, we find that a switched system is invertible over some input set if and only if all the subsystems are invertible and there is no singular pair over the output set.

We also consider the output generation problem for switched systems, which concerns finding inputs and switching signals that can generate a given output from a given initial state even if the switched system is not invertible (Section 2.5). We provide a switching inversion algorithm that produces all inputs and switching signals capable of generating a given function in a finite time interval from a given initial state, when there are finitely many such inputs and switching signals. In the algorithm, the concept of singular pair plays a crucial role as it allows us to identify potential switching times even though the output may be smooth at these times.

2. The dissertation provides new stability results for switched nonlinear systems as well as a new way to characterize the slow switching property of switching signals.

   Previously, switching signals have been characterized using the dwell-time and average dwell-time like concepts [20, 21, 22]. To describe more general classes of switching signals, we propose the use of event profiles – functions that can be nonlinear in general to describe the relationship between the numbers of switches in time intervals and the interval lengths – to characterize switching signals (Section 3.4). Dwell-time and average dwell-time switching become the special cases where the event profiles are uniform and affine.

Unlike the case of switched linear systems where there always exists a *constant switching gain* among the Lyapunov functions of the subsystems, the existence of such a constant is not guaranteed for switched nonlinear systems in general. Using event profiles, we characterize classes of switching signals which guarantee asymptotic stability of autonomous switched nonlinear systems with globally asymptotically stable subsystems without requiring the existence of a constant switching gain. This result significantly extends a previous result in [21] that uses average dwell-time as well as assuming the existence of a constant switching gain. For switched nonlinear systems with inputs, we study the input-to-state stability (ISS) property. We provide conditions on the switching signals to guarantee ISS of switched systems when all the subsystems are ISS, both when a constant switching gain among the ISS-Lyapunov functions of the subsystems exists and when it does not.

Event profiles can also be used to describe properties of sequences of time instants, of which the switching times of a switching signal are one example. As such, event profiles can also be used to characterize impulse times of impulsive systems. Using event profiles, we also obtain stability results for impulsive systems.

3. The dissertation advances adaptive control by showing how switched supervisory control can be used to stabilize uncertain nonlinear systems with disturbances and to stabilize uncertain linear time-varying systems.

   Previously, it was only known that switching supervisory control could stabilize an uncertain nonlinear plant provided that switching stops in finite time with the assumption that there are no disturbances [23, 24]. In the presence of disturbances, switching is not guaranteed to stop. The contribution of the thesis is to show that even if switching does not stop in finite time, it is still possible to achieve boundedness of all the closed-loop continuous signals under some mild assumptions on controller design. This result is built upon our ISS result for switched nonlinear systems in Section 3.3.

We also show how supervisory control can be used to control uncertain linear systems with large time-varying unknown parameters. This result directly extends reported results on supervisory control of uncertain linear plants with a constant unknown parameter [20, 21, 25, 26]. Unlike the case with constant parameters, closed-loop analysis leads to stability analysis of interconnected switched systems, in which two switched systems are interacting with each other via a non-trivial way other than input-output connections. Using the ISS result in Section 3.8, we show that even if the unknown parameter is changing over time, the supervisory control scheme still guarantees boundedness of all the closed-loop states in the presence of noises and disturbances, provided that the parameter varies slowly enough as described by certain event profiles.

1.4 Notations

Functions

Denote by $\mathcal{K}$ the class of continuous functions $[0,\infty) \to [0,\infty)$ which are zero at zero and increasing. Denote by $\mathcal{K}_\infty$ the subclass of $\mathcal{K}$ functions that are unbounded. Denote by $\mathcal{L}$ the class of continuous functions $[0,\infty) \to [0,\infty)$ which are decreasing and converge to 0 as their argument tends to ∞. Denote by $\mathcal{KL}$ the class of functions $[0,\infty)^2 \to [0,\infty)$ which are class $\mathcal{K}$ in the first argument and class $\mathcal{L}$ in the second argument.

Denote by $C^n_{\mathcal{D}}$ the set of n times differentiable functions on a domain $\mathcal{D} \subseteq \mathbb{R}$; when the domain is not relevant, we write C^n. Denote by $\mathcal{F}^{\mathrm{pc}}_{\mathcal{D}}$ the set of *piecewise right-continuous functions* on a domain $\mathcal{D} \subseteq \mathbb{R}$. By piecewise right-continuous functions, we mean that there is a finite number of *jump discontinuities* in any finite interval; there are no hole discontinuities and vertical asymptotes; the function is continuous in between every two consecutive discontinuities; and the function is continuous from the right at discontinuities. To avoid excessive rigidness, we will use the term "piecewise continuous" throughout the paper, and it is understood that "piecewise continuous" means "piecewise right-continuous." When the domain is not important, we write $\mathcal{F}^{\text{pc}}$. Denote by $\mathcal{F}^n$ the subset of $\mathcal{F}^{\text{pc}}$ whose elements are n times differentiable between every two consecutive discontinuities. For a function $u : \mathcal{D} \to \mathbb{R}^n$, denote by $u_{\mathcal{Q}}$ the restriction of u onto a subset $\mathcal{Q}$ of $\mathcal{D}$. Denote by 0 the zero elements; whether 0 being the zero scalar, zero vectors or zero functions is clear from the context.

Operators

Denote by $(\cdot)_{t_0,t}$, $t \geqslant t_0 \geqslant 0$, the segmentation operator such that for a function f,

$$(f)_{t_0,t}(\tau) := \begin{cases} f(\tau) & \tau \in [t_0, t) \\ 0 & \text{else.} \end{cases} \tag{1.7}$$

The segmentation operator is a generalization of the truncation operator where $t_0 = 0$.

For a vector v, denote by $|\cdot|$ the Euclidean norm $|v| := (v^T v)^{\frac{1}{2}}$. Denote by $\|(\cdot)\|_\infty$ the $\mathcal{L}_\infty$ norm $\|x\|_\infty := \sup_{t\in[0,\infty)} |x(t)|$. We say that $x \in \mathcal{L}_\infty$ if $\|x\|_\infty$ exists and we say $x \in \mathcal{L}_{\infty e}$ if $\|(x)_{0,t}\|_\infty$ exists for all $t < \infty$. Define the *exponentially weighted* $\mathcal{L}_2$ *norm* as

$$\|(x)_{t_0,t}\|_{2,\lambda} := \left(\int_{t_0}^{t} e^{-\lambda(t-\tau)} |x(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad t \geqslant t_0. \tag{1.8}$$

Denote by $\|(x)_{t_0,*}\|_{2,\lambda}$ the function of t obtained when we let t be a variable in (1.8). We refer to $\|\cdot\|_{2,\lambda}$ as the $\mathcal{L}_{2,\lambda}$ norm (the 2 refers to the 2-norm of x and λ refers to the exponentially decaying rate) and we say that $x \in \mathcal{L}_{2,\lambda}$ if $\|(x)_{t_0,t}\|_{2,\lambda}$ exists for all $t \geqslant t_0$. These norms are popular in functional analysis of input/output properties of systems (see, e.g., [27, Chapter 3] or [8, Chapter 5]). It is often assumed that

$t_0 = 0$ and hence, truncation is used, but nonzero initial time is important in our analysis and this prompts the use of the segmentation operator. The $\mathcal{L}_{2,\lambda}$ norm has the following properties:

1. Addition and multiplication:

$$\|\alpha(f)_{t_0,t}\|_{2,\lambda} = \alpha\|(f)_{t_0,t}\|_{2,\lambda} \quad \forall f \in \mathcal{L}_{2,\lambda}, \alpha \geqslant 0. \tag{1.9}$$

2. Finite gain for $\mathcal{L}_{\infty e}$ signals:

$$\|(f)_{t_0,t}\|_{2,\lambda} \leqslant \frac{\|(f)_{t_0,t}\|_\infty}{\sqrt{\lambda}} \quad \forall f \in \mathcal{L}_{\infty e}. \tag{1.10}$$

3. Reducing the rate of exponentially decaying functions:

$$\|(e^{-\bar{\lambda}/2})_{t_0,t}\|_{2,\lambda} \leqslant \frac{1}{\sqrt{\bar{\lambda}-\lambda}}(e^{-\lambda/2})_{t_0,t}, \quad \forall \lambda \in (0, \bar{\lambda}). \tag{1.11}$$

4. Composition of norms:

$$\|(\|(f)_{t_0,*}\|_{2,\bar{\lambda}})_{t_0,t}\|_{2,\lambda} \leqslant \frac{1}{\sqrt{\lambda-\bar{\lambda}}}\|(f)_{t_0,t}\|_{2,\lambda} \quad \forall f \in \mathcal{L}_{2,\lambda}, 0 < \lambda < \bar{\lambda} > 0, t \geqslant \tau \geqslant t_0. \tag{1.12}$$

5. Chain rule:

$$\|(f)_{t_0,t}\|^2_{2,\lambda} = \|(f)_{t_0,\tau}\|^2_{2,\lambda}e^{-\lambda(t-\tau)} + \|(f)_{\tau,t}\|^2_{2,\lambda} \quad \forall t \geqslant \tau \geqslant t_0. \tag{1.13}$$

6. Strictly decreasing in λ:

$$\|(f)_{t_0,t}\|_{2,\lambda+\delta} < \|(f)_{t_0,t}\|_{2,\lambda} \quad \forall f \in \mathcal{L}_{2,\lambda}, \delta > 0. \tag{1.14}$$

Properties 1, 2, 5, and 6 can be verified directly from the definitions of $\mathcal{L}_{2,\lambda}$ norm and elementary calculus. Following are proofs of Properties (1.11) and (1.12).

Proof Proofs of (1.11) and (1.12). We have

$$\|(e^{-\bar{\lambda}/2})_{t_0,t}\|^2_{2,2\lambda} = \int_{t_0}^{t} e^{-\lambda(t-\tau)} e^{-\bar{\lambda}\tau} d\tau = e^{-\lambda t}\int_{t_0}^{t} e^{(\lambda-\bar{\lambda})\tau} d\tau = e^{-\lambda t}\frac{e^{(\lambda-\bar{\lambda})t} - e^{(\lambda-\bar{\lambda})t_0}}{\lambda-\bar{\lambda}},$$
$$= \frac{1}{\bar{\lambda}-\lambda}(e^{-\lambda t+(\lambda-\bar{\lambda})t_0} - e^{-\bar{\lambda}t}) \leqslant \frac{1}{\bar{\lambda}-\lambda} e^{-\lambda t+\lambda t_0} = \frac{1}{\bar{\lambda}-\lambda} e^{-\lambda(t-t_0)}.$$

$$\begin{aligned}
\|(\|(f)_{t_0,*}\|_{2,\bar{\lambda}})_{t_0,t}\|^2_{2,\lambda} &= \int_{t_0}^{t} e^{-\lambda(t-s)} \int_{t_0}^{s} e^{-\bar{\lambda}(s-\tau)} |f(\tau)|^2 d\tau ds \\
&= e^{-\lambda t}\int_{t_0}^{t} e^{(\lambda-\bar{\lambda})s}\int_{t_0}^{s} e^{\bar{\lambda}\tau}|f(\tau)|^2 d\tau ds \\
&= e^{-\lambda t}\frac{1}{\lambda-\bar{\lambda}}\int_{t_0}^{t}\left(\int_{t_0}^{s} e^{\bar{\lambda}\tau}|f(\tau)|^2 d\tau\right) de^{(\lambda-\bar{\lambda})s} \\
&= e^{-\lambda t}\frac{1}{\bar{\lambda}-\lambda}\left(e^{(\lambda-\bar{\lambda})s}\int_{t_0}^{s} e^{\bar{\lambda}\tau}|f(\tau)|^2 d\tau - \int_{t_0}^{t} e^{(\lambda-\bar{\lambda})s} e^{\bar{\lambda}s}|f(s)|^2 ds\right) \\
&\leqslant e^{-\lambda t}\frac{1}{\bar{\lambda}-\lambda}\int_{t_0}^{t} e^{(\lambda-\bar{\lambda})s} e^{\bar{\lambda}s}|f(s)|^2 ds \\
&= e^{-\lambda t}\frac{1}{\bar{\lambda}-\lambda}\int_{t_0}^{t} e^{\lambda s}|f(s)|^2 ds =: \frac{1}{\bar{\lambda}-\lambda}\|(f)_{t_0,t}\|^2_{2,\lambda}.
\end{aligned}$$

□

We define the concatenation of two functions as follows: let $f_i \in \mathcal{F}^{\text{pc}}_{[t_i,\tau_i)}$, $i = 1, 2$ (τ_i could be ∞); define the *concatenation map* $\oplus : \mathcal{F}^{\text{pc}} \times \mathcal{F}^{\text{pc}} \to \mathcal{F}^{\text{pc}}$ as

$$(f_1 \oplus f_2)(t) := \begin{cases} f_1(t), & \text{if } t \in [t_1, \tau_1), \\ f_2(t_2+t-\tau_1), & \text{if } t \in [\tau_1, \tau_1+\tau_2-t_2). \end{cases}$$

Basically, $f_1 \oplus f_2$ means putting the two functions together in that order. Note that $f_1 \oplus f_2 = f_1\ \forall f_2$ if $\tau_1 = \infty$, and in general, $f_1 \oplus f_2 \neq f_2 \oplus f_1$. The concatenation of vector functions is defined as pairwise concatenation of the corresponding elements (concatenations of functions of different dimensions are defined exactly the same with the exception that the resulting functions are hybrid functions instead of functions). The concatenation of an element f and a set S is $f \oplus S := \{f \oplus g,\ g \in S\}$. By convention, $f \oplus \emptyset = \emptyset \; \forall f$. Finally, the concatenation of two sets S and T is $S \oplus T := \{f \oplus g,\ f \in S, g \in T\}$; by convention, $S \oplus \emptyset = \emptyset$ and $\emptyset \oplus S = \emptyset \; \forall S$.

Systems

For a linear system Γ, for an initial state x_0 and an input $u \in \mathcal{F}^{\mathrm{pc}}$, denote by $\Gamma_{x_0}(u)$ the trajectory of the system and by $\Gamma^{\mathrm{O}}_{x_0}(u)$ the corresponding output (the trajectory exists (in the sense of Carathéodory) and is unique for every initial condition x_0 and input $u \in \mathcal{F}^{\mathrm{pc}}$). The initial time, which might be nonzero, is specified by the starting time t_0 in u as $t_0 = \inf\{t \in \mathcal{D}\}$ where $\mathcal{D}$ is the domain of u.

For a switched system, denote by $\mathcal{S}$ the set of all admissible switching signals. Denote by σ^p the constant switching signal such that $\sigma^p(t) = p \; \forall t \geqslant 0$. For a switched system Γ_σ, for an initial state x_0, a switching signal σ, and a compatible piecewise-continuous hybrid function u, denote by $\Gamma_{x_0,\sigma}(u)$ the trajectory of the switched system and by $\Gamma^{\mathrm{O}}_{x_0,\sigma}(u)$ the corresponding output. For the subsystem with an index $p \in \mathcal{P}$, for an initial state x_0 and an input $u \in \mathcal{F}^{\mathrm{pc}}$, denote by $\Gamma_{p,x_0}(u)$ the trajectory of the subsystem and by $\Gamma^{\mathrm{O}}_{p,x_0}(u)$ the corresponding output.

CHAPTER 2

INVERTIBILITY OF SWITCHED SYSTEMS

In this chapter, we address the *invertibility problem* for switched systems, which concerns the following question: *What is the condition on the subsystems of a switched system so that, given an initial state x_0 and the corresponding output y generated with some switching signal σ and input u, we can recover the switching signal σ and the input u uniquely?* The aforementioned problem is in the same vein with the classic invertibility problem for nonswitched linear systems, where one wishes to recover the input uniquely knowing the initial state and the output. The invertibility problem for nonswitched linear systems has been studied extensively, first by Brockett and Mesarovic [15], then with other algebraic criteria and the inversion constructions by Silverman and Payne [16, 28] and Sain and Massey [17], and also, a geometric criterion is given by Morse and Wonham [29] (see also the discussions and the references in [30]). However, the invertibility problem for switched systems has not been investigated and it is a research topic of this thesis. As in the classic setting, we start with an output and an initial state, but here, the underlying process constitutes multiple models and we have an extra ingredient to recover apart from the input, namely, the switching signal.

On the one hand, switched systems can be seen as generalizations of nonswitched linear systems (when the switching signal is constant, there is no switching and we recover nonswitched linear systems). In this regard, the invertibility problem for switched systems is an extension of the nonswitched counterpart in the sense that we have to recover the switching signal in addition to the input, based on the output and the initial state. On the other hand, switched systems can be viewed as higher-level abstractions of hybrid systems. Recovering the switching signal for switched systems is equivalent to *mode (or discrete state) identification* for hybrid systems. For autonomous hybrid systems, this mode identification task is a part of the observability problem for hybrid systems, which has been formulated and solved by Vidal et al. [18, 19]. Mode detection for nonautonomous hybrid systems using known inputs and outputs has been studied, for example, in [31, 32] (see also [33]). Here, the difference is that we do not know the input and we wish to do both *mode detection* and *input recovery* at the same time using the outputs of switched systems. To the best of our knowledge, the invertibility problem for switched systems is a new problem formulation that can be seen as a nontrivial extension of the classic invertibility problem to switched systems as well as an extension of the mode identification problem for hybrid systems to the case with unknown inputs.

Our approach to the invertibility problem is similar in spirit to the approach used in the switching observability problem by Vidal et al. [18, 19]. In the observability problem, the objective is to recover the switching signal and the state uniquely from the output of a switched system without inputs. The basic idea is to do mode identification by utilizing relationships among the outputs and the states of the individual subsystems (more details are in Section 2.5). For nonswitched systems without inputs, there is a straightforward relationship between the output and the state involving the observability matrix, which was used to solve the switching observability problem in [19]. For nonswitched systems with inputs, the relationship among the output, the input, and the state is much more complicated and is realized using the structure algorithm [16], with the help of which our results for switched systems are subsequently developed.

A motivating example

Before going into detail in the next section, we present a motivating example to illustrate an interesting aspect of the invertibility problem for switched systems that does not have a counterpart in nonswitched systems (for more details, see Example

2.2 in Section 2.6). Consider a switched system consisting of the two subsystems

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \end{cases} \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} 1 & 2 \end{bmatrix} x. \end{cases}$$

Both Γ_1 and Γ_2 are invertible, which means that for every output y in the output space of Γ_1 and every initial state x_0, there exists a unique input u such that the system Γ_1 with the initial state x_0 and the input u generates the output y, and similarly for Γ_2. (Procedures for checking invertibility of nonswitched linear systems are well-known as discussed previously; for more details on the invertibility of these Γ_1, Γ_2, see Example 2.2 in Section 2.6.) Suppose that one is given the following output:

$$y(t) = \begin{cases} 2e^{2t} - 3e^{t}, & \text{if } t \in [0, t^*), \\ c_1 e^{t} + c_2 e^{2t}, & \text{if } t \in [t^*, T), \end{cases}$$

where $t^* = \ln 3$, $T = 6/5$, $c_1 = 15 + 18\ln(2/3)$, $c_2 = -4/3 - 4\ln(2/3)$, and the initial state $x_0 = (-1, 0)^T$, and is asked to reconstruct the switching signal σ and the input u. Because the output has a discontinuity at time $t = t^*$ and is smooth everywhere else, one can guess that t^* must be a switching time and is the only one (assuming that there is no state jump at switching times and hence, x is always continuous). Following this reasoning, since there are two subsystems, there are only two possible switching signals:

$$\sigma_1(t) = \begin{cases} 1, & t \in [0, t^*), \\ 2, & t \in [t^*, T), \end{cases} \qquad \sigma_2(t) = \begin{cases} 2, & t \in [0, t^*), \\ 1, & t \in [t^*, T). \end{cases}$$

For a fixed switching signal, the active subsystem is known at every instant of time, and so by invertibility of Γ_1 and Γ_2, given the output and the initial state, the input can be reconstructed uniquely. One can then try both the switching signals σ_1 and σ_2

above to see which one gives an input that generates the output y. As it turns out, none of the switching signals σ_1 and σ_2 would give an input that generates y. One might falsely conclude that there are no input and switching signals for the switched system to generate the given y with the initial state x_0. Nonetheless, the output is generated, *uniquely* in this case, by the following switching signal and input:

$$\sigma(t) = \begin{cases} 2, & t \in [0, t_1), \\ 1, & t \in [t_1, t^*), \\ 2, & t \in [t^*, T), \end{cases} \quad u(t) = \begin{cases} 0, & t \in [0, t_1), \\ 6e^{2t} - 6e^{t}, & t \in [t_1, t^*), \\ 0, & t \in [t^*, T), \end{cases}$$

where $t_1 = \ln 2$ (details are in Example 2.2 in Section 2.6). In Section 2.5, we will show how a switch at a later time $t = t^*$ can be used to recover a switch at an earlier time $t = t_1$ even if the output is smooth at $t = t_1$.

2.1 Linear Time-Invariant Systems

We provide some background and literature review on the invertibility of linear time-invariant systems. Consider a linear time-invariant system

$$\Gamma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du. \end{cases} \tag{2.1}$$

The invertibility problem for linear systems (more precisely, the left invertibility property; from here onward, when we say invertibility we mean left invertibility) concerns with finding conditions for a linear time-invariant (LTI) system so that for a given initial state x_0, the input-output map $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ is one-to-one (injective), where $\mathcal{U}$ is the input function space and $\mathcal{Y}$ is the corresponding output function space. There are two major approaches to the problem: one is the *algebraic approach* where conditions are obtained in terms of matrix rank equalities; the other is the *geometric* *approach* that is based on the invariant properties of subspaces.

The algebraic approach relies on the observation that differentiating the output y reveals extra information about the input u via the term $C\dot{x} = CAx + CBu$. If one keeps differentiating the output, one obtains more information about the input u from $y, \dot{y}, \ddot{y}, \ldots$. It turns out that one needs at most n differentiations, where n is the state dimension, in order to conclude whether u can be recovered uniquely or not and thus to establish invertibility of the system. Taking this approach, most well-known results are the rank condition for invertibility by Sain and Massey [17] and the structure algorithm by Silverman [16], where the latter differs from the former in that it only differentiates parts of the output as needed and not the entire output.

The geometric approach is quite different from the algebraic one in that it does not involve output differentiations. Instead, the invertibility property is realized from the fact that (i) H_{x_0} is invertible if and only if the kernel of H_0 is trivial, and (ii) the kernel of H_0 is the same as the set of inputs that yield the set $\mathcal{V}$ of states that are reachable from the origin while keeping the output zero. The set $\mathcal{V}$ can be calculated geometrically (in terms of its basis) using the concepts of invariant subspaces and controllability subspaces (see, e.g., [34] for references on geometric control). Left invertibility is equivalent to $\mathcal{V} \cap B \ker D = \{0\}$ and Ker $B \cap$ Ker $D = \{0\}$ (see, e.g., [35, Ch. 8.5]; see also [29, 36]).

Remark 2.1 *The map H_{x_0} for nonswitched linear systems enjoys the following property thanks to linearity. For nonswitched linear systems, the following are equivalent:*

i. There is a unique u such that $y = \mathrm{H}_{x_0}(u)$ for one particular pair (x_0, y) where y has a domain $\mathcal{D} \subseteq [0, \infty)$.

ii. There is a unique u such that $y = \mathrm{H}_{x_0}(u)$ for every y in the range of Γ_{x_0} for all possible domains and for all $x_0 \in \mathbb{R}^n$.

Thus, for nonswitched linear systems, there is no distinction between unique input recovery for one output and unique input recovery for an output set. Also, the time *domain is irrelevant. As a consequence, the output set $\mathcal{Y}$ can be taken as the set of all output functions that are generated by continuous inputs on one arbitrary time domain or on all possible time domains. As we will see later in Section 2.3, the map* H_{x_0} *for switched systems (as defined in Section 2.2) does not have this property and a careful consideration of the output set $\mathcal{Y}$ is needed.* $\triangleleft$

2.1.1 The structure algorithm and the range theorem

We will pursue the algebraic approach (a geometric approach is equally interesting and could be the topic for future research). In particular, we will employ the *structure algorithm* for nonswitched linear systems by Silverman [16]. In this section, we cover the structure algorithm for nonswitched linear systems, closely following [16] and [28]. The reader is referred to [28] for further technical details and proofs.

Consider the linear system (2.1). Let n be the state dimension, m the input dimension, and ℓ the output dimension. For the moment, assume that the input u is continuous (for piecewise continuous inputs, see Remark 2.4 below). Let $q_0 = \operatorname{rank}(D)$. There exists a nonsingular $\ell \times \ell$ matrix S_0 such that $D_0 := S_0 D = \begin{bmatrix} \overline{D}_0 \\ 0 \end{bmatrix}$, where $\overline{D}_0$ has q_0 rows and rank q_0. Let $y_0 = S_0 y$ and $C_0 = S_0 C$. Thus, we have $y_0 = C_0 x + D_0 u$. Suppose that at step k, we have $y_k = C_k x + D_k u$, where D_k has the form $\begin{bmatrix} \overline{D}_k \\ 0 \end{bmatrix}$; $\overline{D}_k$ has q_k rows and is full rank. Let the partition of C_k be $\begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k \end{bmatrix}$, where $\overline{C}_k$ is the first q_k rows, and the partition of y_k be $\begin{bmatrix} \bar{y}_k \\ \tilde{y}_k \end{bmatrix}$, where $\bar{y}_k$ is the first q_k rows. If $q_k < \ell$, let M_k be the differential operator $M_k := \left[\begin{array}{c|c} I_{q_k} & 0 \\ \hline 0 & I_{\ell - q_k}(d/dt) \end{array}\right]$. Then $M_k y_k = \begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k A \end{bmatrix} x + \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix} u$. Let $q_{k+1} = \operatorname{rank}\left(\begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix}\right)$. There exists a nonsingular $\ell \times$

ℓ matrix S_{k+1} such that $D_{k+1} := S_{k+1} \begin{bmatrix} \overline{D}_k \\ \widetilde{C}_k B \end{bmatrix} = \begin{bmatrix} \overline{D}_{k+1} \\ 0 \end{bmatrix}$, where $\overline{D}_{k+1}$ has q_{k+1} rows and rank q_{k+1}. Let $y_{k+1} := S_{k+1} M_k y_k$, $C_{k+1} := S_{k+1} \begin{bmatrix} \overline{C}_k \\ \widetilde{C}_k A \end{bmatrix}$. Then $y_{k+1} = C_{k+1} x + D_{k+1} u$ and we can repeat the procedure. Let $N_k := \prod_{i=0}^{k} S_{k-i} M_{k-i-1}, \quad k = 0, 1, \ldots (M_{-1} := I)$, $\overline{N}_k := \begin{bmatrix} I_{q_k} & 0_{q_k \times (\ell - q_k)} \end{bmatrix} N_k$, and $\widetilde{N}_k := \begin{bmatrix} 0_{(\ell - q_k) \times q_k} & I_{\ell - q_k} \end{bmatrix} N_k$. Then $y_k = N_k y$, $\bar{y}_k = \overline{N}_k y$, and $\tilde{y}_k = \widetilde{N}_k y$. Notice that since D_k has ℓ rows and m columns, $q_k \leqslant \min\{\ell, m\}$ for all k and since $q_{k+1} \geqslant q_k$, using the Cayley-Hamilton theorem, it was shown in [28] that there exists a smallest integer $\alpha \leqslant n$ such that $q_k = q_\alpha \; \forall k \geqslant \alpha$.

If $q_\alpha = m$, the system is invertible (left-invertible) and an inverse is

$$\Gamma^{-1} : \begin{cases} \bar{y}_\alpha &= \overline{N}_\alpha y, \\ \dot{z} &= (A - B\overline{D}_\alpha^{-1} \overline{C}_\alpha) z + B \overline{D}_\alpha^{-1} \bar{y}_\alpha, \\ u &= -\overline{D}_\alpha^{-1} \overline{C}_\alpha z + \overline{D}_\alpha^{-1} y_\alpha \end{cases} \tag{2.2}$$

with the initial state $z(0) = x_0$. If $q_\alpha < m$, the system is not invertible, and then a generalized inverse is

$$\Gamma^{-1} : \begin{cases} \bar{y}_\alpha &= \overline{N}_\alpha y, \\ \dot{z} &= (A - B\overline{D}_\alpha^{\dagger} \overline{C}_\alpha) z + B \overline{D}_\alpha^{\dagger} \bar{y}_\alpha + BKv, \\ u &= -\overline{D}_\alpha^{\dagger} \overline{C}_\alpha z + \overline{D}_\alpha^{\dagger} \bar{y}_\alpha + Kv \end{cases} \tag{2.3}$$

with the initial state $z(0) = x_0$, where K is a matrix whose columns form a basis for the null space of $\overline{D}_\alpha$ and $\overline{D}_\alpha^{\dagger} := \overline{D}_\alpha^T (\overline{D}_\alpha \overline{D}_\alpha^T)^{-1}$ is a right-inverse of $\overline{D}_\alpha$.[1] The system Γ^{-1} in (2.3) is called a generalized inverse because $y = \Gamma_{x_0}^{\mathrm{O}}(u)$ if and only if $u = \Gamma_{x_0}^{-1,\mathrm{O}}(y_\alpha, v)$ for some v.

[1] $(\overline{D}_\alpha \overline{D}_\alpha^T)^{-1}$ exists because the rows of $\overline{D}_\alpha$ are linearly independent.

From the structure algorithm, it can be seen that $\tilde{y}_k = \widetilde{C}_k x \; \forall k$ and hence,

$$\begin{pmatrix} \widetilde{N}_0 \\ \vdots \\ \widetilde{N}_k \end{pmatrix} y = \begin{pmatrix} \tilde{y}_0 \\ \vdots \\ \tilde{y}_k \end{pmatrix} = \begin{bmatrix} \widetilde{C}_0 \\ \vdots \\ \widetilde{C}_k \end{bmatrix} x =: L_k x \quad \forall k. \tag{2.4}$$

Using the Cayley-Hamilton theorem, Silverman and Payne have shown in [28] that there exists a smallest number β, $\alpha \leqslant \beta \leqslant n$, such that $\operatorname{rank}(L_k) = \operatorname{rank}(L_\beta) \; \forall k \geqslant \beta$. There also exists a number δ, $\beta \leqslant \delta \leqslant n$ such that $\widetilde{C}_\delta = \sum_{i=0}^{\delta-1} P_i \left(\prod_{j=i+1}^{\delta} \widetilde{R}_j \right) \widetilde{C}_i$ for some matrices $\widetilde{R}_j$ from the structure algorithm and some constant matrices P_i (see [28, p.205] for details). The number δ is not easily determined as α and β. The significance of β and δ is that they can be used to characterize the set of all outputs of a linear system as in the range theorem [28, Theorem 4.3]. We include the range theorem from [28] below. Define the differential operators $\mathbf{M}_1 := \left(\frac{d^\delta}{dt^\delta} - \sum_{i=0}^{\delta-1} P_i \frac{d^i}{dt^i} \right) \prod_{j=0}^{\alpha} \widetilde{R}_j$ and $\mathbf{M}_2 := \sum_{j=0}^{\delta} \left(\prod_{\ell=j+1}^{\alpha} \widetilde{R}_\ell \right) K_j \frac{d^{\delta-1}}{dt^{\delta-j}} - \sum_{j=0}^{\delta-1} P_j \times \sum_{k=0}^{j} \left(\prod_{\ell=k+1}^{\alpha} \widetilde{R}_\ell \right) K_k \frac{d^{j-k}}{dt^{j-k}}$ for some matrices K_i from the structure algorithm (see [28] for more details). The notation $|_t$ means "evaluating at t."

Theorem 2.1 *[28] A function $f : [t_0, T) \to \mathbb{R}^\ell$ is in the range of Γ_{x_0} if and only if*

i. f is such that $N_\delta f$ is defined and continuous;

ii. $\widetilde{N}_k f|_{t_0} = \widetilde{C}_k x_0, \; k = 0, \ldots, \beta - 1$;

iii. $(\mathbf{M}_1 - \mathbf{M}_2 \overline{N}_\alpha) f \equiv 0$.

To simplify the presentation of our paper, we paraphrase the range theorem into the following lemma that is easier to understand at the expense of having to define extra notations. Define

- $\mathbf{N} := \begin{pmatrix} \widetilde{N}_0 \\ \ldots \\ \widetilde{N}_{\beta-1} \end{pmatrix}$, $L := L_{\beta-1}$, and

- $\widehat{\mathcal{Y}}$ the set of functions $f : \mathcal{D} \to \mathbb{R}^\ell$ for all $\mathcal{D} \subseteq [0, \infty)$ which satisfy (i) and (iii) of Theorem 2.1. The set $\widehat{\mathcal{Y}}$ is the set of all outputs of the system Γ that are generated by all possible continuous inputs and initial states for all possible durations.

Lemma 2.1 *For a linear system Γ, using the structure algorithm on the system matrices, construct a set $\widehat{\mathcal{Y}}$ of functions, a differential operator $\mathbf{N} : \widehat{\mathcal{Y}} \to \mathcal{C}^0$, and a matrix L. There exists $u \in C^0$ such that $y = \Gamma_{x_0}(u)$ if and only if $y \in \widehat{\mathcal{Y}}$ and $\mathbf{N}y|_{t_0^+} = Lx_0$ where t_0 is the initial time of y.*

Remark 2.2 *A special case is $q_\alpha = \ell$, in which case $\mathbf{M}_1 = \mathbf{M}_2 = 0$ and the condition (iii) becomes trivial. Also, in this case, $\alpha = \beta = \delta$. The set $\widehat{\mathcal{Y}}$ is simplified to the set of functions f for which $N_\delta f$ is defined and continuous. In particular, any C^n function will be in $\widehat{\mathcal{Y}}$. For an invertible system, $q_\alpha = \ell$ if the input and output dimensions are the same.* $\triangleleft$

Remark 2.3 *Note from the structure algorithm that regardless of what the input is, the output and the state are related by the equation $\mathbf{N}y|_t = Lx(t)$ for all $t \geqslant t_0$, not just at the initial time t_0. The relationship between y and x for systems with inputs is much more complicated than that for systems without inputs. For the linear system without input $\dot{x} = Ax, y = Cx$, the output is always smooth and the output and the state are related via the observability matrix $O_{A,C} = \begin{bmatrix} C^T & (CA)^T \dots & (CA^{n-1})^T \end{bmatrix}^T$ such that $(y, \dot{y}, \dots, y^{(n-1)})^T = O_{A,C}x$. For systems with inputs, the output, in general, is not smooth and the output and the state are related via the equation $\mathbf{N}y = Lx$ for some differential operator $\mathbf{N}$. For further discussion, see [28, p.204]).* $\triangleleft$

We found that Lemma 2.1 is better suited for our purpose in the paper than the more detailed Theorem 2.1. The exact formulae of $\mathbf{N}$, L, and $\widehat{\mathcal{Y}}$ are less important in the understanding of the development for checking singularity in Section 2.3 than the facts that $\mathbf{N}$ is some differential operator, L is some matrix, and $\widehat{\mathcal{Y}}$ is some set of functions (when the actual formulae are needed, it is then referred to Theorem 2.1). What is important is the necessary and sufficient condition asserted by Lemma 2.1. Roughly speaking, the set $\widehat{\mathcal{Y}}$ characterizes continuous functions that can be generated by the system from all initial positions (the components of the output must be related in some way via the system matrices A, B, C, D; in some sense, the operators $\mathbf{M}_1$ and $\mathbf{M}_2$ capture the coupling among the output components as one input component may affect several output components). The condition $\mathbf{N}y|_{t_0^+} = Lx_0$ guarantees that the particular y can be generated starting from the particular initial state x_0 at time t_0. We evaluate $\mathbf{N}y$ at t_0^+, which means $\lim_{t \to t_0^+} \mathbf{N}y|_t$, to reflect that y does not need to be defined for $t < t_0$. This is especially useful later when we consider switched systems where inputs and outputs can be piecewise right-continuous.

Remark 2.4 *If we allow piecewise continuous inputs, the invertibility condition for* Γ *is still the same* ($q_\alpha = m$)*. The inverse* (2.2) *and the generalized inverse* (2.3) *are still the same but* v *in* (2.3) *can be piecewise continuous functions. The inverse, the generalized inverse, and Theorem 2.1 are applicable in between every two consecutive discontinuities of the input (on every output segment* $y_{[t,\tau)}$ *in which* $N_\delta y_{[t,\tau)}$ *exists and is continuous).* $\triangleleft$

The differential operator $\mathbf{N}$ is used in Lemma 2.1 to deal with a general output y that may not be differentiable but $\mathbf{N}y$ exists (an example is that $\dot{y}$ is not differentiable so $\ddot{y}$ does not exist, but $\mathbf{N}y = \frac{d}{dt}(y - \dot{y})$ exists and is continuous). As it will be useful in our results for switched systems later, let us take a closer look at $\mathbf{N}$. We have

$$\begin{aligned} M_0 y_0 = \begin{pmatrix} \bar{y}_0 \\ \dot{\tilde{y}}_0 \end{pmatrix} = M_0 S_0 y &= \begin{bmatrix} \overline{S}_0 \\ 0 \end{bmatrix} y + \frac{d}{dt} \begin{bmatrix} 0 \\ \widetilde{S}_0 \end{bmatrix} y \\ &=: K_{0,0} y + \frac{d}{dt} K_{0,1} y. \end{aligned} \tag{2.5}$$

In general, let

$$M_i y_i = K_{i,0}y + \frac{d}{dt}(K_{i,1}y + \ldots + \frac{d}{dt}(K_{i,i}y + \frac{d}{dt}K_{i,i+1}y)). \tag{2.6}$$

Then in view of $M_{i+1}y_{i+1} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} M_i y_i + \frac{d}{dt}\begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} M_i y_i$, we have the $\ell \times \ell$ matrices $K_{i,j}$ defined recursively as follows:

$$K_{i+1,j} = \begin{bmatrix} \overline{S}_{i+1} \\ 0 \end{bmatrix} K_{i,j} + \begin{bmatrix} 0 \\ \widetilde{S}_{i+1} \end{bmatrix} K_{i,j-1},$$

$$0 \leqslant j \leqslant i+2, i \geqslant 0, \tag{2.7}$$

where $K_{i,-1} = 0 \; \forall i$ by convention and $K_{0,0}, K_{0,1}$ are as in (2.5). Using the notation in (2.6), in view of $\widetilde{N}_k y = \widetilde{S}_k M_{i-1} y_{i-1}$, then

$$\mathbf{N}y =: N_0 y + \frac{d}{dt}(N_1 y + \cdots + \frac{d}{dt}(N_{\beta-2}y + \frac{d}{dt}N_{\beta-1}y)), \tag{2.8}$$

where $N_i := \begin{bmatrix} \widetilde{S}_0 K_{-1,i} \\ \widetilde{S}_1 K_{0,i} \\ \vdots \\ \widetilde{S}_{\beta-1} K_{\beta-2,i} \end{bmatrix}$, $0 \leqslant i \leqslant \beta - 1$, $K_{-1,0} = I$ and $K_{j,k} = 0 \; \forall k \geqslant j+2, \forall j$ by convention. We then have the following lemma.

Lemma 2.2 *Consider the linear system* (2.1). *Let* β *be the number and* $\mathbf{N}$ *be the differential operator as described in the structure algorithm. For any* $\kappa \geqslant \beta$, *for every* y *such that* $\mathbf{N}y$ *exists and is continuous, we have* $\mathbf{N}y = N_0 y + \frac{d}{dt}(N_1 + \cdots + \frac{d}{dt}(N_{\kappa-2}y + \frac{d}{dt}N_{\kappa-1}y))$, *where* $N_i, 0 \leqslant i \leqslant \beta - 1$, *are as in* (2.8) *and* $N_i = 0, i \geqslant \beta$.

Note that in general, $\mathbf{N}y$ is calculated by a chain of differentiations and additions as in the lemma. However, whenever $y \in C^{\beta-1}$, calculating $\mathbf{N}y$ can be simplified to the matrix multiplication $[N_0 \ldots N_{\beta-1}]$ by $(y, \ldots, y^{(\beta-1)})^T$.

2.2 Invertibility Problem for Continuous-Time Switched Linear Systems

Consider a *switched linear system*

$$\Gamma_\sigma : \begin{cases} \dot{x} = A_\sigma x + B_\sigma u, \\ y = C_\sigma x + D_\sigma u, \end{cases} \tag{2.9}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is the switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some index set in a real vector space, and A_p, B_p, C_p, D_p, $p \in \mathcal{P}$, are the matrices of the individual subsystems. For the sake of presentation, we assume that the output dimensions of all the subsystems are the same, because detecting switchings between two subsystems with different output dimensions is trivial. However, it is emphasized that it is just a technicality to restate our results presented here to cover the general case where output dimensions are different. The input dimensions of the subsystems can be different.

For the switched system (2.9), denote by n the state dimension, ℓ the output dimension, and $m_p, p \in \mathcal{P}$ the input dimensions of the subsystems. Denote by $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ the (switching signal $\times$ input)-output map for some input set $\mathcal{U}$, switching signal set $\mathcal{S}$, and the corresponding output set $\mathcal{Y}$. We will seek conditions on the subsystem dynamics so that the map H_{x_0} is one-to-one for some sets $\mathcal{S}$,$\mathcal{U}$, and $\mathcal{Y}$ (precise problem formulation is in Section 2.2). We do not yet specify here what the sets $\mathcal{U}$, $\mathcal{S}$, and $\mathcal{Y}$ are as we will see in Section 2.3 that injectivity of the map H_{x_0} depends on particular sets $\mathcal{Y}$ and we cannot simply take $\mathcal{Y}$ to be the set of all possible outputs generated by all possible combinations of inputs and switching signals (cf. Remark 2.1 for nonswitched systems in Section 2.1 below).

The invertibility problem. *Consider a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ for the switched system* (2.9). *Find a largest possible set $\mathcal{Y}$ and a condition on the subsystems, independent of x_0, such that the map H_{x_0} is one-to-one.*

2.3 Singular Pairs

We discuss the invertibility of the map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ for the switched system (2.9) in more detail. We say that H_{x_0} is *invertible at* y if $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \Rightarrow \sigma_1 = \sigma_2, u_1 = u_2$ (similarly for nonswitched systems, $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ is invertible at y if $\mathrm{H}_{x_0}(u_1) = \mathrm{H}_{x_0}(u_2) = y \Rightarrow u_1 = u_2$). We say that H_{x_0} is *invertible on* $\mathcal{Y}$ if it is invertible at y for all $y \in \mathcal{Y}$. We say that the switched system is *invertible on* $\mathcal{Y}$ if H_{x_0} is invertible on $\mathcal{Y}$ for all x_0.

There is a major difference between the maps H_{x_0} for nonswitched systems and for switched systems. The former is a linear map on vector spaces (i.e., the input functions). The latter is a nonlinear map on the domain $\mathcal{S} \times \mathcal{U}$, of which $\mathcal{S}$ is not a vector space. The map H_{x_0} for nonswitched systems has the property that if H_{x_0} is invertible at y for one particular pair (x_0, y), then the map H_{x_0} is invertible on the set of all possible outputs generated by continuous inputs (see Remark 2.1). In contrast, for switched systems, uniqueness of (σ, u) for one pair (x_0, y) does not imply uniqueness for other pairs, and thus, a switched system may be invertible on one output set $\mathcal{Y}_1$ but not invertible on another set $\mathcal{Y}_2$ (which is not the case for nonswitched systems). This situation prompts a more delicate definition of the output set $\mathcal{Y}$ for switched systems, instead of letting $\mathcal{Y}$ be the set generated by all possible combinations of piecewise continuous inputs and switching signals. For the invertibility problem, we look for the largest possible set $\mathcal{Y}$ and a condition on the subsystems so that unique recovery of (σ, u) is guaranteed for all $y \in \mathcal{Y}$ and all $x_0 \in \mathbb{R}^n$.

One special case is $x_0 = 0$, $y \equiv 0$. It is obvious that with $u \equiv 0$ and any switching signal, we always have $y \equiv 0$; i.e., $\mathrm{H}_0(\sigma, u) = 0 \ \forall \sigma$ regardless of the subsystem dynamics and therefore the map H_0 is not one-to-one if the function $0 \in \mathcal{Y}$. Note that the available information is the same for both nonswitched systems and switched systems, namely, the pair (x_0, y), but the domain in the switched system case has been enlarged to $\mathcal{S} \times \mathcal{U}$, compared to $\mathcal{U}$ in the nonswitched system case. For nonswitched systems, under a certain condition on the system dynamics (i.e., when the system is invertible), the information $(x_0, y) = (0, 0)$ is sufficient to determine u uniquely, while for switched systems that information is insufficient to determine (σ, u) uniquely, regardless of what the subsystems are. This illustrates why we cannot take the output set $\mathcal{Y}$ to be all the possible outputs. We call those pairs (x_0, y) for which H_{x_0} is not invertible at y *singular pairs*. Fortunately, $x_0 = 0$ and $y_{[0,\varepsilon)} \equiv 0$ for some $\varepsilon > 0$ are the only type of singular pairs that are independent of the subsystems and for other pairs (x_0, y), the invertibility of H_{x_0} at y depends on the subsystem dynamics and properties of y.

Definition 2.1 *Let $x_0 \in \mathbb{R}^n$ and $y \in C^0$ be a function in $\mathbb{R}^\ell$ on some time interval. The pair (x_0, y) is a* singular pair *of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 such that $\Gamma^{\mathrm{O}}_{p,x_0}(u_1) = \Gamma^{\mathrm{O}}_{q,x_0}(u_2) = y$.*

Essentially, if a state and an output function (the time domain can be arbitrary) form a singular pair, then there exist inputs for the two systems to produce that same output starting from that same initial state. We proceed to develop a formula for checking if (x_0, y) is a singular pair of Γ_p, Γ_q, utilizing the range theorem by Silverman and Payne (Theorem 2.1 in this paper). We will use our notations in Lemma 2.1. For the subsystem indexed by p, denote by $\mathbf{N}_p$, L_p, and $\widehat{\mathcal{Y}}_p$ the corresponding objects of interest as in Lemma 2.1. It follows from Definition 2.1 and Lemma 2.1 that (x_0, y) is a singular pair if and only if $y \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ and

$$\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} = \begin{bmatrix} L_p \\ L_q \end{bmatrix} x_0, \tag{2.10}$$

where t_0 is the initial time of y. For a given (x_0, y), the condition (2.10) can be directly verified since all $\widehat{\mathcal{Y}}_p, \widehat{\mathcal{Y}}_q, \mathbf{N}_p, \mathbf{N}_q, L_p, L_q$ are known. Observe that $0 \in \mathrm{Im} \begin{bmatrix} L_p \\ L_q \end{bmatrix}$

and we can always have (2.10) with $x_0 = 0$ and y such that $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} = 0$. In particular, if $y_{[t_0,t_0+\varepsilon)} \equiv 0$ and $x_0 = 0$, then (2.10) holds regardless of $\mathbf{N}_p, \mathbf{N}_q$, L_p, L_q. Apart from this case, in general, $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} = 0$ depends on $\mathbf{N}_p, \mathbf{N}_q$, and y, and it is possible to find conditions on $\mathbf{N}_p, \mathbf{N}_q, L_p, L_q$, and y so that there is no x_0 satisfying (2.10) if $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} \neq 0$.

Remark 2.5 *The singular pair notion captures the situation where it is not possible to distinguish between two dynamical systems just using the output and the initial state. This property leads to the scenario where there is a switch in the underlying dynamical system yet the output is still smooth at the switching time (as in the motivating example in the introduction). A similar scenario but with a different objective can be found in the context of bumpless switching [37], in which the objective is to design the subsystems so that the output of the switched system is continuous. In the invertibility problem considered here, the subsystems are fixed and the objective is to recover the switching signal and the input.*

2.4 A Solution of the Invertibility Problem

We now return to the invertibility problem. Let $\mathcal{Y}^{\text{all}}$ be the set of outputs of the system (2.9) generated by all possible piecewise continuous inputs and switching signals from all possible initial states for all possible durations (the set $\mathcal{Y}^{\text{all}}$ can be seen as all the possible concatenations of all elements of $\widehat{\mathcal{Y}}_p$, $\forall p \in \mathcal{P}$). Let $\overline{\mathcal{Y}} \subset \mathcal{Y}^{\text{all}}$ be the largest subset of $\mathcal{Y}^{\text{all}}$ such that if $y \in \overline{\mathcal{Y}}$ and $y_{[t_0,t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$ for some $p \neq q, p, q \in \mathcal{P}$, $\varepsilon > 0$, $t_0 \geqslant 0$, then $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y\big|_{t_0^+} \neq 0$. Literally speaking, we avoid functions whose segments can form singular pairs with $x_0 = 0$ (so that even if $x_0 = 0$, there is no $y \in \overline{\mathcal{Y}}$ that can form a singular pair with x_0). Excluding such functions from our output set, we can impose conditions on the subsystems to eliminate the possibility of singular pairs for all $x_0 \in \mathbb{R}^n$ and all $y \in \overline{\mathcal{Y}}$. Note that the singular pair concept in Definition 2.1 is defined for continuous functions. Applying to switched systems, we check for singular pairs for the continuous output segments in between consecutive discontinuities at the output (note that $\widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q \subseteq C^0$ if the intersection is non-empty). We have the following result.

Theorem 2.2 *Consider the switched system* (2.9) *and the output set* $\overline{\mathcal{Y}}$. *The switched system is invertible on* $\overline{\mathcal{Y}}$ *if and only if all the subsystems are invertible and the subsystem dynamics are such that for all* $x_0 \in \mathbb{R}^n$ *and* $y \in \overline{\mathcal{Y}} \cap C^0$, *the pairs* (x_0, y) *are not singular pairs of* Γ_p, Γ_q *for all* $p \neq q, p, q \in \mathcal{P}$.

Proof Sufficiency: Suppose that all the subsystems are invertible and the subsystem dynamics are such that for all $x_0 \in \mathbb{R}^n$ and $y \in \overline{\mathcal{Y}} \cap C^0$, the pairs (x_0, y) are not singular pairs of Γ_p, Γ_q for all $p \neq q, p, q \in \mathcal{P}$.

Suppose that $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \in \overline{\mathcal{Y}}$.

Let $t_1 := \min\{t > 0 : u_1 \text{ or } u_2 \text{ or } \sigma_1 \text{ or } \sigma_2 \text{ is discontinuous at } t\}$. Let $p = \sigma_1(0)$ and $q = \sigma_2(0)$. From Lemma 2.1, we have $\mathbf{N}_p y|_{0^+} = L_p x_0$ and $\mathbf{N}_q y|_{0^+} = L_q x_0$, and $y_{[0,t_1)} \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q$. By definition, $(x_0, y_{[0,t_1)})$ is a singular pair of Γ_p, Γ_q if $p \neq q$. Also by definition, $y_{[0,t_1)} \in \overline{\mathcal{Y}} \cap C^0$. Since there is no singular pair for Γ_p, Γ_q, we must have $p = q$, i.e., $\sigma_1(t) = \sigma_2(t) = p \ \forall t \in [0, t_1)$. Since Γ_p is invertible, $u_{1[0,t_1)} = u_{2[0,t_1)} = u_{[0,t_1)} = \Gamma^{-1,\mathrm{O}}_{p,x_0}(y_{[0,t_1)})$ is uniquely recovered on $[0, t_1)$ (recall from Section 1.4 that $\Gamma^{-1,\mathrm{O}}_{p,x_0}(y_{[0,t_1)})$ is the output of the inverse of Γ_p starting at x_0 with input $y_{[0,t_1)}$). Let $x_1 = x(t_1^-)$. By continuity of the trajectory, we have $x(t_1) = x(t_1^-) = x_1$. If $t_1 = \infty$, we then have $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t) \ \forall t \in [0, \infty)$.

Suppose that $t_1 < \infty$. We have $\mathrm{H}_{x_1}(\sigma_{1[t_1,\infty)}, u_{1[t_1,\infty)}) = \mathrm{H}_{x_1}(\sigma_{2[t_1,\infty)}, u_{2[t_1,\infty)}) = y_{[t_1,\infty)}$. Let $t_2 := \min\{t > t_1 : u_1 \text{ or } u_2 \text{ or } \sigma_1 \text{ or } \sigma_2 \text{ is discontinuous at } t\}$. Repeating the arguments in the previous paragraph, we must have $\sigma_1(t) = \sigma_2(t) = q \ \forall t \in [t_1, t_2)$ for some $q \in \mathcal{P}$, and $u_{1[t_1,t_2)} = u_{2[t_1,t_2)} = u_{[t_1,t_2)} = \Gamma^{-1,\mathrm{O}}_{q,x_0}(y_{[t_1,t_2)})$ is uniquely recovered on $[t_1, t_2)$.

If $t_2 = \infty$, we then have $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t)$ for all $t \in [0, \infty)$; otherwise, repeat the procedure with $y_{[t_2,\infty)}$. Since there cannot be infinitely many discontinuities in a finite interval (in other words, if there are infinitely many discontinuities, the interval must be $[0, \infty)$), we conclude that $\sigma_1(t) = \sigma_2(t)$ and $u_1(t) = u_2(t)$ $\forall t \in [0, \infty)$.

Necessity: Suppose that Γ_p is not invertible for some $p \in \mathcal{P}$. Pick some x_0 and $y \in \overline{\mathcal{Y}} \cap \widehat{\mathcal{Y}}_p$; this is always possible from the definition of the set $\overline{\mathcal{Y}}$. Since $y \in \widehat{\mathcal{Y}}_p$, there exists u such that $y = \Gamma_{p,x_0}(u)$. Since Γ_p is not invertible, there exists $\tilde{u} \neq u$ such that $\Gamma_{p,x_0}(\tilde{u}) = y$ (see Remark 2.1). Then $\mathrm{H}_{x_0}(\sigma^p, u) = \mathrm{H}_{x_0}(\sigma^p, \tilde{u}) = y$. That means the map H_{x_0} is not invertible at y and thus, the switched system is not invertible on $\overline{\mathcal{Y}}$, a contradiction. Therefore, Γ_p must be invertible for all $p \in \mathcal{P}$.

Suppose that there are some $x_0 \in \mathbb{R}^n$, $y \in \overline{\mathcal{Y}} \cap C^0$ and $p \neq q, p, q \in \mathcal{P}$ such that (x_0, y) is a singular pair of Γ_p, Γ_q. This means $\exists\, u_1, u_2 \in C^0$ (not necessarily different) such that $\Gamma_{p,x_0}(u_1) = \Gamma_{q,x_0}(u_2) = y$ and therefore, $\mathrm{H}_{x_0}(\sigma^p, u_1) = \mathrm{H}_{x_0}(\sigma^q, u_2) = y$. Since $\sigma^p \neq \sigma^q$, the foregoing equality implies that H_{x_0} is not invertible at y, and thus, the switched system is not invertible on $\overline{\mathcal{Y}}$, a contradiction. Therefore, for all $x_0 \in \mathbb{R}^n$ and $y \in \overline{\mathcal{Y}} \cap C^0$, (x_0, y) are not singular pairs of Γ_p, Γ_q for all $p \neq q, p, q \in \mathcal{P}$. $\square$

Theorem 2.2 provides a necessary and sufficient condition for a switched system to be invertible on the set $\overline{\mathcal{Y}}$. While checking for singularity for given x_0 and y is feasible using the property (2.10), in general, checking for singularity for all x_0 and all $y \in \overline{\mathcal{Y}}$ (and hence, checking invertibility) is not an easy task. We now develop a rank condition for checking invertibility of switched systems, which is more computationally friendly. In the case when the subsystem input and output dimensions are equal, the rank condition is also necessary. We first have the following lemma. For an index p, let $W_p := [N_{p,0}\ N_{p,1}\ \ldots\ N_{p,n-1}]$ where $N_{p,0}, \ldots, N_{p,n-1}$ are the matrices as in Lemma 2.2 for the subsystem with index p.

Lemma 2.3 *Consider the switched system* (2.9) *and the output set* $\overline{\mathcal{Y}}$. *Consider the* *following two statements:*

S1. The subsystem dynamics are such that for all $x_0 \in \mathbb{R}^n$ *and* $y \in \overline{\mathcal{Y}} \cap C^0$*, the pairs* (x_0, y) *are not singular pairs of* Γ_p, Γ_q *for all* $p \neq q, p, q \in \mathcal{P}$*.*

S2. The subsystem dynamics are such that

$$Rank \begin{bmatrix} W_p & L_p \\ W_q & L_q \end{bmatrix} = Rank \begin{bmatrix} W_p \\ W_q \end{bmatrix} + Rank \begin{bmatrix} L_p \\ L_q \end{bmatrix} \tag{2.11}$$

for all $p \neq q, p, q \in \mathcal{P}$ *such that* $\widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q \neq \{0\}$*.*

Then S2 always implies S1. If the subsystems are invertible and the input and output dimensions are the same, then S1 also implies S2.

Proof S2 ⇒ S1:

Suppose that there exist x_0 and $y \in \overline{\mathcal{Y}} \cap C^0$ such that (x_0, y) is a singular pair of Γ_p, Γ_q for some $p \neq q$. Then we have the equality (2.10). From Lemma 2.2, we have

$$\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y = \begin{bmatrix} N_{p,0} \\ N_{q,0} \end{bmatrix} y + \frac{d}{dt}\left(\begin{bmatrix} N_{p,1} \\ N_{q,1} \end{bmatrix} y + \cdots + \frac{d}{dt}\left(\begin{bmatrix} N_{p,n-2} \\ N_{q,n-2} \end{bmatrix} y + \frac{d}{dt} \begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix} y \right)\right) \tag{2.12}$$

in view of the definition of $N_{p,i}, N_{q,i}, 0 \leqslant i \leqslant n-1$. Since $\begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix} y$ is in the range of $\begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix}$, we must also have $\frac{d}{dt} \begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix} y \in \text{Range} \begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix}$. Then $\begin{bmatrix} N_{p,n-2} \\ N_{q,n-2} \end{bmatrix} y + \frac{d}{dt} \begin{bmatrix} N_{p,n-1} \\ N_{q,n-1} \end{bmatrix} y$ is in the range of $\begin{bmatrix} N_{p,n-2} & N_{p,n-1} \\ N_{q,n-2} & N_{q,n-1} \end{bmatrix}$. Continuing this procedure, we conclude that $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y \in \text{Range} \begin{bmatrix} N_{p,0} & \ldots & N_{p,n-2} & N_{p,n-1} \\ N_{q,0} & \ldots & N_{q,n-2} & N_{q,n-1} \end{bmatrix} = \text{Range} \begin{bmatrix} W_p \\ W_q \end{bmatrix}$. The rank condition (2.11) implies that Range $\begin{bmatrix} W_p \\ W_q \end{bmatrix} \cap$ Range $\begin{bmatrix} L_p \\ L_q \end{bmatrix} = \{0\}$. Therefore, from (2.10), we must have $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y|_{t_0^+} = 0$ if $p \neq q$. But this equality contradicts the fact that $y \in \overline{\mathcal{Y}}$, and hence, there must be no x_0, y that can form a singular pair for some Γ_p, Γ_q, $p \neq q$.

S1 $\Rightarrow$ **S2** if the subsystems are invertible and input and output dimensions are the same:

Suppose that the rank condition is violated for some $p \neq q, p, q \in \mathcal{P}$. That implies that Range $\begin{bmatrix} W_p \\ W_q \end{bmatrix} \cap$ Range $\begin{bmatrix} L_p \\ L_q \end{bmatrix} \neq \{0\}$ and hence, there exist ξ and x_0 such that $\begin{bmatrix} W_p \\ W_q \end{bmatrix} \xi = \begin{bmatrix} L_p \\ L_q \end{bmatrix} x_0 \neq 0$. If the subsystems are invertible and the input and output dimensions are the same, then C^n functions are always in $\widehat{\mathcal{Y}}_p$, $\widehat{\mathcal{Y}}_q$ (see Remark 2.2). There always exists a C^n function y on an interval $[0, \varepsilon)$ for some $\varepsilon > 0$ such that $(y, \dots, y^{(n-1)})^T\big|_0 = \xi$ and $(y, \dots, y^{(n-1)})^T\big|_t \notin \ker \begin{bmatrix} W_p \\ W_q \end{bmatrix}$ $\forall t \in [0, \varepsilon)$. Since $y \in C^n$, $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y|_{0^+} = \begin{bmatrix} W_p \\ W_q \end{bmatrix} (y, \dots, y^{(n-1)})^T\big|_{0^+} = \begin{bmatrix} W_p \\ W_q \end{bmatrix} \xi$ and $\begin{bmatrix} \mathbf{N}_p \\ \mathbf{N}_q \end{bmatrix} y|_t = \begin{bmatrix} W_p \\ W_q \end{bmatrix} (y, \dots, y^{(n-1)})^T\big|_t \neq 0$ $\forall t \in [0, \varepsilon)$, i.e., $y \in \widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q \cap \overline{\mathcal{Y}}$. It follows that x_0, y forms a singular pair for Γ_p, Γ_q, a contradiction. $\square$

From Theorem 2.2 and Lemma 2.3, we arrive at the following result.

Theorem 2.3 *Consider the switched system* (2.9) *and the output set* $\overline{\mathcal{Y}}$. *The switched system is invertible on* $\overline{\mathcal{Y}}$ *if all the subsystems are invertible and the rank condition* (2.11) *holds. If the input and output dimensions of all the subsystems are the same, then invertibility of all the subsystems together with the rank condition is also necessary for invertibility of the switched system.*

Remark 2.6 *If a subsystem has more inputs than outputs, then it cannot be invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case.* $\triangleleft$

When a switched system is invertible, i.e., it satisfies the conditions of Theorem 2.2, a *switched inverse system* can be constructed as follows. Define the *index inversion function* $\overline{\Sigma}^{-1} : \mathbb{R}^n \times \overline{\mathcal{Y}} \to \mathcal{P}$ as

$$\overline{\Sigma}^{-1} : (x_0, y) \mapsto p : y \in \widehat{\mathcal{Y}}_p \text{ and } \mathbf{N}_p y|_{t_0^+} = L_p x_0, \tag{2.13}$$

where t_0 is the initial time of y. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there is no singular pair, so the index p at every time t_0 is uniquely determined from the output y and state x. In the invertibility problem, it is assumed that $y \in \overline{\mathcal{Y}}$ is an output so the existence of p in (2.13) is guaranteed. Then a switched inverse system is

$$\Gamma_\sigma^{-1} : \begin{cases} \sigma(t) &= \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}), \\ \dot{z} &= (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z + (B\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)} y, \\ u &= -(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)}\, z + (\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)}\, y \end{cases} \tag{2.14}$$

where t_0 is the initial time of y. The function $\overline{\Sigma}^{-1}$ is well-defined since there is no singular pair, so the index p at every time t_0 is uniquely determined from the output y and state x. In the invertibility problem, it is assumed that $y \in \overline{\mathcal{Y}}$ is an output so the existence of p in (2.13) is guaranteed. Then a switched inverse system is:

$$\Gamma_\sigma^{-1}: \begin{cases} \sigma(t) = \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}), \\ \dot{z} = (A - B\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)} z + (B\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)} y, \\ u = -(\overline{D}_\alpha^{-1}\overline{C}_\alpha)_{\sigma(t)}\, z + (\overline{D}_\alpha^{-1} N_\alpha)_{\sigma(t)}\, y \end{cases} \tag{2.15}$$

with $z(0) = x_0$, where $\varepsilon > 0$ is sufficiently small. The notation $(\cdot)_{\sigma(t)}$ denotes the object in the parentheses calculated for the subsystem with index $\sigma(t)$. The initial condition $z(0) = x_0$ helps determine the initial active subsystem $\sigma(0) = \overline{\Sigma}^{-1}(x_0, y_{[0,\varepsilon)})$ at $t = 0$, from which time onwards, the switching signal and the input as well as the state are determined uniquely and simultaneously via (2.15). In (2.15), we use $\sigma(t)$ in the right-hand side of $\dot{z}$ and $z(t)$ in the formula of $\sigma(t)$ for notational convenience. Indeed, if t is a switching time, $\overline{\Sigma}^{-1}$ helps recover σ at the time t using small enough ε such that $t + \varepsilon$ is less than the next switching time. If t is not a switching time, σ is constant between t and the next switching time and is equal to σ at the last switching time.

Remark 2.7 *In* (2.15)*, since we use a full order inverse for each subsystem, the state z is exactly the same as the state x of the switched system, and hence, we can use z in the index inversion function $\overline{\Sigma}^{-1}$. If we use a reduced-order inverse for each subsystem (see, e.g., [16]), we still get u but then need to plug this u into the switched system to get x to use in $\overline{\Sigma}^{-1}$.*

Remark 2.8 *Let $\bar{\beta} := \max_{p \in \mathcal{P}}\{\beta_p\}$, where β_p, $p \in \mathcal{P}$, are the β as in Theorem 2.1 for the subsystems. In Lemma 2.3, instead of W_p, we can work with $\overline{W}_p$ where $\overline{W}_p y^{\bar{\beta}} := [N_{p,0} \dots N_{p,\bar{\beta}-1}](y, \dots, y^{(\bar{\beta}-1)})^T$. In general, $\overline{W}_p$ have fewer columns than W_p, which make checking the rank condition for systems with large dimensions simpler.* $\lhd$

Remark 2.9 *Our results can be extended to include the more general case of different output dimensions. For switched systems whose subsystems are of different output dimensions, the definition of singular pairs is unchanged. Definition 2.1 implies that the output dimensions of the two systems must be the same in order for (x_0, y) to be a singular pair. If the output dimensions are different, then it is automatically true that there is no singular pair for the two systems since $\widehat{\mathcal{Y}}_p \cap \widehat{\mathcal{Y}}_q = \emptyset$. One needs* *to use the concept of hybrid functions in Chapter 1 to describe the output set $\overline{\mathcal{Y}}$ of the switched system (which are now not functions but concatenations of functions of different dimensions), but other than that, the statements of Theorem 2.2 and Theorem 2.3 remain the same.* $\triangleleft$

Remark 2.10 *The results in this section can also be extended to include the case when there are state jumps at switching times. Denote by $f_{p,q} : \mathbb{R}^n \to \mathbb{R}^n$ the* jump map *(also called reset map), i.e., if τ is a switching time, $x(\tau) = x(\tau^+) = f_{\sigma(\tau^-),\sigma(\tau)}(x(\tau^-))$. Note that the case of identity jump maps $f_{p,q}(x) = x \ \forall p, q \in \mathcal{P}, \forall x \in \mathbb{R}^n$ is the case considered in this paper. For nonidentity jump maps, the concept of singular pairs changes to "(x_0, y) is a singular pair of Γ_p, Γ_q if $\exists\ u_1, u_2$ such that $\Gamma^{\mathrm{O}}_{p,x_0}(u_1) = \Gamma^{\mathrm{O}}_{q,f_{p,q}(x_0)}(u_2) = y$ or $\Gamma^{\mathrm{O}}_{p,f_{q,p}(x_0)}(u_1) = \Gamma^{\mathrm{O}}_{q,x_0}(u_2) = y$." Equation* (2.10) *becomes:*

$$\mathbf{N}_{pq}y|_{t_0} = \begin{bmatrix} L_p x_0 \\ L_q f_{p,q}(x_0) \end{bmatrix} \ \text{or} \ \mathbf{N}_{pq}y|_{t_0} = \begin{bmatrix} L_p f_{q,p}(x_0) \\ L_q x_0 \end{bmatrix}.$$

There will also be a distinction between identifying the initial switching mode and subsequent switching modes.

Another generalization is to include switching mechanisms, such as switching surfaces*. Denote by $S_{p,q}$ the switching surface for system p changing to system q such that $x(t) = f_{p,q}(x(t^-))$ if $x(t^-) \in S_{p,q}$ and $\sigma(t^-) = p$. Then we only need to check for singularity for $x_0 \in S_{p,q}$ and $x_0 \in S_{q,p}$ instead of $x_0 \in \mathbb{R}^n$ for Γ_p, Γ_q.* $\triangleleft$

Remark 2.11 *It is worth mentioning that the discrete-time case is not a straightforward extension of the continuous-time case. The difficulty arises from the essential difference between invertibility of discrete-time nonswitched linear systems and their continuous-time counterpart, although the invertibility conditions are exactly the same for the two cases. For the continuous-time case, the information in an infinitesimally small interval $y_{[t,t+\delta)}$, $\delta > 0$, completely determines $u(t)$. In contrast, for the* *discrete-time case, generally we need* d *output samples* $y(k), \ldots, y(k+d-1)$ *to determine the input* $u(k)$*. The delay behavior in the invertibility property of discrete-time nonswitched systems causes difficulties in solving the invertibility problem for discrete-time switched systems. If we utilize the output after a switch to determine the input before the switch, this in turns leads to a combinatorial mixing of the subsystem dynamics around switching times (cf. concatenation, not mixing, in the continuous-time case). The combinations of the individual subsystems around switching times can lead to many possibilities; for example, for a fixed switching signal, one can uniquely recover* u *even if the individual subsystems are not invertible (which is not the case for continuous-time switched systems). One possible research direction is to reduce the complexity of the invertibility problem for discrete-time switched systems by letting the set* $\mathcal{S}$ *be the set of dwell-time switching signals.* $\triangleleft$

2.5 Output Generation

In the previous section, we considered the invertibility question of whether one can recover (σ, u) uniquely for all y in some output set $\overline{\mathcal{Y}}$. In this section, we address a different but closely related problem which concerns finding (σ, u) (there maybe more than one) such that $\mathrm{H}_{x_0}(\sigma, u) = y$ for a given function y and a state x_0. For the invertibility problem, we find conditions on the subsystems and the output set $\mathcal{Y}$ so that the map H_{x_0} is injective for all x_0. Here, we are given one particular pair (x_0, y) and wish to find the preimage of the map H_{x_0}.

For the switched system (2.9), denote by $\mathrm{H}_{x_0}^{-1}(y)$ the *preimage* of an output y under H_{x_0},

$$\mathrm{H}_{x_0}^{-1}(y) := \{(\sigma, u) : \mathrm{H}_{x_0}(\sigma, u) = y\}. \tag{2.16}$$

By convention, $\mathrm{H}_{x_0}^{-1}(y) = \emptyset$ if y is not in the image set of H_{x_0}. In general, $\mathrm{H}_{x_0}^{-1}(y)$ is a set for a given y (when $\mathrm{H}_{x_0}^{-1}(y)$ is a *singleton*, the map H_{x_0} is invertible at y). We want to find conditions and an algorithm to generate $\mathrm{H}_{x_0}^{-1}(y)$ when $\mathrm{H}_{x_0}^{-1}(y)$ is a finite set.

We need the individual subsystems to be invertible because if this is not the case, then the set $\mathrm{H}_{x_0}^{-1}(y)$ will be infinite by virtue of the following lemma.

Lemma 2.4 *Consider the nonswitched linear system* Γ *in* (2.1) *with piecewise continuous input. Suppose that* Γ *is not invertible. Let* $[a, b]$ *be an arbitrary interval. For every* $u \in \mathcal{F}_{[a,b]}^{\mathrm{pc}}$ *and* $x_a \in \mathbb{R}^n$, *there are infinitely many different* $v \in \mathcal{F}_{[a,b]}^{\mathrm{pc}}$, $v \neq u$, *such that* $\Gamma_{x_a}^{\mathrm{O}}(v) = \Gamma_{x_a}^{\mathrm{O}}(u)$ *and* $\Gamma_{x_a}(v)\big|_b = \Gamma_{x_a}(u)\big|_b$.

Proof If Γ is not invertible, then $K \neq 0$ in the generalized inverse (2.3). Then the controllable subspace $\bar{\mathcal{C}}$ of $(A - B\overline{D}_\alpha^\dagger \overline{C}_\alpha, BK)$ is nontrivial. Pick any $\xi \in \bar{\mathcal{C}}$, $\xi \neq 0$ and $T_1, T_2 > 0$ such that $T_1 + T_2 = b - a =: T$. Since $0, \xi$ are in the controllable subspace of $(A - B\overline{D}_\alpha^\dagger \overline{C}_\alpha, BK)$, there exist a nonzero input $w_1 \in \mathcal{C}_{[0,T_1]}^0$ such that $\Gamma_0^{-1}(0, w_1)\big|_{T_1} = \xi$ and $w_2 \in \mathcal{C}_{[0,T_2]}^0$ such that $\Gamma_\xi^{-1}(0, w_2)\big|_{T_2} = 0$. By the time-invariant property, we then have $\Gamma_0^{-1}(0, w)\big|_T = 0$ where $w = w_1 \oplus w_2$. Let $\bar{u} = \Gamma_0^{-1,\mathrm{O}}(0, w)$; then $\Gamma_0(\bar{u})\big|_T = 0$ and $\Gamma_0^{\mathrm{O}}(\bar{u}) \equiv 0$. Also by the time-invariant property, if we shift the function $\bar{u}$ (which is defined on the interval $[0, T]$ since $w \in \mathcal{F}_{[0,T]}^{\mathrm{pc}}$) to become the function $\hat{u}$ defined on $[a, b]$ such that $\hat{u}(t) := \bar{u}(t - a)$, then the state trajectory with input $\bar{u}$ staring at x_a at time 0 is exactly the same as the trajectory with input $\hat{u}$ starting at x_a at time a, and therefore, $\Gamma_0(\hat{u})\big|_b = 0$ and $\Gamma_0^{\mathrm{O}}(\hat{u}) \equiv 0$. Let $v = u + \hat{u}$. Linearity of Γ implies $\Gamma_{x_a}(v) = \Gamma_{x_a}(u) + \Gamma_0(\hat{u})$ and $\Gamma_{x_a}^{\mathrm{O}}(v) = \Gamma_{x_a}^{\mathrm{O}}(u) + \Gamma_0^{\mathrm{O}}(\hat{u})$, which means $\Gamma_{x_a}(v)\big|_b = \Gamma_{x_a}(u)\big|_b$ and $\Gamma_{x_a}^{\mathrm{O}}(v) = \Gamma_{x_a}^{\mathrm{O}}(u)$. Since there are infinitely many $\xi \in \bar{\mathcal{C}}$, we have infinitely many such v. $\square$

Lemma 2.4 states that for a noninvertible nonswitched linear system, if an output is generated by some input with some initial state, there always exist infinitely many different inputs (which might be discontinuous; note that inputs of the subsystems can be piecewise continuous functions) that generate the same output with the same initial state and moreover, with the same terminal state at any arbitrary terminal time $T > 0$. It is then clear that for a switched system with noninvertible active subsystems, there are infinitely many inputs that generate the same output starting from the initial state and with the same switching signal. The fact that invertibility of all the subsystems is a necessary condition for invertibility of switched systems (see Theorem 2.2) can also be inferred from Lemma 2.4. For our output generation problem, we have the following corollary which states that if one active subsystem is not invertible, then the set $\mathrm{H}_{x_0}^{-1}(y)$ will be infinite.

Corollary 2.1 *Consider a function $y = \mathrm{H}_{x_0}(\sigma, u)$ for some σ, u. Let $\mathcal{Q} \subseteq \mathcal{P}$ be the set of values of σ. If there exists $q \in \mathcal{Q}$ such that Γ_q is not invertible, then there exist infinitely many u such that $y = \mathrm{H}_{x_0}(\sigma, u)$.*

The previous discussion motivates us to introduce the following assumption.

Assumption 2.1 *The individual subsystems Γ_p are invertible for all $p \in \mathcal{P}$.*

However, we have no other assumption on the subsystem dynamics and the switched system may not be invertible as the subsystems may not satisfy the invertibility condition in the previous section. Since we look for an algorithm to find $\mathrm{H}_{x_0}^{-1}(y)$, we only consider the functions y of finite intervals (and hence, there is a finite number of switches) to avoid infinite loop reasoning when there are infinitely many switchings.

At a quick glance, it seems that one solution is straightforward: switchings occur at times when the output loses continuity and since we have finite numbers of systems and switches, there is a finite number of possible switching signals and an exhaustive search would suffice (cf. the motivating example in the introduction). However, this is far from being the only case because we can have a switching and the output is still smooth at that switching time, and likewise, we can have no switching even if the output looses continuity (because we allow discontinuous inputs for the subsystems and we allow y to depend directly on u through D). As we shall see, it is possible to use a switch at a later time to recover a "hidden switch" earlier (e.g., a switch at which the output is smooth) and this phenomenon is peculiar to switched systems with no counterpart in nonswitched systems.

We now present a switching inversion algorithm for switched systems that returns $\mathrm{H}_{x_0}^{-1}(y)$ for a function $y \in \mathcal{F}_{\mathcal{D}}^{\mathrm{pc}}$ where $\mathcal{D}$ is a finite interval, when $\mathrm{H}_{x_0}^{-1}(y)$ is a finite set. The parameters to the algorithm are $x_0 \in \mathbb{R}^n$ and $y \in \mathcal{F}_{\mathcal{D}}^{\mathrm{pc}}$, and the return is $\mathrm{H}_{x_0}^{-1}(y)$ as in (2.16). In the algorithm, "$\leftarrow$" reads "assigned as," and ":=" reads "defined as." Define the *index-matching map*[2] $\Sigma^{-1} : \mathbb{R}^n \times \mathcal{F}^{\mathrm{pc}} \to 2^{\mathcal{P}}$ as

$$\Sigma^{-1}(x_0, y) := \{p : y \in \widehat{\mathcal{Y}}_p \text{ and } \mathbf{N}_p y|_{t_0^+} = L_p x_0\}, \tag{2.17}$$

where t_0 is the initial time of y. The index-matching map returns the indexes of the subsystems that are capable of generating y starting from x_0. If the returned set is empty, no subsystem is able to generate that y starting from x_0. Note that the index-matching map Σ^{-1} in (2.17) is defined for every pair (x_0, y) and always returns a set, whereas the index inversion function $\overline{\Sigma}^{-1}$ in (2.13) is defined for nonsingular pairs only and returns an element of $\mathcal{P}$.

Algorithm

Begin of Function $\mathrm{H}_{x_0}^{-1}(y)$

Let the domain of y be $[t_0, T)$.

Let $\overline{\mathcal{P}} := \{p \in \mathcal{P} : y_{[t_0, t_0+\varepsilon)} \in \widehat{\mathcal{Y}}_p$ for some $\varepsilon > 0\}$.

Let $t^* := \min\{t \in [t_0, T) : y_{[t, t+\varepsilon)} \notin \widehat{\mathcal{Y}}_p$

for some $p \in \overline{\mathcal{P}}, \varepsilon > 0\}$ or $t^* = T$ otherwise.

Let $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{[t_0, t_0+\varepsilon)})$ for sufficiently small ε.

If $\mathcal{P}^* \neq \emptyset$,

Let $\mathcal{A} := \emptyset$.

For each $p \in \mathcal{P}^*$,

Let $u := \Gamma_{p, x_0}^{-1, \mathrm{O}}(y_{[t_0, t^*)})$,

[2]The symbol $2^{\mathcal{P}}$ denotes the set of all subsets of a set $\mathcal{P}$.

$\mathcal{T} := \{t \in (t_0, t^*) : (x(t), y_{[t,t^*)})$ is

a singular pair of Γ_p, Γ_q for some $q \neq p\}$.

If $\mathcal{T}$ is a finite set,

For each $\tau \in \mathcal{T}$, let $\xi := \Gamma_p(u)(\tau)$.

$\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,\tau)}, u_{[t_0,\tau)}) \oplus H^{-1}_\xi(y_{[\tau,T)})\}$

End For each

Else If $\mathcal{T} = \emptyset$ and $t^* < T$, let $\xi = \Gamma_p(u)(t^*)$.

$\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,t^*)}, u) \oplus H^{-1}_\xi(y_{[t^*,T)})\}$

Else If $\mathcal{T} = \emptyset$ and $t^* = T$,

$\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma^p_{[t_0,T)}, u)\}$

Else

$\mathcal{A} := \emptyset$

End If

End For each

Else

$\mathcal{A} := \emptyset$

End If

Return $H^{-1}_{x_0}(y) := \mathcal{A}$

End of Function

The idea behind the algorithm is based on the following relationship:

$$\begin{aligned} \mathrm{H}^{-1}_{x_0}(y_{[t_0,T)}) = \{(\sigma, u) \oplus \mathrm{H}^{-1}_{\mathrm{H}_{x_0}(\sigma,u)(t)}(y_{[t,T)}), \\ (\sigma, u) \in \mathrm{H}^{-1}_{x_0}(y_{[t_0,t)})\} \quad \forall t \in [t_0, T), \end{aligned} \tag{2.18}$$

which follows from the fact that $\mathrm{H}_{x_0}(\sigma_{[t_0,T)}, u_{[t_0,T)})(t) = \mathrm{H}_{x(s)}(\sigma_{[s,T)}, u_{[s,T)})$, $x := \Gamma_{x_0,\sigma_{[t_0,s)}}(u_{[t_0,s)})$ $\forall s \in [t_0, T)$, $\forall t \geqslant s$ (this is known as the *semigroup property* for trajectories of dynamical systems). Basically, (2.18) means that we can do the preimage determination on a subinterval $[t_0, t)$ of $[t_0, T)$ and then concatenate these with the corresponding preimage on the rest of the interval $[t, T)$. Now, if t in (2.18) is the first switching time after t_0, then we can find $\mathrm{H}_{x_0}^{-1}(y_{[t_0,t)})$ by singling out which subsystems are capable of generating $y_{[t_0,t)}$ using the index-matching map (2.17). So the problem comes down to determining the first switching time t (and then the procedure is repeated for the function $y_{[t,T)}$). The concept of singular pairs helps to identify possible switching times when there is no irregularity at the output (e.g., when the output is smooth at a switching time).

In light of the discussion in the previous paragraph, it is noted that the switching inversion algorithm is a recursive procedure calling itself with different parameters within the main algorithm (e.g., the function $H_{x_0}^{-1}(y)$ uses the returns of $H_{\xi}^{-1}(y_{[\tau,T)})$). There are three stopping conditions: it terminates either when $\mathcal{P}^* = \emptyset$, in which case there is no subsystem that can generate the output y at time t_0 starting from x_0, or when $\mathcal{T}$ is not a finite set, in which case we cannot proceed because of infinitely many possible switching times, or when $\mathcal{T}$ is an empty set and $t^* = T$, in which case the switching signal is a constant switching signal.

If the algorithm returns a nonempty set, the set must be finite and this set contains pairs of switching signals and inputs that generate the given function y starting from x_0. If the algorithm returns an empty set, it means that there is no switching signal and input that generate y, or there is an infinite number of possible switching times (it is possible to further distinguish between these two cases by using an extra variable in the algorithm that is assigned different values for the different cases). Notice the utilization of the concatenation notation here: if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $f \oplus \emptyset = \emptyset$.

Remark 2.12 *When $y \in \mathcal{F}^n$ and the input and output dimensions are the same, then t^* in the algorithm can be simplified to be the first discontinuity of y in $[t_0, T)$ (or $t^* = T$ if y is continuous). That is because in this case, C^n functions are always* *in $\widehat{\mathcal{Y}}_p$ for all $p \in \mathcal{P}$ (see Remark 2.2) and thus, $\overline{\mathcal{P}} = \mathcal{P}$, and $y_{[t,t+\varepsilon)} \notin \widehat{\mathcal{Y}}_p$ only if y is discontinuous at t.* $\triangleleft$

2.6 Examples

In this section, we provide detailed examples to illustrate our results. The first one is an example of an invertible switched system. The second example illustrates the use of the switching inversion algorithm on a noninvertible switched system.

Example 2.1 Consider the switched system generated by the following two subsystems:

$$\Gamma_1 : \begin{cases} \dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x, \end{cases} \quad \Gamma_2 : \begin{cases} \dot{x} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \\ y = \begin{bmatrix} 0 & 2 \end{bmatrix} x. \end{cases}$$

Using the structure algorithm, we can check that the two systems are invertible. In this case, the operators $\mathbf{N}_1, \mathbf{N}_2$ are matrix operators and

$$\mathbf{N}_1 = W_1 = \begin{bmatrix} 1 \end{bmatrix}, \ L_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$\mathbf{N}_2 = W_2 = \begin{bmatrix} 1 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

In this example, the input and output dimensions are the same. We have

$$\text{Rank} \begin{bmatrix} W_1 & L_1 \\ W_2 & L_2 \end{bmatrix} = \text{Rank} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} + \text{Rank} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

By Theorem 2.2, we conclude that the switched system generated by $\{\Gamma_1, \Gamma_2\}$ is invertible on $\overline{\mathcal{Y}} := \{y \in \mathcal{F}^{\mathrm{pc}} : \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} y|_{t^+} \neq 0 \ \forall t\} = \{y \in \mathcal{F}^{\mathrm{pc}} : y(t) \neq 0 \ \forall t\}$.

Example 2.2 We return to the example in the introduction. Using the structure algorithm, we can check that the two systems are invertible. We have

$$\mathbf{N}_1 = W_1 = \begin{bmatrix} 1 \end{bmatrix}, \; L_1 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$\mathbf{N}_2 = W_2 = \begin{bmatrix} 1 \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

The rank condition is violated (Rank $\begin{bmatrix} W_{1,2} & L_{1,2} \end{bmatrix} = 2 \neq \text{Rank } W_{1,2} + \text{Rank } L_{1,2} = 3$), and hence, the switched system generated by Γ_1, Γ_2 is not invertible. Consider the output

$$y(t) = \begin{cases} 2e^{2t} - 3e^{t}, & \text{if } t \in [0, t^*), \\ c_1 e^{t} + c_2 e^{2t}, & \text{if } t \in [t^*, T), \end{cases}$$

where $t^* = \ln 3$, $T = 6/5$, $c_1 = 15 + 18\ln(2/3)$, $c_2 = -4/3 - 4\ln(2/3)$ and the initial state $x_0 = (-1, 0)^T$.

We now illustrate how the inversion algorithm works. In this example, y is piecewise smooth and the subsystems' input and output dimensions are the same, so t^* in the algorithm coincides with t^* in the definition of y (see Remark 2.12). We have $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{[0,t^*)}) = \{2\}$ by using the index-matching procedure (2.17) with x_0 and $y(0) = -1$. The structure algorithm for Γ_2 on $[0, t^*)$ yields the inverse

$$\Gamma_2^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \dot{y}, & \quad t \in [0, t^*) \\ u(t) = \dot{y} - \begin{bmatrix} -1 & 4 \end{bmatrix} z, \end{cases}$$

with $z(0) = x_0$, which then gives

$$\begin{aligned} z(t) &= \begin{pmatrix} -e^t \\ -e^t + e^{2t} \end{pmatrix} =: x(t), \\ u(t) &= 0, \end{aligned} \qquad t \in [0, t^*). \tag{2.19}$$

We want to find $\mathcal{T} = \{t \leqslant t^* : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving $W_1 y(t) = L_1 x(t)$ (since we already have $W_2 y(t) = L_2 x(t)$ $\forall t \in [0, t^*)$). This leads to the equation

$$2e^{2t} - 3e^t = x_2(t) = -e^t + e^{2t}, \quad t \in [0, t^*).$$

The foregoing equation has a solution $t = \ln 2 =: t_1 < t^*$, and hence, $\mathcal{T} = \{t_1\}$, which is a finite set. We have $\xi = x(t_1) = (-2, 2)^T$ and we repeat the procedure for the initial state ξ and the output $y_{[t_1,T)}$. Now $\mathcal{P}^* = \Sigma^{-1}(\xi, y_{[t_1,t^*)}) = \{1, 2\}$.

- **Case 1**: $p = 1$. Using the structure algorithm, we obtain the inverse system of Γ_1,

$$\Gamma_1^{-1} : \begin{cases} \dot{z} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{y}, \\ u = \dot{y} - \begin{bmatrix} 0 & -1 \end{bmatrix} z, \end{cases}$$

with the initial state $z(t_1) = \xi$, which yields

$$\begin{aligned} z(t) &= \begin{pmatrix} (-13 + 6\ln 2)e^t + 6e^{2t} - 6te^t \\ 2e^{2t} - 3e^t \end{pmatrix} =: x(t), \\ u(t) &= 6e^{2t} - 6e^t, \end{aligned} \qquad t \geqslant t_1.$$

We find $\mathcal{T} = \{t_1 < t \leqslant t^* : (x(t), y_{[t,t^*)})$ is a singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving

$$\begin{aligned} W_2 y(t) &= 2e^{2t} - 3e^t \\ &= L_2 x(t) = (-19 + 6\ln 2)e^t + 10e^{2t} - 6te^t \end{aligned}$$

for $t_1 < t \leqslant t^*$. It is not difficult to check that the foregoing equation does not have a solution. We then let $\xi = x(t^*) = (15 + 18\ln(2/3), 9)$ and repeat the procedure with ξ and $y_{[t^*,T)}$, which yields the unique solution $\sigma = \sigma^2_{[t^*,T)}$ and $u_{[t^*,T)} = 0$.

- **Case 2**: $p = 2$. This case means that t_1 is not a switching time. Then $u(t) = 0$ up to time t^* by the structure algorithm, and hence,

$$x(t) = \begin{pmatrix} -e^t \\ -e^t + e^{2t} \end{pmatrix}, \quad \tau \leqslant t \leqslant t^*,$$

in view of (2.19). We then repeat the procedure with $\xi = x(t^*) = (-3, 6)$ and $y_{[t^*,T)}$. We have $y(t^*) = 33 + 18\ln(2/3)$, and since $L_1\xi \notin W_1 y(t^*)$ and $L_2\xi \notin W_2 y(t^*)$, the index-matching map returns an empty set, $\Sigma^{-1}(\xi, y_{[t^*,T)}) = \emptyset$.

Thus, the switching inversion algorithm returns $\{(\sigma, u)\}$, where

$$\begin{aligned} \sigma(t) &= \sigma^2_{[0,t_1)} \oplus \sigma^1_{[t_1,t^*)} \oplus \sigma^2_{[t^*,T)}, \\ u(t) &= \begin{cases} 0, & \text{if } 0 \leqslant t < t_1, \\ 6e^{2t} - 6e^t, & \text{if } t_1 \leqslant t < t^*, \\ 0, & \text{if } t^* \leqslant t \leqslant T. \end{cases} \end{aligned}$$

As we can see, the output loses smoothness at t^* and t^* is a switching time. However, there is another switching at $t_1 < t^*$ whilst the output is smooth at t_1. Thus, without the concept of singular pairs, one might falsely conclude that there is no switching signal and input after trying all the obvious combinations of the switching signals (i.e., $\sigma = \sigma^i_{[0,t^*)} \oplus \sigma^j_{[t^*,T)}$, $i, j \in \{1, 2\}$). Here, using the switching inversion algorithm, we can recover the switching signal and in this case, the switching signal and input are unique. This clearly demonstrates the usefulness of the singular-pair concept.

2.7 Robust Invertibility

Note that the index matching function (2.13) requires exact knowledge of the initial state x_0 and the output y. To to handle disturbances, we replace this inversion map (2.13) with the following map that returns the index with minimum cost. Define the function

$$\widetilde{\Sigma}^{-1} : (x_0, y) \mapsto \operatorname{argmin}_{p \in \mathcal{P}} \|\mathbf{N}_p y|_{t_0^+} - L_p x_0\|. \tag{2.20}$$

We say that the switched system is robustly invertible with radius ε if

1. every subsystem has a stable inverse, and
2. a small disturbance of magnitude ε does not affect our detection algorithm, i.e.,

$$\widetilde{\Sigma}^{-1}(\hat{x}_0, \hat{y}) = \overline{\Sigma}^{-1}(x_0, y) \tag{2.21}$$

   for all $x_0 \in \mathbb{R}^n, y \in \mathcal{Y}, \|\hat{x} - x\| < \varepsilon, \|\mathbf{N}_p \hat{y} - \mathbf{N}_p y\| < \varepsilon, p \in \mathcal{P}$.

Now (2.21) implies that we are able to recover the correct switching signal even in the face of disturbances. If the subsystem has a stable inverse, then the state estimation error $|z - x|$, where z as in (2.15), will asymptotically go to zero even if the initial state $z(0) \neq x_0$, provided that the initial error is small enough and the switching signal is slow enough in the dwell-time or average dwell-time sense (see Section 3.2 for the definition of average dwell-time).

Proposition 2.1 *Consider the switched system* (2.9) *and the output set* $\overline{\mathcal{Y}}$. *Suppose that switched system is robustly invertible with radius* ε. *There exists a number* $\delta > 0$ *and a class of switching signal* $\mathcal{S}_{average}[\tau_a, N_0]$ *such that if the initial state error is less than* δ, *then the switching signal can be reconstructed exactly and the input can be recovered asymptotically for all switching signals in* $\mathcal{S}_{average}[\tau_a, N_0]$.

2.8 Discrete-Time Switched Systems

In this section, we address the invertibility problems for discrete-time switched linear systems. Consider a discrete-time switched linear system Γ_σ with the family of subsystems $\{\Gamma_p, p \in \mathcal{P}\}$,

$$\Gamma_\sigma : \begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \\ y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k). \end{cases} \tag{2.22}$$

We assume that the the individual subsystems live in the same state space $\mathbb{R}^n$. The discrete-time switching signal σ is a function $\sigma : \{0, 1, 2, \dots\} \to \mathcal{P}$; switching times are k such that $\sigma(k) \neq \sigma(k-1),\ k \geqslant 1$. Denote by $\mathcal{S}$ the set of discrete-time switching signals. Unlike the case of continuous-time switching signals, we do not have further restrictions on σ because a discrete-time signal already implies that there can only be finitely many switches in any finite interval. The input set $\mathcal{U}$ is the set of functions mapping $\{0, 1, 2, \dots\}$ into a value in the input spaces. A switching signal $\sigma \in \mathcal{S}$ and an input $u \in \mathcal{U}$ are compatible if $\dim(u(k)) = m_{\sigma(k)}$, where $m_i, i \in \mathcal{P}$ are the input dimensions of the individual subsystems. We assume that there is no jump at switching times. For each initial state $x_0 \in \mathbb{R}^n$, a compatible (σ, u), $\sigma \in \mathcal{S}$, $u \in \mathcal{U}$, a unique trajectory of the switched system exists and is denoted by $\Gamma_{\sigma,x_0}(u)$; the corresponding output is $\Gamma^{\mathrm{O}}_{\sigma,x_0}(u)$.

2.8.1 Input invertibility

In this subsection, we consider the input invertibility problem, which concerns with recovering the input from the output of switched systems with *known* switching signals. For the continuous-time case, as we have seen, input invertibility is equivalent to invertibility of the individual subsystems (a consequence of Lemma 2.4). For the discrete-time case, we do not have the analogy of Lemma 2.4 (unless the switching signal has the minimum dwell-time no less than $2n$), and invertibility of the individual subsystems is neither necessary nor sufficient (see Example 2.4 below).

We now present an extension of the structure algorithm to discrete-time switched linear systems, which shall be called the *hybrid structure algorithm*. This algorithm is essentially the structure algorithm for time-varying linear systems and is complimentary to the other approach on input invertibility of discrete-time systems [17, 38] which use the whole past output data, whereas here we only use partial past output data. The name *hybrid structure algorithm* reflects the fact that the time-varying system matrices only take a finite number of possible matrices, compared to the case of general time-varying systems where there is no restriction on the system matrices. When the switching signal is a constant switching signal, the algorithm reduces to those of discrete-time nonswitched systems, which is numerically the same as those for continuous-time algorithms except that differentiations are replaced by delay operators.

The hybrid structure algorithm

Let $q_\sigma(\ell, 0) = \mathrm{rank}(D_{\sigma(\ell)})$. There exists a nonsingular matrix $S_\sigma(\ell, 0)$ such that $S_\sigma(\ell, 0) D_{\sigma(\ell)} = \begin{bmatrix} \overline{D}_\sigma(\ell, 0) \\ 0 \end{bmatrix}$, where $\overline{D}_\sigma(\ell, 0)$ has $q_\sigma(\ell, 0)$ rows and rank $q_\sigma(\ell, 0)$. Let

$z_\sigma(\ell,0) = S_\sigma(\ell,0)y(\ell)$ and $C_\sigma(\ell,0) = S_\sigma(\ell,0)C_{\sigma(\ell)}$. Thus, we have

$$z_\sigma(\ell,0) = \begin{pmatrix} \bar{z}_\sigma(\ell,0) \\ \tilde{z}_\sigma(\ell,0) \end{pmatrix} = \begin{bmatrix} \overline{C}_\sigma(\ell,0) \\ \widetilde{C}_\sigma(\ell,0) \end{bmatrix} x(\ell) + \begin{bmatrix} \overline{D}_\sigma(\ell,0) \\ 0 \end{bmatrix} u(\ell). \tag{2.23}$$

We also have

$$z_\sigma(\ell+1,0) = \begin{pmatrix} \bar{z}_\sigma(\ell+1,0) \\ \tilde{z}_\sigma(\ell+1,0) \end{pmatrix} = \begin{bmatrix} \overline{C}_\sigma(\ell+1,0) \\ \widetilde{C}_\sigma(\ell+1,0) \end{bmatrix} x(\ell+1) + \begin{bmatrix} \overline{D}_\sigma(\ell+1,0) \\ 0 \end{bmatrix} u(\ell+1),$$

and hence,

$$\tilde{z}_\sigma(\ell+1,0) = \widetilde{C}_\sigma(\ell+1,0)A_{\sigma(\ell)}x(\ell) + \widetilde{C}_\sigma(\ell+1,0)B_{\sigma(\ell)}u(\ell). \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\begin{aligned} \begin{pmatrix} \bar{z}_\sigma(\ell,0) \\ \tilde{z}_\sigma(\ell+1,0) \end{pmatrix} &= \begin{bmatrix} \overline{C}_\sigma(\ell,0) \\ \widetilde{C}_\sigma(\ell+1,0)A_{\sigma(\ell)} \end{bmatrix} x(\ell) + \begin{bmatrix} \overline{D}_\sigma(\ell,0) \\ \widetilde{C}_\sigma(\ell+1,0)B_{\sigma(\ell)} \end{bmatrix} u(\ell) \\ &=: C_\sigma(\ell,1)x(\ell) + D_\sigma(\ell,1)u(\ell). \end{aligned}$$

Let $q_\sigma(\ell,1) = \text{rank}D_\sigma(\ell,1)$ and $S_\sigma(\ell,1)$ such that $S_\sigma(\ell,1)D_\sigma(\ell,1) = \begin{bmatrix} \overline{D}_\sigma(\ell,1) \\ 0 \end{bmatrix}$, where the rows of $\overline{D}_\sigma(\ell,1)$ are linear independent. We have

$$z_\sigma(\ell,1) := S_\sigma(\ell,1) \begin{pmatrix} \bar{z}_\sigma(\ell,0) \\ \tilde{z}_\sigma(\ell+1,0) \end{pmatrix} = \begin{bmatrix} \overline{C}_\sigma(\ell,1) \\ \widetilde{C}_\sigma(\ell,1) \end{bmatrix} x(\ell) + \begin{bmatrix} \overline{D}_\sigma(\ell,1) \\ 0 \end{bmatrix} u(\ell). \tag{2.25}$$

The foregoing equation also implies

$$z_\sigma(\ell+1,1) = \begin{bmatrix} \overline{C}_\sigma(\ell+1,1) \\ \widetilde{C}_\sigma(\ell+1,1) \end{bmatrix} x(\ell+1) + \begin{bmatrix} \overline{D}_\sigma(\ell+1,1) \\ 0 \end{bmatrix} u(\ell+1).$$

and hence,

$$\tilde{z}_\sigma(\ell+1,1) = \widetilde{C}_\sigma(\ell+1,1)A_{\sigma(\ell)}x(\ell) + \widetilde{C}_\sigma(\ell+1,1)B_{\sigma(\ell)}u(\ell). \tag{2.26}$$

From (2.25) and (2.26), we obtain

$$\begin{pmatrix} \bar{z}_\sigma(\ell,1) \\ \tilde{z}_\sigma(\ell+1,1) \end{pmatrix} = \begin{bmatrix} \overline{C}_\sigma(\ell,1) \\ \widetilde{C}_\sigma(\ell+1,1)A_{\sigma(\ell)} \end{bmatrix} x(\ell) + \begin{bmatrix} \overline{D}_\sigma(\ell,1) \\ \widetilde{C}_\sigma(\ell+1,1)B_{\sigma(\ell)} \end{bmatrix} u(\ell).$$

Continuing the described procedure, we have the following recursive definition of the structure algorithm starting at time ℓ and after k steps:

$$\begin{aligned}
D_\sigma(\ell,0) &:= D_{\sigma(\ell)}, \quad C_\sigma(\ell,0) := C_{\sigma(\ell)} \\
D_\sigma(\ell,k) &:= \begin{bmatrix} \overline{D}_\sigma(\ell,k-1) \\ \widetilde{C}_\sigma(\ell+1,k-1)B_\sigma(\ell) \end{bmatrix}, \quad q_\sigma(\ell,k) := \mathrm{rank} D_\sigma(\ell,k) \\
S_\sigma(\ell,k)D_\sigma(\ell,k) &= \begin{bmatrix} \overline{D}_\sigma(\ell,k) \\ 0 \end{bmatrix}, \quad \mathrm{rank}\ \overline{D}_\sigma(\ell,k) = q_\sigma(\ell,k) \\
C_\sigma(\ell,k) &:= \begin{bmatrix} \overline{C}_\sigma(\ell,k-1) \\ \widetilde{C}_\sigma(\ell+1,k-1)A_{\sigma(\ell)} \end{bmatrix}, \\
S_\sigma(\ell,k)C_\sigma(\ell,k) &:= \begin{bmatrix} \overline{C}_\sigma(\ell,k) \\ \widetilde{C}_\sigma(\ell,k) \end{bmatrix}, \quad \overline{C}_\sigma(\ell,k) \text{ is the first } q_\sigma(\ell,k) \text{ rows of } S_\sigma(\ell,k)C_\sigma(\ell,k) \\
z_\sigma(\ell,k) &:= S_\sigma(\ell,k) \begin{pmatrix} \bar{z}_\sigma(\ell,k-1) \\ \tilde{z}_\sigma(\ell+1,k-1) \end{pmatrix} = \begin{bmatrix} \overline{C}_\sigma(\ell,k) \\ \widetilde{C}_\sigma(\ell,k) \end{bmatrix} x(\ell) + \begin{bmatrix} \overline{D}_\sigma(\ell,k) \\ 0 \end{bmatrix} u(\ell) \\
z_\sigma(\ell,k) &:= \begin{pmatrix} \bar{z}_\sigma(\ell,k) \\ \tilde{z}_\sigma(\ell,k) \end{pmatrix}, \quad \bar{z}_\sigma(\ell,k) \text{ is the first } q_\sigma(\ell,k) \text{ rows of } z_\sigma(\ell,k) \\
z_\sigma(\ell,0) &:= S_\sigma(\ell,0)y(\ell).
\end{aligned} \tag{2.27}$$

Remark 2.13 *For nonswitched systems,* $\sigma(\ell) = p \; \forall \ell$ *for some* $p \in \mathcal{P}$*. Then*

$$C_\sigma(\ell, 0) = C, \; D_\sigma(\ell, 0) = D \quad \forall \ell.$$

It follows that $D_\sigma(\ell, k)$ *is independent of* ℓ*, and hence,* $S_\sigma(\ell, k)$ *and* $q_\sigma(\ell, k)$ *are also independent of the starting time* ℓ*. In particular,* $\widetilde{C}_\sigma(\ell + 1, k - 1) = \widetilde{C}_\sigma(\ell, k - 1)$*, which means that we do not need to calculate* $\widetilde{C}_\sigma(\ell+1, k-1)$ *but can use the available* $\widetilde{C}_\sigma(\ell, k - 1)$ *(this is the nested feature in the structure algorithm for nonswitched systems). In this case, the hybrid structure algorithm* (2.27) *is exactly the same as the structure algorithm described in the appendix.*

The hybrid structure algorithm produces a sequence of integers $q_\sigma(\ell, 0) \leqslant q_\sigma(\ell, 1) \leqslant q_\sigma(\ell, 2) \leqslant \dots$. The following notion of *relative degree* is an extension of the same notion for nonlinear systems by Singh in [39] to discrete-time switched linear systems.

Definition 2.2 *The relative degree* $\Gamma^r_\sigma(\ell)$ *of the switched system* (2.22) *at a time* ℓ *is the least integer* k *such that* $q_\sigma(\ell, k) = m_{\sigma(\ell)}$ *or* $\Gamma^r_\sigma(\ell) = \infty$ *if* $q_\sigma(\ell, k) < m_{\sigma(\ell)} \; \forall k = 0, 1, \dots,$ *where* $m_{\sigma(\ell)}$ *is the input dimension of the system* $\Gamma_{\sigma(\ell)}$*.*

For nonswitched systems (i.e., σ is a constant switching signal), because of the time-invariant property, the relative degree is time-invariant (see Remark 2.13) and is denoted by Γ^r. For switched systems, the relative degree is not time-invariant; it is time-dependent so we must talk of a relative degree at a time. Also, for nonswitched systems, there is a lower bound $\bar{k}$ on k such that $q(k) = q(\bar{k}) \; \forall k \geqslant \bar{k}$. In fact, $\bar{k} \leqslant n$, where n is the state dimension (this follows from the Cayley-Hamilton theorem). This means that we can determine whether $\Gamma^r < \infty$ or $\Gamma^r = \infty$ after at most n steps in the algorithm. In contrast, for switched systems, we do not have such a lower bound on k and in general, $\Gamma^r = \infty$ means we have infinite steps in the algorithm (in certain cases, for example, when there is a pattern in σ, we can still verify that $\Gamma^r_\sigma(\ell) = \infty$; see Example 2.3 below). We have the following lemma.

Lemma 2.5 *For a fixed switching signal σ, the discrete-time switched system* (2.22) *is invertible if* $\Gamma_\sigma^r(\ell) < \infty \; \forall \ell = 0, 1, \ldots$.

Proof The proof is straightforward from the hybrid structure algorithm and the definition of a relative degree. If $\Gamma_\sigma^r(\ell) < \infty$, we have $q_\sigma(\ell, k) = m_i$ for some $k \geqslant 0$, which implies that the input $u(\ell)$ is uniquely recovered as $u(\ell) = \overline{D}_\sigma^{-1}(\ell, k)(\bar{z}_\sigma(\ell, k) - \overline{C}_\sigma(\ell, k)x(\ell))$, where $\overline{D}_\sigma^{-1}(\ell, k)$ is the inverse of $\overline{D}_\sigma(\ell, k)$. Since $x(0) = x_0$ is known, we can recover $u(0)$. Then $x(1)$ is known. Since $\Gamma_\sigma^r(1) < \infty$, $u(1)$ can be recovered uniquely. It follows that $x(\ell), u(\ell)$ are recovered uniquely for all $\ell \geqslant 0$. $\square$

Lemma 2.5 is a sufficient condition for input invertibility. Whether this condition is necessary or not remains an open question. Notice that, in the case of nonswitched systems, $\Gamma^r < \infty$ is both necessary and sufficient for invertibility ($\Gamma^r < \infty$ is equivalent to $q_\alpha = m$ in the appendix).

Example 2.3 Consider the following two systems:

$$\Gamma_1 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases},$$

$$\Gamma_2 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(k) \end{cases}.$$

For the nonswitched system Γ_1, we have $C_1 B_1 = 1$, and hence Γ_1 is invertible. Similarly, Γ_2 is invertible. Consider the switched system generated by Γ_1, Γ_2 with the switching signal $\sigma = \{1, 2, 1, 2, 1, 2 \ldots\}$.

Using the hybrid structure algorithm, we can see that $C(0, k) = C_1, D(0, k) = 0$, and $q_\sigma(0, k) = 0 \; \forall k \geqslant 0$. Hence, $\Gamma_\sigma^r(0) = \infty$. It can be shown that $\Gamma_\sigma^r(\ell) = \infty \; \forall \ell \geqslant 0$. The switched system is not invertible because it can be checked by direct substitutions that $\Gamma^{\mathrm{O}}_{\sigma,x_0}(u) = y$ for all u where $y(\ell) := \begin{bmatrix} 1 & 0 \end{bmatrix} x_0 \ \forall \ell \geqslant 0$.

Example 2.4 Consider the following two subsystems:

$$\Gamma_1 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{cases},$$

$$\Gamma_2 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases}.$$

The system Γ_1 is not invertible. To see this, we have $y(1) = C_1 x(1) = C_1 A_1 x_0 + C_1 B_1 u_0 = C_1 x_0$, and it can be shown that $y(k) = C_1 x_0 \ \forall k \geqslant 0$. Thus, the system Γ_1 is not invertible. By a similar analysis, it can be shown that Γ_2 is not invertible. Consider the switched system generated by Γ_1, Γ_2 with the switching signal $\sigma = \{1, 2, 1, 2, \ldots\}$. Using the hybrid structure algorithm, we can calculate that $\Gamma^r_\sigma(0) = 1 < \infty$. Similarly, we calculate that $\Gamma^r_\sigma(1) = 1$. It can then be verified that $\Gamma^r_\sigma(\ell) = 1 < \infty \ \forall \ell \geqslant 0$ and by Lemma 2.5, $u(\ell), x(\ell)$ can be recovered uniquely for all ℓ. Here, we see that the input of the switched system can be recovered uniquely for all output y even though the individual subsystems are not invertible.

2.8.2 Invertibility

The invertibility problem for discrete-time switched systems is formulated similarly as those for the continuous-time case but there is a difference in that, for discrete-time systems, delay plays an important role in invertibility.

Problem 2.1 *Consider a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$ for the discrete-time switched system* (2.22)*. For a number $d > 0$, find a largest possible set $\mathcal{Y}$ and a condition on the subsystems, independent of x_0, such that $y^1_{[0:k]} = y^1_{[0:k]} \Rightarrow$*

$u^1_{[0:k-d]} = u^2_{[0:k-d]}$ *for all* $k \geqslant d$.

It is worth mentioning that the case of discrete-time switched linear systems is not parallel to the continuous-time counterpart, and we have not yet obtained a necessary and sufficient condition for invertibility of discrete-time switched linear systems (we do have a sufficient condition and a necessary condition separately). The difficulty arises from the essential difference between invertibility of discrete-time nonswitched linear systems and their continuous-time counterparts, although the invertibility conditions are exactly the same for the two cases. For the continuous-time case, the information in an *infinitesimally small* interval $y_{[t,t+\delta)}$, $\delta > 0$, completely determines $u(t)$. In contrast, for the discrete-time case, generally we need d-samples, $1 \leqslant d \leqslant n$, $y(k), \ldots, y(k+d-1)$, to determine the input $u(k)$. The delay behavior in the invertibility property of discrete-time nonswitched systems causes difficulties in solving the invertibility problem for discrete-time switched systems. That is because it is not always possible to recover all the inputs in between consecutive switching times just using the outputs in the same interval when the individual subsystems are invertible; i.e., $u(k), \ldots, u(k+N)$ cannot all be recovered from $y(k), \ldots, y(k+N)$ given $\sigma(k) = \cdots = \sigma(k+N) = p$ (cf. the continuous-time case). If we utilize the output after a switch to determine the input before switching, this in turns leads to a *combinatorial mixing* of the subsystem dynamics around switching times (cf. the concatenation (not mixing) in the continuous-time case). The combinations of the individual subsystems are hard to characterize as they can lead to many possibilities: for a fixed switching signal, one can lose invertibility of the switched system even if the individual subsystems are invertible, and one can gain invertibility even if the individual subsystems are not invertible (see Examples 2.3 and 2.4 below; cf. Theorem 2.2 for the continuous-time case).

A necessary condition that is easy to see (by picking constant switching signals) is that the individual subsystems must be invertible.

Lemma 2.6 *If the map* H_{x_0} *is invertible, it is necessary that* Γ_p *are invertible for all* $p \in \mathcal{P}$.

A sufficient condition is more complicated. We consider the case $d = 0$. The *instant identification* property is characterized as follows, which is the discrete-time counterpart of the singular pair notion of continuous-time systems.

Definition 2.3 *A pair* (x_0, z), $x_0 \in \mathbb{R}^n, z \in \mathbb{R}^m$ *is a singular pair for the two systems* Γ_p, Γ_q *with zero delay if the output dimensions of the two systems are both* m *and further, there exist* $u_1(0), u_2(0)$ *such that* $C_p x_0 + D_p u_1(0) = C_q x_0 + D_q u_2(0) = z$.

Similarly to the continuous-time case, a singular pair for discrete-time systems means that we cannot distinguish between the two systems just by looking at the current output and the current state (it may be possible to distinguish between the two if more outputs are taken into account but here we aim for instant identification). Conversely, if (x_0, z) is not a singular pair, we can immediately determine which system is able to generate z starting from x_0 (cf. the continuous-time case). We now develop a formula to check if (x_0, z) is a singular pair with zero delay. The equation $C_p x_0 + D_p u_1(0) = C_q x_0 + D_q u_2(0) = z$ contains u, so we have to find a way to eliminate u from the equations. From the structure algorithm, we have

$$\begin{bmatrix} \overline{S}_p \\ \widetilde{S}_p \end{bmatrix} z = \begin{bmatrix} \overline{S}_p \\ \widetilde{S}_p \end{bmatrix} C_p x_0 + \begin{bmatrix} \overline{S}_p \\ \widetilde{S}_p \end{bmatrix} D_p u_1(0) =: \begin{bmatrix} \overline{C}_p \widetilde{C}_p \end{bmatrix} x_0 + \begin{bmatrix} \overline{D}_p \\ 0 \end{bmatrix} u_1(0)$$

for some nonsingular matrix S_p such that $S_p D_p = \overline{D}_p$, where $\operatorname{rank}(\overline{D}_p) = \operatorname{rank}(D_p)$ (S_p is the matrix $S_p(0,0)$ in the algorithm). So if $\widetilde{S}_p z = \widetilde{C}_p x_0$, we can always find $u_1(0)$ such that $C_p x_0 + D_p u_1(0) = z$ (since $\overline{D}_p$ is full rank). Define $\widetilde{S}_q, \widetilde{C}_q$ similarly for the system Γ_q. Then (x_0, z) is a singular pair iff it is a solution to the following equation:

$$\begin{bmatrix} \widetilde{S}_p \\ \tilde{S}_q \end{bmatrix} z = \begin{bmatrix} \widetilde{C}_p \\ \widetilde{C}_q \end{bmatrix} x_0. \tag{2.28}$$

Similarly to the continuous-time case, we define the set $\mathcal{R}_{p,q}(z)$ be the set of points $x \in \mathbb{R}^n$ such that (x, z) forms a singular pair,

$$\mathcal{R}_{p,q}(z) := \{x \in \mathbb{R}^n : (x, z) \text{ is a singular pair for } \Gamma_p, \Gamma_q\}. \tag{2.29}$$

By convention, $\mathcal{R}_{p,q}(z) = \emptyset$ if the output dimensions of Γ_p, Γ_q are different. We have described the instant identification property. The *instant recovery* property, which shall be called *invertible with order 0*, is formally defined as follows.

Definition 2.4 *The system Γ_p is invertible with order 0 if rank(D_p) is equal the input dimension.*

For the sake of presentation, assume that the output dimensions of the individual subsystems are the same. Let $\mathcal{Y}$ be the set of output functions such that $\mathcal{Y} = \{y : y(k) \in \mathcal{Y}'\}$. We then now have a sufficient condition for invertibility of discrete-time switched linear systems.

Lemma 2.7 *Consider a discrete-time switched systems generated by $\{\Gamma_p, p \in \mathcal{P}\}$. The switched system is invertible on $\mathcal{Y}$ if every subsystem is invertible with order 0 and $\mathcal{R}_{p,q}(z) = \emptyset \; \forall p, q \in \mathcal{P}, p \neq q, \; \forall z \in \mathcal{Y}'$.*

Proof Because $\mathcal{R}_{p,q}(z) = \emptyset \; \forall p, q \in \mathcal{P}, p \neq q \forall z \in \mathcal{Y}'$, it follows that $(x_0, y(0))$ is not a singular pair for any Γ_p, Γ_q, $p \neq q$. Therefore, the index $\sigma(0) = p$ is recovered uniquely such that $\widetilde{S}_p y(0) = \widetilde{C}_p x_0$. Then $u(0)$ is recovered uniquely from x_0 and $y(0)$ since Γ_p is invertible with order 0. We then have $x(1) = A_p x_0 + B_p u(0)$ is known. Since $(x(1), y(1))$ is not a singular pair for any Γ_p, Γ_q, $p \neq q$, the index $\sigma(1) = q$ is recovered uniquely such that $\widetilde{S}_q y(1) = \widetilde{C}_q x(1)$. Then $u(1)$ is recovered uniquely from

$x(1)$ and $y(1)$ since Γ_q is invertible with order 0. Repeating this procedure, we can see that $\sigma(\ell)$, $u(\ell)$, and $x(\ell)$ are recovered uniquely for all $\ell \geqslant 0$. $\square$

Remark 2.14 *We have presented a sufficient condition for invertibility of discrete-time switched systems with respect to the whole switching signal* $\mathcal{S}$ *(i.e., arbitrary switching), using the instant identification and instant recovery properties. The instant identification property can be seen as* 0*-order identification, and the instant recovery property can be seen as* 0*-order invertibility. These notions can be extended to* k*-order identification and* k*-order invertibility where one can use up to* k *additional output samples instead of only the current output. To employ these properties, one may need to consider the invertibility problem on a smaller class of switching signals, such as those with the minimum dwell-time of at least* k *samples. We do not have a result on this and this can be a topic of further research.*

We present an example of an invertible discrete-time switched linear system below.

Example 2.5 Consider switched systems generated by the two subsystems

$$\Gamma_1 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \end{cases},$$

$$\Gamma_2 : \begin{cases} x(k+1) &= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \end{cases}.$$

We have $\widetilde{S}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $\widetilde{S}_2 = \begin{bmatrix} -1 & 1 \end{bmatrix}$, $\widetilde{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\widetilde{C}_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}$. We then calculate

$\mathcal{R}_{p,q}(z)$,

$$\mathcal{R}_{p,q}(z) = \left\{ x : \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} z = \begin{pmatrix} z_2 \\ -z_1 + z_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}.$$

We can see that $\mathcal{R}_{p,q}(z) = \emptyset$ if $z_2 \neq 2(-z_1 + z_2) \Leftrightarrow 2z_1 \neq z_2$. Thus, the switched system is invertible on $\mathcal{Y} := \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : 2y_1(\ell) - y_2(\ell) \neq 0 \right\}$ by Lemma 2.7.

2.9 Applications

Invertibility of switched systems – the ability to reconstruct both continuous inputs and discrete states of switched systems from partial measurements – can find application in various domains. We discuss some potential application scenarios below. Note that the invertibility result presented in this thesis is for switched linear system only and may not be readily applicable to scenarios where the plants are switched nonlinear systems. Invertibility of switched nonlinear systems as well as applications of invertibility of switched systems will be a thrust for future research in invertibility of switched systems.

Vehicle systems Multimodal aircraft can be modeled as switched systems (see the aircraft example in Section 1.2) where the switching signals indicate which mode the aircraft is currently. The invertibility property relates to the problem of detecting the flying mode of the aircraft using the measured information, which could be the path and/or the velocity of the aircraft on ground radar equipment. Another possible application of invertibility of switched systems is to recover topology of networked multiagent systems with time-varying topology (see also the networked system example in Section 1.2). Invertibility in this case relates to detecting the group formation (which is identified via the switching signal) using partial measurements of the collective state in the presence of disturbances, which act as unknown inputs.

Biological systems Systems approach to biology is one of the fastest growing trends in biology, bringing the systematic view of engineering into the wealth of empirical data available in biology in order to better understand how biological systems behave within themselves and among a collection of systems [40, 41]. Among dynamical system tools, hybrid dynamical systems are particularly suited to model biological systems due to the inherent presence of switching mechanisms in biological systems [42, 43, 44]. For example, in cell biology, gene activation and inhibition alter the dynamics of chemical concentrations in cells, giving rise to hybrid models that involve switching among multiple dynamics [45, 46, 47]. One potential application of hybrid invertibility in the context of cell biology means the possibility of detecting certain genes's behaviors according to the active mode of a hybrid system model (which is identified by the switching signal) in the presence of disturbances (which act as unknown inputs) using measurements of concentration of proteins, mRNA, or metabolites as the output.

Power networks The use of hybrid modeling tools in power networks can be found in [48, 49, 50]. Switches are inherent in power networks and when there is a significant change in the power load, for example when there is power surge or power outage, switches/fuses will be active resulting in changes of the network dynamics. In such events, it is essential to detect when power dynamics change and adjust power distribution to recover stability. Understanding the invertibility property of hybrid systems could lend a hand on the problem of identifying times of critical events using partial measurements of power or voltage (which act as outputs).

Communication networks A communication network consists of a number of nodes and there are data packets flowing along communication links between nodes. To avoid congestion, network traffic is regulated by transmission control protocols (TCPs), which tell nodes to drop packets if there is excessive incoming data compared to outgoing data. When dropping occurs, dynamics of the nodes change abruptly and this fact leads to the use of hybrid system to model communication networks [13, 14].

From the analysis point of view, one may want to detect changes in network dynamics, maybe due to intrusion and attacks, using observed information. From security point of view, one may want to design a hybrid system in such a way that the system is not invertible to avoid letting others know its operating modes.

2.9.1 Topology recovery in multiagent networked systems

We present one application of invertibility of switched systems in recovering topology of multiagent networked systems (see the multiagent example in Section 1.2 for background and more discussion). Consider a network of n dynamical agents. Let x_i be the state of agent i and its dynamics be $\dot{x} = f_i(x_i, u_i)$ where u_i is the input. In networked systems, the input action on an agent can depend not only on the state of the agent but also the states of other agents in the network. The agents can communicate with each other and the communication among the agents is captured by a graph G — also known as a topology of the network — whose nodes are agents and edges denote communication links between the agents. The graph G can be a directed graph or not. Denote by $\mathcal{N}_i$ the neighborhood of agent i which consists of all the agents who have communication with agent i. The control action on agent i is then of the form $u_i = g(x_i, \{x_j, j \in \mathcal{N}_i\})$. The collective dynamics of the network of all agents are obtained by putting all the dynamics of the agents together and the state of the networked multiagent system is $x = (x_1, \dots, x_n)$. We will consider the topology recovery problem for two different types of networked multiagent systems in the following examples.

Cooperative consensus network

Consider a network of agents with undirected topologies. Assume the agents are static when there are no external inputs: $\dot{x}_i = u_i + d_i$, where d_i is a disturbance, and the control action on the agent is a consensus protocol $u_i = \sum_{j \in \mathcal{N}_i}(x_j - x_i)$ with the aim of reaching a common value x^* among all the agents, $x_i(t) \to x^*$ as $t \to \infty$. The collective dynamics of the network are

$$\dot{x} = -L(G)x + I_n d, \tag{2.30}$$

where $L(G) = [l_{ij}]$, $l_{ij} = \begin{cases} -1 & \text{if } j \in \mathcal{N}_i, j \neq i, \\ |\mathcal{N}_i| & i = j \end{cases}$ is the graph Laplacian of G, I_n is the identity matrix with dimension n, and $d = (d_1, \ldots, d_n)$.

Suppose that the network topology is varying with time and taking values in a set of topologies such that $G : [0, \infty) \to \{G_1, \ldots, G_m\}$. We then write the network dynamics as a switched system

$$\dot{x} = A_\sigma x + I_n d, \tag{2.31}$$

where $\sigma(t) := p : G(t) = G_p$ and $A_p := L(G_p)$, $p \in \{1, \ldots, m\} =: \mathcal{P}$. Suppose that we also have output observation coming out of the network

$$y = C_\sigma x, \tag{2.32}$$

where $y = C_p x$ is the observation corresponding to topology G_p, $p \in \mathcal{P}$. Our invertibility result implies that if the switched system described by (2.31) and (2.32) is invertible, then it is guaranteed that we can recover the active topology at every time using only output measurement and the initial state, even if there are disturbances.

For example, consider a network switching between two topologies $G_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (the element a_{ij} of the matrix is 1 if there is a link between node i and node j

and the element is 0 otherwise) and $G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. If the output observation with the topology G_1 is $y = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_3 \end{pmatrix}$ and the output with the topology G_2 is $y = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_1 \end{pmatrix}$, then the rank condition (2.11) is violated. However, if the output observation with the topology G_1 is $y = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_3 \end{pmatrix}$ and the output with the topology G_2 is $y = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_1 + x_2 \end{pmatrix}$, then the rank condition is satisfied and we can guarantee topology recovery in the presence of disturbance for all output y such that $y(t) \neq 0 \forall t$.

Cooperative work processing

Consider the scenario where a network of agents cooperate to process work (which is viewed as "fluid"). There is a stream of work going into the network at rate $u(t)$ and the network returns processed work via an output y. There is a central unit that divides the incoming work into portions and delegates them to a number of agents. Suppose that an agent can process work at a rate f_i. An agent can also pass work to other agents (to share work load or because the work needs to be done through several stages, each of which is managed by particular agents) and it can also receive work from others. The relationship among the agents is captured by a directed graph $G = [g_{ij}]$; $g_{ij} = 1$ when there is a directed link from node i to node j and $g_{ij} = 0$

otherwise; $g_{ii} = 0$ for all i. Then an approximated fluid model of agent i is

$$\dot{x}_i = -f_i - \sum_{j:g_{ij}>0} z_{ij} + u_i, \tag{2.33}$$

where x_i is the buffer length, f_i is the processing rate, z_{ij} is the work agent i send to agent j per unit time, and u_i is the coming work rate to buffer i,

$$u_i = \beta_i B_i u + \sum_{j:g_{ji}>0} z_{ji}, \tag{2.34}$$

where $\beta_i B_i u$ is a portion of the incoming work u delegated to agent i from the central unit (if $\beta_i = 0$ or $B_i = 0$, agent i does not receive work from the central unit). The constants β_i satisfy the constraint $\sum_i \beta_i B_i u = u$. We assume that the agent can process work at a constant rate (there are other models for the work processing rate and f_i is a function of x_i in general)

$$f_i = \lambda_i I_{x_i>0} \tag{2.35}$$

and work passing among the agents is at a rate proportional to the buffer length,

$$z_{ij} = w_{ij} x_i. \tag{2.36}$$

Let $\hat{G} = [\hat{g}_{ij}], \hat{g}_{ij} = g_{ij} w_{ij}$ be the weighted directed graph (or digraph) obtained by associating the weight w_{ij} on the graph G. Assuming that $x_i > 0$ for all i, from (2.33), (2.34), and (2.36), the overall dynamics of the network are then

$$\dot{x} = A(\hat{G})x + Bu + \lambda, \tag{2.37}$$

where $x = (x_1, \ldots, x_n)$, $A(\hat{G}) = [a_{ij}]$, $a_{ii} = -\sum_{j:\hat{g}_{ij}>0} w_{ij}$ and $a_{ij} = \hat{g}_{ji}, j \neq i$, $B = \begin{bmatrix} \beta_1 B_1 \\ \vdots \\ \beta_n B_n \end{bmatrix}$, and $\lambda := (\lambda_1, \ldots, \lambda_n)$. The processed work is then returned to the central unit and the central unit puts processed work out of the network as an output

$$y = Cx. \tag{2.38}$$

We call a tuple $(\hat{G}, B, C, \lambda)$ a network configuration, denoted by CF. Now, suppose that the central unit has several configurations $\{CF_1, \ldots, CF_m\}$ and the network can change its configuration according to some criteria (such as work load). Then the network can be represented by the switched system

$$\begin{aligned} \dot{x} &= \bar{A}_\sigma x + B_\sigma u + \lambda_\sigma, \\ y &= C_p x, \end{aligned} \tag{2.39}$$

where $\bar{A}_p = A(\hat{G}_p)$ and $B_p = B$, $p = 1, \ldots, m$. If the switched system is invertible, then one can recover the network configuration using just the output y and an initial state, without knowing u.

Invertibility of networked systems has an interpretation in a security context. If an intruder has access to both u and y, he or she may be able to do hybrid system identification (see, e.g., [33]). Now, suppose that an intruder only knows the data rate y directly coming out of the network and the initial state $x(0)$ and have no knowledge of u and σ. The intruder may know the configurations from product specifications (e.g., from vendors). The intruder then wants to figure out under which configuration the network is currently operating and utilizes this type of information to carry out network attacks. In other words, the intruder wants to recover σ from y and $x(0)$. This scenario fits into the invertibility problem for switched systems described above.

CHAPTER 3

STABILITY OF SWITCHED SYSTEMS

3.1 Stability Definitions

One of the fundamental issues in control is stability of systems. There are different kinds of stabilities (such as bounded input-bounded output stability or periodic-cycle stability), but here we concern ourselves with stability of equilibria, in particular asymptotic stability for autonomous switched systems and input-to-state stability for switched systems with inputs.

Trajectories of autonomous nonswitched systems are parameterized by the initial state. However, for autonomous switched systems, trajectories depend on both the initial state and the switching signal. This fact leads to the major difference in the study of switched system properties compared to nonswitched system properties in that switched system properties need to be characterized over all possible initial states as well as all possible switching signals in some class of switching signals. Otherwise, if a property holds for only one fixed switching signal, a switched system becomes a time-varying nonlinear system and switching behavior plays a minimal role.

We first cover the concept of equilibria for switched systems. Consider the autonomous switched system

$$\Gamma_\sigma : \dot{x} = f_\sigma(x) \tag{3.1}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ for some index set $\mathcal{P}$. Assume that the switched system state space is $\mathbb{R}^n$.

Definition 3.1 *A state* $x^\star$ *is an* equilibrium *of the switched system* (3.1) *if it has the property that whenever the switched system starts at* $x^\star$*, the state will remain at*

$x^\star$ for all future time and for all possible switching signals.

From the definition, it is clear that $x^\star$ is an equilibrium of a switched system if and only if it is a *common equilibrium* of all the subsystems. Therefore, $x^\star$ is an equilibrium of (3.1) if and only if $f_p(x^\star) = 0 \ \forall p \in \mathcal{P}$.

We assume that the switched system (3.1) has an equilibrium $x^\star$ and without loss of generality, we assume that $x^\star = 0$ (otherwise, the change of variable $x \leftarrow x - x^\star$ will move the equilibrium to the origin). Asymptotic stability for switched systems is defined as follows.

Definition 3.2 *Consider the switched system* (3.1) *with the origin being an equilibrium. The origin is* asymptotically stable *(AS) over a class of switching signal $\mathcal{S}$ if there exists a class $\mathcal{KL}$ function β and $\bar{x}_0 > 0$ such that $|x(t)| \leqslant \beta(|x(0)|, t) \ \forall t \geqslant 0$ for all $|x(0)| \leqslant \bar{x}_0$ and for all $\sigma \in \mathcal{S}$.*

We say that the origin is globally asymptotically stable (GAS) if we can take $\bar{x}_0 = \infty$.

We also have input-to-state properties for switched systems with inputs. Consider a switched system with input

$$\Gamma_\sigma : \ \dot{x} = f_\sigma(x, v). \tag{3.2}$$

Assume that the unforced switched system with $v = 0$ has an equilibrium $x^\star$ and without loss of generality, assume that $x^\star = 0$. Denote by $\mathcal{U}$ the set of all piecewise continuous hybrid functions that can be inputs to the switched system (3.2). Local input-to-state stability (local ISS) for switched systems is defined as follows (see [51] for the definition of local ISS for nonswitched systems):

Definition 3.3 *Consider the switched system* (3.2) *with the origin being an equilibrium of the unforced system. The origin is* locally input-to-state stable (locally ISS) *over a class of switching signals $\mathcal{S}$ if there exist $\beta \in \mathcal{KL}$, $\alpha, \gamma \in \mathcal{K}_\infty$, and $\bar{x}_0, \bar{u} > 0$ such that for all $x(0)$ such that $|x(0)| \leqslant \bar{x}_0$, for all $\sigma \in \mathcal{S}$, and for all compatible* *hybrid functions $v \in \mathcal{U}$ such that $|v(t)| \leqslant \bar{u}\ \forall t$, we have*

$$\alpha(|x(t)|) \leqslant \beta(|x(0)|, t) + \gamma\left(\|v\|_{[0,t)}\right) \quad \forall\, t \geqslant 0. \tag{3.3}$$

When we can take $\bar{x}_0 = \infty$ and $\bar{u} = \infty$, we say that the origin is ISS. The function α can be taken to be the identity function without loss of generality; see, e.g., [8, Lemma 4.2]. We use the above formulation of ISS for the sake of consistency with other input-to-state definitions below. The following definitions introduce a quantitative measure of the "stability margin" via exponentially weighted signals.

Definition 3.4 *Consider the switched system* (3.2) *with the origin being an equilibrium of the unforced system. The origin is locally* $\mathrm{e}^{\lambda t}$-weighted input-to-state stable (locally $\mathrm{e}^{\lambda t}$-weighted ISS) *over a class of switching signal $\mathcal{S}$ for some $\lambda > 0$ if $\exists$ $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ and $\bar{x}_0, \bar{u} > 0$ such that for all $x(0)$ such that $|x(0)| \leqslant \bar{x}_0$, for all $\sigma \in \mathcal{S}$, and for all compatible hybrid functions $v \in \mathcal{U}$ such that $|v(t)| \leqslant \bar{u}\ \forall t \geqslant 0$, we have*

$$\mathrm{e}^{\lambda t}\alpha_1(|x(t)|) \leqslant \alpha_2(|x(0)|) + \sup_{s\in[0,t)}\left\{\mathrm{e}^{\lambda s}\gamma(|v(s)|)\right\}\ \forall\, t \geqslant 0. \tag{3.4}$$

Definition 3.5 *Consider the switched system* (3.2) *with the origin being an equilibrium of the unforced system. The origin is locally* $\mathrm{e}^{\lambda t}$-weighted integral input-to-state stable (locally $\mathrm{e}^{\lambda t}$-weighted iISS) *over a class of switching signal $\mathcal{S}$ for some $\lambda > 0$ if $\exists$ $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$, and $\bar{x}_0, \bar{u} > 0$, such that for all $x(0)$ such that $|x(0)| \leqslant \bar{x}_0$, for all $\sigma \in \mathcal{S}$, and for all compatible hybrid functions $v \in \mathcal{U}$ such that $|v(t)| \leqslant \bar{u}\ \forall t \geqslant 0$, we have*

$$\mathrm{e}^{\lambda t}\alpha_1(|x(t)|) \leqslant \alpha_2(|x(0)|) + \int_0^t \mathrm{e}^{\lambda\tau}\gamma(|v(\tau)|)d\tau\ \forall\, t \geqslant 0. \tag{3.5}$$

When we can take $\bar{x}_0 = 0$ and $\bar{u} = 0$, we say that the origin is $\mathrm{e}^{\lambda t}$-weighted ISS/ $\mathrm{e}^{\lambda t}$-weighted iISS. The $\mathrm{e}^{\lambda t}$-weighted ISS and $\mathrm{e}^{\lambda t}$-weighted iISS properties generalize ISS and iISS properties[1] in the spirit of exponentially weighted induced norms considered in [21]. While the ISS property characterizes stability in general, the $e^{\lambda t}$-weighted ISS and $e^{\lambda t}$-weighted iISS properties characterize stability with a "stability margin" λ (similarly to stability margin of linear systems), which is useful in quantitative analysis (e.g., in supervisory control as we will see later).

If in a definition, the set $\mathcal{S}$ is the set of all admissible switching signals, then we say that the property is true for *arbitrary switching*. It has been shown [53] that GAS of the switched system (3.1) for arbitrary switching is equivalent to the existence of a *common Lyapunov function* $V : \mathbb{R}^n \to [0, \infty)$ such that $\forall x$,

$$\begin{aligned} \alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|) \\ \frac{\partial V}{\partial x} f_p(x) \leqslant -\gamma(|x|) \quad \forall p \in \mathcal{P}, \end{aligned}$$

for some functions $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$. Likewise, ISS of the switched system (3.1) for arbitrary switching is equivalent to the existence of a *common ISS-Lyapunov function* $V : \mathbb{R}^n \to [0, \infty)$ such that $\forall x, u$,

$$\begin{aligned} \alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|) \\ \frac{\partial V}{\partial x} f_p(x) \leqslant -\rho(|x|) + \gamma(|u|) \quad \forall p \in \mathcal{P}, \end{aligned}$$

for some functions $\alpha_1, \alpha_2, \gamma, \rho \in \mathcal{K}_\infty$ [54].

Finding conditions on the subsystems to guarantee stability for arbitrary switching is one of the research directions in switched systems. For autonomous switched nonlinear systems, it has been shown that if the vector fields f_p of the subsystems are commutative, then one can construct a common Lyapunov function and thus conclude asymptotic stability of the switched system for arbitrary switching [55]. Other more general conditions in terms of Lie algebra have been found; see [56] for details and the references therein. In general, requiring the subsystems to have a common

[1] See [52] for the original definition of iISS.

Lyapunov function places a restriction on the structure of the subsystems.

Another research direction is to find classes of switching signals that still guarantee stability of switched systems but with fewer requirements on the subsystems. Earlier work leads to the class of dwell-time switching signals that guarantee GAS for any switched system with asymptotically stable linear subsystems [20]. For switching signals with dwell-time τ_d, every switching interval is at least τ_d or longer. Later, dwell-time switching has been extended to average dwell-time switching [21], in which one can have arbitrarily small switching intervals as long as fast switching is compensated by slower switching later so that on average, the dwell-time is sufficiently large. It has been shown [21] that under average dwell-time switching, one still can guarantee GAS of autonomous switched linear systems with GAS subsystems. With large enough average dwell-time, one can also guarantee ISS, $e^{\lambda t}$-ISS, and $e^{\lambda t}$-iISS of switched linear systems with ISS subsystems [21]. Several results on stability of switched nonlinear systems over the classes of dwell-time and average dwell-time switching have been obtained [7, 21, 57, 58]. Results with dwell-time switching [7, 58] do not impose conditions on the subsystem dynamics compared to those with average dwell-time switching [21, 57], but the set of switching signals with dwell-time switching for stability of switched systems is smaller than that with average dwell-time switching. This chapter aims to obtain stability for switched nonlinear systems for larger classes of switching signals and with fewer requirements on the subsystems than the results currently reported in the literature.

3.2 Dwell-Time and Average Dwell-Time Switching

A switching signal has dwell-time τ_d if two consecutive switches are separated by at least τ_d units of time. Dwell-time switching has been used to guarantee asymptotic stability of switched linear systems with asymptotically stable subsystems in the following way: since each subsystem is asymptotically stable, for a subsystem Γ_p,

$p \in \mathcal{P}$, if the subsystem starts at an initial state x_0, there exists a finite time τ_p so that for all $t \geqslant \tau$, the subsystem state $|x(t)| \leqslant \rho|x_0|$ for some constant $\rho < 1$. It follows that if we dwell on the active subsystems for at least τ_d where τ_d is the supremum of all τ_p of the subsystems, then the switched system state will decay to zero. Specifically, if $|x(t)| \leqslant Ce^{-\lambda t}|x(0)|$ for all the subsystems, then for all switching signals with dwell-time $\tau_d \geqslant \ln C/(\lambda - \lambda_0)$ for any $\lambda_0 \in (0, \lambda)$, the state of the switched system satisfies $|x(t)| \leqslant e^{-\lambda_0 t}|x(0)|$ in view of the fact that $Ce^{-(\lambda-\lambda_0)\tau_d} \leqslant 1$ (see [20, Lemma 2]).

Dwell-time switching has the advantage of having simple implementation. In practice, switching with a dwell-time τ_d can be easily attained by not allowing a switch to happen if the elapsed time from the last switch is less than τ_d. The drawback is that sometimes, dwelling on for constant amounts of time may lead to undesirable results such as spikes in signal magnitude. Average dwell-time switching alleviates the need of being in a particular mode for longer time in case one is able to switch to another mode in evidence of new data being available. Shorter switching intervals are compensated by longer switching intervals so that on average, the switching interval is large enough. Specifically, a switching signal has an *average dwell-time* τ_a [21] if

$$N_\sigma(T, t) \leqslant N_0 + \frac{T - t}{\tau_a} \quad \forall T \geqslant t \tag{3.6}$$

for some number $N_0 \geqslant 1$. The number N_0 is called a chatter bound. Note that $N_0 = 1$ implies that there is no more than one switched in interval less than τ_a, and hence, is corresponding to dwell-time switching.

Denote by $\mathcal{S}_{dwell}[\tau_d]$ the class of switching signals with dwell-time τ_d and by $\mathcal{S}_{average}[\tau_a, N_0]$ the class of switching signals with average dwell-time τ_a and chatter bound N_0.

3.3 Input-to-State Properties With Average Dwell-Time Switching

For autonomous switched nonlinear systems with *average dwell-time switching signals*, the switched system state is asymptotically stable if the subsystems are asymptotically stable and certain assumptions on the Lyapunov functions of the subsystems hold [21]. For switched nonlinear systems with inputs and dwell-time switching signals, a switched system is *input-to-state stable* (ISS) if the individual subsystems are ISS [58]; see also [59, Section 5]. If the individual subsystems are *integral input-to-state stable* (iISS), the switched system remains iISS with state-dependent dwell-time switching signals [7].

We extend the nonlinear results in [21] to switched nonlinear systems with inputs, similarly to the results for switched linear systems with inputs (also in [21]). When the individual subsystems of a switched system are ISS and their ISS-Lyapunov functions satisfy a suitable condition (which was also used in [21]), we show that for switching signals with sufficiently large average dwell-time, the switched system has *ISS*, *exponentially weighted-ISS*, and *exponentially weighted-iISS* properties. Unlike the ISS result in [58] which relies on dwell-time switching, our result only requires average dwell-time switching, which is a less stringent requirement. Compared to state-dependent dwell-time switching employed in [7] which requires the knowledge of the state, average dwell-time switching can be achieved using simple hysteresis-based switching logics [21, 60].

Theorem 3.1 *Consider the switched system* (3.2)*. Suppose that there exist continuously differentiable functions* $V_p : \mathbb{R}^n \to [0, \infty)$*,* $p \in \mathcal{P}$*, class* $\mathcal{K}_\infty$ *functions* $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\gamma}$*,* *and numbers* $\lambda_0 > 0, \mu \geqslant 1$ *such that* $\forall\, \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$*, and* $\forall\, p, q \in \mathcal{P}$*, we have*

$$\overline{\alpha}_1(|\xi|) \leqslant V_p(\xi) \leqslant \overline{\alpha}_2(|\xi|), \tag{3.7}$$

$$\frac{\partial V_p}{\partial \xi} f_p(\xi, \eta) \leqslant -\lambda_0 V_p(\xi) + \overline{\gamma}(|\eta|), \tag{3.8}$$

$$V_p(\xi) \leqslant \mu V_q(\xi). \tag{3.9}$$

Let σ *be a switching signal having average dwell-time* τ_a*.*

(i) *If* $\tau_a > \dfrac{\ln \mu}{\lambda_\circ}$*, then the switched system* (3.2) *is ISS.*

(ii) *If* $\tau_a > \dfrac{\ln \mu}{\lambda_0 - \lambda}$ *for some* $\lambda \in (0, \lambda_\circ)$*, then the switched system* (3.2) *is* $\mathrm{e}^{\lambda t}$*-weighted ISS.*

(iii) *If*

$$\tau_a \geqslant \frac{\ln \mu}{\lambda_0 - \lambda} \tag{3.10}$$

for some $\lambda \in (0, \lambda_\circ)$*, then the switched system* (3.2) *is* $\mathrm{e}^{\lambda t}$*-weighted iISS.*

Proof For notational brevity, define $G_a^b(\lambda) := \int_a^b \mathrm{e}^{\lambda s} \overline{\gamma}(|v(s)|) ds$. Let $T > 0$ be an arbitrary time. Denote by $\tau_1, \ldots, \tau_{N_\sigma(T,0)}$ the switching times on the interval $(0, T)$ (by convention, $\tau_0 := 0$, $\tau_{N_\sigma(T,0)+1} := T$). Consider the piecewise continuously differentiable function

$$W(s) := \mathrm{e}^{\lambda_0 s} V_{\sigma(s)}(x(s)). \tag{3.11}$$

On each interval $[\tau_i, \tau_{i+1})$, the switching signal is constant. From (3.11) and (3.8), $\dot{W}(s) \leqslant \mathrm{e}^{\lambda_0 s} \overline{\gamma}(|v(s)|)\ \forall\, s \in [\tau_i, \tau_{i+1})$. Integrating both sides of the foregoing inequality from τ_i to τ_{i+1}^- and using (3.9) we arrive at

$$W(\tau_{i+1}) \leqslant \mu W(\tau_{i+1}^-) \leqslant \mu(W(\tau_i) + G_{\tau_i}^{\tau_{i+1}}(\lambda_0)). \tag{3.12}$$

Iterating (3.12) from $i = 0$ to $i = N_\sigma(T, 0)$, we get

$$W(T^-) \leqslant \mu^{N_\sigma(T,0)} \left(W(0) + \sum_{k=0}^{N_\sigma(T,0)} \mu^{-k} G_{\tau_k}^{\tau_{k+1}}(\lambda_0) \right). \tag{3.13}$$

From (3.10), for every $\delta \in [0, \lambda_0 - \lambda - \ln \mu/\tau_a]$, we have $\tau_a \geqslant \ln \mu/(\lambda_0 - \lambda - \delta)$, and by virtue of (3.6), we get

$$N_\sigma(T, s) \leqslant N_\circ + (\lambda_0 - \lambda - \delta)(T - s)/\ln \mu \quad \forall\, s \in [0, T].$$

Since $N_\sigma(T, 0) - k - 1 \leqslant N_\sigma(T, \tau_{k+1})$, it follows that

$$\mu^{N_\sigma(T,0)-k} \leqslant \mu^{1+N_\circ} \mathrm{e}^{(\lambda_0-\lambda-\delta)(T-\tau_{k+1})}, \tag{3.14}$$

for all $k = 0, \ldots, N_\sigma(T, 0)$. Since $\lambda + \delta < \lambda_0$, we have

$$G_{\tau_k}^{\tau_{k+1}}(\lambda_0) \leqslant \mathrm{e}^{(\lambda_0-\lambda-\delta)\tau_{k+1}} G_{\tau_k}^{\tau_{k+1}}(\lambda + \delta). \tag{3.15}$$

From (3.13), (3.14), and (3.15), we then arrive at

$$\overline{\alpha}_1(|x(T)|) \leqslant c\mathrm{e}^{-(\lambda+\delta)T}(\overline{\alpha}_2(|x_0|) + G_0^T(\lambda + \delta)), \tag{3.16}$$

$$c := \mu^{1+N_\circ}, \tag{3.17}$$

by virtue of (3.11), (3.7) and since x is continuous. Letting $\delta = 0$ in (3.16), we obtain (3.5) with $\alpha_1 := \overline{\alpha}_1$, $\alpha_2 := c\overline{\alpha}_2$, $\gamma := c\overline{\gamma}$. From the definition of $G_a^b(\lambda)$, we have

$$G_0^T(\lambda + \delta) \leqslant \frac{c_1}{c} \mathrm{e}^{(\lambda+\delta-\bar{\lambda})T} \sup_{\tau\in[0,T)} \left\{ \mathrm{e}^{\bar{\lambda}\tau} \overline{\gamma}(|v(\tau)|) \right\} \tag{3.18}$$

for every $\bar{\lambda} \in [0, \lambda+\delta)$ where $c_1 := c/(\lambda+\delta-\bar{\lambda})$. From (3.18) and (3.16), we obtain

$$\begin{aligned} \overline{\alpha}_1(|x(T)|) &\leqslant c\mathrm{e}^{-(\lambda+\delta)T}\overline{\alpha}_2(|x_0|) \\ &\quad + c_1\mathrm{e}^{-\bar{\lambda}T}\sup_{\tau\in[0,T)}\left\{\mathrm{e}^{\bar{\lambda}\tau}\overline{\gamma}(|v(\tau)|)\right\} \quad \forall T \geqslant 0. \end{aligned} \tag{3.19}$$

Picking some $\delta \in (0, \lambda_0 - \lambda - \ln\mu/\tau_a)$, and letting $\bar{\lambda} = \lambda$ in (3.19), we have the property (3.4) with $\alpha_1 := \overline{\alpha}_1$, $\alpha_2 := c\overline{\alpha}_2$, and $\gamma := c_1\overline{\gamma}$. If we let $\bar{\lambda} = 0, \delta = 0$ in (3.19), we have the property (3.3) with $\alpha := \overline{\alpha}_1$, $\beta(r,s) := c\mathrm{e}^{-\lambda s}\overline{\alpha}_2(r)$, and $\gamma := c\overline{\gamma}/\lambda$ by the fact that $\sup_{\tau\in[0,T)}\overline{\gamma}(|v(\tau)|) \leqslant \overline{\gamma}\left(\|v\|_{[0,T)}\right)$. $\square$

An immediate consequence of Theorem 3.1 is that for a switched system satisfying (3.7), (3.8), and (3.9), if the input v is bounded, then the state x is bounded for average dwell-time switching signals with average dwell-time τ_a satisfying (3.10). If the input goes to zero, the average dwell-time switching implies global asymptotic stability of the switched system.

Remark 3.1 *If each individual subsystem is ISS, then for every $p \in \mathcal{P}$ there exist functions $\overline{\alpha}_{1,p}, \overline{\alpha}_{2,p}, \overline{\gamma}_p \in \mathcal{K}_\infty$, numbers $\lambda_{\circ,p} > 0$, and ISS-Lyapunov functions V_p, satisfying $\overline{\alpha}_{1,p}(|\xi|) \leqslant V_p(\xi) \leqslant \overline{\alpha}_{2,p}(|\xi|)$ and $\frac{\partial V_p}{\partial \xi}f_p(\xi) \leqslant -\lambda_{\circ,p}V_p(\xi) + \overline{\gamma}_p(|\eta|)\ \forall\, \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$; see [61, 51]. If the set $\mathcal{P}$ is finite or if the set $\mathcal{P}$ is compact and suitable continuity assumptions on $\left\{\overline{\alpha}_{1,p}, \overline{\alpha}_{2,p}, \overline{\gamma}_p\right\}_{p\in\mathcal{P}}$ and $\left\{\lambda_{\circ,p}\right\}_{p\in\mathcal{P}}$ with respect to p hold, then* (3.7) *and* (3.8) *follow. The set of possible ISS-Lyapunov functions is restricted by* (3.9). *This inequality does not hold, for example, if V_p is quadratic for one value of p and quartic for another. If $\mu = 1$,* (3.9) *implies that $V = V_p, p \in \mathcal{P}$ is a common ISS-Lyapunov function for the family of the systems, and the switched system is ISS for* arbitrary switching *(also called* uniformly input-to-state stable *[54]).*

The result in this section relies on the existence of a constant μ as in (3.9). While this constant is always possible for a family of quadratic ISS-Lyapunov functions (such as in the case of linear subsystems), in general, requiring such constant μ for a family of ISS-Lyapunov functions of nonlinear subsystems seems rather restrictive. We will relax the constant μ to a function μ of $|x|$ and want to find classes of switching signals that still guarantee certain input-to-state properties of the switched systems. In doing so, we introduce a more general way of classifying switching signals using event profiles (compared to average dwell-time) and obtain stability results that are more general than those reported in the literature.

3.4 Event Profiles

We wish to quantify switching signals without taking into account the actual switching intervals and the switching times. The rationale behind this approach stems from the need to characterize switching signals using coarser information while still being useful in assuring certain properties of switched systems. Obviously, one has more information if the exact switching times are utilized, but this leads to a restrictive description of switching signals that may not be very useful in implementation.

In light of that, we propose to quantify the relationship between the number of switches in an interval and the interval length as $N_\sigma(t,s) \leqslant \alpha(t,s)$, $\forall t \geqslant s$, for some function α. For every fixed s, a function $\alpha(t,s)$ can be written as a function of $t-s$ as $\alpha_s(t-s) := \alpha(t,s)$. For each fixed s, the function $\alpha_s(t-s)$ must have the following properties:

P1. $\alpha_s(t-s)$ is bounded for bounded t. This is a direct consequence from the non-Zeno property.

P2. $\alpha_s(t-s)$ must be nondecreasing in t since the number of switches cannot decrease if we allow a longer interval.

P3. Since $N_\sigma(s,s) \geqslant 0$ $\forall \sigma$, we must have $\alpha_s(0) \geqslant 0$ $\forall s$.

Actually, relationship of the form $N_\sigma(t,s) \leqslant \alpha(t,s)$ can be used to describe admissible switching signals in general. From the ongoing reasoning, we see that if

$N_\sigma(t,s) \leqslant \alpha(t,s)$ and $\alpha_s(t-s)$ has the properties P1-P3 for all $s \geqslant 0$, then the switching signal σ is admissible. Conversely, the existence of such a function α is guaranteed for every admissible switching signal as we can always have $\alpha(t,s) = N_\sigma(t,s)$ and this $\alpha(t,s)$ has the property P1 from admissibility of σ and the properties P2-P3 from the properties of $N_\sigma(t,s)$. This leads us to formally define admissible switching signals as follows.

Definition 3.6 *A switching signal $\sigma \in \mathcal{S}$ is said to have an event profile (or profile for short) α_s at $s \geqslant 0$ if*

$$N_\sigma(t,s) \leqslant \alpha_s(t-s) \quad \forall t \geqslant s \tag{3.20}$$

for some nondecreasing function $\alpha_s : [0,\infty) \to [0,\infty)$. A switching signal is admissible if it has some profile α_s at s for every $s \geqslant 0$.

Remark 3.2 *Note that in general, α is a function of both t and s. We can say a profile $\alpha(t,s)$ or a profile $\alpha_s(t-s)$ at s. Also note that for a given switching signal, a profile is not unique (similarly to the case that switching signals with an average dwell-time τ_a also have an average dwell-time $\tau < \tau_a$ from the definition).*

The function $\alpha(t,s)$ characterizes the switching times only without regarding the actual values of the switching signal (hence the name event profile for switching events). The concept and definition do not change if we replace switching signals by sets of discrete times. Specifically, for a discrete set S of positive numbers, define $N_S(T,t_0)$ be the number of elements of S inside the interval $[t_0,T)$. We say S has a profile α_{t_0} at t_0 if $N_S(T,t_0) \leqslant \alpha_{t_0}(T-t_0)$ $\forall T \geqslant t_0$. This formulation is helpful when we have sets of discrete times but not the switching signal (for example, sets of impulse times for impulsive systems).

Straightforward as it seems, the concept of event profiles is useful because it enables us to describe classes of switching signals that share the common property (3.20) without resorting to the exact switching times/switching intervals. As we will see later in Section 3.7, the property described by (3.20) can be used to analyze stability of switched nonlinear systems and leads to broader results than those obtained using only the dwell-time/*average dwell–time concepts (which, indeed, are uniform affine profiles).* Moreover, event profiles also allow us to compare switching signals in a more general way than what is currently available in the literature.

Remark 3.3 *We point out here the conceptual similarity between using profiles to describe admissible switching signals and using class $\mathcal{K}_\infty$ functions to describe Lyapunov stability of nonlinear systems. Here, admissibility is equivalent to the existence of α_s such that $N_\sigma(t,s) \leqslant \alpha_s(t-s)$, while Lyapunov stability is equivalent to the existence of $\gamma \in \mathcal{K}$ such that $|x(t)| \leqslant \gamma(|x(s)|)$. Similarly to the Lyapunov stability case where one can infer about stability without referring to the exact trajectory, we can use event profiles to characterize switching signals without taking into account the exact switching times.*

Below are some classes of event profiles:

1. A profile α is a *uniform profile* if α does not depend on s, i.e., $\alpha(t,s) = \alpha(t-s)$. This means that the switching signal has the same profile at every starting time.

2. A profile α_s is an *affine profile* if $\alpha_s(t-s)$ is affine in $t-s$. For example, $\alpha_s(t-s) = a_s(t-s) + b_s$ for some number a_s, b_s is an affine profile. Uniform affine profiles characterize average dwell-time switching as we shall see in the next subsection.

We define the *chatter bound* $\overline{N}_\alpha$ at s as

$$\overline{N}_\alpha(s) := \alpha_s(0), \tag{3.21}$$

which is the upper bound on the number of switches in an arbitrarily small interval $[s, s+\delta)$.

A uniform profile has a constant chatter bound $N_\circ$ at every time. A uniform profile also has a *generalized dwell-time*, which is defined as[2] $\tau_g := \sup\{t : \alpha(t) < N_\circ + 1\}$. Then $N_\sigma(t+s,s) \leqslant \alpha(t) \leqslant \alpha(\tau_g) < N_\circ + 1\ \forall s, \forall s \leqslant t < s+\tau_g$, which implies that we can have no more than $\lceil N_\circ \rceil$ switches in every interval of length less than τ_g. For a *uniform affine profile* $\alpha(t,s) = a(t-s)+b$, the generalized dwell-time becomes the average dwell-time $\tau_a := 1/a$, and the chatter bound is $N_\circ = b$. If $b = 1$, we have a *dwell-time profile* because we cannot have more than 1 switches on every interval of length less than $1/a$.

Remark 3.4 *As we can see, the statement "to have no more than $N_\circ$ switches on any interval of length less than τ" does* not *imply that the switching signal is an average dwell-time switching signal. For example, the uniform event profile $\alpha(t,s) = 3 + (t-s)^{1/2}$ also implies that there are no more than 3 switches in any interval with length less than 1 but this switching is not an average dwell-time switching. Here, the notion of event profile provides some insight into the average dwell-time concept.*

We can compare two event profiles as follows. We say that profile α is *not slower* than profile β at s (or β_s is not faster than α_s) if

$$\alpha_s(t) \geqslant \beta_s(t) \quad \forall t \geqslant s. \tag{3.22}$$

If the strict inequality holds, we say that α_s is *faster* than β_s. We say that profile α is not slower than profile β if (3.22) holds for all $s \geqslant 0$. When α_s is not slower than β_s, we write $\alpha_s \geqslant \beta_s$ (and similarly, $\alpha \geqslant \beta$). Literally, $\alpha_s \geqslant \beta_s$ means that for all t, in the same duration $[s,t)$, switching signals having profile α are allowed (but not necessarily always) to have more switches than signals having profile β do.

We say that profile α_s *dominates* profile β_s if there is a time $\tau \geqslant s$ such that

$$\alpha_s(t) \geqslant \beta_s(t) \quad \forall t \geqslant \tau. \tag{3.23}$$

[2] The supremum always exists because α is nondecreasing in t.

If this is the case, we write $\alpha_s \gg \beta_s$. It is clear that $\alpha_s \geqslant \beta_s$ implies $\alpha_s \gg \beta_s$, but not the other way round. For example, the profile $\alpha(t,s) = 1 + (t-s)$ dominates the profile $\beta(t,s) = 3 + (t-s)/2$ even though in the interval $[s, s+1)$, profile β allows more switches than profile α.

3.5 Gain Functions

In studying stability of switched systems, we usually work with a collection of Lyapunov functions (or ISS-Lyapunov functions) of the subsystems. The gain when switching from one Lyapunov function to another plays an important role in stability analysis. For a family of quadratic Lyapunov functions, the gain can be bounded by a constant for all the pairs of the subsystems. In general, these gains depend on the state at switching times and may be unbounded as the state norm goes to infinity. We formally define *gain functions* as the class of nondecreasing scalar functions from $[0,\infty)$ to $[1,\infty)$. The nondecreasing requirement makes gain functions useful in our stability analysis, but this requirement is not a limitation as we can always find such a nondecreasing gain function as we will see below.

For a set of continuous functions $V_p : \mathbb{R}^n \to [0,\infty)$, $p \in \mathcal{P}$ such that $\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|)\ \forall x \in \mathbb{R}^n, p \in \mathcal{P}$, a gain function μ is called a *common gain function* for the family of the functions $V_p, p \in \mathcal{P}$ over a set $X \subseteq \mathbb{R}^n$ if

$$V_p(x) \leqslant \mu(V_q(x))V_q(x) \quad \forall x \in X, \forall p, q \in \mathcal{P}. \tag{3.24}$$

If gain functions exist for all pairs of V_p, V_q, then existence of a common gain function for all $p, q \in \mathcal{P}$ as in (3.24) is guaranteed if the set $\mathcal{P}$ is a finite set or the gain functions depend continuously on the index and the set $\mathcal{P}$ is compact. A common gain function may not exist if we require $X = \mathbb{R}^n$ since a finite gain may not exist as $|x| \to 0$ (for example, if V_p is quadratic and V_q is quartic). But if we let $X = \{x : |x| \geqslant \delta\}$ for any $\delta > 0$, then a gain function μ as in (3.24) always exists as one can define $\mu(r) :=$

$\sup_{p,q} \sup_{x:V_q(x)=r} V_p(x)/r$ where $r := \min_{q\in\mathcal{P},|x|=\delta} V_q(x)$ and $\mu(\tau) := \mu(r)\ \forall\tau \in [0, r)$.

The concept of a common gain function as in (3.24) is new and generalizes the concept of a constant gain among the functions V_p [21, 57]. Prior to this work, it was always assumed that, in (3.24),

1. μ is a constant,

2. $X = \mathbb{R}^n$.

These two assumptions are always true for switched linear systems. When the subsystems are linear and asymptotically stable, we have quadratic Lyapunov functions for the subsystems. Then $\alpha_1(|x|) = a_1|x|^2$ and $\alpha_2(|x|) = a_2|x|^2$ for some number a_1 and a_2, and therefore, for switched linear systems, we can always bound the function μ by a_2/a_1 for all $x \in \mathbb{R}^n$. Earlier research in stability of switched nonlinear systems [21, 57] also assumes that μ is a constant and $X = \mathbb{R}^n$ in (3.24) for switched nonlinear systems. Clearly, these assumptions are restrictive for switched nonlinear systems, and in general, we must consider gain functions that may be unbounded as well as the set X other than the entire state space $\mathbb{R}^n$.

Example 3.1 In this example, we show how one can have μ as in (3.24) even though a global and bounded μ does not exist for a given family of Lyapunov functions in (3.28). Consider a switched system with two asymptotically stable subsystems in $\mathbb{R}^2$ whose Lyapunov functions are $V_1(x) = x_1^2 + x_2^2$ and $V_2(x) = x_1^2 + x_2^4$. The available average dwell-time result [21] is not applicable in this case because there is no constant μ such that $V_1(x) \leqslant \mu V_2(x)$ and $V_2(x) \leqslant \mu V_1(x)\ \forall x$. For these V_1, V_2, we can define a common gain function μ as follows:

$$\tilde{\rho}(r) := \max\left\{ \sup_{s\in[0,r]} \frac{s^2 - s + r}{r}, \sup_{s\in[0,r]} \frac{s + (r-s)^{1/2}}{r} \right\}, r > 0,$$

$$\mu(r) := \sup_{z\in[\delta,r)} \tilde{\rho}(z)$$

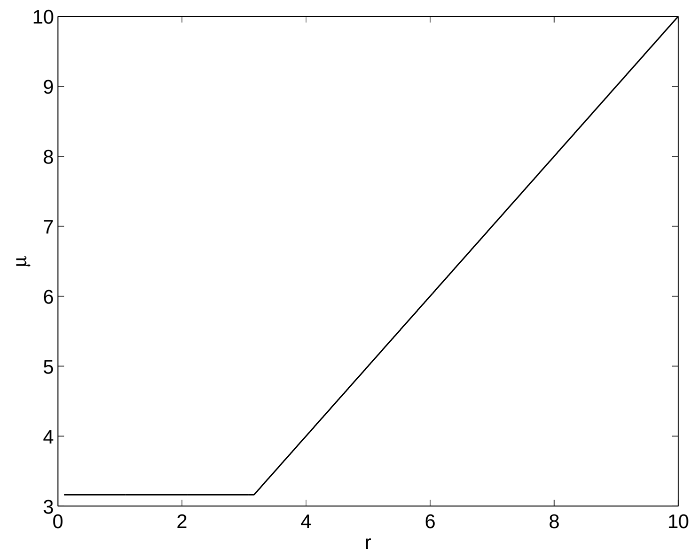

Figure 3.1: Function μ

for some arbitrary $\delta > 0$. We can chose $\alpha_1(r) = \min\{1, \inf_{s\in[0,r^2]}(r^2+s^2-s)/r^2\}r^2, r \geqslant \delta,\ \alpha_2(r) = r^2 + r^4$ so that $\alpha_1(|x|) \leqslant V_1(x), V_2(x) \leqslant \alpha_2(|x|),\ |x| \geqslant \delta$. For $\delta = 0.1$, we plot μ in Fig. 3.1. $\triangleleft$

3.6 Class $\mathcal{KL}_S$, $\mathcal{KL}_{S2}$, and $\mathcal{KL}^\beta$ Functions

In this section, we specify subclasses of $\mathcal{KL}$ functions that will be useful in our stability analysis. We say a function $\beta \in \mathcal{KL}_S$ if $\beta \in \mathcal{KL}$ and further, it has the following properties:

S1. Semigroup property:

$$\beta(\beta(r,t), s) = \beta(r, t+s),\ \forall r, t, s \geqslant 0. \tag{3.25}$$

In fact, we can always set $\beta(z(0), t) := z(t)$ being the solution of the stable scalar system $\dot{z} = -\varphi(z), z(0) = r$, but one can bound $z(t)$ by a function instead of finding the exact solution and obtain a different function β in (3.29). Note that $\dot{z} = -\varphi(z)$ is scalar so in a large number of cases, we can obtain the β function without much difficulties.

S2. Scalable from the initial state:

$$\beta(ar,t) \leqslant a\beta(r,t) \quad \forall r,t \geqslant 0, \forall a \geqslant 1. \tag{3.26}$$

One example of such β is the exponentially decaying function $\beta(r,t) = e^{-\lambda t}r$ (or more generally, those of the form $\beta(r,t) = \phi(t)r$). Another (nonlinear) example is the solution of the stable system $\dot{z} = -z^3$, which gives $\beta(r,t) = (2t + r^{-2})^{-1/2}$ that satisfies (3.26) (note that we only require (3.26) for $a \geqslant 1$, not for all $a \geqslant 0$).

We say a function $\beta \in \mathcal{KL}_{S2}$ if $\beta \in \mathcal{KL}_S$ and further, it has

S3. Subadditive property:

$$\beta(r_1+r_2,t) \leqslant \beta(r_1,t)+\beta(r_2,t) \quad \forall r_1, r_2 \geqslant 0. \tag{3.27}$$

Remark 3.5 *$\beta(r,t) = e^{-\lambda t}r$ satisfying all the three properties S1, S2, and S3 (also, functions of the form $\beta(r,t) = \phi(t)r$). An example of β that is nonlinear in r is $\beta(r,t) = (2t + r^{-2})^{-1/2}$ as the solution of $\dot{z} = -z^3, z(0) = r$. It is straightforward to check that β has the properties S1, S2, and S3.*

We say that a function $\bar{\beta} \in \mathcal{KL}^{\beta}$, $\beta \in \mathcal{KL}$, if $\bar{\beta} \in \mathcal{KL}$ and further,

B1. $\bar{\beta}(r,t)/\beta(r,t)$ is nondecreasing in t for a fixed r.

As we shall see later, $\bar{\beta}$ is an envelope of a switched system state and β is an envelope for the nonswitched subsystems. The quantity $\bar{\beta}(r,t)/\beta(r,t)$ relates to the number of switches in the interval $[0,t)$ so this nondecreasing property ensures that we cannot have less switches in a longer interval, which is consistent with Property P2 in Subsection 3.4.

B2. $\bar{\beta}(r,t)/\beta(r,t)$ is nonincreasing in r for a fixed t.

Property B2 means that for the same envelope $\bar{\beta}$ of the switched system state, a larger initial state allows for less switching than a smaller initial state. This follows from the assumption that a gain function μ in (3.24) is increasing, so a larger state implies a larger switching gain, resulting in fewer switches if the upper bound is still the same.

B3. $\bar{\beta}(r,0)/\beta(r,0) \geqslant 1 \; \forall r$.

Properties B3 and B1 together imply that $\bar{\beta}(r,t)/\beta(r,t) \geqslant 1 \; \forall t, r$, which means that an envelope of the switched system state cannot be smaller than those of the nonswitched subsystems. This fact follows from the assumption that gain functions $\mu \geqslant 1$ (thus, we do not gain stability by switching[3]).

3.7 Asymptotic Stability

3.7.1 Problem formulation

We now use the concept of event profiles to study the stability problem for switched systems, which concerns with finding classes of switching signals that guarantee asymptotic stability of a switched system given that all the subsystems are asymptotically stable. The problem formulation is as follows.

Problem 3.1 *Consider the autonomous switched nonlinear system* (3.1)*. Assume that all the subsystems are globally asymptotically stable around the origin. Find a class $\mathcal{S}$ of switching signals so that for the switched system, the origin is asymptotically stable over $\mathcal{S}$.*

Since the individual subsystems are GAS, then by the converse Lyapunov theorem (see, e.g., [8]), there always exist positive definite continuously differentiable functions

[3] In fact, we can gain stability by switching even among unstable subsystems, but the problem considered in this chapter is for the switched system to inherit the subsystem stability property, and so we assume that switching only has destabilizing effects.

$V_p : \mathbb{R}^n \to [0, \infty)$ such that $\forall x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \tag{3.28a}$$

$$\frac{\partial V_p(x)}{\partial x} f_p(x) \leqslant -\varphi(V_p(x)) \tag{3.28b}$$

for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \varphi$. The existence of common functions $\alpha_1, \alpha_2, \varphi$ is guaranteed if the set $\mathcal{P}$ is discrete or if the function $\alpha_{1,p}, \alpha_{2,p}$ and φ_p depends continuously in p for every p and the set $\mathcal{P}$ is compact. The inequality (3.28b) implies that there exists a $\mathcal{KL}$ function β such that

$$V_p(x(t)) \leqslant \beta(V_p(x(t_0)), t - t_0) \quad \forall t \geqslant t_0 \geqslant 0. \tag{3.29}$$

Obviously, one can let $\beta(r, t)$ be the solution of the nonlinear system $\dot{z} = -\varphi(z), z(0) = r$. However, Equation (3.29) allows flexibility in choosing β, such that β can be an upper bound on the solution of $\dot{z} = -\varphi(z), z(0) = r$.

Remark 3.6 *In fact, one can assert the existence of exponentially decaying Lyapunov functions for GAS systems (see, e.g., [61]) such that* $\forall x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \tag{3.30a}$$

$$\frac{\partial V_p(x)}{\partial x} f_p(x) \leqslant -\lambda_0 V_p(x) \tag{3.30b}$$

for some class $\mathcal{K}_\infty$ *functions* α_1, α_2 *and* $\lambda_0 > 0$.

Remark 3.7 *We can require that* (3.28) *holds true only for* $|x| \geqslant \delta$ *instead of for all* $x \in \mathbb{R}^n$*, thereby relaxing the requirement of GAS of the individual subsystems to be globally practical stability. The results will be similar and for the sake of clarity of the presentation, we stay with the GAS assumption.*

3.7.2 Asymptotic stability of switched systems

Given functions $\beta \in \mathcal{KL}$, $\bar{\beta} \in \mathcal{KL}^{\beta}$, a gain function μ, and a number $r \geqslant \delta$, we have the following procedure to construct a function $\alpha_{\beta,\bar{\beta},\mu,r}$ (which will serve as a event profile in Theorem 3.2 below). First, define the discrete function $\mathbf{G} : \mathbb{Z}^+ \to [0,\infty)$ recursively as follows:

$$\begin{aligned} \mathbf{G}_0 &:= 1, \\ \mathbf{G}_{k+1} &:= \mu(\min\{\mathbf{G}_k r, \bar{\beta}(r,0)\})\mathbf{G}_k, \quad k \geqslant 0. \end{aligned} \tag{3.31}$$

We then "connect the dots" among $\mathbf{G}_k$ to get a continuous function $\mathrm{G} : [0,\infty) \to [1,\infty)$ in the following way: $\mathrm{G}(t) = \mathbf{G}_k$ $\forall t = k, k \in \mathbb{Z}^+$; between every two consecutive points $\mathbf{G}_k$ and $\mathbf{G}_{k+1}$, $\mathrm{G}(s) = f_k(s),\ s \in [k, k+1]$ for any nondecreasing function f_k such that $f_k(k) = \mathbf{G}_k$ and $f_k(k+1) = \mathbf{G}_{k+1}$. Then the function G will be nondecreasing by construction and the fact that $\mathbf{G}_{k+1} \geqslant \mathbf{G}_k$ (since $\mu \geqslant 1$). There are many such functions f_k; for the sake of presentation, and also in order to get a result that align with the average dwell-time case, we chose the function G as follows:

$$\mathrm{G}(s) = (\mathbf{G}_{k+1}/\mathbf{G}_k)^{s-k}\mathbf{G}_k, \quad s \in [k, k+1), k \in \mathbb{Z}. \tag{3.32}$$

Since G is nondecreasing, define $\mathrm{G}^{-1}(a) := \sup\{t \geqslant 0 : \mathrm{G}(t) \leqslant a\}$ and then $\mathrm{G}(\mathrm{G}^{-1}(a)) \leqslant a$ (note that G^{-1} becomes the inverse of G if G is strictly increasing). Finally, define the profile

$$\alpha_{\bar{\beta},\beta,\mu,z_0}(t) = \mathrm{G}^{-1}(\bar{\beta}(z_0,t)/\beta(z_0,t)). \tag{3.33}$$

We have the following result, which says that outside the ball of radius δ where δ as in (3.24), the switched system state is decaying with the envelope $\bar{\beta}(r,t)$ if the switching signal is slower than some predefined event profile.

Theorem 3.2 *Consider the switched system* (3.1)*. Assume that $\beta \in \mathcal{KL}_S$, where β*

is as in (3.29)*. Let μ be a common gain function among Lyapunov functions of the subsystems over the set $\{x : |x| \geqslant \delta\}$ for some $\delta > 0$ as in* (3.28)*. Let τ_k^{o}, $k \geqslant 1$ be the times $x(t)$ leaves the ball B_δ (by convention, $\tau_1^{\mathrm{o}} = 0$ if $|x(0)| > \delta$). Let $\bar{\beta} \in \mathcal{KL}^\beta$ and $\bar{x}_0 > 0$. Let $r_1 := \alpha_2(\max\{\bar{x}_0, \delta\})$ and $r_2 := \alpha_2(\delta)$. Then for all $|x(0)| \leqslant \bar{x}_0$ and for all switching signals that are not faster than $\alpha_{\bar{\beta},\beta,\mu,r_1}$ at time* 0 *and not faster than $\alpha_{\bar{\beta},\beta,\mu,r_2}$ at times τ_k^{o}, $k \geqslant 2$, we have*

$$|x(t)| \leqslant \max\left\{ \max\{\delta, x_0\}, \alpha_1^{-1}(\bar{\beta}(r_1, t - \tau_1^{\mathrm{o}}))\right\} \quad \forall t \in [0, \tau_2^{\mathrm{o}}) \tag{3.34a}$$

$$|x(t)| \leqslant \max\left\{\delta, \alpha_1^{-1}(\bar{\beta}(r_2, t - \tau_k^{\mathrm{o}}))\right\} \quad \forall t \in [\tau_k^{\mathrm{o}}, \tau_{k+1}^{\mathrm{o}}) \; \forall k \geqslant 2. \tag{3.34b}$$

Proof Let τ_k^{i} be the first time $x(t)$ re-enters the ball B_δ after the time τ_k^{o}. We will show that

$$|x(t)| \leqslant \max\{\delta, x_0\}, \quad t \in [0, \tau_1^{\mathrm{o}}) \tag{3.35a}$$

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(r_1, t - \tau_1^{\mathrm{o}})), \quad t \in [\tau_1^{\mathrm{o}}, \tau_1^{\mathrm{i}}) \tag{3.35b}$$

$$|x(t)| \leqslant \delta \quad \forall t \in [\tau_k^{\mathrm{i}}, \tau_{k+1}^{\mathrm{o}}), \quad k \geqslant 1 \tag{3.35c}$$

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(r_2, t - \tau_k^{\mathrm{o}})), \quad \forall t \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}}), \; k \geqslant 2, \tag{3.35d}$$

from which (3.34a) is the combination of (3.35a), (3.35b), and (3.35c); (3.34b) is the combination of (3.35c) and (3.35d).

If $|x(0)| > \delta$, then $\tau_1^{\mathrm{o}} = 0$ and therefore (3.35a) holds. If $|x(0)| \leqslant \delta$, from the definition of τ_1^{o}, we have $|x(t)| \leqslant \delta \; \forall t \in [0, \tau_1^{\mathrm{o}})$, and hence, we have (3.35a). Likewise, the inequality (3.35c) follows from the definitions of $\tau_k^{\mathrm{i}}, \tau_k^{\mathrm{o}}$.

To establish (3.35b) and (3.35d), we utilize the Lyapunov-like function $V_{\sigma(t)}(x(t))$ to show that between the times $\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}}$, we have $V_{\sigma(t)}(x(t)) \leqslant \bar{\beta}(r_2, t - \tau_k^{\mathrm{o}}))$ if the switching signal is not faster than the profile $\alpha_{\bar{\beta},\beta,\mu,r_1}$ at time 0 and is not faster than the profiles $\alpha_{\bar{\beta},\beta,\mu,r_2}$ at times τ_k^{o}, $k \geqslant 2$.

Consider an interval $[\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ for some arbitrary k. For all $s \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$, we have

$|x(s)| \geqslant \delta$ by definition. Let $t \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ be arbitrary. Denote by s_j, $j = 1, \ldots, N_\sigma(t, \tau_k^{\mathrm{o}+})$ the switching times on (t_0, t); by convention, $s_0 = \tau_k^{\mathrm{o}}$ and $s_{N_\sigma(T, t_0^+)+1} = t$. Define $v_k := V_{\sigma(s_k)}(x(s_k))$ and $v_k^- := V_{\sigma(s_k^-)}(x(s_k^-))$. From (3.29), we have

$$v_{j+1}^- \leqslant \beta(v_j, s_{j+1} - s_j) \quad \forall j. \tag{3.36}$$

Since $|x(s)| \geqslant \delta \; \forall s \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$, letting $p = \sigma(s_k)$ and $q = \sigma(s_k^-)$ in (3.24), in view of $x(s_k) = x(s_k^-)$, we obtain

$$v_j \leqslant \mu(v_j^-) v_j^-. \tag{3.37}$$

From (3.36) and (3.37), we get

$$v_{j+1}^- \leqslant \beta(\mu(v_j^-) v_j^-, s_{j+1} - s_j) \leqslant \mu(v_j^-) \beta(v_j^-, s_{j+1} - s_j), \tag{3.38}$$

where the second inequality follows from the property (3.26). Iterating (3.38) for $j = 1$ to $j = N_\sigma(t, t_0^+)$ and using (3.36) with $j = 0$, in view of (3.25), we obtain

$$V_{\sigma(t^-)}(t^-) \leqslant G_{N_\sigma(t, t_0^+)} \beta(v_0, t - \tau_k^{\mathrm{o}}), \tag{3.39}$$

where $G_\ell := \prod_{j=1}^{\ell} \mu(v_j^-)$.

Consider $k \geqslant 2$. From (3.39), letting $t = s_{j+1}$, we get $v_{j+1}^- \leqslant G_j \bar{\beta}(v_0, 0) \leqslant \bar{\beta}(r_2, 0) \; \forall j$ since $v_0 = V_{\sigma(\tau_k^{\mathrm{o}})}(x(\tau_k^{\mathrm{o}})) \leqslant \alpha_2(\delta) = r_2$. We also have $v_{j+1}^- \leqslant G_j v_0 \leqslant G_j r_2$. Therefore,

$$G_{\ell+1} = \mu(v_{\ell+1}^-) G_\ell \leqslant \mu(\min\{G_\ell r_2, \bar{\beta}(r_2, 0)\}) G_\ell. \tag{3.40}$$

From (3.31) and (3.40), it is not difficult to see that $G_\ell \leqslant \mathbf{G}_\ell \; \forall \ell$ by induction starting with $G_0 \geqslant 1 = \mathbf{G}_0$. Since $\gamma_{\tau_k^{\mathrm{o}}} \leqslant \alpha_{\bar{\beta}, \beta, \mu, r_2}$ for all $t \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$, we have $N_\sigma(t, \tau_k^{\mathrm{o}}) \leqslant \gamma_{\tau_k^{\mathrm{o}}}(t - \tau_k^{\mathrm{o}}) \leqslant \alpha_{\bar{\beta}, \beta, \mu, r_2}(t - \tau_k^{\mathrm{o}})$, which yields that $\mathbf{G}_{N_\sigma(t, \tau_k^{\mathrm{o}})} \leqslant \mathbf{G}_{N_\sigma(t, \tau_k^{\mathrm{o}})} = \mathrm{G}(N_\sigma(t, \tau_k^{\mathrm{o}})) \leqslant$

$G(\alpha_{\bar{\beta},\beta,\mu,r_2}(t-\tau_k^{\text{o}})) \leqslant \bar{\beta}(r_2, t-\tau_k^{\text{o}})/\beta(r_2, t-\tau_k^{\text{o}}) \leqslant \bar{\beta}(v_0, t-\tau_k^{\text{o}})/\beta(v_0, t-\tau_k^{\text{o}})$, where the penultimate inequality follows from the definition of $\alpha_{\bar{\beta},\beta,\mu,r_2}$ as in (3.33) and the last equality follows from property B2 and the fact $v_0 \leqslant r_2$. Since $G_{N_\sigma(t,\tau_k^{\text{o}}+)} \leqslant G_{N_\sigma(t,\tau_k^{\text{o}})} \leqslant \mathbf{G}_{N_\sigma(t,\tau_k^{\text{o}})}$, we then have

$$G_{N_\sigma(t,\tau_k^{\text{o}}+)} \leqslant \bar{\beta}(v_0, t-\tau_k^{\text{o}})/\beta(v_0, t-\tau_k^{\text{o}}). \tag{3.41}$$

From (3.39) and (3.41), we get $V_{\sigma(t^-)}(x(t^-)) \leqslant \bar{\beta}(v_0, t-\tau_k^{\text{o}}) \leqslant \bar{\beta}(r_2, t-\tau_k^{\text{o}})\ \forall t \in [\tau_k^{\text{o}}, \tau_k^{\text{i}})$. The foregoing inequality and (3.30a) yield (3.35d) in view of the fact that $x(t)$ is continuous.

For $k=1$, note that $|x(\tau_1^{\text{o}})| \leqslant \max\{x_0, \delta\}$ from (3.35a) and the continuity of $x(t)$. In the previous paragraph, replacing r_2 by r_1 and replacing $v_0 = V_{\sigma(\tau_k^{\text{o}})}(x(\tau_k^{\text{o}})) \leqslant r_2$ by $V_{\sigma(\tau_1^{\text{o}})}(x(\tau_1^{\text{o}})) \leqslant r_1$, we obtain (3.35b). □

Remark 3.8 *The common gain function is only defined for $|x(t)| \geqslant \delta$, in which region we have* asymptotic practical stability[4] *if the event profile of the switching signal is not faster than $\alpha_{\bar{\beta},\beta,\mu,r_2}$. The cases* (3.35a), (3.35b) *deal with whether $|x(0)| > \delta$ or not; if $|x(0)| \leqslant \delta$,* (3.35a) *and* (3.35b) *coincide with* (3.35c) *and* (3.35d)*, respectively.*

A special case is when $\delta = 0$. Then $r_1 = \alpha_2(x_0)$, $\tau_1^{\text{o}} = 0, \tau_1^{\text{i}} = \infty\ \forall x(0)$ by definition. Applying Theorem 3.2 with $\delta = 0$, we obtain the following corollary, which gives us asymptotic stability on $[0, \infty)$. Note that for asymptotic stability, we only require the switching signal to have a slower profile at the starting time 0, not at every time (since $\tau_1^{\text{i}} = \infty$).

Corollary 3.1 *Consider the switched system* (3.1)*. Let the subsystems have the family of Lyapunov functions as in* (3.28) *and the function μ as in* (3.24) *with $\delta = 0$. Let $\beta \in \mathcal{KL}^\beta$ and $x_0 > 0$. Then for all $|x(0)| \leqslant \bar{x}_0$ and for all switching signals with*

[4]Asymptotic practical stability is the property that $|x(0)| > x_0 \Rightarrow \exists T : |x(t)| \leqslant \beta(x(0), t)\ \forall t \in [0, T)$.

profiles not faster than $\alpha_{\bar{\beta},\beta,\mu,\alpha_2(\bar{x}_0)}$ *at time* 0*, we have*

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(\alpha_2(x_0), t)), \quad \forall t \geqslant 0. \tag{3.42}$$

In Theorem 3.2, we require the switching signal to be not faster than certain profiles at the time 0 and the times τ_k^{o}. When neglecting these actual times τ_k^{o} in the event profiles, we require the switching signals to have uniform profiles and obtain the following result.

Corollary 3.2 *Consider the switched system* (3.1)*. Let the subsystems have the family of Lyapunov functions as in* (3.28) *and the function* μ *as in* (3.24)*. Let* $\bar{\beta} \in \mathcal{KL}^{\beta}$ *and* $\bar{x}_0 > 0$*. Let* $r := \alpha_2(\max\{\bar{x}_0, \delta\})$*. Then for all* $|x(0)| \leqslant \bar{x}_0$ *and for all switching signals with uniform profile not faster than the profile* $\alpha_{\bar{\beta},\beta,\mu,r}$*, we have*

$$|x(t)| \leqslant \max\left\{r, \alpha_1^{-1}(\bar{\beta}(r, t - t^{\mathrm{o}}))\right\} \quad \forall t \geqslant 0, \tag{3.43}$$

where $t^{\mathrm{o}} \leqslant t$ *is the latest time less than or equal* t *such that* $|x(t^{\mathrm{o}})| = r$*.*

Remark 3.9 *The result in Corollary 3.2 differs from the current results in the literature in that it treats both the case of practical stability*[5] *as well as a more general asymptotic property (cf. exponential stability in [21]), while not imposing the existence of a constant* μ *in* (3.24) *(cf. practical stabilities with constant* μ *in [62, 63]; exponential stabilities with constant* μ *in [21]).*

Remark 3.10 *For the exponentially decaying Lyapunov functions as in* (3.30)*, one can utilize the available average dwell-time results (e.g., [21]) by letting* $\bar{\beta}(z_0, t) = Ce^{-\lambda t} z_0$*,* $C > 1$*,* $\lambda \in (0, \lambda_0)$ *and replacing the function* μ *by its bound* $\mu(Cz_0)$*. One then obtains the profile* $\tilde{\alpha}(t) = N_\circ + t/\tau$ *where* $N_\circ = \ln C / \ln \mu(Cz_0)$ *and* $\tau = \ln \mu(Cz_0)/(\lambda_0 - \lambda)$*. However, for the same* $\bar{\beta}$ *function and* z_0*, the result in*

[5] Practical stability is the property that $|x(0)| \leqslant \alpha \Rightarrow |x(t)| \leqslant \beta \; \forall t \geqslant 0$.

Theorem 3.2 gives faster event profiles ($\alpha > \tilde{\alpha}$). See Example 3.2 below for numerical illustration.

Remark 3.11 *We consider the class of switching signals with small dwell-time. The dwell-time is small enough so that the dwell-time alone is not sufficient to guarantee asymptotic stability of the impulsive system. However, we show how this small dwell-time allows for a faster switching behavior (such as smaller average dwell-time) compared to the case of arbitrarily small switching intervals. The dwell-time assumption is realistic as it is what to expect in the actual implementation, where it takes a certain amount of time for the computation before the next switch can happen (see [20, p. 1418]).*

Assume that the minimum dwell-time is τ_δ. From (3.36), we have $v_{k+1}^- \leqslant \bar{\beta}(v_0, \tau_\delta) \leqslant \bar{\beta}(v_0, \tau_\delta)$. Then in (3.31), we have

$$\mathbf{G}_{k+1} := \mu(\bar{\beta}(\min\{\mathbf{G}_k v_0, \bar{\beta}(v_0, \tau_\delta)\}, \tau_\delta))\mathbf{G}_k, \quad k \geqslant 0. \tag{3.44}$$

We then use this (3.44) to construct the profile α as in (3.33). The statement of Theorem 3.2 remains exactly the same since the arguments in the proof using event profiles are carried over without any modifications.

3.7.3 Asymptotic stability of impulsive systems

Our results in this section can also be applied to *impulsive systems*, which can be cast into the framework of (3.28) and (3.24). Consider the impulsive system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ x(\tau) &= g(x(\tau^-)) \quad \forall \tau \in \mathcal{T} \end{aligned} \tag{3.45}$$

where f is locally Lipschitz and $g : \mathbb{R} \to \mathbb{R}$ is an *impulse map*, and $\mathcal{T} := \{\tau_1, \tau_2, \dots\}$ is the set of impulse times; $\tau_0 = 0$ by convention. Assuming GAS of the nonimpulsive system, we want to see under what conditions that asymptotic stability of the impulsive system is guaranteed.

Since the nonimpulsive system in (3.45) is GAS, there exists a positive definite function $V(x)$ such that $\forall x$,

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|) \tag{3.46a}$$

$$\frac{\partial V(x)}{\partial x} f(x) \leqslant -\varphi(V(x)) \tag{3.46b}$$

for some functions $\alpha_1, \alpha_2, \varphi \in \mathcal{K}_\infty$. We assume that impulse map g is well-behaved such that the impulse effect does not have an infinite gain for arbitrary small z; in particular, assume that $\sup_{x\in[0,r]} \alpha_2(|g(x)|)/\alpha_1(r) < \infty \ \forall r < \infty$. Define

$$\mu(r) := \max\{1, \sup_{r_1 \leqslant r} \sup_{x:V(x)=r_1} \alpha_2(|g(x)|)/r_1\}. \tag{3.47}$$

The function μ is nondecreasing and $\mu \geqslant 1$ by construction. From the definition of μ, we have

$$\begin{aligned} V(x(\tau_k)) &\leqslant \alpha_2(|g(x(\tau_k^-))|) \\ &\leqslant \mu(V(x(\tau_k^-)))V(x(\tau_k^-)) \quad \forall \tau_k \in \mathcal{T}. \end{aligned} \tag{3.48}$$

We can see that (3.88) and (3.91) are almost identical to (3.28) and (3.24) (except the subscripts). If we let the set $\mathcal{P}$ for the system in (3.28) be a singleton (i.e., it has only one element) and view the switching signal in (3.1) as the set of switching times (the switching values do not play a role in the stability analysis), then we will obtain the relationship (3.88) and (3.91) for the impulsive system. It then follows that we will have stability results as in Theorem 3.2, Corollary 3.1, and Corollary 3.2 for impulsive systems.

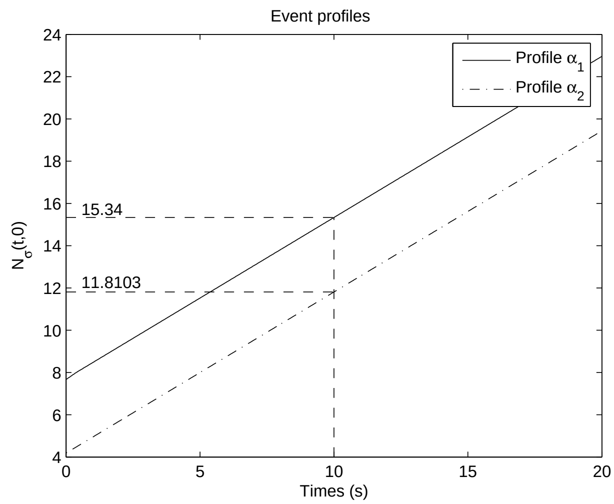

Figure 3.2: The profiles in Example 3.2

3.7.4 Examples

Example 3.2 Consider the switched systems with exponentially decaying Lyapunov functions in (3.30) with the common gain function $\mu(r) = 1 + r$. We have $\beta(t, r) = e^{-\lambda_0 t} r$ and hence, $\eta(a) = a$ and $\mu = \mu$. Consider $\bar{\beta}(t, r) = Ce^{-\lambda t} r$, $C \geqslant 1$. For $z_0 = V_{\sigma(0)}(0) = 0.1$, $C = 3$, $\lambda_0 = 1, \lambda = 0.8$, we plot in Fig. 3.2 the event profile α_1 as in (3.33) and the profile α_2 obtained by using the current available average dwell-time result (see Remark 3.10). As we can see, the event profile α_1 given by Corollary 3.1 is faster than the profile α_2 obtained by the current average dwell-time result. For example, α_1 allows for up to 15 switches in the interval $[0, 10)$ while α_2 only allow at most 11 switches. $\triangleleft$

Example 3.3 Consider the same switched system and the same gain function μ in Example 3.2. Consider the decaying function $\bar{\beta}(z_0, t) = (1/t)z_0$ if $t \geqslant 1$ and $\bar{\beta}(z_0, t) = (2 - t)z_0$ if $t \in [0, 1]$. For $z_0 = 0.1$, $\lambda_0 = 1$, we plot the event profile α_1 as in (3.33) in Fig. 3.3. (Profile α_2 is the affine profile as in Example 3.2). As we can see, profile α_1 is strictly above α_2 and α_1 has a steeper slope for large t than α_2. $\triangleleft$

Example 3.4 For the system in Example 3.2 with the same z_0, β, λ_0 and λ, now

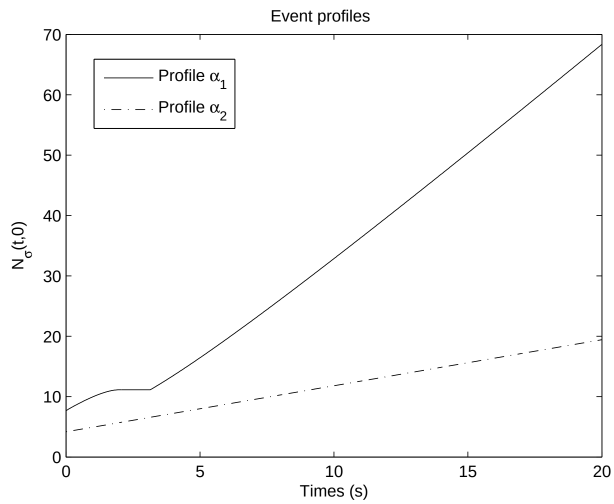

Figure 3.3: The profiles in Example 3.3

assume that the minimum dwell-time is $\tau_\delta = 0.5 < \tau_d = \ln(\mu(Cz_0))/(\lambda_0 - \lambda)$ (which means the dwell-time alone does not guarantee asymptotic stability). We plot the event profile constructed with $\mathbf{G}$ as in (3.44) in Fig. 3.4. The simulation shows that, unlike the case without minimum dwell-time, the slope of α_1 is smaller, which implies that for the same number of switches, α_1 allows a smaller interval compared to α_2 (this can be thought of as a smaller average dwell-time). $\lhd$

3.8 Input-to-State Stability

3.8.1 Problem formulation

Problem 3.2 *Consider the switched system* (3.2)*. Assume that all the subsystems are ISS around the origin. Find a class $\mathcal{S}$ of switching signal such that for the switched system, the origin is locally ISS.*

Since the subsystems are ISS, there always exist (see, e.g., [51]) positive definite

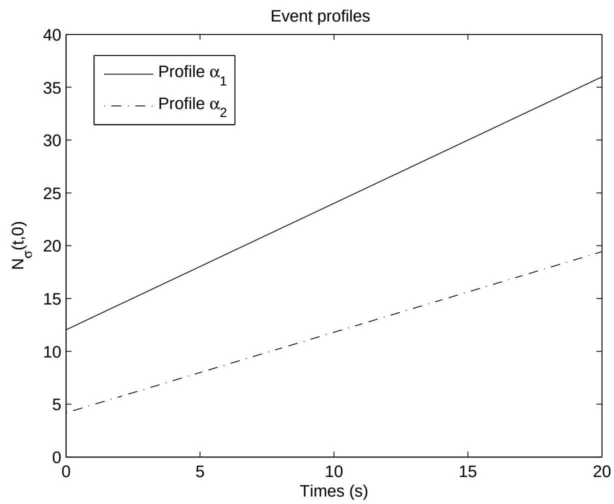

Figure 3.4: The profiles in Example 3.4

continuously differentiable functions $V_p : \mathbb{R}^n \to [0, \infty)$ such that $\forall x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \tag{3.49a}$$

$$\frac{\partial V_p(x)}{\partial x} f_p(x) \leqslant -\varphi(V_p(x)) + \gamma(|u|) \tag{3.49b}$$

for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \varphi$, and γ. Like before, the existence of common functions $\alpha_1, \alpha_2, \varphi$, and γ is guaranteed if the set $\mathcal{P}$ is discrete or if the family of functions $\alpha_{1,p}, \alpha_{2,p}, \varphi_p$, and γ_p for the subsystems depend continuously in p and the set $\mathcal{P}$ is compact. The property (3.49b) implies that there exists $\beta \in \mathcal{KL}$ such that

$$V_p(x) \leqslant \beta(V_p(t_0), t - t_0) + \chi(\|u\|_{[t_0,t)}) \tag{3.50}$$

for some $\chi \in \mathcal{K}$, where $\|u\|_{[0,t)} := \sup_{s\in[0,t)} |u|$. Going from (3.49b) to (3.50) is a standard technique in the ISS literature (see, e.g., [64] and also the example in Remark 3.5).

Remark 3.12 *In fact, one can assert the existence of ISS-Lyapunov functions with the exponentially decaying property for ISS systems (see, e.g., [61]) such that* $\forall x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \tag{3.51a}$$

$$\frac{\partial V_p(x)}{\partial x} f_p(x) \leqslant -\lambda_0 V_p(x) + \gamma(|u|) \tag{3.51b}$$

for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \gamma$ and $\lambda_0 > 0$.

Example 3.5 We illustrate how to arrive at (3.50) from (3.49b). Suppose that $\dot{V} \leqslant -2V^3 + |u|^2$ in (3.49b). We write $\dot{V} \leqslant -V^3 + |u|^2 - V^3$, and so $\dot{V} \leqslant -V^3$ if $|u|^2 < V^3$. Thus, it is either $V(t) \leqslant (\|u\|_{[0,t)})^{2/3}$ or $V(t) \leqslant \beta(V(0), t)$ if $(\|u\|_{[0,t)})^2 < V^3$, where $\beta(r,t) = (2t + r^{-2})^{-1/2}$ is the solution of $\dot{z} = -z^3$. Therefore, $V(t) \leqslant \beta(V(0), t) + (\|u\|_{[0,t)})^{2/3}$. It is straightforward to check that β has Properties S1, S2, and S3. $\lhd$

3.8.2 Input-to-state property of scalar jumped variables: The general case

Before stating our stability result for switched system, we will present a general technical lemma for a scalar variable (with properties similar to those of $V_{\sigma(t)}(x(t))$) and a set of event times (recall the definition of set of event times in Section 3.4). Let $v : [t_0, T) \to [0, \infty)$ be a piecewise right-continuous scalar function. For a set σ of event times, for any number $t \geqslant 0$, denote $t_\sigma := \max\{\tau \in \sigma : \tau \leqslant t\}$ the latest switching time before t; by convention, $\max \emptyset := 0^-$. Suppose that v has the following properties for all $t \in [t_0, T)$:

$$v(t) \leqslant \begin{cases} \beta(v_0, t - t_0) + \chi(u(t)) & \text{if } t_\sigma \leqslant t_0 \\ \mu(v(t_\sigma^-))\beta(v(t_\sigma^-), t - t_\sigma) + \chi(u(t)) & \text{else} \end{cases} \tag{3.52}$$

for some set σ of event times, a number $v_0 > 0$ (not necessarily equals $v(t_0)$), and some functions $\beta \in \mathcal{KL}$ with properties (3.25) and (3.26), $\chi \in \mathcal{K}_\infty$, $u : [t_0, T) \to [0, \infty)$, and μ is a gain function. The first case of (3.52) describes the situation from the time

t_0 up to the first event time after t_0 (if $t_\sigma \leqslant t_0$, then there is no event time in $(t_0, t]$). The number v_0 may be different from $v(t_0)$ to allow flexibility as we have the same β function in both the cases of (3.52). The second case of (3.52) describes the situation from the first event time onwards. It will be clear later that $V_{\sigma(t)}(x(t))$ satisfies (3.52) with the switching signal σ, in view of (3.50) and (3.24) (for stability results, the terms "switching signal" and "sets of event times" are completely interchangeable). The lemma is of independent interest as we encounter variables with property (3.52) in situations other than the switched Lyapunov-like function $V_{\sigma(t)}(x(t))$, for example, in impulsive systems or in switching supervisory control of time-varying plants.

Suppose that $|u(t)| \leqslant \bar{u}\ \forall t \geqslant 0$ and we want

$$v(t) \leqslant \bar{\beta}(v_0, t) + D \quad \forall t \geqslant 0 \tag{3.53}$$

for some given function $\bar{\beta} \in \mathcal{KL}^\beta$ and some number $D > 0$ that may depend on $\bar{\beta}$ (cf. the condition $\bar{\beta} \in \mathcal{KL}^\beta$ for the case without inputs in the previous section). Define

$$(\bar{\beta}/\beta)_r^{-1}(a) := \inf\left\{t : (\bar{\beta}/\beta)(r, t) \geqslant a\right\}, \tag{3.54}$$

where $(\bar{\beta}/\beta)(r,t) := \bar{\beta}(r,t)/\beta(r,t)$. Recall that $\bar{\beta} \in \mathcal{KL}^\beta$ so $(\bar{\beta}/\beta)_r^{-1}$ is well-defined. For reasons that will be clear later in the analysis, we further assume that the function $\bar{\beta}$ satisfies

$$\nu_r(\bar{u}) := \chi(\bar{u}) + \sum_{i=1}^{\infty} \bar{\mu}^i \beta(\chi(\bar{u}), (\bar{\beta}/\beta)_r^{-1}(\bar{\mu}^i)) \leqslant D, \tag{3.55}$$

where χ as in (3.50) and $\bar{\mu} := \mu(\bar{\beta}(r, 0) + D)$ for some positive numbers r and D. Note that $\bar{\beta}$ is given and D is not given but inferred from (3.55), if such one exists. Note that $\bar{\mu}$ is a function of D and so (3.55) is actually an equality having D in both sides. In practice, for given β and $\bar{\beta}$, numerical D satisfying (3.55) can be found by comparing the graphs of the left-hand side and the right-hand side of (3.55) (see

Example 3.6). Intuitively, (3.55) gives a condition among β, $\bar{\beta}$, and χ so that the decaying of β and $\bar{\beta}$ can "accommodate" the increase in the state by the input via χ. Note that $\nu_{v_0} \in \mathcal{K}_\infty$ since $\beta \in \mathcal{KL}$ and $\chi \in \mathcal{K}_\infty$, and $\nu_{v_0}(\bar{u})$ is nondecreasing in v_0 for a fixed $\bar{u}$, i.e.,

$$r_1 \leqslant r_2 \Rightarrow \nu_{r_1} \leqslant \nu_{r_2}, \tag{3.56}$$

which follows from the fact that $\bar{\beta} \in \mathcal{KL}^\beta$. For the functions $\beta, \bar{\beta}, \mu$, and numbers $r, D > 0$, define

$$\bar{\alpha}_{\bar{\beta},\beta,\mu,r,D}(t,s) := \frac{\ln(\bar{\beta}/\beta)(r, t-s)}{\ln \mu(\bar{\beta}(r,0) + D)}. \tag{3.57}$$

Lemma 3.1 *Let $v : [t_0, T) \to [0, \infty)$ be a pieceswise right-continuous scalar function satisfying* (3.52) *and assume that $\beta \in \mathcal{KL}_{S2}$. Let $\bar{\beta} \in \mathcal{KL}^\beta$. Suppose that $u(t) \leqslant \bar{u}$ $\forall t \in [0, T)$ and suppose that there exist positive numbers D and $\bar{v}_0$ such that $\nu_{\bar{v}_0}(\bar{u}) \leqslant D$ where ν is as in* (3.55)*. Then for all $v_0 \leqslant \bar{v}_0$ and for all sets of event times σ that are not faster than the uniform profile $\bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{v}_0,D}$ where $\bar{\alpha}$ is as in* (3.57)*, we have*

$$v(t^-) \leqslant \bar{\beta}(v_0, t) + \eta(\|u\|_{t_0,t}) \quad \forall t \in [t_0, T) \tag{3.58}$$

for some $\eta \in \mathcal{K}_\infty$.

Proof Let $t \in [t_0, T)$ be an arbitrary time. Denote by τ_k, $k = 1, \ldots, N_\sigma(t, t_0^+)$ the event times of σ inside the interval (t_0, t) in the increasing order. By convention, $\tau_0 = t_0$ and $\tau_{N_\sigma(t,0^+)+1} = t$. From the second case of (3.52), we get

$$v(\tau_{i+1}^-) \leqslant \mu(v(\tau_i^-))\beta(v(\tau_i^-), \tau_{i+1} - \tau_i) + \chi(u(\tau_{i+1})) \tag{3.59}$$

in view of $\tau_\sigma = \tau_i$. For any number $\ell \leqslant N_\sigma(t, 0^+) + 1$, iterating (3.59) for $i = 1$ to ℓ and then using the first case of (3.52), in view of (3.25), (3.26), (3.27), and the fact that $u(\tau) \leqslant \|u\|_{t_0,t}\ \forall \tau \in [t_0, t]$, we then have $\forall \tau \in [\tau_\ell, \tau_{\ell+1})$,

$$v(\tau) \leqslant G_{\ell,1}\beta(v_0, \tau) + \chi_\ell(\|u\|_{t_0,\tau}), \tag{3.60}$$

where

$$G_{\ell,j} := \prod_{i=j}^{\ell} \mu(v(\tau_i^-)),\ 1 \leqslant j \leqslant \ell, \tag{3.61}$$

$$\chi_\ell(\|u\|_{t_0,t}) := \chi(\|u\|_{t_0,t}) + \sum_{j=0}^{\ell-1} G_{\ell,j+1}\beta(\chi(\|u\|_{t_0,t}), t - \tau_{j+1}). \tag{3.62}$$

By convention $G_{\ell,j} = 1\ \forall \ell < j$.

We will show by induction that $v(\tau_j^-) \leqslant \bar{\beta}(v_0, 0) + D\ \forall j,\ 1 \leqslant j \leqslant N_\sigma(t, t_0^+) + 1$.

1. In (3.60), letting $\ell = 0$, we have $v(\tau_1^-) \leqslant \beta(v_0, \tau_1 - t_0) + \chi(\|u\|_{t_0,\tau_1}) \leqslant \bar{\beta}(v_0, 0) + D$ since $\beta(v_0, \tau_1 - t_0) \leqslant \bar{\beta}(v_0, \tau_1 - t_0) \leqslant \bar{\beta}(v_0, 0)$ and since $\chi(\|u\|_{t_0,\tau_1}) \leqslant D$ in view of the assumption $u \leqslant \bar{u}$ and (3.55).

2. Suppose that we have $v(\tau_j^-) \leqslant \bar{\beta}(v_0, 0) + D\ \forall j, 1 \leqslant j \leqslant \ell$ for some number $\ell \leqslant N_\sigma(t, t_0)$.

3. Let $\bar{\mu} := \mu(\bar{\beta}(\bar{v}_0, 0) + D)$. Since the set σ of event times is not faster than $\bar{\alpha}_{\beta,\bar{\beta},\mu,\bar{v}_0,D}$, we have $N_\sigma(\tau, s) \leqslant \bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{v}_0,D}(\tau, s)\ \forall \tau \geqslant s$, which, in view of (3.57), implies that

$$\bar{\mu}^{N_\sigma(\tau,s)} \leqslant (\bar{\beta}/\beta)(r, \tau - s), \quad \forall \tau \geqslant s. \tag{3.63}$$

Note that $N_\sigma(\tau, \tau_{j+1}) = \ell - j\ \forall \tau \in [\tau_\ell, \tau_{\ell+1}),\ 0 \leqslant j \leqslant \ell$. We then have $\forall \tau \in [\tau_\ell, \tau_{\ell+1})$,

$$G_{\ell,j+1} \leqslant \bar{\mu}^{\ell-j} = \bar{\mu}^{N_\sigma(\tau,\tau_{j+1})} \leqslant (\bar{\beta}/\beta)(\bar{v}_0, \tau - \tau_{j+1}) \leqslant (\bar{\beta}/\beta)(v_0, \tau - \tau_{j+1}), \tag{3.64}$$

where the first inequality follows from (3.61) and the fact that $\mu(v(\tau_i^-)) \leqslant \bar{\mu}\ \forall i =$

$1, \ldots, \ell$ (from the induction hypothesis and the assumption $v_0 \leqslant \bar{v}_0$); the second inequality follows from (3.63); the last inequality follows from Property B2 of $\bar{\beta}$ and the fact $v_0 \leqslant \bar{v}_0$. From (3.64), we get $G_{\ell,1} \leqslant (\bar{\beta}/\beta)(v_0, \tau - \tau_1) \leqslant (\bar{\beta}/\beta)(v_0, \tau)$ $\forall \tau \in [\tau_\ell, \tau_{\ell+1})$ in view of Property B1 of $\bar{\beta}$. We then have $\forall \tau \in [\tau_\ell, \tau_{\ell+1})$,

$$G_{\ell,1}\beta(v_0, \tau) \leqslant \bar{\beta}(v_0, \tau). \tag{3.65}$$

From (3.63), in view of (3.54), we get

$$\tau - s \geqslant (\bar{\beta}/\beta)_r^{-1}(\bar{\mu}^{N_\sigma(\tau,s)}) \quad \forall \tau \geqslant s. \tag{3.66}$$

Substituting (3.66) into (3.62) by letting $\tau - s$ in (3.66) play the role of $\tau - \tau_{j+1}$ in (3.62), we obtain $\forall \tau \in [\tau_\ell, \tau_{\ell+1})$,

$$\begin{aligned} \chi_\ell(\|u\|_{t_0,\tau}) &\leqslant \chi(\|u\|_{t_0,\tau}) + \sum_{j=1}^{\ell-1} \bar{\mu}^{\ell-j}\beta(\chi(\|u\|_{t_0,\tau}), (\bar{\beta}/\beta)_r^{-1}(\bar{\mu}^{\ell-j})) \\ &\leqslant \chi(\|u\|_{t_0,\tau}) + \sum_{j=1}^{\infty} \bar{\mu}^{j}\beta(\chi(\|u\|_{t_0,\tau}), (\bar{\beta}/\beta)_r^{-1}(\bar{\mu}^{j})) = \nu_{\bar{v}_0}(\|u\|_{t_0,\tau}). \end{aligned} \tag{3.67}$$

From (3.60), (3.65), and (3.67), we arrive at

$$v(\tau) \leqslant \bar{\beta}(v_0, \tau) + \nu_{\bar{v}_0}(\|u\|_{t_0,\tau}) \quad \forall \tau \in [\tau_\ell, \tau_{\ell+1}). \tag{3.68}$$

From (3.68) and the fact $\|u\|_{0,\tau} \leqslant \bar{u}$, in view of the assumption (3.55), we get $v(\tau_{\ell+1}^-) \leqslant \bar{\beta}(v_0, 0) + D$. By induction, we conclude that $v(\tau_j^-) \leqslant \bar{\beta}(v_0, 0) + D$ $\forall j$, $1 \leqslant j \leqslant N_\sigma(t, t_0^+)$.

In (3.68), letting $\ell = N_\sigma(t, t_0^+)$ and $\tau = \tau_{\ell+1}^- = t^-$, we get (3.78) where η is some class $\mathcal{K}_\infty$ function (one can take $\eta := \nu_{\bar{v}_0}$). □

3.8.3 Input-to-state property of scalar jumped variables: The exponentially decaying case

We provide another ISS result for scalar variables having property (3.52) when the β function in (3.52) has an exponentially decaying form.

Consider a variable W satisfying an inequality of the form

$$W(t) \leqslant \rho(W(t_s^-))W(t_s^-)e^{-\lambda(t-t_s)} + \alpha_3 \tag{3.69}$$

for all $t_s > 0$ and $W(t) \leqslant W_0 e^{-\lambda t} + \alpha_3$ $\forall t \in [0, t_1)$, where ρ is a nondecreasing function and $W_0 = \rho(W(0))W(0)$.

Define the function

$$\begin{aligned} h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) := {} & \rho^{\overline{N}}(M) W_0 e^{-(\lambda - \ln\rho(M)/\bar{\tau})\delta_d} \\ & + \alpha_3 + \alpha_3 \frac{\rho^{\overline{N}-1}(M)}{1 - e^{-(\lambda - \ln\rho(M)/\bar{\tau})\delta_d}} - M. \end{aligned} \tag{3.70}$$

This h_ρ function stems from stability analysis of W later. Stability proof of W will require $h_\rho \leqslant 0$. We will next quantify conditions on $\overline{N}$, δ_d, and τ_a to guarantee $h_\rho \leqslant 0$.

- Fix a δ_d. Define the set $\mathcal{A}_{\rho,\delta_d}$ parameterized by W_0 as

$$\begin{aligned} \mathcal{A}_{\rho,\delta_d}(W_0) := \{ & (\overline{N}, \bar{\tau}) : \overline{N} \geqslant 1, \bar{\tau} \geqslant \delta_d > 0, \text{ and } \exists M : \\ & \tau > \ln\rho(M)/\lambda \text{ and } h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) \leqslant 0\}. \end{aligned} \tag{3.71}$$

  Note that for any ρ and W_0, we can always have $h_{\rho,\delta_d}(M, \overline{N}, \bar{\tau}, W_0) < 0$ with $\overline{N} = 1$ and $M > 2\alpha_3$ if δ_d is large enough.

  Since the function h_ρ is increasing in $\overline{N}$ and decreasing in $\bar{\tau}$, in view of (3.71), there exists a function $\bar{\tau} = \phi_{\rho,\delta_d,W_0}(\overline{N})$ that is the lower boundary of $\mathcal{A}_{\rho,\delta_d}(W_0)$ such that $\mathcal{A}_{\rho,\delta_d}(W_0) := \{(n, t) : 1 \leqslant n \leqslant N_{max}, t > \phi_{\rho,\delta_d,W_0}(n)\}$ for some N_{max} (N_{max} can be ∞). We will call ϕ_{ρ,δ_d,W_0} an *average dwell-time vs. chatter bound* *curve*. It is not easy to characterize the function ϕ_{ρ,δ_d,W_0} analytically, but the function can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, δ_d, and W_0 (up to approximation errors; the simplest way is using brute force: fix a $\overline{N} \geqslant 1$; starting from $\bar{\tau} \geqslant \tau_d$ and in increment of δ, check for existence of M in some range $(0, M_{max})$. The first $\bar{\tau}$ that gives existence of M is an approximated value of $\phi_{\rho,\delta_d,W_0}(N)$. Repeat for new $\overline{N}$ with small increment). For example, for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 0.01, W(0) = 0.01, \kappa = 1.5, \lambda = 0.1, \delta_d = 0.5$, we plot an approximation of ϕ_{ρ,δ_d,W_0} in Fig. 3.5.

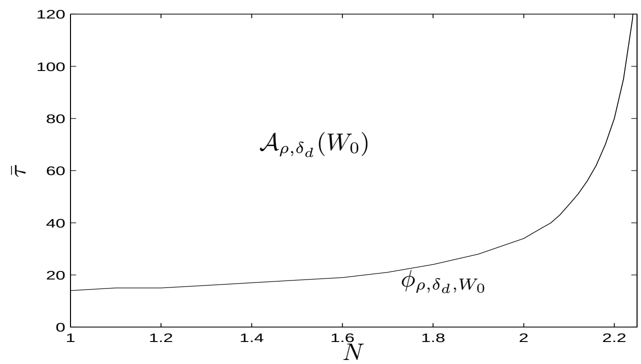

Figure 3.5: Average dwell-time vs. chatter bound curve

- Fix a $\overline{N}$. Define the set $\mathcal{B}_{\rho,\overline{N}}$ parameterized by W_0 as

$$\begin{aligned}\mathcal{B}_{\rho,\overline{N}}(W_0) := \{(\bar{\tau}, \delta_d) : \bar{\tau} \geqslant \delta_d > 0, \text{ and } \exists M : \\ \tau > \ln \rho(M)/\lambda \text{ and } h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) \leqslant 0\}. \qquad (3.72)\end{aligned}$$

Since the function h is decreasing in δ_d and also decreasing in $\bar{\tau}$, in view of (3.72), there exists a function $\bar{\tau} = \psi_{\rho,\overline{N},W_0}(\delta_d)$ that is the lower boundary of $\mathcal{B}_{\rho,\overline{N}}(W_0)$ such that $\mathcal{B}_{\rho,\overline{N}}(W_0) := \{(t,d) : \delta_d^{min} \leqslant d \leqslant \delta_d^{max}, t > \psi_{\rho,\overline{N},W_0}(d)\}$ for some $\delta_d^{max} > \delta_d^{min}$. We call $\psi_{\rho,\overline{N},W_0}$ an *average dwell-time vs. dwell-time curve*. The function $\psi_{\rho,\overline{N},W_0}$ is not easy to characterize analytically but can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, $\overline{N}$, and W_0. For example, for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 0.01, W(0) = 0.01, \kappa = 1.5, \lambda = 0.1, \overline{N} = 2$, we plot an approximation of $\psi_{\rho,\overline{N},W_0}$ in Fig. 3.6.

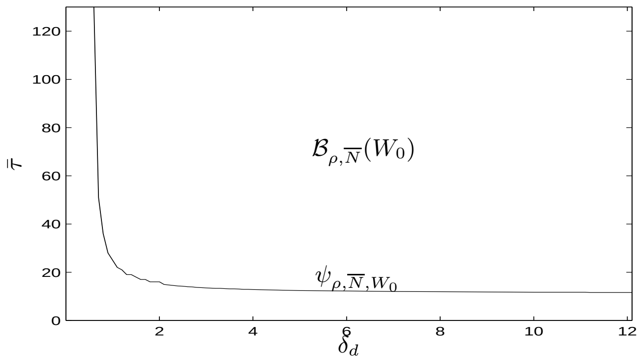

Figure 3.6: Average dwell-time vs. dwell-time curve

Remark 3.13 *When ρ is bounded, say $\rho(M) \leqslant \bar{\rho}\ \forall M$, for every $W_0 \geqslant 0$, $\delta_d > 0$, $\overline{N} \geqslant 1$, and $\tau \geqslant \ln \rho/\lambda$, we can always have $h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) < 0$ for a large enough M. Therefore, $\mathcal{A}_{\rho,\delta_d}(W_0) = \{(\overline{N}, \bar{\tau}) : \overline{N} \geqslant 1, \bar{\tau} > \ln \bar{\rho}/\lambda\}$, which does not depend on W_0 and δ_d, and also, $\mathcal{B}_{\rho,\overline{N}}(W_0) = \{(t,d) : d > 0, t > \ln \bar{\rho}/\lambda, t \geqslant d\}$, which does not depend on $\overline{N}$ and W_0. Both the sets $\mathcal{A}$ and $\mathcal{B}$ can be characterized by a single number $\ln \bar{\rho}/\lambda$, which is exactly the lower bound on average dwell-time for stability of W (as reported in [21, 57]; see also [65]). In general, when ρ is an unbounded function, the set $\mathcal{A}$ is characterized by an average dwell-time vs. chatter bound curve (the curve is a horizonal line when ρ is constant) and the set $\mathcal{B}$ is characterized by an average dwell-time vs. dwell-time curve (the curve is a horizonal line when ρ is constant).*

Now, for a function ρ, numbers δ_d, W_0, define a function

$$f_{\rho,W_0}(\delta_d, \overline{N}) := \max\{\phi_{\rho,\delta_d,W_0}(N), \psi_{\rho,\overline{N},W_0}(\delta_d)\}. \tag{3.73}$$

Construct a uniform event profile $\alpha_{\rho,W_0,\delta_d,\overline{N}}$ parameterized by δ_d and $\overline{N}$ as follows:

$$\alpha_{\rho,W_0,\delta_d,\overline{N}}(t,s) = \begin{cases} 1 + \dfrac{t-s}{\tau_d} & \text{if } s \leqslant t < s + \tau_d, \\ \overline{N} + \dfrac{t-s}{f_{\rho,W_{0},(\delta_d,\overline{N})}} & \text{if } t \geqslant s + \tau_d. \end{cases} \tag{3.74}$$

The first case of (3.74) implies that two consecutive switches must be separated by at least τ_d. The second case of (3.74) implies that the switching signal has a chatter bound $\overline{N}$ and an average dwell-time $f_{\rho,W_0}(\delta_d,\overline{N})$.

Stability

Let $\rho(M) := \alpha_1 M^\kappa + \alpha_2$ where $\alpha_1, \alpha_2, \kappa$ are the constants as in Lemma 4.5. Suppose that the initial state is bounded by X_0: $|x(0)|^2 \leqslant X_0$. Let $W_0 := \rho(\frac{c_1 a_0}{\hat{\lambda}-\lambda} X_0)\frac{c_1 a_0}{\hat{\lambda}-\lambda} X_0$ where c_1, a_0 are as in (4.61).

Lemma 3.2 *The variable W in* (3.69) *satisfies the inequality*

$$W(t) \leqslant \gamma_1(W(0))e^{-\bar{\lambda}t} + \gamma_2(\alpha_3) \quad \forall t \geqslant 0 \tag{3.75}$$

for some $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and some $\bar{\lambda} > 0$ for all $|x(0)|^2 \leqslant X_0$ and for every switching signal s not faster than $\alpha_{\rho,W_0,\delta_d,\overline{N}}$ where f_{ρ,W_0} is as in (3.73) *for some $\delta_d \geqslant 0, \overline{N} \geqslant 1$.*

Proof Denote by $t_1, t_2, \ldots$ the switching times of s. Since $\bar{\tau} > \phi_{\rho,\delta_d,W_0}(\overline{N})$, from the definition of ϕ_{ρ,δ_d,W_0}, there exists M such that $\bar{\tau} > \ln\mu/\lambda$ and

$$\bar{\mu}^{\overline{N}} W_0 e^{-(\lambda - \ln\bar{\mu}/\bar{\tau})\delta_d} + \alpha_3 + \alpha_3 \bar{\mu}^{\overline{N}-1}\frac{1}{1-e^{-(\lambda-\ln\bar{\mu}/\bar{\tau})\delta_d}} \leqslant M \tag{3.76}$$

where $\bar{\mu} := \alpha_1 M^\kappa + \alpha_2$.

Let T be the number such that $T := \sup\{t \geqslant 0 : W(t) \leqslant M\}$. Consider an arbitrary interval $[0,t) \subseteq [0,T)$ and let N be the number of switches in $(0,t)$. Then

$$W(t_{i+1}^-) \leqslant \bar{\mu} W(t_i^-)e^{-\lambda(t_{i+1}-t_i)} + \alpha_3,\ i = 0, \ldots, N$$

where by convention $t_0 = 0$ and $t_{N+1} = t$. Iterating the foregoing inequality for $i = 0$

to $i = N$, we have

$$W(t) \leqslant \bar{\mu}^N \rho(W(t_0))W(t_0)e^{-\lambda(t-t_0)} + \alpha_3 + \alpha_3 \sum_{i=0}^{N-1} \bar{\mu}^{N-i-1} e^{-\lambda(t-t_{i+1})}.$$

Since the switching signal s has the property $N_s(b,a) \leqslant \overline{N} + (b-a)/\bar{\tau}$ for all $b \geqslant a,\ a,b \in [0,T)$ and since there are $N-i$ switches in the interval $[t_{i+1}, t)$, we have $N - i \leqslant \overline{N} + (t-t_{i+1})/\bar{\tau}$ and so, $\bar{\mu}^{N-i-1} \leqslant \bar{\mu}^{\overline{N}-1}\bar{\mu}^{(t-t_{i+1})/\bar{\tau}}, i = 0, \ldots, N$. Then

$$W(t) \leqslant \bar{\mu}^{\overline{N}} \rho(W(t_0))W(t_0)e^{-(\lambda - \ln \bar{\mu}/\bar{\tau})t} + \alpha_3 + \alpha_3 \bar{\mu}^{\overline{N}-1} \sum_{i=0}^{N} e^{-(\lambda - \ln\bar{\mu}/\bar{\tau})(t-t_{i+1})}.$$

Let $\bar{\lambda} := \lambda - \ln \bar{\mu}/\bar{\tau}$. Since $t_{i+1} - t_i > \delta_d$, $t - t_{i+1} > (N-i-1)\delta_d + (t - t_N)$, $i = 0, \ldots, N-1$, and hence, $\sum_{i=0}^{N-1} e^{-\bar{\lambda}(t-t_{i+1})} \leqslant \sum_{i=0}^{N-1} e^{-\bar{\lambda}\delta_d(N-i-1)} \leqslant 1/(1-e^{-\bar{\lambda}\delta_d})$ since $\bar{\tau} > \ln \bar{\mu}/\lambda$. Then

$$W(t) \leqslant \bar{\mu}^{\overline{N}} \rho(W(t_0))W(t_0)e^{-\bar{\lambda}t} + \alpha_3 + \alpha_3 \bar{\mu}^{\overline{N}-1} \frac{1}{1 - e^{-\bar{\lambda}\delta_d}} \tag{3.77}$$

for all $t \in [0,T)$. Since $|x(0)|^2 \leqslant X_0$, we have $W(t_0) \leqslant \frac{c_1 a_0}{\hat{\lambda} - \lambda} X_0$ and $\rho(W(t_0))W(t_0) \leqslant W_0$ from the definition of W_0. Since $e^{-\bar{\lambda}t} \leqslant e^{-\bar{\lambda}\delta_d}\ \forall t \geqslant t_1$, the inequalities (3.76) and (3.77) imply that indeed $W(t) \leqslant M\ \forall t \in [t_1^-, T)$. Clearly, $W(t) \leqslant \rho(W(0))W(0) + c_3 \leqslant W_0 + c_3 \leqslant M\ \forall t \in [0, t_1^-)$ in view of (3.76). Therefore, $W(t) \leqslant M\ \forall t \in [0,T)$ and we must have $T = \infty$ from the definition of T. We then have (3.75) where $\gamma_1(r) := \bar{\mu}^{\overline{N}}\rho(r)r$ and $\gamma_2(r) := r + r\bar{\mu}^{\overline{N}-1}\frac{1}{1-e^{-\bar{\lambda}\delta_d}}$. □

Remark 3.14 *The switching signal s is characterized by a dwell-time δ_d, an average dwell-time $\bar{\tau}$, $\bar{\tau} > \delta_d$, and a chatter bound $\overline{N}$. For the variable W having property* (4.62)*, it is not possible to guarantee stability using only average dell-time (we need $\delta_d > 0$). If using dwell-time alone ($\overline{N} = 1$), a stability result will be more restrictive (the dwell-time will be greater than δ_d whereas average dwell-time switching still allows switching interval as small as δ_d).*

3.8.4 Input-to-state stability of switched systems

Theorem 3.3 *Consider the switched system* (3.2)*. Assume that* $\beta \in \mathcal{KL}_{S2}$ *where* β *is as in* (3.50)*. Let* μ *be a common gain function among ISS-Lyapunov functions* (3.49) *of the subsystems over the set* $\{x : |x| \geqslant \delta\}$ *for some* $\delta > 0$*. Let* $r := \alpha_2(\max\{\bar{x}_0, \delta\})$ *for some* $\bar{x}_0 > 0$*. Let* $\bar{\beta} \in \mathcal{KL}^{\beta}$*. Suppose that the input* u *is such that* $|u(t)| \leqslant \bar{u}\ \forall t \geqslant 0$ *and suppose that there exists* $D > 0$ *such that* $\nu_r(\bar{u}) \leqslant D$ *where* ν *is as in* (3.55)*. Then for all* $|x(0)| \leqslant \bar{x}_0$ *and for all switching signals that are not faster than the uniform profile* $\bar{\alpha}_{\bar{\beta},\beta,\mu,r,D}$ *where* $\bar{\alpha}$ *is as in* (3.57)*, we have*

$$|x(t)| \leqslant \max\left\{\delta, \alpha_1^{-1}(\bar{\beta}(r, t-t^{\mathrm{o}}) + \eta(\|u\|_{[t^{\mathrm{o}},t)})\right) \quad \forall t \geqslant 0 \tag{3.78}$$

for some $\eta \in \mathcal{K}_\infty$*, where* $t^{\mathrm{o}} \leqslant t$ *is the latest time less than or equal* t *such that* $|x(t^{\mathrm{o}})| = r$ *and* $|x(t^{\mathrm{o}+})| > r$*.*

Proof Let $\tau_k^{\mathrm{o}},\ k \geqslant 1$ be the times $x(t)$ leaves the ball B_δ (by convention, $\tau_1^{\mathrm{o}} = 0$ if $|x(0)| > \delta$). Let τ_k^{i} be the first time $x(t)$ re-enters the ball B_δ after the time τ_k^{o}. We will show that $\forall k$,

$$|x(t)| \leqslant \delta, \quad t \in [\tau_k^{\mathrm{i}}, \tau_{k+1}^{\mathrm{o}}) \tag{3.79a}$$

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(r, t-\tau_k^{\mathrm{o}}) + \eta(\|u\|_{[\tau_k^{\mathrm{o}},t)})),\ t \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}}), \tag{3.79b}$$

from which (3.78) is the combination of (3.79a) and (3.79b).

The inequality (3.79a) follows directly from the definitions of $\tau_k^{\mathrm{i}}, \tau_k^{\mathrm{o}}$.

Consider an interval $[\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ for some k. Let t be an arbitrary time in $[\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$. Let $v : [\tau_k^{\mathrm{i}}, \tau_k^{\mathrm{o}}) \to [0, \infty)$ be defined as $v(t) := V_{\sigma(t)}(x(t))$. In view of the property (3.50), we have $v(\tau) \leqslant \beta(v(\tau_\sigma), \tau - \tau_\sigma) + \chi(\|u\|_{\tau_\sigma,\tau})$ for all $\tau \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ by the fact that the switching signal σ is constant in the time interval $[\tau_\sigma, \tau)$. Since $|x(\tau)| \geqslant \delta$ for all $\tau \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ by the definitions of $\tau_k^{\mathrm{i}}, \tau_k^{\mathrm{o}}$, we have the property $v(\tau) \leqslant \mu(v(\tau_\sigma^-))v(\tau_\sigma^-)$ for all $\tau \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$ in view of the existence of a gain function μ as in (3.24). In view of (3.50), we have $\forall \tau \in [\tau_k^{\mathrm{o}}, \tau_k^{\mathrm{i}})$

$$v(\tau) \leqslant \begin{cases} \beta(v(\tau_k^{\mathrm{o}}), \tau - \tau_k^{\mathrm{o}}) + \chi(\|u\|_{\tau_k^{\mathrm{o}},\tau}) & \text{if } \tau_\sigma \leqslant \tau_k^{\mathrm{o}} \\ \mu(v(\tau_\sigma^-))\beta(v(\tau_\sigma^-), \tau - \tau_\sigma) + \chi(\|u\|_{\tau_\sigma,\tau}) & \text{else.} \end{cases} \tag{3.80}$$

Therefore, the constructed function v satisfies the property (3.52) with $v_0 = v(\tau_k^{\mathrm{o}})$. In view of (3.49a) and the fact that $|x(\tau_k^{\mathrm{o}})| \leqslant \max\{|x(0)|, \delta\}$, we have $v(\tau_k^{\mathrm{o}}) = V_{\sigma(\tau_k^{\mathrm{o}})}(x(\tau_k^{\mathrm{o}})) \leqslant r$ where r is defined as in the theorem statement. Applying Lemma 3.1, we then have

$$V_{\sigma(t)}(x(t)) \leqslant \bar{\beta}(V_{\sigma(\tau_k^{\mathrm{o}})}(x(\tau_k^{\mathrm{o}})), t - \tau_k^{\mathrm{o}}) + \eta(\|u\|_{\tau_k^{\mathrm{o}},t})$$

if the switching signal is not faster than the profile $\bar{\alpha}_{\beta,\bar{\beta},\mu,r,D}$. The foregoing inequality together with (3.49a) yields (3.79b). $\square$

If $\delta = 0$, then $\tau_1^{\mathrm{o}} = 0$, $\tau_1^{\mathrm{i}} = \infty$, and $r = x_0$. We have the following corollary.

Corollary 3.3 *Consider the switched system* (3.2)*. Let the subsystems having the family of ISS-Lyapunov functions* (3.49) *with the gain function μ as in* (3.24) *over $\mathbb{R}^n$ (i.e., $\delta = 0$). Let $r := \alpha_2(x_0)$. Let $\bar{\beta} \in \mathcal{KL}^{\beta}$. Suppose that the input u satisfies $|u(t)| \leqslant \bar{u}$ $\forall t \geqslant 0$ and suppose that there exists $D > 0$ such that $\nu_r(\bar{u}) \leqslant D$ where ν is as in* (3.55)*. Then for all $|x(0)| \leqslant x_0$ and for all switching signals with uniform profile γ such that $\gamma \leqslant \bar{\alpha}_{\bar{\beta},\beta,r,D}$ where $\bar{\alpha}$ is as in* (3.57)*, we have*

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(\alpha_2(x(0)), t) + \eta(\|u\|_{[0,t)})), \ \forall t \geqslant 0 \tag{3.81}$$

for some $\eta \in \mathcal{K}_\infty$.

Remark 3.15 *Our result, which is made possible with the concept of event profiles, differs from the available results in the literature (e.g., [57]) in two ways. First, we consider a general function μ in* (3.51) *while reported results always assume that μ is* *constant. Second, we allow $\bar{\beta}$ and β to be general class $\mathcal{KL}$ functions while available results in the literature assume that $\bar{\beta}$ and β are of the form $Ce^{-\lambda t}r$ (which is a special case of our result in which our event profiles become affine profiles; see Subsection 3.8.5 below).*

3.8.5 Affine profiles

A case of special interest is when we have exponentially decaying ISS-Lyapunov functions as in (3.51). Then $\beta(r,t) = e^{-\lambda_0 t}r$ clearly satisfies Properties S1, S2, and S3. We also have an explicit formula for the function χ in (3.50), which is $\chi(\|u\|_{[t,s)}) = \gamma(\|u\|_{[t,s)})/\lambda_0 \; \forall t \geqslant s$.

If we chose $\bar{\beta}(r,t) = Ce^{-\lambda t}r$, $\lambda \in (0,\lambda_0)$, $C > 1$, then $\bar{\beta} \in \mathcal{KL}^{\beta}$. Then $g(t-s) = Ce^{(\lambda_0-\lambda)(t-s)}$ and $g^{-1}(a) = (\ln a - \ln C)/(\lambda_0 - \lambda)$. Then $\nu(\bar{u}) = \chi(\bar{u})(1 + \sum_{i=1}^{\infty} \bar{\mu}^i \bar{\mu}^{-\lambda_0/(\lambda_0-\lambda)i} C^{\lambda_0/(\lambda_0-\lambda)}) = \chi(\bar{u})(1 + c\sum_{i=1}^{\infty} a^i)$ where $c = C^{\lambda_0/(\lambda_0-\lambda)} > 1$ and $a = \bar{\mu}^{-\lambda/(\lambda_0-\lambda)} < 1$. The profile $\bar{\alpha}_{\bar{\beta},\beta,r,D}$ in (3.57) becomes $\ln(e^{\lambda_0(t-s)}Ce^{-\lambda(t-s)})/\ln\bar{\mu} = \ln C/\ln\bar{\mu} + (\lambda_0-\lambda)(t-s)/\ln\bar{\mu}$, so we have the linear uniform profile

$$\bar{\alpha}(t,s) = \frac{\ln C}{\ln\bar{\mu}} + \frac{\lambda_0 - \lambda}{\ln\bar{\mu}}(t-s), \tag{3.82}$$

which leads to the average dwell-time switching with $\tau_a = \ln\bar{\mu}/(\lambda_0-\lambda)$.

Since $1 + c\sum_{i=1}^{\infty} a^i \leqslant 1 - c + c/(1-a)$, we have $\nu_{\bar{v}_0}(\bar{u}) \leqslant \chi(\bar{u})(1-c+c/(1-a))$ so we need to find if there is such a number D that

$$\chi(\bar{u})(1-c+c/(1-(\mu(C\bar{v}_0+D))^{-\lambda/(\lambda_0-\lambda)})) \leqslant D. \tag{3.83}$$

If μ is bounded above by some constant, say $\bar{\mu}$, then the left-hand side of (3.83) is bounded by some constant and for every $\bar{u}$, C, and $\bar{v}_0$, we can always pick D large enough in order for (3.83) to be true, which implies that switching with average dwell time not less than τ_a always guarantees ISS of the switched system for arbitrary initial conditions.

For unbounded μ, (3.83) can be used to find D numerically. Notice that the left-hand side of (3.83) is a decreasing function of D while the right-hand side of (3.83) is an increasing function of D. By direct expansion, it is not difficult to see that the left-hand side of (3.83) is positive for $D = 0$. Thus, we can always find D satisfying (3.83) for any given C, z_0, and $\bar{u}$.

Lemma 3.3 *For any given C, $\bar{v}_0$, and $\bar{u}$, there always exists $D > 0$ satisfying* (3.83).

However, notice that the chatter bound $N_0 = \ln C / \ln \bar{\mu}$, so to avoid triviality, we need $N_0 \geqslant 1$ (otherwise, $N_0 < 1$ means no switching). We then have the following problem: *for a given $\bar{v}_0$ and $\bar{u}$, find $C > 1$ and $D > 0$ satisfying* (3.83) *and such that* $C > \mu(C\bar{v}_0 + D)$. This problem can be solved numerically (see Example 3.7).

An interesting observation of (3.83) is that the left-hand side of (3.83) can be made arbitrarily small by choosing small enough $\bar{u}$. Thus, for any given C and D, there always exists a number $\bar{u}$ such that the switched system has the property (3.92) for all u such that $|u(t)| \leqslant \bar{u}\ \forall t \geqslant 0$. If $\bar{u} = 0$, we recover asymptotic stability. This can be seen as a *robust stability property* of average dwell-time switching that is strictly slower than the profile $\bar{\alpha}$ in (3.82) for switched systems without inputs with Lyapunov functions in (3.30) (cf. ISS as robust stability for nonswitched nonlinear systems [51]).

3.8.6 Asymptotic stability for vanishing disturbances

The result in the previous subsection implies that for a bounded input, the states will be bounded under our event profiles. However, the inequality (3.93) is not really the ISS property as in the case of nonswitched systems as we do not have the property that $|u| \to 0$ implies $|x| \to 0$. That is because our event profile depends on the initial state, so that if we shift the initial time while keeping the same event profile, (3.93) is no longer applicable.

To have the asymptotic stability for vanishing disturbances, indeed we will need event profiles that are slower than $\bar{\alpha}_{\bar{\beta},\beta,\mu,r,D}$ in Corollary 3.3. In particular, we replace the profile $\bar{\alpha}_{\bar{\beta},\beta,\mu,\alpha_2(x_0),D}$ in Corollary 3.3 by the profile $\bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{z},D}$ where $\bar{z} = \bar{\beta}(z_0,0)+D$, and we arrive at the following Theorem.

Theorem 3.4 *Consider the switched system* (3.2). *Let the subsystems have the family of ISS-Lyapunov functions* (3.49) *with the gain function μ as in* (3.24) *over $\mathbb{R}^n$ (i.e., $\delta = 0$). Let $\bar{\beta} \in \mathcal{KL}^{\beta}$. Suppose that the input u is such that $|u(t)| \leqslant \bar{u} \forall t$ and suppose that there exists $D > 0$ such that $\nu_{\bar{r}}(\bar{u}) \leqslant D$ where ν is as in* (3.55) *and $\bar{r} := \bar{\beta}(\alpha_2(x_0), 0) + D$. Then for all $|x(0)| \leqslant x_0$ and for all switching signals with uniform profile γ such that $\gamma \leqslant \bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{r},D}$ where $\bar{\alpha}$ is as in* (3.57)*, we have*

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(\alpha_2(x(0)),t)+\eta(\|u\|_{[0,t)})), \quad \forall t \geqslant 0 \tag{3.84}$$

for some $\eta \in \mathcal{K}$ and moreover, $|x(t)| \to 0$ as $t \to \infty$ if $|u(t)| \to 0$.

Proof The proof has two parts: the first part is to establish (3.84) and is a consequence of Corollary 3.3; the second part is to show asymptotic stability if u is vanishing.

Since g in (3.63) is nonincreasing in z_0, μ is nondecreasing and $\bar{\beta} \in \mathcal{KL}$, from (3.57), we have that $\bar{\alpha}_{\bar{\beta},\beta,\mu,r,D}$ is nonincreasing in r. Since $\bar{r} \geqslant \alpha_2(x_0) = r$, we then have that $\bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{r},D} \leqslant \bar{\alpha}_{\bar{\beta},\beta,\mu,r,D}$ and by Corollary 3.3, we obtain (3.93).

Suppose that $|u(t)| \to 0$. To show $|x(t)| \to 0$ as $t \to \infty$, we will show that for every $\varepsilon > 0$, there exists T such that $|x(t)| \leqslant \varepsilon \; \forall t \geqslant T$. Since $|u(t)| \to 0$, there exists T_1 such that $|u(t)| \leqslant \eta^{-1}(\alpha_1(\varepsilon)/2) \; \forall t \geqslant T_1$. In view of the fact $\nu(\|u\|_{[t,s)}) \leqslant D$, we have $z(t) \leqslant \bar{\beta}(\alpha_2(x_0), 0) + D = \bar{r} \; \forall t \geqslant 0$. In particular, $z(T_1) \leqslant \bar{r}$. Since the profile $\gamma \leqslant \bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{r},D}$, we have

$$\bar{\mu}^{N_\sigma(t,T_1)} \leqslant (\bar{\beta}/\beta)(\bar{r}, t - T_1) \; \forall t \geqslant T_1. \tag{3.85}$$

From (3.85) and the fact $z(T_1) \leqslant \bar{r}$, in view of (3.63), we obtain $G_{N_\sigma(t,T_1),1}\beta(z(T_1), t-$

$T_1) \leqslant \bar{\beta}(\bar{r}, t - T_1)$. In view of (3.60), we get $z(t) \leqslant \bar{\beta}(\bar{r}, t - T_1) + \eta(\|u\|_{[T_1,t)})\ \forall t \geqslant T_1$. Since $\bar{\beta} \in \mathcal{KL}$, there exists $T_2 > 0$ such that $\bar{\beta}(\bar{r}, T_2) < \alpha_1(\varepsilon)/2\ \forall t \geqslant T_2$. Therefore, for all $t \geqslant T_1 + T_2$, we have $z(t) \leqslant \alpha_1(\varepsilon)/2 + \eta(\eta^{-1}(\alpha_1(\varepsilon)/2)) = \alpha_1(\varepsilon)$ and hence, $|x(t)| \leqslant \varepsilon\ \forall t \geqslant T_1 + T_2$ in view of (3.49a). Thus, $|x(t)| \to 0$ as $t \to \infty$. $\square$

Remark 3.16 *Instead of a time-invariant μ in* (3.24)*, we can also have a time-dependent gain function $\mu : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that*

$$V_q(x(t)) \leqslant \mu(V_p(x(t)), t)V_p(x(t)) \quad \forall p, q \in \mathcal{P}, \forall t \geqslant 0 \tag{3.86}$$

provided that $\mu(V_p(x(t)), t) \leqslant \bar{\mu}\ \forall t \geqslant 0, V_p(x(t)) \leqslant \bar{z}$. That is because we bound μ by $\bar{\mu}$ and as long as we can assure $\mu \leqslant \bar{\mu}$ even if μ is time-dependent, all of our analysis in the previous subsection still holds without any modification.

3.8.7 Input-to-state stability of impulsive systems

Consider the impulsive system

$$\dot{x} = f(x, u) \tag{3.87a}$$

$$x(\tau) = g(x(\tau^-)) \quad \forall \tau \in \mathcal{T} \tag{3.87b}$$

where f is locally Lipschitz and $g : \mathbb{R} \to \mathbb{R}$ is an *impulse map*, and $\mathcal{T} := \{\tau_1, \tau_2, \dots\}$ is the set of impulse times; $\tau_0 = 0$ by convention. Assuming ISS of the nonimpulsive system, we want to see under what conditions on the set of impulse times $\mathcal{T}$ the impulsive system is ISS.

Since the nonimpulsive system (3.87a) is ISS, there exists a positive definite function $V(x)$ such that $\forall x$,

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|) \tag{3.88a}$$

$$\frac{\partial V(x)}{\partial x} f(x) \leqslant -\varphi(V(x)) + \gamma(u) \tag{3.88b}$$

for some functions $\alpha_1, \alpha_2, \varphi, \gamma \in \mathcal{K}_\infty$. Condition (3.88) also implies that there exist $\beta \in \mathcal{KL}$ and $\chi \in \mathcal{K}_\infty$ such that

$$V(x(t)) \leqslant \beta(V(x(t_0)), t) + \chi(\|u\|_{[t_0,t)}) \quad \forall t \geqslant t_0. \tag{3.89}$$

We assume that impulse map g is well-behaved such that the impulse effect does not have an infinite gain for arbitrary small z; in particular, assume that

$$\sup_{x \in [0,r]} \frac{\alpha_2(|g(x)|)}{\alpha_1(r)} < \infty \; \forall r < \infty.$$

Define

$$\mu(r) := \max \left\{ 1, \sup_{r_1 \leqslant r} \sup_{x : V(x) = r_1} \frac{\alpha_2(|g(x)|)}{r_1} \right\}. \tag{3.90}$$

The function μ is nondecreasing and $\mu \geqslant 1$ by construction, and thus is a gain function. From the definition of μ, we have

$$\begin{aligned} V(x(\tau_k)) &\leqslant \alpha_2(|g(x(\tau_k^-))|) \\ &\leqslant \mu(V(x(\tau_k^-)))V(x(\tau_k^-)) \quad \forall \tau_k \in \mathcal{T}. \end{aligned} \tag{3.91}$$

We can see that in view of (3.88) and (3.91), $V(x)$ satisfies (3.52). In view of Lemma 3.1, we have the following result.

Theorem 3.5 *Consider the impulsive system* (3.87)*. Let the nonimpulsive system have the ISS-Lyapunov function* (3.88) *with the gain function* μ *as in* (3.91)*. Let* $\bar{\beta} \in \mathcal{KL}^\beta$ *where* β *as in* (3.89)*. Suppose that the input* u *satisfies* $|u(t)| \leqslant \bar{u} \; \forall t \geqslant 0$ *and suppose that there exists* $x_0, D > 0$ *such that* $\nu_r(\bar{u}) \leqslant D$ *where* ν *is as in* (3.55) *and* $r = \alpha_2(x_0)$*. Then for all* $|x(0)| \leqslant x_0$ *and for all switching signals with uniform* *profile* γ *such that* $\gamma \leqslant \bar{\alpha}_{\bar{\beta},\beta,r}$ *where* $\bar{\alpha}$ *is as in* (3.57)*, we have*

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(\alpha_2(x(0)), t) + \eta(\|u\|_{[0,t)})), \ \forall t \geqslant 0 \tag{3.92}$$

for some $\eta \in \mathcal{K}_\infty$.

Theorem 3.6 *Consider the impulsive system* (3.87)*. Let the nonimpulsive system have the ISS-Lyapunov function* (3.88) *with the gain function* μ *as in* (3.91)*. Let* $\bar{\beta} \in \mathcal{KL}^\beta$ *where* β *as in* (3.89)*. Suppose that the input* u *is such that* $|u(t)| \leqslant \bar{u} \forall t$ *and suppose that there exists* $x_0, D > 0$ *such that* $\nu_{\bar{r}}(\bar{u}) \leqslant D$ *where* ν *is as in* (3.55) *and* $\bar{r} := \bar{\beta}(\alpha_2(x_0), 0) + D$*. Then for all* $|x(0)| \leqslant x_0$ *and for all impulse times* $\mathcal{T}$ *with uniform profile* γ *such that* $\gamma \leqslant \bar{\alpha}_{\bar{\beta},\beta,\mu,\bar{r},D}$ *where* $\bar{\alpha}$ *is as in* (3.57)*, we have*

$$|x(t)| \leqslant \alpha_1^{-1}(\bar{\beta}(\alpha_2(x(0)), t) + \eta(\|u\|_{[0,t)})), \quad \forall t \geqslant 0 \tag{3.93}$$

for some $\eta \in \mathcal{K}$ *and moreover,* $|x(t)| \to 0$ *as* $t \to \infty$ *if* $|u(t)| \to 0$.

Remark 3.17 *ISS property of impulsive systems is considered in [65], where the impulse map also depends on the input. Here, we assume that the impulse map does not depend on the input. Neglecting this difference, our result gives local ISS with less requirement on the ISS-Lyapunov function* V *in* (3.88)*, whereas [65] obtains ISS with a stronger assumption on the ISS-Lyapunov function (in particular, the function* V *in* (3.88) *is exponential with* $\varphi(r) = -cr$ *for some* $c > 0$*, and the gain function* μ *in* (3.91) *is constant).*

3.8.8 Examples

Example 3.6 We will illustrate how to find D in (3.55) for a general $\bar{\beta}$ function even though we have an infinite summation (recall that for the special case of exponential decaying β and $\bar{\beta}$, we have an explicit formula for the upper bound of $\nu(\bar{u})$ as in Subsection 3.8.5).

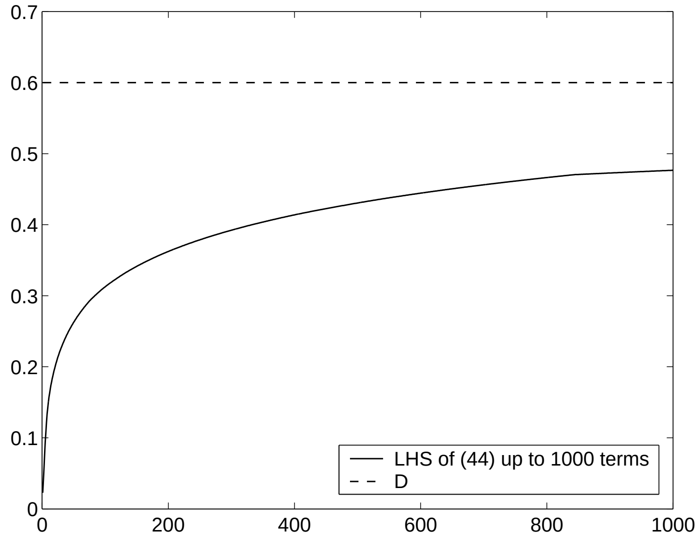

Figure 3.7: Example 3.6

Suppose $\mu(r) = 1 + 0.1r$, $\beta(r,t) = e^{-2t}r$, $v(r) = r^2$ and $\bar{u} = 0.1$. Consider the function $\bar{\beta}(r,t) = (2-t)r$, $t \in [0,1)$ and $\bar{\beta}(t,r) = (1/t)r$, $t \geqslant 1$. It can be checked that $\bar{\beta} \in \mathcal{KL}^{\beta}$.

Let $z(0) = 1$. Now $\bar{z} = \bar{\beta}(z(0),0) + D = 2 + D$ and so, $\bar{\mu}(D) = 1.2 + 0.1D$. We will find D numerically: we start from some initial value, for example $D_0 = 0.1$, and increment D by some constant step $D_{k+1} = D_k + 0.1$. For each D_k, we calculate the left-hand side of (3.55) for summation of $N = 1000$ terms and see if the sum converges graphically and if the sum is less than D_k. If this is true, D_k is the value of D we want[6] and we then verify mathematically by letting $s_i = \bar{\mu}^i e^{-\lambda_0 g^{-1}(\bar{\mu}^i)}$ and check if the series $\sum s_i$ converge (for example, using the root test $|s_i|^{1/i} < 1$ as $i \to \infty$ or the ratio test $|s_{i+1}/s_i| < 1$ as $i \to \infty$), and if so, the expression in the left-hand side of (3.55) indeed converges.

Using the above procedure, we find that $D = 0.6$ satisfies (3.55); see Fig. 3.7. $\lhd$

Example 3.7 In this example, we consider the special case of average dwell-time in Subsection 3.8.5. Let $\lambda_0 = 2, \lambda = 1, z_0 = 0.1$, $v(r) = r^2$, $\bar{u} = 1$, $\mu(r) = 1 + r$. Using numerical search, we found $C = 10$ and $D = 1$ that satisfies (3.83) (see Fig. 3.8).

[6]We do not go into details of the programming of the algorithm here; the algorithm may not always return a D and we may want use a large upper bound on D_k to avoid infinite loop in the program.

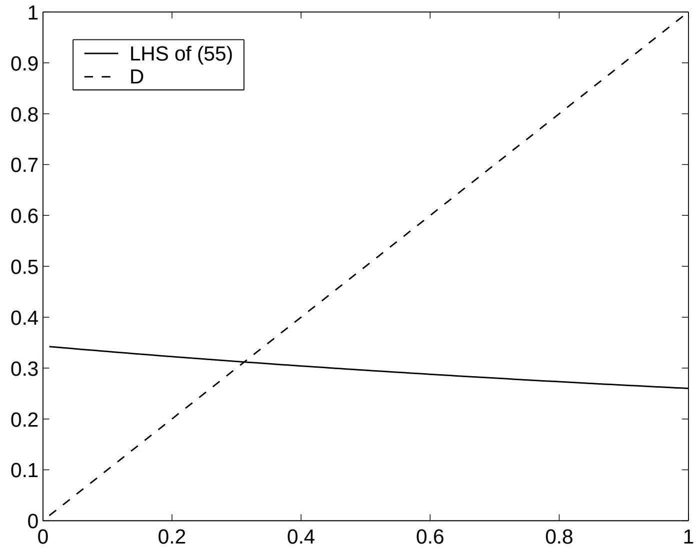

Figure 3.8: Example 3.7

Then $\tau_a \approx 1.6095$ and $N_0 \approx 1.4307$. $\triangleleft$

CHAPTER 4

SUPERVISORY CONTROL

4.1 The Supervisory Control Framework

We quickly review here the supervisory control framework for adaptively controlling plants with large uncertainty; for details, see e.g. [9, Chapter 6] and the references therein. In supervisory control, the basic idea is to discretized the uncertainty set into a finite number of nominal values and then employ a family of controllers, one for each nominal value. Switching among the controllers is orchestrated by a *supervisor* in such a way that closed-loop stability is guaranteed. The benefits gained by this approach include (i) simplicity and modularity in design: controller design amounts to controller design for known linear time-invariant systems for which various computationally efficient controller design tools are available; (ii) ability to handle larger classes of systems than is possible with other approaches (see [5] for more discussion).

Consider an uncertain plant $\mathbb{P}$ that is parameterized by a parameter p^*, which can be a constant or time-varying parameter,

$$\begin{aligned} \dot{x} &= f(x, u, p^\star, d), \\ y &= h(x, p^*, w), \end{aligned}$$

where x, y, u, d, w are the state, output, input, disturbance, and noise, respectively. The goal is to stabilize the plant without knowing p^*. The basic idea of supervisory control is to partition the uncertainty set (which p^* belongs to) into a finite number of subsets, design a controller for each subset, and at every time, chose an active controller in certain way so that it can be proved that the closed-loop plant is stable. The overall scheme is depicted in Fig. 4.1.

Suppose that the value of p^* (or the range of p^* if p^* is time-varying) is in a

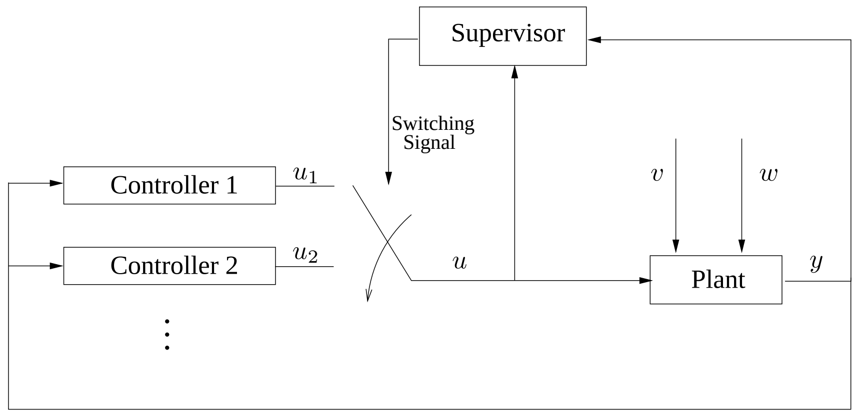

Figure 4.1: The supervisory adaptive control framework

compact set Ω. We partition Ω into m subsets $\Omega_i, i = 1, \ldots, m$. How to partition and into how many subsets is a question of interest in its own but is not pursued here. We just assume that each subset of the partition is small enough in some sense so that for each subset Ω_i, we can pick a nominal value $p_i \in \Omega_i$ such that there exists a controller for the plant with the parameter p_i that stabilizes all the plants with constant parameters in Ω_i. Let $\mathcal{P} = \{p_1, \ldots, p_m\}$.

A family of *candidate controllers*

$$\begin{aligned} \dot{x}_{\mathbb{C}} &= g_q(x_{\mathbb{C}}, y, u), \\ u_q &= r_q(x_{\mathbb{C}}, y), \end{aligned} \qquad q \in \mathcal{P}, \tag{4.1}$$

are designed such that the q^{th} controller stabilizes the plant with index q. The decision maker comprises three subsystems (see Fig. 4.2):

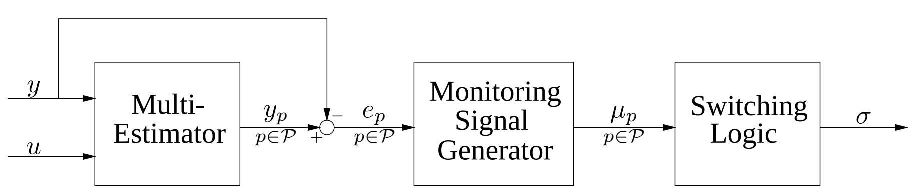

Figure 4.2: The decision maker

(i) The first subsystem is a *multiestimator*:

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= F(x_{\mathbb{E}}, y, u), \\ y_p &= h_p(x_{\mathbb{E}}), \end{aligned} \qquad p \in \mathcal{P}. \tag{4.2}$$

Let $e_p = y_p - y,\ p \in \mathcal{P}$ be the estimation errors. The multiestimator has the following property.

Assumption 4.1 *There exists a constant $c_0 > 0$ such that $|e_{p^\star}(t)| \leqslant c_0 \quad \forall\, t \geqslant 0$.*

There is a family of *injected systems*, where the injected system indexed by $q \in \mathcal{P}$ comprises the multiestimator and the corresponding controller (see Fig. 4.3):

$$\dot{x}_{\mathbb{CE}} = \begin{bmatrix} g_q(x_{\mathbb{C}}, y, r_q(x_{\mathbb{C}}, y)) \\ F(x_{\mathbb{E}}, y, r_q(x_{\mathbb{C}}, y)) \end{bmatrix} =: f_q(x_{\mathbb{CE}}, e_q, w)$$

by virtue of $y = h_q(x_{\mathbb{E}}, w) - e_q\ \forall q \in \mathcal{P}$, where $x_{\mathbb{CE}} := \begin{bmatrix} x_{\mathbb{C}}^{\mathrm{T}}\ x_{\mathbb{E}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is the state of the injected system; $x_{\mathbb{CE}} \in \mathbb{R}^n$; $e_q \in \mathbb{R}^\ell$.

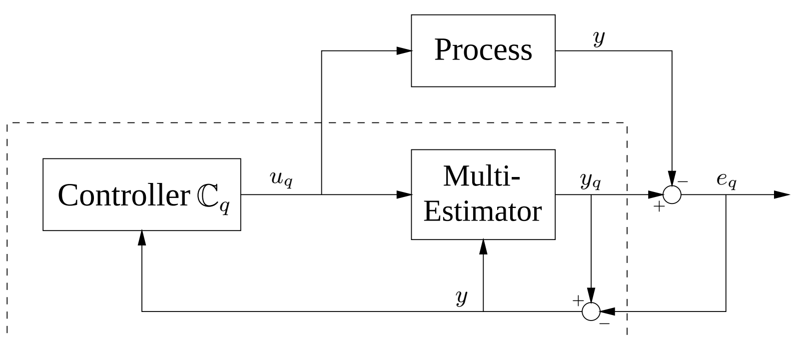

Figure 4.3: Injected system

The *switched injected system* is generated by the above family of injected systems and some switching signal σ defined in (iii) below.[1]

[1]By switched injected system we mean that there are no jumps in $x_{\mathbb{CE}}$ at switching instants. When implement-

(ii) The second subsystem is the *monitoring signal generator*, which generates the *monitoring signals* $\mu_p, p \in \mathcal{P}$:

$$\begin{aligned} \dot{z}_p &= -\lambda z_p + \overline{\gamma}(|e_p|), \quad z_p(0) = 0, \\ \mu_p(t) &= \varepsilon + z_p(t), \end{aligned} \tag{4.3}$$

for some $\varepsilon > 0$, $\lambda \in (0, \lambda_0)$, where λ_0, $\overline{\gamma}$ are in (3.8).

(iii) The third subsystem is a *switching logic*. We use the *scale-independent hysteresis switching logic*:

$$\sigma(t) := \begin{cases} \operatorname*{argmin}_{q\in\mathcal{P}} \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else,} \end{cases} \tag{4.4}$$

where $h > 0$ is called a *hysteresis constant* and is a design parameter.

Note that the above hysteresis switching logic is scale-independent—the switching signal σ is unaltered when we multiply all the monitoring signals by a positive scalar. Let $\bar{\mu}_p(t) := e^{\lambda t}\mu_p(t),\ t \geqslant 0,\ p \in \mathcal{P}$, be the scaled version of μ_p. From (4.3), for each $p \in \mathcal{P}$, we have

$$\bar{\mu}_p(t) = \varepsilon e^{\lambda t} + \int_0^t e^{\lambda s}\overline{\gamma}(|e_p(s)|)ds, \qquad t \geqslant 0, \tag{4.5}$$

which indicates that $\bar{\mu}_p$ is continuous and monotonically nondecreasing. Lemma 4.1 below provides a property of the switching signals generated by the switching logic (4.4) (cf. [60, Theorem 1]); the proof is along the lines of [60] (with more careful counting).

ing (4.1), at each switching instant τ_i, we can ensure that $x_{\mathbb{C}}(\tau_i^-) = x_{\mathbb{C}}(\tau_i)$, and thus $x_{\mathbb{C}}$ is continuous; $x_{\mathbb{E}}$ is continuous in view of (4.2).

Lemma 4.1 *For every index $q \in \mathcal{P}$ and arbitrary $t \geqslant t_0 \geqslant 0$, we have*

$$N_\sigma(t, t_0^+) \leqslant m + \frac{m}{\ln(1+h)} \ln\left(\frac{\bar{\mu}_q(t)}{\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)} \right), \tag{4.6}$$

$$\begin{aligned} \sum_{k=0}^{N_\sigma(t,t_0^+)} & \left(\bar{\mu}_{\sigma(\tau_k)}(\tau_{k+1}) - \bar{\mu}_{\sigma(\tau_k)}(\tau_k) \right) \\ & \leqslant m \left((1+h)\bar{\mu}_q(t) - \min_{p \in \mathcal{P}} \bar{\mu}_p(t_0) \right), \end{aligned} \tag{4.7}$$

where $\tau_1, \tau_2, \ldots, \tau_{N_\sigma(t,t_0)}$ are the discontinuities of σ on (t_0, t) and $\tau_{N_\sigma(t,t_0^+)+1} := t$, $\tau_0 := t_0$.

4.2 Supervisory Control of Uncertain Nonlinear Plants

This section considers the problem of stabilizing uncertain nonlinear systems in the presence of disturbances via *switching supervisory control* [5, 20, 25, 66]. This control scheme with the *scale-independent hysteresis switching logic* has been applied successfully to linear systems in the presence of modeling uncertainty and disturbances [67], and a recent research direction is to extend the result to nonlinear plants. For nonlinear plants with the same switching logic, it has been shown that if there are no disturbances, then switching stops in finite time and the states converge to zero [24, 23]. However, in the presence of disturbances, switching is not guaranteed to stop and the states can diverge. We show that using supervisory control with the scale-independent hysteresis switching logic, the states of an uncertain nonlinear plant can be kept bounded for arbitrary initial conditions and bounded disturbances when the controllers provide the ISS property with respect to the estimation errors.

We consider the case of exact matching, i.e., the unknown parameter p^* is a constant and p^* is a member of the set $\mathcal{P}$. Let the uncertain plant be

$$\dot{x} = f(x, u, p^\star, d),$$

$$y = h(x, p^*).$$

(For the sake of presentation, assume that there is no measurement noise). The hysteresis constant h is designed such that

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu}{\lambda_0 - \lambda}. \tag{4.8}$$

We have the following result on supervisory control of nonlinear plants with disturbances.

Theorem 4.1 *Suppose that*

(i) *the state x of the process $\mathbb{P}$ is bounded when the input u, output y, and disturbance d are bounded,*

(ii) *the multiestimator is designed such that Assumption 4.1 holds,*

(iii) *the candidate controllers are designed such that the hypotheses of Theorem 3.1 hold for the switched injected system.*

Then under the supervisor with scale-independent hysteresis switching logic, all continuous states of the closed-loop system are bounded for arbitrary initial conditions and bounded disturbances.

Proof From hypothesis (iii) and the condition on average dwell-time (4.8), it follows from Theorem 3.1 that the state of switched injected system $x_{\mathbb{CE}}$ has the $e^{\lambda t}$-weighted iISS property:

$$\mathrm{e}^{\lambda t}\alpha_1(|x_{\mathbb{CE}}(t)|) \leqslant \alpha_2(|x_{\mathbb{CE}}(0)|) + \int_0^t \mathrm{e}^{\lambda s}\overline{\gamma}(|e_\sigma(s)|)ds \;\forall t \geqslant 0 \tag{4.9}$$

for some $\mathcal{K}_\infty$ functions α_1, α_2 and $\overline{\gamma}$.

Letting $p = p^\star$ in (4.5), in view of Assumption 4.1, we get

$$\bar{\mu}_{p^\star}(t) \leqslant \kappa \mathrm{e}^{\lambda t}, \quad \kappa := \varepsilon + \overline{\gamma}(c_0)/\lambda. \tag{4.10}$$

Since $\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0) \geqslant \varepsilon \mathrm{e}^{\lambda t_0}\ \ \forall\, t_0 \geqslant 0$, (4.6) with $q = p^\star$ and (4.10) yield $N_\sigma(t, t_0) \leqslant N_\circ + \dfrac{t - t_0}{\tau_a}$, where $N_\circ := m + m\ln(\kappa/\varepsilon)/\ln(1+h)$, and $\tau_a := \ln(1+h)/(\lambda m)$. With $q = p$ in (4.7), from (4.5) and (4.10), we arrive at

$$\begin{aligned} &\int_0^t \mathrm{e}^{\lambda s}\overline{\gamma}(|e_{\sigma(s)}(s)|)ds + \varepsilon\mathrm{e}^{\lambda t} - \varepsilon = \\ &\sum_{k=0}^{N_\sigma(t,t_0)} \big(\bar{\mu}_{\sigma(\tau_k)}(\tau_{k+1}) - \bar{\mu}_{\sigma(\tau_k)}(\tau_k)\big) \leqslant m(1+h)\kappa\mathrm{e}^{\lambda t}. \end{aligned} \tag{4.11}$$

The inequalities (4.9) and (4.11) yield

$$|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(\alpha_2(|x_{\mathbb{CE}}(0)|) + cm(1+h)\kappa) =: c_2 \quad \forall t \geqslant 0. \tag{4.12}$$

We have $\forall\, q \in \mathcal{P},\ \forall\, t \geqslant 0,\ |y_q(t)| = |h_q(x_{\mathbb{E}}(t))| \leqslant \sup_{p\in\mathcal{P}, |\xi|\leqslant c_2}\{|h_p(\xi)|\} =: c_3$. Since $y = y_{p^\star} - e_{p^\star}$ and $|e_{p^\star}(t)| \leqslant c_0\ \forall\, t \geqslant 0$ (by Assumption 4.1), it follows that $|y(t)| \leqslant c_3 + c_0 =: c_4\ \forall\, t \geqslant 0$. Also $e_q = y_q - y$, and therefore $|e_q(t)| \leqslant c_0 + 2c_3 =: c_5\ \forall\, q \in \mathcal{P},\ \forall\, t \geqslant 0$. Further, we have $\forall\, t \geqslant 0,\ |u(t)| \leqslant \sup_{q\in\mathcal{P}, |\xi|\leqslant c_2, |\eta|\leqslant c_4}\{|r_q(\xi, \eta)|\} =: c_6$. Since d, u, and y are bounded, the state x is bounded in view of hypothesis (i). Finally, every monitoring signal $\mu_q, q \in \mathcal{P}$, is bounded since $|e_q|$ is bounded $\forall\, q \in \mathcal{P}$. $\square$

Remark 4.1 *Hypothesis* (i) *of Theorem 4.1 holds, for example, when the plant is input-output-to-state stable (see [68] for the definition). Hypothesis* (ii) *requires that at least one estimator provides a bounded estimation error in the presence of disturbances. This is more or less a standard assumption in multiestimator design; a similar assumption was used in [23] for plants without disturbances. Hypothesis* (iii) *stipulates that the injected systems are ISS (which was also an assumption in [23]); the design of ISS injected systems is nontrivial, and is a topic of ongoing research (cf. [69]). All three hypotheses can be completely characterized via detectability and stabilizability of the plant for linear systems [20], but characterizing the nonlinear plants for which these hypotheses hold is still an open question. However, there are* *certain nonlinear systems for which these conditions hold (see Example 4.1 below).*

Remark 4.2 *If the disturbance d is vanishing and in Assumption 4.1 we replace the constant bound c_0 with a time-varying bound $c_0(t) \to 0$ as $t \to \infty$, and further, if the plant is IOSS, then we can have $|x(t)| \to 0$ as $t \to \infty$ if we use a nonnegative decaying $\varepsilon(t)$ in the monitoring signal generator such that $\varepsilon(t) \to 0$ and $\overline{\gamma}(c_0(t))/\varepsilon(t) < \infty$ as $t \to \infty$ (which means ε should decay more slowly than $\overline{\gamma}(c_0)$). If this is the case, $\kappa \to 0$ in (4.10) and the chatter bound $N_\circ < \infty$. Then the iISS property of the switched injected system together with (4.11) yields $|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(e^{-\lambda t}\alpha_2(|x_{\mathbb{CE}}(0)|) + cm(1 + h)\kappa(t)) \to 0$ as $t \to \infty$; thus, c_2 in (4.12) becomes a time-varying $c_2(t) \to 0$ as $t \to \infty$. It then follows that $c_3(t), c_4(t), c_5(t), c_6(t) \to 0$ as $t \to \infty$ where $c_3(t), c_4(t), c_5(t), c_6(t)$ are the time-varying bounds in places of c_3, c_4, c_5, c_6 in the proof of the theorem. Since $|u(t)| \to 0$, $|y(t)| \to 0$ and the plant is IOSS, the state norm $|x(t)| \to 0$ as $t \to \infty$.*

4.2.1 Boundedness under weaker hypotheses

As noted in Remark 3.1, the existence of μ as in (3.9) for all ξ restricts the set of possible ISS-Lyapunov functions. We now assume that we only have μ such that the inequality (3.9) holds in some annulus $\Omega := \{\xi \in \mathbb{R}^n : r_1 \leqslant |\xi| \leqslant r_2\}$, for some numbers $r_2 > r_1 \geqslant 0$.

Consider the switched injected system described in the previous section. Suppose $\mu \geqslant 1$ such that $V_p(\xi) \leqslant \mu V_q(\xi)$, $\forall\, r_1 \leqslant |\xi| \leqslant r_2 \;\forall\, p, q \in \mathcal{P}$. We can set $x_{\mathbb{CE}}(0) = 0$. Let $\hat{t}_1 := \inf\{t \geqslant 0 : |x_{\mathbb{CE}}(t)| > r_1\}$. If $\hat{t}_1 = \infty$, then $|x_{\mathbb{CE}}(t)| \leqslant r_1 \;\forall t \geqslant 0$. Otherwise, let $\hat{t}_2 := \inf\{t \geqslant \hat{t}_1 : |x_{\mathbb{CE}}(t)| > r_2\}$ and $\check{t}_1 := \inf\{t \geqslant \hat{t}_1 : |x_{\mathbb{CE}}(t)| < r_1\}$ and $\bar{t} := \min\{\hat{t}_2, \check{t}_1\}$. Since $r_1 \leqslant |x_{\mathbb{CE}}(t)| \leqslant r_2 \;\forall t \in [\hat{t}_1, \bar{t})$, it follows from (4.12) that $\forall t \in [\hat{t}_1, \bar{t})$, we have

$$|x_{\mathbb{CE}}(t)| \leqslant \alpha_1^{-1}(\alpha_2(r_1) + cm(1+h)\kappa) =: c_2 \tag{4.13}$$

with c and κ as in (3.17), (4.10).

Let $\bar{x}_0$ and $\bar{d}$ be the bounds on the plant initial state and disturbance, respectively. Then the bound c_0 in Assumption 4.1 depends on $\bar{x}_0$ and $\bar{d}$ only. Suppose that $\bar{x}_0$ and $\bar{d}$ are sufficiently small such that $c_2 \leqslant r_2$. This inequality together with (4.13) and the definition of $\hat{t}_2$ implies that we must have $\hat{t}_2 = \infty$. If $\check{t}_1 = \infty$, then $\bar{t} = \infty$ and hence, $|x_{\mathbb{CE}}(t)| \leqslant c_2 \; \forall\, t \geqslant 0$. If $\check{t}_1 < \infty$, then $\bar{t} = \check{t}_1$, and let $\hat{t}_3 := \inf\{t \geqslant \check{t}_1 : |x_{\mathbb{CE}}(t)| > r_1\}$. If $\hat{t}_3 = \infty$, then $|x_{\mathbb{CE}}(t)| \leqslant r_1 \leqslant c_2 \; \forall t \geqslant \check{t}_1$, and hence, $|x_{\mathbb{CE}}(t)| \leqslant c_2 \; \forall t \geqslant 0$; otherwise, repeat the current argument with $\hat{t}_3$ playing the role of $\hat{t}_1$. We can then conclude that $|x_{\mathbb{CE}}(t)| \leqslant c_2 \; \forall t \geqslant 0$. From the boundedness of $x_{\mathbb{CE}}$, we can prove that all continuous states are bounded using similar arguments as in the proof of Theorem 4.1. We then have the following result:

Theorem 4.2 *Suppose that*

(i) *the state x of the process $\mathbb{P}$ is bounded when the input u, output y, and disturbance d are bounded,*

(ii) *the multiestimator is designed such that Assumption 4.1 holds,*

(iii) *the candidate controllers are designed such that hypotheses* (3.7), (3.8) *of Theorem 3.1 hold for the switched injected system for some family of ISS-Lyapunov functions $\{V_p\}_{p \in \mathcal{P}}$,*

(iv) *there exist positive numbers r_1, r_2, μ, such that $V_q(\xi) \leqslant \mu V_q(\xi) \; \forall\, r_1 \leqslant |\xi| \leqslant r_2$ and positive numbers $\bar{x}_0, \bar{d}$ such that $c_2 \leqslant r_2$ for some $\varepsilon > 0, h > 0, 0 < \lambda < \lambda_0$ where c_2 is as in* (4.13).

Then under the supervisor with the scale-independent hysteresis switching logic, with hysteresis constant h, all continuous states of the closed-loop system are bounded for bounded disturbances $|d(t)| \leqslant \bar{d},\ t \geqslant 0$ whenever the initial plant state $|x(0)| \leqslant \bar{x}_0$.

On the one hand, when ISS-Lyapunov functions satisfying (3.9) are not available, Theorem 4.2 can provide a way to achieve local boundedness of the plant state. There are more choices of ISS-Lyapunov functions, which can lead to simpler controller and multiestimator designs, but it may be difficult to find the positive numbers in hypothesis (iv) in Theorem 4.2. Also, the hysteresis constant h cannot be chosen arbitrarily small since λ cannot be arbitrarily small (c_2 increases when λ decreases). On the other hand, if we can find ISS-Lyapunov functions satisfying (3.9), Theorem 4.1 provides a global boundedness result. It also provides the flexibility to choose a small hysteresis constant h, which can be made arbitrarily small by reducing λ (see (4.8)), and a smaller h possibly leads to a better performance.

4.2.2 Example

We consider examples to illustrate our result on supervisory control of nonlinear uncertain plant with disturbances.

Example 4.1 Consider a scalar nonlinear plant

$$\dot{y} = y^2 + p^\star u + d, \tag{4.14}$$

where $p^\star$ is an unknown constant belonging to a finite index set $\mathcal{P}$ of m elements, $\mathcal{P} := \{p_1, \dots, p_m\}$, and d is a disturbance. Our objective is to keep the state bounded in the presence of a bounded disturbance. The unknown parameter enters as the input gain, which makes the problem challenging to solve in the framework of conventional adaptive control when the sign of $p^\star$ is unknown.

The multiestimator and the candidate controllers are

$$\left.\begin{aligned} \dot{y}_p &= -(y_p - y) - (y_p - y)^3 + pu + y^2, \\ u_p &= \frac{1}{p}(-y - y^2 - y^3), \end{aligned}\right. \qquad p \in \mathcal{P}.$$

The injected system for the controller with an index q is

$$\dot{y}_p = -(y_p - y) - (y_p - y)^3 + \frac{p}{q}(-y - y^2 - y^3) + y^2, \ \ p \in \mathcal{P}. \tag{4.15}$$

Consider the candidate ISS-Lyapunov functions

$$V_q(x_{\mathbb{CE}}) := a_1 y_q^4 + b_1 y_q^2 + \sum_{p\neq q, p\in\mathcal{P}} a_0 y_p^4 + b_0 y_p^2,\ q \in \mathcal{P},$$

where $x_{\mathbb{CE}} := [y_{p_1}, \ldots, y_{p_m}]^{\mathrm{T}}$ is the state of the injected system, for some $a_1, b_1, a_0, b_0 > 0$ to be determined. One can pick $\mu := \max\left\{\frac{a_1}{a_0}, \frac{a_0}{a_1}, \frac{b_1}{b_0}, \frac{b_0}{b_1}\right\}$. The derivative of V_q along the q^{th} injected system is $\dot{V}_q = 4a_1 y_q^3 \dot{y}_q + 2b_1 y_q \dot{y}_q + \sum_{p\neq q, p\in\mathcal{P}} 4a_0 y_p^3 \dot{y}_p + 2b_0 y_p \dot{y}_p$. Substituting (4.15) into the foregoing $\dot{V}_q$, after some expansions and simplifications, we arrive at

$$\begin{aligned}\dot{V}_q &\leqslant -a_1 y_q^6 - 4a_1 y_q^4 - 2b_1 y_q^2 + \\ &\sum_{\substack{p\neq q,\\ p\in\mathcal{P}}} -a_0 y_p^6 - 4a_0 y_p^4 - 2b_0 y_p^2 + (4a_0 y_p^3 + 2b_0 y_p)\kappa_{pq} g(y),\end{aligned} \tag{4.16}$$

where $\kappa_{pq} := (1 - p/q)$ and $g(y) := y + y^2 + y^3$. Define $\kappa_{\max} := \max\{|\kappa_{pq}| : p, q \in \mathcal{P}\}$. Using completions of the squares with $-a_0 y_p^6 - b_0 y_p^2 + (4a_0 y_p^3 + 2b_0 y_p^2)\kappa_{pq} g(y)$ and using the triangle inequality with $|g(y)|^2$ in (4.16), after some computations, we obtain

$$\begin{aligned}\dot{V}_q \leqslant &-V_q - (a_1 - 256(4a_0 + b_0)m\kappa_{\max}^2)y_q^6 \\ &- (b_1 - 16(4a_0 + b_0)m\kappa_{\max}^2)y_q^2 \\ &+ (4a_0 + b_0)m\kappa_{\max}^2(16e_q^2 + 256e_q^6).\end{aligned}$$

If b_0, a_0 are chosen such that $(4a_0 + b_0)m\kappa_{\max}^2 \leqslant 1$, $a_1 \geqslant 256, b_1 \geqslant 16$, we then get $\dot{V}_q \leqslant -V_q + \overline{\gamma}(|e_q|)$, where $\overline{\gamma}(r) := 16r^2 + 256r^6$ is a class $\mathcal{K}_\infty$ function. The foregoing inequality shows that for each fixed controller with index q, the corresponding injected system is ISS with respect to the output error e_q. By Theorem 4.1, all the continuous states are bounded for arbitrary initial conditions and bounded disturbances under the supervisor with scale-independent hysteresis switching logic for a large enough h satisfying (4.8).

For $\mathcal{P} = \{-2, -1, 1, 2\}$, $p^\star = 1$, numerical values are $m = 4$, $\kappa_{\max} = 3$, $a_0 =$

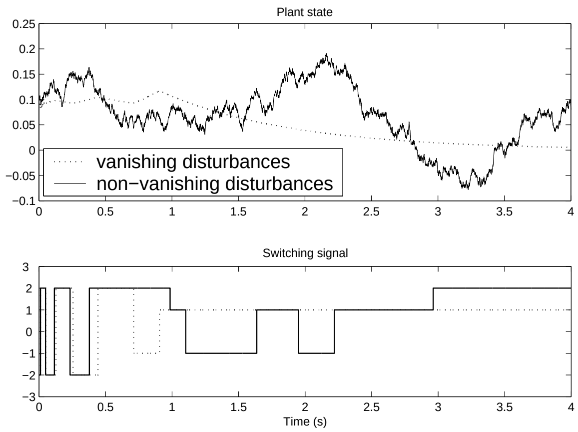

Figure 4.4: Example 4.1

6.5×10^{-3}, $b_0 = 0.5 \times 10^{-3}$, $a_1 = 256$, $b_1 = 16$, $\mu = 3.94 \times 10^4$. Choose $\varepsilon = 10^{-6}$, $\lambda = 2 \times 10^{-4}$. The hysteresis constant $h = 0.02$ satisfies the condition on average dwell-time (4.8). Simulation results in MATLAB® with disturbance uniformly distributed between -5 and 5 (solid lines) and exponentially decaying disturbance (dotted lines), and $x_0 = 0.1$, $x_{\mathbb{E}}(0) = 0$ are plotted in Fig. 4.4. The simulation shows that the state is bounded for the bounded disturbance and is decaying for the vanishing disturbance (indeed, the bound in the simulation is much smaller than those given by the analysis).

$\lhd$

Example 4.2 Consider the scalar nonlinear plant in Example 4.1, and the following simpler multiestimator and candidate controllers:

$$\begin{aligned} \dot{y}_p &= -(y_p - y) + y^2 + pu, \\ u_p &= -\frac{1}{p}(y + y^2), \end{aligned} \qquad p \in \mathcal{P}.$$

The injected system with the controller indexed by q is

$$\dot{y}_p = -(y_p - y) + y^2 + \frac{p}{q}(-y - y^2), \qquad p \in \mathcal{P}.$$

Using the candidate ISS-Lyapunov function $V_q := b_1 y_q^4 + b_2 y_q^2 + a \sum_{\substack{p \neq q, \\ p \in \mathcal{P}}} y_p^2$, it can be checked that for each fixed controller indexed by $q \in \mathcal{P}$, the injected system is ISS:

$$\dot{V}_q = -4b_1 y_q^4 - 2b_2 y_q^2 + 2a \sum_{p \neq q, p \in \mathcal{P}} y_p(-y_p + \kappa_{pq} y + \kappa_{pq} y^2)$$
$$\leqslant -\lambda_0 V_q + \overline{\gamma}(|e_q|),$$

with $y^2 := |y_q - e_q|^2 \leqslant 2(y_q^2 + e_q^2)$ and $y^4 \leqslant 8(y_q^4 + e_q^4)$ for some $0 < \lambda_0 < 2$, $a_1, a_2 > 0$, such that $a_1 + a_2 = 2 - \lambda_0$, where $\kappa_{pq} := (1 - p/q)$, $\kappa_{\max} := \max_{p,q \in \mathcal{P}}\{|\kappa_{pq}|\}$, $b_3 := a(m-1)\kappa_{\max}^2$, $\overline{\gamma}(r) := b_3(2r^2/a_1 + 8r^4/a_2)$, and b_1, b_2, a such that $b_3 < \min\{(4 - \lambda_0)b_1 a_2/8, (2 - \lambda_0)b_2 a_1/2\}$.

The ISS-Lyapunov functions V_q have the property (3.7), (3.8); however, there is no global μ as in (3.9) because V_q is quartic in y_q while $V_{p,p\neq q}$ are quadratic in y_q. Nevertheless, we can obtain a stability result using Theorem 4.2.

We can choose $\overline{\alpha}_1(r) := \min\{b_2, a\}r^2 =: \eta_1 r^2$ and $\overline{\alpha}_2(r) := \max\{(b_1 r_2^2 + b_2), a\}r^2 =: \eta_2 r^2$. Then $\mu := \eta_2/\eta_1$. The error dynamics for $p = p^\star$ are $\dot{e}_{p^\star} = -e_{p^\star} - d$ and hence the bound on $e_{p^\star}$ is $|e_{p^\star}(t)| \leqslant |e_{p^\star}(0)| + \bar{d} \leqslant \bar{x}_0 + \bar{d}$ since $|e_{p^\star}(0)| = |y_{p^\star}(0) - y(0)| \leqslant \bar{x}_0$ by virtue of $y_p(0) = 0 \; \forall p \in \mathcal{P}$. Now, $c_2 = \left(\mu^{1+N_\circ}(\eta_2 r_1^2 + m(1+h)\kappa)/\eta_1\right)^{1/2} < r_2$ if r_1 and $\bar{d}$ are small enough. Choosing the hysteresis constant h to satisfy the average dwell-time condition, we conclude that all the continuous states $x, x_{\mathbb{CE}}$ are bounded.

For $\mathcal{P} = \{-2, -1, 1, 2\}$, $p^\star = 1$, then $m = 4$, $\kappa_{\max} = 3$. Let $r_2 = 0.1$, $r_1 = 10^{-8}$, $b_1 = 2.96 \times 10^{-9}, b_2 = 1.3 \times 10^{-10}$, $a = 8 \times 10^{-12}$, $a_1 = 1.75$, $a_2 = 0.15$, $\lambda_0 = 0.1$. Then $\mu = 19.95$. Choose $h = 0.05$, $\lambda = 0.0003$, $\varepsilon = 3.2914 \times 10^{-21}$. Then $N_\circ = 4.0819$. If $|x(0)| < 10^{-9}$ and $\bar{d} < 10^{-9}$, then all the states are bounded by $c_2 = 0.0836$ for all time. Simulation results are in Fig. 4.5. $\triangleleft$

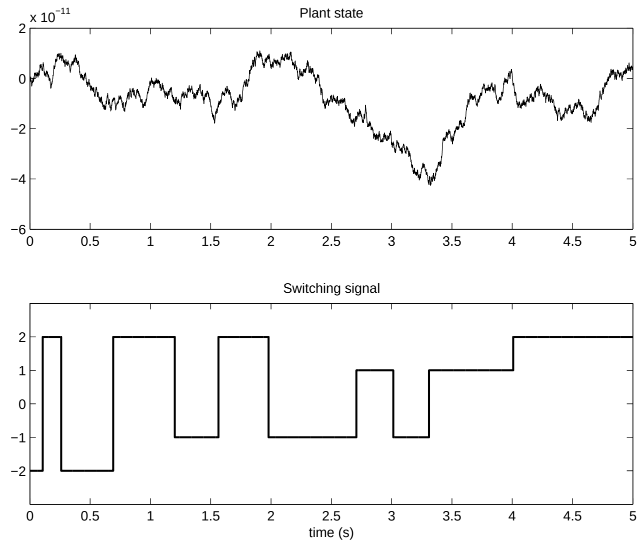

Figure 4.5: Example 4.2

4.3 Supervisory Control of Uncertain Linear Time-Varying Plants

Adaptive control of time-varying uncertain plants is a challenging control problem which has attracted a considerable amount of research in the last few decades. For time-varying uncertain plants, doing off-line parameter identification and control separately is difficult, and as such, an adaptive control approach is often necessary. Various robust adaptive control solutions for linear time-varying systems have been proposed, including direct model reference adaptive control [70], indirect adaptive pole placement control [71, 72, 73, 74], and back-stepping adaptive control [75, 76] (see also e.g., [77, 78, 79]). These works and a majority of the literature on adaptive control of time-varying systems more or less employ a continuously parameterized control law in combination with a continuously estimated parameter. A notably different approach is [80], where the strategy is to approximate the control input directly using sampled output data. The reader is referred to the introduction in [80] for more extensive literature review on adaptive control.

We present here a new approach to the adaptive control problem of time-varying plants, using supervisory control. The supervisory adaptive control scheme has been successfully applied to linear time-invariant systems with constant unknown parameters in the presence of unmodeled dynamics and noise [25, 26, 66, 67]. Nonetheless, the supervisory adaptive control scheme has not been applied to time-varying systems and it is the objective of this paper to explore this topic (see also a related problem of identification and control of time-varying systems using multiple models [81]). When the parameter's variation is small such that the time-varying plant can be approximated by a system with constant parameter and small (time-varying) unmodeled dynamics, the robustness result in [67] can be applied. However, when the parameter's variation is large such that the previous approximation is not valid, the result in [67] is no longer applicable. The main contribution of this paper is to show that supervisory adaptive control is capable of stabilizing plants with large parameter variation, provided that the parameters vary slowly enough in a certain sense. Further, this can be achieved in the presence of unmodeled dynamics with bounded disturbances and measurement noise, provided that the unmodeled dynamics are small enough.

Consider uncertain time-varying plants of the following form:

$$\mathbf{P} : \begin{cases} \dot{x} &= A(t)x + B(t)u + g_\Delta(z, x, u, t) + v, \\ y &= C(t)x + h_\Delta(z, x, u, t) + w, \\ \dot{z} &= f_\Delta(z, x, u, t), \end{cases} \tag{4.17}$$

where $x \in R^n$ is the state, $u \in \mathbb{R}^\ell$ is the input, $y \in \mathbb{R}^r$ is the output, v and w are disturbance and measurement noise, respectively, and $z \in \mathbb{R}^{n_2}$ is the state of the unmodeled dynamic. The functions A, B, C can be piecewise continuous and $f_\Delta, g_\Delta, h_\Delta$ are continuous (thus existence and uniqueness of solution of (4.17) for a given initial condition is guaranteed). The plant is uncertain in the sense that we do not completely know $A(t), B(t), C(t)$.

Assumption 4.2 *We know a compact set* $\Omega \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{r\times n}$ *such that* $(A(t), B(t), C(t)) \in \Omega\ \forall t$.

The existence of such Ω is guaranteed, for example, if A, B, C are globally bounded. We consider the following adaptive stabilization problem: *asymptotically stabilize the uncertain plant* (4.17) *in the absence of noise and unmodeled dynamics and guarantee boundedness of all closed-loop states in the presence of noise and unmodeled dynamics.*

We will approximate the time-varying system (4.17) by a switched system plus unmodeled dynamics in the following way. We divide Ω into a finite number of nonoverlapping subsets such that $\bigcup_{i\in\mathcal{P}} \Omega_i = \Omega$, where $\mathcal{P} = \{1, \ldots, m\}$ and m is the number of subsets. How to divide and what the number of subsets is are interesting research questions of their own and are not pursued here (see [26]). Intuitively, we want the set Ω_i small in some sense. Define a piecewise constant signal $s : [0, \infty) \to \mathcal{P}$ such that

$$s(t) := i : (A(t), B(t), C(t)) \in \Omega_i. \tag{4.18}$$

Since $A(t), B(t), C(t)$ are not completely known, $s(t)$ is also not known a priori. For every subset Ω_i, $i \in \mathcal{P}$, pick a nominal value $(A_i, B_i, C_i) \in \Omega_i$. We can rewrite the plant (4.17) as

$$\mathbf{P} : \begin{cases} \dot{x} &= A_{s(t)}x + B_{s(t)}u + \delta_A(t)x + \delta_B(t)u + g_\Delta(z, x, u, t) + v, \\ y &= C_{s(t)}x + \delta_C(t)x + h_\Delta(z, x, u, t) + w, \\ \dot{z} &= f_\Delta(z, x, u, t), \end{cases} \tag{4.19}$$

where $\delta_A(t) := A(t) - A_{s(t)}$, $\delta_B(t) := B(t) - B_{s(t)}$, and $\delta_C(t) := C(t) - C_{s(t)}$. Equation (4.19) represents a switched system with unmodeled dynamics (note that even in the case the original plant (4.17) does not include the unmodeled dynamics g_Δ, h_Δ, and f_Δ, the approximation of a time-varying plant by a switched system still results in a switched system with unmodeled dynamics as in (4.19) via $\delta_A, \delta_B, \delta_C$).

Assumption 4.3 (A_i, B_i) *are controllable and* (A_i, C_i) *are detectable* $\forall i \in \mathcal{P}$.

4.3.1 Supervisory control design

- **Multiestimator**: A multiestimator with state $x_{\mathbb{E}} = (\hat{x}_1, \ldots, \hat{x}_m) \in \mathbb{R}^{mn}$ is constructed as follows:

$$\dot{\hat{x}}_q = A_q\hat{x}_q + B_q u + L_q(C_q\hat{x}_q - y) \quad q \in \mathcal{P}, \tag{4.20}$$

where L_q are such that $A_q + L_qC_q$ are Hurwitz for all $q \in \mathcal{P}$. We set $\hat{x}_q(0) = 0\ \forall q \in \mathcal{P}$. Let $\tilde{x}_q := x - \hat{x}_q$ and $\tilde{y}_q := y - C\tilde{x}_q$.

- **Multicontrollers**: A family of *candidate feedback gains* $\{K_q\}$ is designed such that $A_q + B_qK_q$ are Hurwitz for all $q \in \mathcal{P}$. The *family of controllers* is

$$u_q = K_q\hat{x}_q \quad q \in \mathcal{P}. \tag{4.21}$$

The *injected system* with index $p \in \mathcal{P}$ for some k, $1 \leqslant p \leqslant m$, is formed by the combination of the multiestimator and the controller with index p:

$$\dot{x}_{\mathbb{E}} = \bar{A}_p x_{\mathbb{E}} + \bar{B}(C\hat{x}_p - y), \tag{4.22}$$

where $\bar{B} = [L_{p_1}^T \ldots L_{p_m}^T]^T$ and $\bar{A}_p$ is the square matrix of dimension nm whose structure is $\bar{A}_P = [A_{ij}]$, $A_{ij} \in \mathbb{R}^{n\times n}$, $1 \leqslant i, j \leqslant m$, $A_{ii} = A_i + L_iC_i$, $A_{ip} = B_iK_p - L_iC_p$ for $1 \leqslant i \leqslant m$ and $i \neq p$, $A_{pp} = A_p + B_pK_p$, and $A_{ij} = 0_{nm\times nm}$ for all other i, j. From the fact that $A_q + B_qK_q$ and $A_q + L_qC_q$ are Hurwitz for all $q \in \mathcal{P}$, it follows that $\bar{A}_p$ are Hurwitz for all $p \in \mathcal{P}$.

There exists a family of quadratic Lyapunov functions $V_q(x_{\mathbb{E}}) = x_{\mathbb{E}}^T P_q x_{\mathbb{E}}, P_q^T = P_q > 0$ such that $\forall p \in \mathcal{P}$,

$$a_1|x_{\mathbb{E}}|^2 \leqslant V_p(x_{\mathbb{E}}) \leqslant a_2|x_{\mathbb{E}}|^2 \tag{4.23a}$$

$$\frac{\partial V_p(x_{\mathbb{E}})}{\partial x}(\bar{A}_p x_{\mathbb{E}} + \bar{B}\tilde{y}_p) \leqslant -\lambda_0 V_p(x_{\mathbb{E}}) + \gamma|\tilde{y}_p|^2 \tag{4.23b}$$

for some constants $a_1, a_2, \lambda_0, \gamma_0 > 0$ (the existence of such common constants for the family of injected systems is guaranteed since we have a finite number of systems). There exists a number $\mu \geqslant 1$ such that

$$V_q(x) \leqslant \mu V_p(x) \quad \forall x \in \mathbb{R}^n, \forall p, q \in \mathcal{P}. \tag{4.24}$$

We can always pick $\mu = a_2/a_1$ but there may be other smaller μ satisfying (4.24) (for example, $\mu = 1$ if V_p are the same for all p even though $a_2/a_1 > 1$).

- **Monitoring signals**: The *monitoring signals* $\mu_p, p \in \mathcal{P}$ are generated as

$$\dot{\hat{\mu}}_p = -\lambda \hat{\mu}_p + \gamma |\tilde{y}_p|^2, \; \hat{\mu}_p(0) = 0, \tag{4.25}$$

$$\mu_p = \varepsilon + \hat{\mu}_p, \tag{4.26}$$

for some $\varepsilon > 0$, $\lambda \in (0, \lambda_0)$, where λ_0, γ are as in (4.23b).

- **Switching logic**: A switching logic produces a switching signal that indicates at every time the active controller. The switching signal is produced by the *scale-independent hysteresis switching logic* [21]:

$$\sigma(t) := \begin{cases} \underset{q \in \mathcal{P}}{\operatorname{argmin}}\, \mu_q(t) \text{ if } \exists\, q \in \mathcal{P} \text{ such that} \\ \qquad\qquad\qquad (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) \qquad\qquad \text{else,} \end{cases} \tag{4.27}$$

where $h > 0$ is called a *hysteresis constant*. At every switching time τ of σ, we replace the current controller by the new controller with index $\sigma(\tau)$ and we set $x_{\mathbb{CE}}(\tau) = x_{\mathbb{CE}}(\tau-)$.

4.3.2 Design parameters

The design parameters h and λ must satisfy certain conditions to ensure closed-loop stability. Let $-\hat{\lambda}/2$ be the maximum eigenvalue (negative) of the matrices $A_p + L_p C$

over all $p \in \mathcal{P}$. For the proof of stability,

- the parameter $h > 0$ is chosen such that

$$\ln(1 + h) \leqslant m \ln \mu, \tag{4.28}$$

- then choose $\lambda > 0$ such that

$$(\kappa + 1)\lambda < \lambda_0, \ \kappa := \frac{m \ln \mu}{\ln(1 + h)}, \tag{4.29a}$$

$$\lambda < \hat{\lambda}. \tag{4.29b}$$

Remark 4.3 *We can give the conditions* (4.28), (4.29a), *and* (4.29b) *the following interpretations:* (4.28) *means that the switching logic must be active enough (smaller h) to cope with changing parameters in the plant;* (4.29a) *implies that the "learning rate" λ of the monitoring signal generator must be slower in some sense than the "convergence rate" λ_0 of the injected systems; and* (4.29b) *can be seen to mean that the "learning rate" λ must be slower than the "estimation rate" $\hat{\lambda}$ of the multiestimator. For the case of time-invariant plants (i.e., A, B, C are constant matrices), we only need the condition* (4.29a)*, not the extra conditions* (4.29b) *and* (4.28)*, to prove stability of the closed-loop system [66] (the condition* (4.29a) *can be rewritten as* $\ln(1 + h)/(\lambda m) > \ln \mu/(\lambda_0 - \lambda)$ *exactly as in [66]).*

4.3.3 The closed-loop structure

There are two switched systems in the closed loop:

1. **The switched system Γ_s:** The first switched system arises from the error dynamics. Recall that $\tilde{x}_q := \hat{x}_q - x$ is the state estimation error of the q^{th} subsystem of the multiestimator, $q \in \mathcal{P}$. Then from (4.20), since s is constant

in $[t_s, t)$, we have

$$\dot{\tilde{x}}_{s(t_s)}(t) = (A_{s(t_s)} + L_{s(t_s)}C_{s(t_s)})\tilde{x}_{s(t_s)}(t) + (\delta_A(t) + L_{s(t_s)}\delta_C(t))x + \delta_B(t)u$$
$$+ g_\Delta(z, x, u, t) + v(t) - L_{s(t_s)}w(t).$$

Let $\mathbf{A}_p := A_p + L_pC_p$, $\delta_1(t) := \delta_A(t) + L_{s(t_s)}\delta_C(t)$, and $v_1(t) = v(t) - L_{s(t_s)}w(t)$; $\mathbf{A}_p$ are Hurwitz for all $p \in \mathcal{P}$. In view of $u = K_{\sigma(t)}\hat{x}_{\sigma(t_\sigma)}$ and $x = \tilde{x}_{s(t_s)} + \hat{x}_{s(t_s)}$, we have

$$\dot{\tilde{x}}_{s(t_s)}(t) = \mathbf{A}_{s(t_s)}\tilde{x}_{s(t_s)}(t) + \delta_1(t)\tilde{x}_{s(t_s)}(t) + \delta_2(t)x_{\mathbb{CE}}(t) + g_\Delta(z, x, u, t) + v_1(t), \tag{4.30}$$

where $\delta_2(t)x_{\mathbb{CE}} \equiv \delta_1(t)\hat{x}_{s(t_s)} + \delta_B(t)K_{\sigma(t_s)}\hat{x}_{\sigma(t_\sigma)}$ (recall that $\hat{x}_p$ is a component of $x_{\mathbb{CE}}$ for all $p \in \mathcal{P}$). Equation (4.30) is rewritten as a switched system

$$\dot{\zeta} = \mathbf{A}_s\zeta + \delta_1\zeta + \delta_2 x_{\mathbb{CE}} + \Delta_1 + v_1 \tag{4.31}$$

where $\zeta(t) := \tilde{x}_{s(t_s)}$ and $\Delta_1 := g_\Delta(z, x, u, t)$. For the purpose of analysis later, we will augment ζ by the variable $\xi(t) = \|(x_{t_s})_{0,t}\|^2_{2,\lambda}$ (which is the $e^{\lambda t}$-weighted $\mathcal{L}_{2,\lambda}$ norm of the $[0, t)$ segment of x_{t_s}) to arrive at the following switched system with jumps:

$$\Gamma_s : \begin{cases} \dot{\zeta} & = \mathbf{A}_s\zeta + \Delta_1 + v_1, \\ \dot{\xi} & = -\lambda\xi + |\zeta|^2, \\ \begin{pmatrix} \zeta(t_s) \\ \xi(t_s)) \end{pmatrix} & = \varphi\left(\begin{pmatrix} \zeta(t_s^-) \\ \xi(t_s^-)) \end{pmatrix}\right) \ \forall t \end{cases} \tag{4.32a}$$

for some $\varphi : R^{n+1} \to R^{n+1}$.

2. **The switched system** Π_σ: The second switched system is the switched injected system from (4.22):

$$\Pi_\sigma : \begin{cases} \dot{x}_{\mathbb{CE}} & = \mathbf{E}_\sigma x_{\mathbb{CE}} + \bar{B} u_1 \\ x_{\mathbb{CE}}(t_\sigma) & = x_{\mathbb{CE}}(t_\sigma^-) \end{cases} \quad \forall t \tag{4.33}$$

where $\mathbf{E}_p := \bar{A}_p$ and $u_1 := -\tilde{y}_p$. The matrices $\mathbf{E}_p$ are Hurwitz for all $p \in \mathcal{P}$.

These two switched systems interact as follows:

1. **Constraint 1** This constraint tells how the state of the switched system Γ_s is bounded in terms of the state of switched system Π_σ: for all $t \geqslant 0$,

$$|\zeta(t_s)|^2 \leqslant 2|\zeta(t_s^-)|^2 + 4|x_{\mathbb{CE}}(t_s)|^2, \tag{4.34a}$$

$$\xi(t_s) \leqslant 2\xi(t_s^-) + 4\|(x_{\mathbb{CE}})_{0,t}\|_{2,\lambda}^2. \tag{4.34b}$$

Proof Since $\tilde{x}_p(t) + \hat{x}_p(t) = \tilde{x}_q(t) + \hat{x}_q(t) = x(t)\ \forall t, \forall p, q \in \mathcal{P}$, we have

$$\begin{aligned} |\tilde{x}_p(t)|^2 &\leqslant 2|\tilde{x}_q(t)|^2 + 2|\hat{x}_p - \hat{x}_q|^2 \\ &\leqslant 2|\tilde{x}_q(t)|^2 + 4|x_{\mathbb{CE}}(t)|^2\ \forall t \end{aligned} \tag{4.35}$$

in view of $x_{\mathbb{CE}} = (\hat{x}_1, \ldots, \hat{x}_q)^T$. Therefore, in view of $\zeta(t) = \tilde{x}_{s(t)}(t)$, we have (4.34a). Also from (4.35), $\|(\tilde{x}_p)_{0,t}\|_{2,\lambda}^2 \leqslant 2\|(\tilde{x}_q)_{0,t}\|_{2,\lambda}^2 + 4\|(x_{\mathbb{CE}})_{0,t}\|_{2,\lambda}^2$, we get (4.34b). □

2. **Constraint 2**: This constraint tells how the switching signal Π_σ is bounded in terms of the state of Γ_s:

$$N_\sigma(t, t_0) \leqslant N_0(\xi(t)) + \frac{t - t_0}{\tau_a}, \tag{4.36}$$

$$\|(u_1)_{0,t}\|_{2,\lambda}^2 \leqslant \frac{m(1+h)}{\gamma}\eta(t), \tag{4.37}$$

where

$$\tau_a := \ln(1+h)/(m\lambda), \tag{4.38a}$$

$$N_0(\xi(t)) := m - \frac{m}{\ln(1+h)}\ln\varepsilon + \frac{m}{\ln(1+h)}\ln\eta(t), \tag{4.38b}$$

$$\eta(t) := \Delta_2 + \gamma\bar{c}_2\xi(t) + v_2, \tag{4.38c}$$

$$v_2(t) := \varepsilon + 4\gamma\|(w)_{0,t}\|^2_{2,\lambda}, \tag{4.38d}$$

$$\Delta_2 := 4\gamma\bar{c}_1\|(x_{\mathbb{CE}})_{0,t}\|^2_{2,\lambda} + 4\gamma\|(h_\Delta(z,x,u,\tau)_{0,t}\|^2_{2,\lambda} \tag{4.38e}$$

where $\bar{c}_1 := 2(\|\delta_C\|^2_{[0,\infty)} + \max_{p,q\in\mathcal{P}}\|C_p - C_q\|^2)$ and $\bar{c}_2 > 0$.

Proof We have $\tilde{y}_p(t) = C_{s(t_s)}x + \delta_C(t)x + h_\Delta(z,x,u,t) + w - C_p\hat{x}_p = (\delta_C(t) + C_{s(t_s)} - C_p)\hat{x}_p(t) + (C_{s(t_s)} + \delta_C(t))\tilde{x}_p(t) + h_\Delta(z,x,u,t) + w \; \forall p\in\mathcal{P}, t\geqslant 0$. In view of the definition of $x_{\mathbb{CE}}$, we get

$$|\tilde{y}_p(t)|^2 \leqslant 4\bar{c}_1|x_{\mathbb{CE}}(t)|^2 + 4\bar{c}_2|\tilde{x}_p|^2 + 4|h_\Delta(z,x,u,t)|^2 + 4|w(t)|^2 \tag{4.39}$$

where $\bar{c}_2 := 2(\|\delta_C\|^2_{[0,\infty)} + \max_{p\in\mathcal{P}}\|C_p\|^2)$. Taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of both sides of the foregoing inequality, in view of $\mu_p(t) = \varepsilon + \gamma\|(\tilde{y}_p)_{0,t}\|^2_{2,\lambda}$ from (4.25), we obtain

$$\mu_p(t) \leqslant 4\Delta_2 + 4\gamma\bar{c}_2\|(\tilde{x}_p)_{0,t}\|^2_{2,\lambda} + v_2(t) \tag{4.40}$$

for all $p\in\mathcal{P}$.

The hysteresis switching logic has the following properties ([57, Lemma 4.2] with $\bar{\mu}_p = e^{\lambda t}(\varepsilon + \mu_p(t)$; see also [60]): For every index $q\in\mathcal{P}$ and arbitrary $t\geqslant t_0\geqslant 0$, we have $\forall t\geqslant t_0$,

$$N_\sigma(t,t_0) \leqslant m + \tfrac{m}{\ln(1+h)}\ln\left(\tfrac{\mu_q(t)}{\varepsilon}\right) + \frac{m\lambda(t-t_0)}{\ln(1+h)}, \tag{4.41}$$

$$\|(\tilde{y}_\sigma)_{t_0,t}\|^2_{2\lambda} \leqslant \frac{m(1+h)}{\gamma}\mu_q(t). \tag{4.42}$$

From (4.41), we have (4.36) in view of (4.40) and the definition of $\xi(t)$. From (4.42), the definition of u_1 in (4.33), and (4.40), we have (4.37). $\square$

4.3.4 Interconnected switched systems

To avoid cumbersome notations, in this section, the symbols and notations (such as x, u, etc.) are not related to the symbols in the previous sections i.e., this section is self-contained.

In order to make it easier to understand the closed-loop structure in the previous section, we consider a formalism of two switched systems in the loop which is called an interconnected switched system. The two switched systems are interconnected in the following way (see Fig. 4.6):

- The input u of the second switched system Π_σ is bounded in terms of the state x of the first switched system Γ_s.
- The switching signal σ of Π_σ is bounded in terms of the state x of Γ_s.
- The jump map of Γ_s is influenced by the state z of Π_σ.

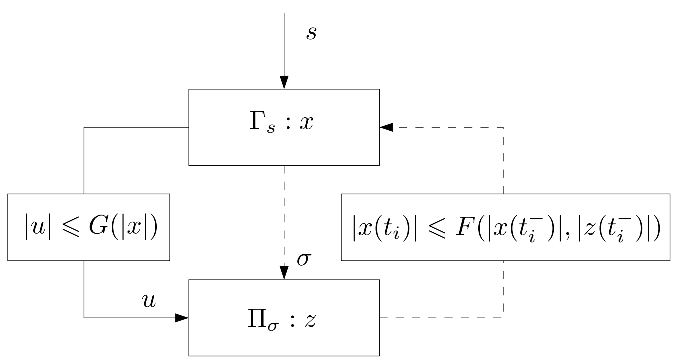

Figure 4.6: Interconnected switched system

Assuming that the subsystems of both switched systems are affine and zero-input exponentially stable, we want to study stability of the closed loop.

Stability of certain interconnected switched systems has been studied in [82, 83, 84]. In these works, the connection between the two switched systems in a loop is the usual input-output connection. In [82], a small-gain theorem for interconnected switched systems is provided. References [83, 84] give passivity theorems for interconnected switched systems and hybrid systems. However, for the loop in Fig. 4.6, the small-gain theorem in [82] and the passivity theorem in [83, 84] are not directly applicable because it is difficult to quantify input-output relationship/input-output gains of the two switched systems, in which the first switched system's jump map is affected by the second switched system and the second switched system's switching signal is constrained by the first switched system. We provide here tools for analyzing such interconnected switched systems.

For the first switched system Γ_s, the switching signal is $s : [0, \infty) \to \mathcal{P}$, where $\mathcal{P}$ is the index set. The dynamics of the subsystems of Γ_s are

$$\Gamma_s : \begin{cases} \dot{x} & = A_p x + v, \\ \dot{\xi} & = -\lambda \xi + |x|^2, \\ \begin{pmatrix} \zeta(t_s) \\ \xi(t_s)) \end{pmatrix} & = \varphi \left(\begin{pmatrix} \zeta(t_s^-) \\ \xi(t_s^-)) \end{pmatrix} \right) \quad \forall t \\ y & = x, \end{cases} \qquad p \in \mathcal{P}, \tag{4.43}$$

where (x, ξ) is the state ($x \in \mathbb{R}^n$, ξ is scalar), v is a bounded disturbance, $\lambda > 0$, and φ is a jump map. Assume that $\xi(0) = 0$ (ξ will later play the role of ξ in (4.32a) where $\xi(0) = 0$ by the definition of ξ).

The second switched system is denoted by Π_σ whose switching signal is $\sigma : [0, \infty) \to \mathcal{P}$ for the same index set $\mathcal{P}$. The subsystems of Π_σ are

$$\Pi_\sigma : \begin{cases} \dot{z} & = E_p z + B_p u, \\ z(t_\sigma) & = z(t_\sigma^-) \ \forall t \end{cases} \qquad p \in \mathcal{P}, \tag{4.44}$$

where u is an input.

The two switched systems interact in the following way. For the first switched system, at a switching time t_i of the switching signal s, the state satisfies

$$|x(t_i)|^2 \leqslant c_1|x(t_i^-)|^2 + c_2|z(t_i^-)|^2, \tag{4.45a}$$

$$\xi(t_i) \leqslant c_1\xi(t_i^-) + c_2\|(z)_{0,t}\|_{2,\lambda}^2 \tag{4.45b}$$

for some constants $c_1, c_2 > 0$.

For the second switched system, the $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of the input u is bounded in terms of ξ:

$$\|(u)_{t_0,t}\|_{2,\lambda}^2 \leqslant f(\xi(t)) \quad \forall t \geqslant t_0 \geqslant 0 \tag{4.46}$$

for some increasing function $f_u : [0,\infty) \to [0,\infty)$. The switching signal σ satisfies the following inequality:

$$N_\sigma(t, t_0) \leqslant N_0(\xi(t)) + \frac{t - t_0}{\tau_a} \quad \forall t \geqslant t_0, \tag{4.47}$$

where $\tau_a > 0$ is some number and N_0 is an increasing function. If N_0 is globally bounded (or equivalently, is a constant), then the bound is known as a chatter bound and τ_a is known as an *average dwell-time* (see Section 3.2). Here, N_0 is not bounded a *priori* and strictly speaking, τ_a is not an average dwell-time (as defined in [21]). However, without causing confusion, we will abuse terminology and also call τ_a in (4.47) an *average dwell-time* and $N_0(\xi(t))$ a *chatter bound* (even though we do not know whether N_0 is bounded or not because it depends on ξ).

We assume that the chatter bound N_0 in (4.47) has the following form:

$$N_0(\xi(t)) := c_3 + c_4 \ln \eta(t), \quad \eta(t) := d + \bar{c}\xi(t) \tag{4.48}$$

for some numbers $c_3, c_4, d, \bar{c} > 0$. We also assume that the function f in (4.46) has the form

$$f(\xi(t)) := c_5\eta(t) \tag{4.49}$$

for some $c_5 > 0$. The reason for these forms is that they stem from the analysis of supervisory adaptive control later.

Assume that A_p, E_p are Hurwitz for all $p \in \mathcal{P}$. Then there exists a family of Lyapunov function $V_p = z^T P_p z$ such that $\dot{V}_p \leqslant -\lambda_0 V_p + \gamma|u|^2$ for some $\lambda_0, \gamma > 0$. Let $\mu > 1$ be a number such that $V_p(z) \leqslant \mu V_q(z)\ \forall z, \forall p \in \mathcal{P}$. Such μ always exists since V_p are quadratic. Assume that τ_a in (4.47) satisfies

$$\tau_a > \frac{\ln \mu}{\lambda_0 - \lambda}. \tag{4.50}$$

If the switching signal s is a constant signal, then (4.45a) and (4.45b) do not come into effect and Γ_s is a nonswitched stable linear system. Therefore, $|x(t)|$ will be bounded for all t. Then ξ is bounded and the chatter bound N_0 is also bounded. It follows from (4.46) that s is an average dwell-time switching signal with a bounded chatter bound. The input u is bounded in view of (4.46). Using the stability result with average dwell-time switching [21], we can conclude that z is bounded if τ_a is large enough, regardless of N_0.

However, the situation is more complicated when s is not a constant signal. The stability results [21, 57] for switched systems without jumps are not applicable here because Γ_s has jumps. Moreover, the jump map of Γ_s involves the state of the second switched system, while the input as well as the switching signal of the second switched system is affected by the state of the first system. This type of mutual interaction makes the analysis of the closed loop's behavior between switching times difficult and the stability result for impulsive systems [65] is also not applicable here (we are not able to find a Lyapunov function as in [65]).

Before going into details, we outline the steps for proving stability of the interconnected switched system.

1. We establish an input-to-state-like property of the switched system Γ_s in terms of the disturbance v and the state z of Π_σ between consecutive switches of s (Lemma 4.2).
2. We establish an input-to-state-like property of the switched system Π_σ with respect to η (which relates to ξ of the first switched system) for arbitrary time intervals using the property of N_σ (Lemma 4.4).
3. We define a Lyapunov-like function W which depends on the states of Γ_s and Π_σ and their norms, and analyze the behavior of W between consecutive switches of s (Lemma 4.5).
4. We establish boundedness of W using a slow switching signal s (as described by event profiles) and conclude boundedness of all continuous signals in the loop.

The switched system Γ_s

Let $-\lambda_0/2 < 0$ be the largest real part of the eigenvalues of E_p for all $p \in \mathcal{P}$ and $-\hat{\lambda}/2 < 0$ be the largest real part of the eigenvalues of A_p for all $p \in \mathcal{P}$.

The state of the first switched system is characterized by the following lemma. The lemma quantifies how the state of the first switched system is bounded in terms of all the states of the closed loop at the latest switching time and the disturbance v.

Lemma 4.2 *For every $\lambda < \hat{\lambda}$, for all $t \geqslant 0$,*

$$|x(t)|^2 \leqslant a_0(c_1|x(t_s^-)|^2 + c_2|z(t_s)|^2)e^{-\hat{\lambda}(t-t_s)} + \hat{\gamma}\|(v)_{t_s,t}\|^2_{2,\hat{\lambda}} \tag{4.51}$$

$$\xi(t) \leqslant f_1(t_s)e^{-\lambda(t-t_s)} + \frac{\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda} \tag{4.52}$$

where $f_1(t_s) := c_1 a_0 \xi(t_s^-) + c_2 a_0 \|(z)_{0,t_s}\|_{2,\lambda}^2 + \frac{c_1 a_0}{\hat{\lambda}-\lambda}|x(t_s^-)|^2 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}|z(t_s)|^2$ *for some constants* $a_0, c_1, c_2, \hat{\gamma} > 0$.

Proof Since s is constant in $[t_s, t)$ and $A_{s(t_s)}$ is Hurwitz, we have

$$|x(t)|^2 \leqslant a_0 e^{-\hat{\lambda}(t-t_s)}|x(t_s)|^2 + \hat{\gamma}\|(v)_{t_s,t}\|_{2,\hat{\lambda}}^2$$

for some $a_0, \hat{\gamma}$. Using (4.45a), we then have (4.51). Taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of (4.51), in view of (1.11) and (1.12), we get

$$\|(|x|)_{t_s,t}\|_{2,\lambda}^2 \leqslant (c_1|x(t_s^-)|^2 + c_2|z(t_s)|^2)\frac{a_0}{\hat{\lambda}-\lambda}e^{-\lambda(t-t_s)} + \frac{\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|_{2,\lambda}^2.$$

Combining the foregoing inequality with (4.45a), we get (4.52) in view of the fact that $\xi(t) = \xi(t_s)e^{-\lambda(t-t_s)} + \|(v_2)_{t_s,t}\|_{2\lambda}^2$. □

The switched system Π_σ

We now characterize the state z of the second switched system. The lemma below characterizes the state z during any interval $[t_0, t)$.

Lemma 4.3 *For every* $\lambda < \lambda_0$,

$$|z(t)|^2 \leqslant \gamma_1 \eta^\kappa(t) e^{\lambda_0 - \lambda\kappa}|z(t_0)|^2 + \gamma_2 \eta^{\kappa+1}(t) \quad \forall t \geqslant t_0 \tag{4.53}$$

for some constants $\gamma_1, \gamma_2 > 0, \kappa > 1$ *and* $\eta(t)$ *is as in* (4.48).

Proof Let $\tau_k, k = 1, \ldots, N_\sigma(t, t_0^+)$ be the switching times of σ in (t_0, t). Let $V_\sigma(t) := V_{\sigma(t)}(z(t))$ where $V_p(z) = z^T P_p z$, $E_p^T P_p + P_p E_p < 0$, is a Lyapunov function for the subsystem in (4.44) with index p. Then $V_\sigma(t) \leqslant e^{-\lambda_0(t-\tau_k)}V_\sigma(\tau_k) + \gamma\|(u)_{\tau_k,t}\|_{2,\lambda_0}^2$ for some γ. Let μ be a constant such that $V_p(z) \leqslant \mu V_q(z)\ \forall p, q \in \mathcal{P}$. Such μ exists because $V_p(z)$ are quadratic in z. Then $V_\sigma(t) \leqslant e^{-\lambda_0(t-\tau_k)}\mu V_\sigma(\tau_k^-) + \gamma\|(u)_{\tau_k,t}\|_{2,\lambda_0}^2$. Letting $t = \tau_{k+1}$ and iterating the foregoing inequality for $k = 1$ to $N_\sigma(t, t_0^+) - 1$, we then have

$$V_\sigma(t) \leqslant \mu^{N_\sigma(t,t_0^+)} e^{-\lambda_0(t-t_0)} V_\sigma(t_0) + \gamma \sum_{i=0}^{N_\sigma(t,t_0^+)} g_i \|(u)_{\tau_i,\tau_{i+1}}\|_{2,\lambda_0}^2 \tag{4.54}$$

where $g_i := \mu^{N_\sigma(t,t_0^+)-i} e^{-\lambda_0(t-\tau_{i+1})}$. Using (4.47), (4.48), and (4.50), we have

$$\mu^{N_\sigma(t,t_0^+)} \leqslant \mu^{c_3+c_4 \ln \eta(t)} \mu^{\frac{t-t_0}{\tau_a}} \leqslant \gamma_0 \eta^\kappa(t) e^{-\lambda\kappa(t-t_0)} \tag{4.55}$$

where $\gamma_0 = \mu^{c_3}$ and $\kappa = c_4 \ln \mu$. Since $\lambda < \lambda_0$, we have

$$\|(u)_{\tau_i,\tau_{i+1}}\|_{2,\lambda_0}^2 \leqslant e^{(\lambda_0-\lambda)\tau_{i+1}} \|(u)_{\tau_i,\tau_{i+1}}\|_{2,\lambda}^2$$

The foregoing inequality and (4.55) give

$$\sum_{i=0}^{N_\sigma(t,t_0^+)} g_i \|(u)_{\tau_i,\tau_{i+1}}\|_{2,\lambda_0}^2 \leqslant \gamma_0 \eta^\kappa(t) \|(u)_{t_0,t}\|_{2,\lambda}^2. \tag{4.56}$$

From (4.54), (4.56), (4.46) and (4.49), we obtain

$$V_\sigma(t) \leqslant \gamma_0 e^{-\lambda_0(t-t_0)} V_\sigma(t_0) + \gamma\gamma_0 c_5 \eta^{\kappa+1}(t).$$

Since V_p are quadratic, we have $\bar{\alpha}_1|x|^2 \leqslant V_p(x) \leqslant \bar{\alpha}_2|x|^2$ for all x, $p \in \mathcal{P}$ for some $\bar{\alpha}_1, \bar{\alpha}_2 > 0$. We then obtain (4.53) where $\gamma_1 := \gamma_0\bar{\alpha}_2/\bar{\alpha}_1$ and $\gamma_2 := \gamma\gamma_0 c_5/\bar{\alpha}_1$. □

The following lemma characterizes z in terms of all of the states of the closed loop at the latest switching time and the disturbance v.

Lemma 4.4 *For every $\lambda < \bar{\lambda}$,*

$$|z(t)|^2 \leqslant g(t) e^{-\lambda(t-t_s)} + \gamma_3 \nu^{\kappa+1}(t) \tag{4.57}$$

$$\|(z)_{0,t}\|_{2,\lambda}^2 \leqslant \|(z)_{0,t_s}\|_{2,\lambda}^2 e^{-\lambda(t-t_s)} + \frac{1}{\bar{\lambda}-\lambda} g(t) e^{-\lambda(t-t_s)} + \gamma_3 \|(\nu^{\frac{\kappa+1}{2}})_{t_s,t}\|_{2,\lambda}^2 \tag{4.58}$$

for all $t \geqslant 0$ where $\bar{\lambda} := \min\{\lambda_0 - \lambda\kappa, (\kappa+1)\lambda\}$, $\nu(t) := d + \frac{\bar{c}\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|_{2,\lambda}^2$, and

$g(t) := (\gamma_1 2^{\kappa-1}\bar{c}^{\kappa} f_1^{\kappa}(t_s) + \gamma_1 2^{\kappa-1}\|\nu\|_{[t_s,t]}^{\kappa})|z(t_s)|^2 + \gamma_2\hat{\gamma}2^{\kappa}\bar{c}^{\kappa+1} f_1^{\kappa+1}(t_s)$ *and* f_1 *is as in Lemma 4.2.*

Proof From the definition of η in (4.53) and (4.52), we have

$$\eta(t) \leqslant d + \bar{c} f_1(t_s)e^{-\lambda(t-t_s)} + \frac{\bar{c}\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|_{2,\lambda}^2. \tag{4.59}$$

For the convex function $a^r, r \geqslant 1$, we have $(a/2+b/2)^r \leqslant (1/2)(a^r+b^r)$ and so $(a+b)^r \leqslant 2^{r-1}(a^r+b^r)\ \forall a, b \geqslant 0$. Using the foregoing inequality, for all $\tau \in [t_s, t)$, for all $r > 1$, we have

$$\eta^r(\tau) \leqslant 2^{r-1}\bar{c}^r f_1^r(t_s)e^{-r\lambda(\tau-t_s)} + 2^{r-1}\nu^r(\tau), \tag{4.60}$$

in view of the definition of ν as in the lemma. Using (4.60) with $r=\kappa$ and $r=\kappa+1$ subsequently, from (4.59), (4.60), and (4.53), we get

$$\begin{aligned}|z(t)|^2 \leqslant &\gamma_1 2^{\kappa-1}\bar{c}^{\kappa} f_1^{\kappa}(t_s)e^{-\lambda_0(t-t_s)}|x_{\mathbb{CE}}(t_s)|^2\\ &+\gamma_1 2^{\kappa-1}\nu^{\kappa}(t)e^{-(\lambda_0-\lambda\kappa)(t-t_s)}|z(t_s)|^2\\ &+\gamma_2 2^{\kappa}\bar{c}^{\kappa+1} f_1^{\kappa+1}(t_s)e^{-(\kappa+1)\lambda(t-t_s)} + \gamma_2 2^{\kappa}\nu^{\kappa+1}(t).\end{aligned}$$

The foregoing inequality leads to (4.57) in view of the definition of g and the fact $\nu(\tau) \leqslant \|\nu\|_{[t_s,t]}\ \forall\tau \in [t_s, t)$. Taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of (4.57), we get $\|(z)_{t_s,t}\|_{2,\lambda}^2 \leqslant \frac{g(t)}{\hat{\lambda}-\lambda}e^{-\lambda(t-t_s)} + \gamma_3\|(\nu^{\frac{\kappa+1}{2}})_{t_s,t}\|_{2\lambda}^2$ where $\gamma_3 := \gamma_2 2^{\kappa}$. The foregoing inequality and the fact that $\|(z)_{0,t}\|_{2\lambda}^2 \leqslant \|(z)_{0,t_s}\|_{2,\lambda}^2 e^{-\lambda(t-t_s)} + \|(z)_{t_s,t}\|_{2,\lambda}^2$ lead to (4.58). ☐

Lyapunov-like function

We now introduce a Lyapunov-like function for the closed loop. Let

$$W(t) := c_1 a_0\xi(t) + c_2 a_0\|(z)_{0,t}\|_{2,\lambda}^2 + \frac{c_1 a_0}{\hat{\lambda}-\lambda}|x(t)|^2 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}|z(t)|^2. \tag{4.61}$$

By convention, $W(0^-) = W(0)$.

Lemma 4.5 *Suppose that $|v(t)| \leqslant \bar{v}\ \forall t$. We then have*

$$W(t) \leqslant (\alpha_1 W^{\kappa}(t_s^-) + \alpha_2) W(t_s^-) e^{-\lambda(t-t_s)} + \alpha_3 \quad \forall t \geqslant 0 \tag{4.62}$$

where κ is as in (4.53) *and*

$$\alpha_1 := (\frac{c_2 a_0}{\bar{\lambda}-\lambda} + \frac{c_2 a_0}{\hat{\lambda}-\lambda})\gamma_4, \tag{4.63a}$$

$$\alpha_2 := c_1 a_0 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_5 + \frac{c_2 a_0 c_1 a_0}{\hat{\lambda}-\lambda} + \frac{c_2 a_0 \gamma_5}{\bar{\lambda}-\lambda}\frac{\hat{\lambda}-\lambda}{c_2 a_0}, \tag{4.63b}$$

$$\alpha_3 := \frac{c_1 a_0 \hat{\gamma}\bar{v}^2}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_3 \bar{\nu}^{\kappa+1} + \frac{c_1 a_0 \hat{\gamma}\bar{v}^2}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_2 a_0 \gamma_3 \bar{\nu}^{\kappa+1}}{\lambda}, \tag{4.63c}$$

$$\bar{\nu} := d + \frac{\bar{c}\hat{\gamma}}{(\bar{\lambda}-\lambda)\lambda}\bar{v}^2, \tag{4.63d}$$

where $\gamma_3 := \gamma_2 2^{\kappa}, \gamma_4 := \gamma_1 2^{\kappa-1}\bar{c}^{\kappa}\frac{\hat{\lambda}-\lambda}{c_2 a_0} + \gamma_2\hat{\gamma}2^{\kappa}\bar{c}^{\kappa+1},\ \gamma_5 := \gamma_1 2^{\kappa-1}\bar{\nu}^{\kappa}$ and γ_1, γ_2 are as in (4.53).

Proof We have $\nu(t) \leqslant d + \frac{\bar{c}\hat{\gamma}}{(\bar{\lambda}-\lambda)\lambda}\bar{v}^2 = \bar{\nu}$. Notice that $W(t_s^-) = f_1(t_s)$ and $|z(t_s)|^2 \leqslant \frac{\hat{\lambda}-\lambda}{c_2 a_0}W(t_s^-)$ from (4.61). We then have

$$\begin{aligned} g(t) &\leqslant \gamma_1 2^{\kappa-1}\bar{c}^{\kappa}\tfrac{\hat{\lambda}-\lambda}{c_2 a_0 \gamma}W^{\kappa+1}(t_s^-) + \gamma_1 2^{\kappa-1}\bar{v}_2^{\kappa}|z(t_s)|^2 + \gamma_2\hat{\gamma}\bar{2}^{\kappa}c^{\kappa+1}W^{\kappa+1}(t_s^-) \\ &=: \gamma_4 W^{\kappa+1}(t_s^-) + \gamma_5|z(t_s)|^2 \end{aligned} \tag{4.64}$$

where $\gamma_4 := \gamma_1 2^{\kappa-1}\bar{c}^{\kappa}\frac{\hat{\lambda}-\lambda}{c_2 a_0} + \gamma_2\hat{\gamma}2^{\kappa}\bar{c}^{\kappa+1}$ and $\gamma_5 := \gamma_1 2^{\kappa-1}\bar{\nu}^{\kappa}$. From (4.57) and (4.64), we get

$$|z(t)|^2 \leqslant (\gamma_4 W^{\kappa+1}(t_s^-) + \gamma_5|z(t_s)|^2)e^{-\lambda(t-t_s)} + \gamma_3\bar{\nu}^{\kappa+1}. \tag{4.65}$$

From (4.58) and (4.64), we have

$$\begin{aligned} \|(z)_{0,t}\|_{2,\lambda}^2 \leqslant (\|(z)_{0,t_s}\|_{2,\lambda}^2 &+ \frac{\gamma_5}{\bar{\lambda}-\lambda}|z(t_s)|^2)e^{-\lambda(t-t_s)} \\ &+ \frac{\gamma_4}{\bar{\lambda}-\lambda}W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} + \frac{\gamma_3\bar{\nu}^{\kappa+1}}{\lambda}. \end{aligned} \tag{4.66}$$

From (4.52), we get

$$\xi(t) \leqslant W(t_s^-)e^{-\lambda(t-t_s)} + \frac{\hat{\gamma}\gamma\bar{v}^2}{(\hat{\lambda}-\lambda)\lambda}. \tag{4.67}$$

From (4.65), (4.66), and (4.67), in view of (4.61), we have

$$\begin{aligned} W(t) \leqslant{}& c_1 a_0 W(t_s^-)e^{-\lambda(t-t_s)} + c_5 W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\ &+ (c_6|z(t_s)|^2 + \tfrac{c_1 a_0}{\hat{\lambda}-\lambda} a_0 c_1 |x(t_s^-)|^2 + c_2 a_0 \|(z)_{0,t_s}\|_{2,\lambda}^2)e^{-\lambda(t-t_s)} + c_7, \end{aligned} \tag{4.68}$$

where $c_5 := \frac{c_2 a_0 \gamma_4}{\lambda-\lambda} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_4$, $c_6 := \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_5 + \frac{c_2 a_0 c_1 a_0}{\hat{\lambda}-\lambda} + \frac{c_2 a_0 \gamma_5}{\lambda-\lambda}$, and $c_7 := \frac{c_1 a_0}{\hat{\lambda}-\lambda}\frac{\hat{\gamma}\bar{v}^2}{\hat{\lambda}} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_3\bar{\nu}^{\kappa+1} + \frac{c_1 a_0 \hat{\gamma}\bar{v}^2}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_2 a_0 \gamma_3 \bar{\nu}^{\kappa+1}}{\lambda}$. Let $c_8 := \max\{c_6 \frac{\hat{\lambda}-\lambda}{c_2 a_0 \gamma}, c_1 a_0, 1\} = c_6 \frac{\hat{\lambda}-\lambda}{c_2 a_0}$. Then from (4.68), we get

$$\begin{aligned} W(t) \leqslant{}& c_1 W(t_s^-)e^{-\lambda(t-t_s)} + c_5 W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\ &+ c_8 W(t_s^-)e^{-\lambda(t-t_s)} + c_7. \end{aligned}$$

We then have (4.62) where $\alpha_1 := c_5, \alpha_2 := c_1 a_0 + c_8$, and $\alpha_3 := c_7$. □

Stability of the closed-loop

Let $W_0(X_0, Z_0) := \rho(\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}Z_0)(\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}Z_0)$ for some $X_0, Z_0 > 0$. We then have the following theorem.

Theorem 4.3 *Consider the interconnected switched system described in Section 4.3.4. Let $X_0, Z_0 > 0$. Then for all $|x(0)|^2 \leqslant X_0$, $|z(0)|^2 \leqslant Z_0$, and for every switching signal s not faster than $\alpha_{\rho,W_0(X_0,Z_0),\delta_d,\overline{N}}$ where $\alpha_{\rho,W_0(X_0,Z_0),\delta_d,\overline{N}}$ is as in* (3.74), *we have*

$$\max\{|x(t)|^2, |z(t)|^2\} \leqslant \bar{\gamma}_1(W_0(X_0, Z_0))e^{-\lambda} + \bar{\gamma}_2(\alpha_3) \quad \forall t \tag{4.69}$$

for some $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}_\infty$ and $\delta_d \geqslant 0, \overline{N} \geqslant 1$.

Proof Using Lemma 3.2 with W in (4.61) and then letting $\bar{\gamma}_1 := \bar{c}_0\gamma_1, \bar{\gamma}_2 := \bar{c}_0\gamma_2$ where $\bar{c}_0 := \max\{\frac{\hat{\lambda}-\lambda}{c_1 a_0}, \frac{\hat{\lambda}-\lambda}{c_2 a_0}\}$, we will obtain (4.69). □

4.3.5 Interconnected switched systems with unmodeled dynamics

In the previous section, we considered interconnected switched systems without unmodeled dynamics to convey the idea behind the stability proof of supervisory adaptive control. To handle the unmodeled dynamics in supervisory adaptive control, we now consider interconnected switched systems with unmodeled dynamics (see Fig. 4.7).

The uncertainties enter the interconnected switched system as follows. The dynamics of Γ_s are

$$\Gamma_s : \begin{cases} \dot{x} & = A_\sigma x + \delta_1 x + \delta_2 z + \Delta_1(\theta, x, z) + v, \\ \dot{\xi} & = -\lambda\xi + |x|^2, \\ \dot{\theta} & = f_\Delta(\theta, x, z, t) \\ \begin{pmatrix} \zeta(t_s) \\ \xi(t_s)) \end{pmatrix} & = \varphi\left(\begin{pmatrix} \zeta(t_s^-) \\ \xi(t_s^-)) \end{pmatrix}\right) \ \forall t \end{cases} \tag{4.70}$$

where $\delta_1(t)x + \delta_2(t)z$ is the additive uncertainty, δ_1, δ_2 are matrices, and θ is the state of the multiplicative uncertainty. The dynamics of Π_σ are the same as in (4.44). However, the signal η in (4.48) becomes

$$\eta := \Delta_2(z, x, z) + \bar{c}\xi + d. \tag{4.71}$$

The unmodeled dynamics Δ_1, Δ_2 are small in the following sense:

$$|\Delta_1(t)|^2 \leqslant \delta\gamma^0_{\Delta_1}(|z(0)|) + \delta\gamma^1_{\Delta_1}(\|x\|_{[0,t)}) + \delta\gamma^2_{\Delta_1}(\|z\|_{[0,t)}) \tag{4.72a}$$

$$|\Delta_2(t)|^2 \leqslant \delta\gamma^0_{\Delta_2}(|z(0)|) + \delta\gamma^1_{\Delta_2}(\|x\|_{[0,t)}) + \delta\gamma^2_{\Delta_2}(\|z\|_{[0,t)}) \tag{4.72b}$$

for some number $\delta \geqslant 0$ and $\gamma^i_{\Delta_1}, \gamma^i_{\Delta_2} \in \mathcal{K}_\infty, i = 0, 1, 2$. We say that the unmodeled dynamics are bounded by δ.

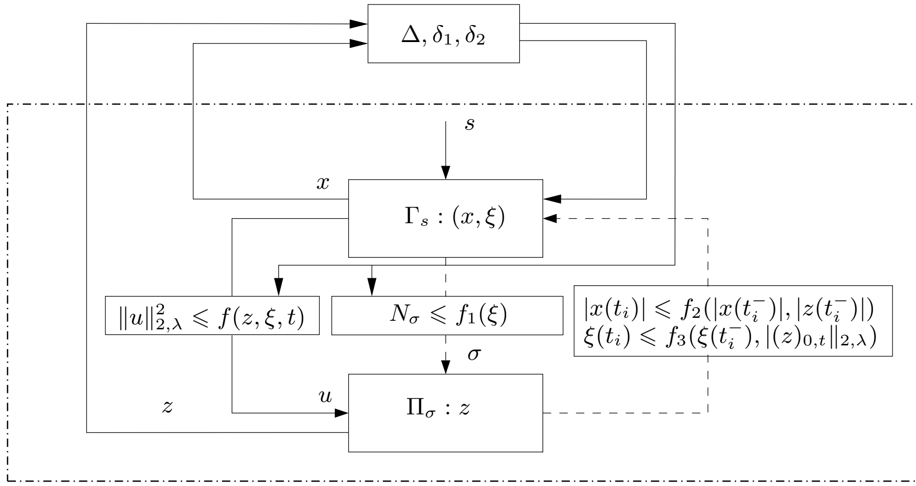

Figure 4.7: Interconnected switched systems with unmodeled dynamics

Compared to the interconnected system in Fig. 4.6, the configuration in Fig. 4.7 has the extra block Δ and extra signals from Δ coming into the function N_0 and f. The dash-dot box (Γ_s, Π_σ) denotes the interconnected switched system without unmodeled dynamics.

We assume that $\delta_1(t)$ is small enough in the norm as follows. Since A_p are Hurwitz for all $p \in \mathcal{P}$, from linear system theory, there always exists a $\bar{\delta} > 0$ such that for all $\|\delta_1\|_{[0,\infty)} \leqslant \bar{\delta}$ and for all $p \in \mathcal{P}$, the system $\dot{x} = (A_p + \delta_1(t))x$ is uniformly exponentially stable i.e., $|x(t)|^2 \leqslant ce^{-\hat{\lambda}(t-t_0)}|x(t_0)|^2 \ \forall t \geqslant t_0$ for some $c, \hat{\lambda} > 0$. Denote by $\bar{\delta}_1^{max}$ the supremum of all such $\bar{\delta}$. Assume that $\|\delta_1\|_{[0,\infty)} \leqslant \bar{\delta}_1$ for some $0 < \bar{\delta}_1 \leqslant \bar{\delta}_1^{max}$.

The lemmas below are the extensions of corresponding lemmas for the case without unmodeled dynamics in the previous section.

Lemma 4.6 *For every $\lambda < \hat{\lambda}$, for all $t \geqslant 0$,*

$$\begin{aligned} |x(t)|^2 \leqslant a_0(c_1|x(t_s^-)|^2 + c_2|z(t_s)|^2)e^{-\hat{\lambda}(t-t_s)} + \hat{\gamma}_0\bar{\delta}_2\|(z)_{t_s,t}\|_{2,\hat{\lambda}}^2 \\ + \hat{\gamma}_1\|(\Delta_1)_{t_s,t}\|_{2,\hat{\lambda}}^2 + \hat{\gamma}_2\|(v)_{t_s,t}\|_{2,\hat{\lambda}}^2 \end{aligned} \tag{4.73}$$

$$\xi(t) \leqslant f_1(t_s)e^{-\lambda(t-t_s)} + \tfrac{\hat{\gamma}_0}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|_{2,\lambda}^2 + \tfrac{\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_1)_{t_s,t}\|_{2,\lambda}^2 + \tfrac{\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|_{2,\lambda}^2 \tag{4.74}$$

where $f_1(t_s) := c_1a_0\xi(t_s^-) + c_2a_0\|(z)_{0,t_s}\|_{2,\lambda}^2 + \frac{c_1a_0}{\hat{\lambda}-\lambda}|x(t_s^-)|^2 + \frac{c_2a_0}{\hat{\lambda}-\lambda}|z(t_s)|^2$ *for some con-*

stants $a_0, \hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2 > 0$ *and* $\bar{\delta}_2 := \|\delta_2\|_{[0,\infty)}$.

Proof Since s is constant in $[t_s, t)$, we have

$$|x(t)|^2 \leqslant a_0 e^{-\hat{\lambda}(t-t_s)}|x(t_s)|^2 + \hat{\gamma}_0\bar{\delta}_2\|(z)_{t_s,t}\|^2_{2,\hat{\lambda}} + \hat{\gamma}_1\|(\Delta_1)_{t_s,t}\|^2_{2,\hat{\lambda}} + \hat{\gamma}_2\|(v)_{t_s,t}\|^2_{2,\hat{\lambda}}$$

for some $a_0, \hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2 > 0$. Using (4.45a), we then have (4.73). Taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of (4.73), in view of (1.11) and (1.12), we get

$$\begin{aligned}\|(|x|)_{t_s,t}\|^2_{2,\lambda} \leqslant{}& (c_1|x(t_s^-)|^2 + c_2|z(t_s)|^2)\frac{a_0}{\hat{\lambda}-\lambda}e^{-\lambda(t-t_s)} \\ &+ \frac{\hat{\gamma}_0\bar{\delta}_2}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|^2_{2,\lambda} + \frac{\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_1)_{t_s,t}\|^2_{2,\lambda} + \frac{\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda}.\end{aligned}$$

Combining the foregoing inequality with (4.45a) and the fact that $\xi(t) = \xi(t_s)e^{-\lambda(t-t_s)} + \|(v_2)_{t_s,t}\|^2_{2\lambda}$, we get (4.74). $\square$

Lemma 4.7 *For every* $\lambda < \frac{\lambda_0}{\kappa+1}$, $\bar{\alpha} \in (0,1)$,

$$|z(t)|^2 \leqslant g_{\bar{\alpha}}(t)e^{-\bar{\lambda}(t-t_s)} + \gamma_3\nu^{\kappa+1}(t) \tag{4.75}$$

$$\|(z)_{0,t}\|^2_{2,\lambda} \leqslant \|(z)_{0,t_s}\|^2_{2,\lambda}e^{-\lambda(t-t_s)} + \frac{g_{\bar{\alpha}}(t)}{\bar{\lambda}-\lambda}e^{-\lambda(t-t_s)} + \gamma_3\|(\nu^{\frac{\kappa+1}{2}})_{t_s,t}\|^2_{2,\lambda},\ \forall t \geqslant 0 \tag{4.76}$$

where $\nu(t) := d + |\Delta_2(t)| + \frac{\bar{c}\hat{\gamma}_0\bar{\delta}_2}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|^2_{2,\lambda} + \frac{\bar{c}\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_1)_{t_s,t}\|^2_{2,\lambda} + \frac{\bar{c}\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda}$, $g_{\bar{\alpha}}(t) := (\gamma_1\bar{\alpha}^{1-\kappa}\bar{c}^{\kappa}f_1^{\kappa}(t_s) + \gamma_1(1-\bar{\alpha})^{1-\kappa}\|\nu\|^{\kappa}_{[t_s,t]})|z(t_s)|^2 + \gamma_2\bar{\alpha}^{-\kappa}\bar{c}^{\kappa+1}f_1^{\kappa+1}(t_s)$ *and* f_1 *is as in Lemma 4.6, and* $\bar{\lambda} := \min\{\lambda_0 - \lambda\kappa, (\kappa+1)\lambda\}$.

Proof From the definition of η in (4.71) and (4.74), we have

$$\begin{aligned}\eta(t) \leqslant{}& d + |\Delta_2(t)| + \bar{c}f_1(t_s)e^{-\lambda(t-t_s)} \\ &+ \tfrac{\bar{c}\hat{\gamma}_1}{\hat{\lambda}-\lambda}\|(\Delta_1)_{t_s,t}\|^2_{2,\lambda} + \tfrac{\bar{c}\hat{\gamma}_2}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda}.\end{aligned} \tag{4.77}$$

For the convex function $a^r, r \geqslant 1$ and any number $\bar{\alpha} \in (0,1)$, we have $(a+b)^r = (\bar{\alpha}a/\bar{\alpha} + (1-\bar{\alpha})b/(1-\bar{\alpha}))^r \leqslant \bar{\alpha}^{1-r}a^r + (1-\bar{\alpha})^{1-r}b^r\ \forall a, b \geqslant 0$. Pick an $\bar{\alpha} \in (0,1)$, using the foregoing inequality, for all $\tau \in [t_s, t)$, for all $r > 1$, we have

$$\eta^r(\tau) \leqslant \bar{\alpha}^{1-r}\bar{c}^r f_1^r(t_s)e^{-r\lambda(\tau-t_s)} + (1-\bar{\alpha})^{1-r}\nu^r(\tau), \tag{4.78}$$

in view of the definition of ν as in the lemma. Using (4.78) with $r = \kappa$ and $r = \kappa + 1$ subsequently, from (4.77), (4.78), and (4.53), we get

$$\begin{aligned}|z(t)|^2 \leqslant{}& \gamma_1\bar{\alpha}^{1-\kappa}\bar{c}^\kappa f_1^\kappa(t_s)e^{-\lambda_0(t-t_s)}|z(t_s)|^2\\ &+ \gamma_1(1-\bar{\alpha})^{1-\kappa}\nu^\kappa(t)e^{-(\lambda_0-\lambda\kappa)(t-t_s)}|z(t_s)|^2\\ &+ \gamma_2\bar{\alpha}^{-\kappa}\bar{c}^{\kappa+1}f_1^{\kappa+1}(t_s)e^{-(\kappa+1)\lambda(t-t_s)}\\ &+ \gamma_2(1-\bar{\alpha})^{-\kappa}\nu^{\kappa+1}(t).\end{aligned}$$

The foregoing inequality leads to (4.75) in view of the definition of g and the fact $\nu(\tau) \leqslant \|\nu\|_{[t_s,t]}\ \forall\tau \in [t_s, t)$. Taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of (4.75), we get $\|(z)_{t_s,t}\|^2_{2,\lambda} \leqslant \frac{g(x,\xi,z,\nu,t)}{\bar{\lambda}-\lambda}e^{-\lambda(t-t_s)} + \gamma_3\|(\nu^{\frac{\kappa+1}{2}})_{t_s,t}\|^2_{2\lambda}$ where $\gamma_3 := \gamma_2 2^\kappa$. The foregoing inequality and the fact that $\|(z)_{0,t}\|^2_{2\lambda} \leqslant \|(z)_{0,t_s}\|^2_{2,\lambda}e^{-\lambda(t-t_s)} + \|(z)_{t_s,t}\|^2_{2,\lambda}$ leads to (4.76). $\square$

Let

$$W(t) := c_1 a_0\xi(t) + c_2 a_0\|(z)_{0,t}\|^2_{2,\lambda} + \tfrac{c_1 a_0}{\hat{\lambda}-\lambda}|x(t)|^2 + \tfrac{c_2 a_0}{\hat{\lambda}-\lambda}|z(t)|^2. \tag{4.79}$$

Lemma 4.8 *Suppose that* $|v(t)| \leqslant \bar{v}$, $\frac{\bar{c}\hat{\gamma}_0\bar{\delta}_2}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|^2_{2,\lambda} \leqslant \bar{\delta}_z\ \forall t \geqslant 0$, $|\Delta_1(t)| \leqslant \bar{\Delta}_1$, *and* $|\Delta_2(t)| \leqslant \bar{\Delta}_2\ \forall t \in [0, T)$ *for some* $T > 0$. *Suppose that* $0 < \lambda < \min\{\hat{\lambda}, \bar{\lambda}, \lambda_\Delta\}$. *We have*

$$W(t) \leqslant (\alpha_1 W^\kappa(t_s^-) + \alpha_2)W(t_s^-)e^{-\lambda(t-t_s)} + \alpha_3\ \forall t \in [0, T), \tag{4.80}$$

where

$$\alpha_1 := \frac{c_2 a_0}{\bar{\lambda}-\lambda}\gamma_4 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_4 \tag{4.81a}$$

$$\alpha_2 := 2c_1 a_0 + \gamma_1 2^{\kappa-1}\bar{\nu}^{\kappa} + \gamma_1 2^{\kappa-1}\bar{\nu}^{\kappa}\frac{\hat{\lambda}-\lambda}{\bar{\lambda}-\lambda} \tag{4.81b}$$

$$\alpha_3 := \frac{c_1 a_0}{\hat{\lambda}-\lambda}\frac{\hat{\gamma}\bar{v}^2}{\hat{\lambda}} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_3\bar{\nu}^{\kappa+1} + \frac{c_1 a_0\hat{\gamma}_2\bar{v}^2}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_2 a_0\gamma_3\bar{\nu}^{\kappa+1}}{\lambda} + \frac{c_1 a_0}{\hat{\lambda}-\lambda}\hat{\gamma}_1\bar{\delta}_1 + \frac{c_1 a_0\hat{\gamma}_1}{(\hat{\lambda}-\lambda)\hat{\lambda}}\bar{\delta}_1 \tag{4.81c}$$

$$\bar{\nu} := d + \bar{\delta}_z + \bar{\Delta}_2 + \frac{\bar{c}\hat{\gamma}_1}{(\hat{\lambda}-\lambda)\lambda}\bar{\Delta}_1 + \frac{\bar{c}\hat{\gamma}_2}{\hat{\lambda}-\lambda}\bar{v}^2. \tag{4.81d}$$

Proof In view of ν as in Lemma 4.7 and the lemma hypothesis, we have $|\nu(t)| \leqslant \bar{\nu}$.

Notice that $W(t_s^-) = f_1(t_s)$ and $|z(t_s)|^2 \leqslant \frac{\hat{\lambda}-\lambda}{c_2 a_0}W(t_s^-)$ from (4.79). We then have

$$\begin{aligned} g_{\bar{\alpha}}(t) &\leqslant \gamma_1\bar{\alpha}^{1-\kappa}\bar{c}^{\kappa}\tfrac{\hat{\lambda}-\lambda}{c_2 a_0}W^{\kappa+1}(t_s^-) + \gamma_1(1-\bar{\alpha})^{1-\kappa}\bar{\nu}^{\kappa}|z(t_s)|^2 \\ &\quad + \gamma_2\bar{\alpha}^{-\kappa}\bar{c}^{\kappa+1}W^{\kappa+1}(t_s^-) \\ &=: \gamma_4 W^{\kappa+1}(t_s^-) + \gamma_5|z(t_s)|^2, \end{aligned} \tag{4.82}$$

where $\gamma_4 := \gamma_1\bar{\alpha}^{1-\kappa}\bar{c}^{\kappa}\frac{\hat{\lambda}-\lambda}{c_2 a_0} + \gamma_2\bar{\alpha}^{-\kappa}\bar{c}^{\kappa+1}$ and $\gamma_5 := \gamma_1(1-\bar{\alpha})^{1-\kappa}\bar{\nu}^{\kappa}$. From (4.75) and (4.82), we get

$$|z(t)|^2 \leqslant (\gamma_4 W^{\kappa+1}(t_s^-) + \gamma_5|z(t_s)|^2)e^{-\bar{\lambda}(t-t_s)} + \gamma_3\bar{\nu}^{\kappa+1}, \tag{4.83}$$

$$\begin{aligned} \|(z)_{t_s,t}\|_{2,\lambda}^2 &\leqslant \tfrac{1}{\bar{\lambda}-\lambda}(\gamma_4 W^{\kappa+1}(t_s^-) + \gamma_5|z(t_s)|^2)e^{-\lambda(t-t_s)} \\ &\quad + \frac{\gamma_3}{\lambda}\bar{\nu}^{\kappa+1}. \end{aligned} \tag{4.84}$$

From (4.76) and (4.82), we have

$$\begin{aligned} \|(z)_{0,t}\|_{2,\lambda}^2 &\leqslant (\|(z)_{0,t_s}\|_{2,\lambda}^2 + \frac{\gamma_5}{\bar{\lambda}-\lambda}|z(t_s)|^2)e^{-\lambda(t-t_s)} \\ &\quad + \frac{\gamma_4}{\bar{\lambda}-\lambda}W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} + \frac{\gamma_3\bar{\nu}^{\kappa+1}}{\lambda}. \end{aligned} \tag{4.85}$$

From (4.73), we get

$$
\begin{aligned}
|x(t)|^2 \leqslant\ & a_0(c_1|x(t_s^-)|^2+c_2|z(t_s)|^2)e^{-\hat{\lambda}(t-t_s)} \\
& + \tfrac{\hat{\gamma}_0\bar{\delta}_2\gamma_5}{\bar{\lambda}-\lambda}|z(t_s)|^2 e^{-\lambda(t-t_s)} + \tfrac{\hat{\gamma}_0\bar{\delta}_2\gamma_4}{\bar{\lambda}-\lambda} W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\
& + \tfrac{\hat{\gamma}_0\bar{\delta}_2\gamma_3}{(\hat{\lambda}-\lambda)\lambda}\bar{\nu}^{\kappa+1} + \hat{\gamma}_1\bar{\Delta}_1 + \tfrac{\hat{\gamma}_2}{\lambda}\bar{v}^2.
\end{aligned}
\tag{4.86}
$$

From (4.74), we get

$$
\begin{aligned}
\xi(t) \leqslant\ & W(t_s^-)e^{-\lambda(t-t_s)} + \tfrac{\hat{\gamma}_0\bar{\delta}_2\gamma_5}{(\hat{\lambda}-\lambda)(\bar{\lambda}-\lambda)}|z(t_s)|^2 e^{-\lambda(t-t_s)} \\
& + \tfrac{\hat{\gamma}_0\bar{\delta}_2\gamma_4}{(\hat{\lambda}-\lambda)(\bar{\lambda}-\lambda)} W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\
& + \frac{\hat{\gamma}_0\bar{\delta}_2\gamma_3}{(\hat{\lambda}-\lambda)\lambda}\bar{\nu}^{\kappa+1} + \frac{\hat{\gamma}_1\bar{\Delta}_1}{(\hat{\lambda}-\lambda)\lambda} + \frac{\hat{\gamma}_2\bar{v}^2}{(\hat{\lambda}-\lambda)\lambda}.
\end{aligned}
\tag{4.87}
$$

From (4.83), (4.85), and (4.87), in view of (4.79), we have

$$
\begin{aligned}
W(t) \leqslant\ & c_1 a_0 W(t_s^-)e^{-\lambda(t-t_s)} + c_5 W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} + c_7 \\
& + (c_6|z(t_s)|^2 + \tfrac{c_1^2 a_0^2}{\hat{\lambda}-\lambda}|x(t_s^-)|^2 + c_2 a_0\|(z)_{0,t_s}\|_{2,\lambda}^2)e^{-\lambda(t-t_s)},
\end{aligned}
\tag{4.88}
$$

where $c_5 := \frac{c_1 a_0\hat{\gamma}_0\bar{\delta}_2\gamma_4}{(\hat{\lambda}-\lambda)(\bar{\lambda}-\lambda)} + \frac{c_2 a_0\gamma_4}{\bar{\lambda}-\lambda} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_4$, $c_6 := \frac{c_1 a_0\hat{\gamma}_0\bar{\delta}_2\gamma_5}{(\hat{\lambda}-\lambda)(\bar{\lambda}-\lambda)} + \frac{c_2 a_0\gamma_5}{\bar{\lambda}-\lambda} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_5 + (c_2 a_0 + \frac{\hat{\gamma}_0\bar{\delta}_2\gamma_5}{\bar{\lambda}-\lambda})\frac{c_1 a_0}{\hat{\lambda}-\lambda}$, and $c_7 := \frac{c_1 a_0}{\hat{\lambda}-\lambda}\hat{\gamma}_1\bar{\delta}_1 + \frac{c_1 a_0}{\hat{\lambda}-\lambda}\frac{\hat{\gamma}\bar{v}^2}{\hat{\lambda}} + \frac{c_2 a_0}{\hat{\lambda}-\lambda}\gamma_3\bar{\nu}^{\kappa+1} + \frac{c_1 a_0\hat{\gamma}_1\bar{\delta}_1}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_1 a_0\hat{\gamma}_2\bar{v}^2}{(\hat{\lambda}-\lambda)\hat{\lambda}} + \frac{c_2 a_0\gamma_3\bar{\nu}^{\kappa+1}}{\lambda}$. Let $c_8 := \max\{c_6\frac{\hat{\lambda}-\lambda}{c_2 a_0\gamma}, c_1 a_0, 1\} = c_6\frac{\hat{\lambda}-\lambda}{c_2 a_0}$. Then from (4.88), we get

$$
\begin{aligned}
W(t) \leqslant\ & c_1 W(t_s^-)e^{-\lambda(t-t_s)} + c_5 W^{\kappa+1}(t_s^-)e^{-\lambda(t-t_s)} \\
& + c_8 W(t_s^-)e^{-\lambda(t-t_s)} + c_7.
\end{aligned}
$$

We then have (4.80) where $\alpha_1 := c_5, \alpha_2 := c_1 + c_8$, and $\alpha_3 := c_7$. $\square$

The following result quantifies the robustness property of a class of switching signals $\mathcal{S}$ in the following sense: every $s \in \mathcal{S}$ implies stability of the interconnected switched system without unmodeled dynamics and the same switching signal still guarantees stability in the presence of unmodeled dynamics provided that the unmodeled dynamics are small enough. The set $\mathcal{S}$ depends on the bound of the initial state and the bound on the unmodeled dynamics also depends on the bound of the initial state of the interconnected switched system.

Let $a = \bar{\Delta}_1$ and $b = \bar{\delta}_z + \bar{\Delta}_2$ in (4.81d) and define the functions $\bar{\alpha}_2(a,b) := \alpha_2$ and $\bar{\alpha}_3(a,b) := \alpha_3$ with α_2 as in (4.81b) and α_3 as in (4.81c). Let $\rho_{a,b}(M) := \alpha_1 M^\kappa + \bar{\alpha}_2(a,b)$ where κ is the constant as in Lemma 4.8 and α_1 is as in (4.81a). Let $W_{a,b}(X_0, Z_0) := \rho_{a,b}(\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}Z_0)(\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0 + \frac{c_2 a_0}{\hat{\lambda}-\lambda}Z_0)$ for some $X_0, Z_0 > 0$. Recall that δ is the bound on the unmodeled dynamics of θ as in (4.72) and $\bar{\delta}_1$, $\bar{\delta}_2$ are the bounds on $\mathcal{L}_\infty$-norm of $\delta_1(t)$ and $\delta_2(t)$, respectively.

Theorem 4.4 *Consider the interconnected switched system described in Section 4.3.5. Let $\bar{\delta}_a, \bar{\delta}_b, X_0, Z_0 > 0$. Then for every $E > 0$, there exists a number $\bar{\delta} > 0$ such that if $\max\{\delta, \bar{\delta}_1, \bar{\delta}_2\} \leqslant \bar{\delta}$ and $|\theta(0)| < E$, then we have*

$$\max\{|x(t)|^2, |z(t)|^2\} \leqslant \bar{\gamma}_1(W_{\bar{\delta}_a,\bar{\delta}_b}(X_0, Z_0))e^{-\lambda} + \bar{\gamma}_2(\bar{\alpha}_3(\bar{\delta}_a, \bar{\delta}_b)) \quad \forall t \tag{4.89}$$

for all $|x(0)|^2 \leqslant X_0, |z(0)|^2 \leqslant Z_0$ and every s not faster than $\alpha_{\rho,W_0,\delta_d,\overline{N}}$ where $\alpha_{\rho,W_0,\delta_d,\overline{N}}$ is as in (3.74) for some $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}_\infty$ and $\delta_d \geqslant 0, \overline{N} \geqslant 1$.

Proof Let $T = \sup\{t : |\Delta_1(t)| \leqslant \bar{\Delta}_1, |\Delta_2(t)| \leqslant \bar{\Delta}_2, \frac{\bar{c}\hat{\gamma}_0\bar{\delta}_2}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|^2_{2,\lambda} \leqslant \bar{\delta}_2\}$. Then applying Lemma 3.2 using W as in (4.80), we have that $W(t) \leqslant \gamma_1(W_{\bar{\delta}_a,\bar{\delta}_b}(X_0, Z_0))e^{-\bar{\lambda}t} + \gamma_2(\alpha_3)$ $\forall t \in [0,T)$ for some $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. Let $\overline{W} := \gamma_1(W_{\bar{\delta}_a,\bar{\delta}_b}(X_0, Z_0)) + \gamma_2(\alpha_3)$. From the definition of $W(t)$, we then have $|x(t)|^2 \leqslant \frac{\hat{\lambda}-\lambda}{c_1 a_0}\overline{W}$, $|z(t)|^2 \leqslant \frac{\hat{\lambda}-\lambda}{c_2 a_0}\overline{W}$. In view of (4.72a), (4.72b), and the definition of $\bar{\delta}_z$, if $\delta \leqslant \min\{(\gamma^0_{\Delta_1}(E) + \gamma^1_{\Delta_1}(\frac{\hat{\lambda}-\lambda}{c_1 a_0}\overline{W}) + \gamma^2_{\Delta_1}(\frac{\hat{\lambda}-\lambda}{c_2 a_0}\overline{W}))^{-1}\bar{\delta}_a, (\gamma^0_{\Delta_2}(E) + \gamma^1_{\Delta_2}(\frac{\hat{\lambda}-\lambda}{c_1 a_0}\overline{W}) + \gamma^2_{\Delta_2}(\frac{\hat{\lambda}-\lambda}{c_2 a_0}\overline{W}))^{-1}\bar{\delta}_b/2, \frac{\bar{\delta}_b}{2}\frac{c_2 a_0}{\overline{W}\bar{c}\hat{\gamma}_0}, \bar{\delta}_1^{\max}\} =: \bar{\delta}$, then $\bar{\delta}_1 \leqslant \delta \leqslant \bar{\delta}_1^{max}$, $|\Delta_1(t)| \leqslant \bar{\delta}_a, |\Delta_2(t)| + \frac{\bar{c}\hat{\gamma}_0\bar{\delta}_2}{\hat{\lambda}-\lambda}\|(z)_{t_s,t}\|^2_{2,\lambda} \leqslant \bar{\delta}_b$ $\forall t \in [0,T)$. From the definition of T, we conclude that $T = \infty$. Thus, if the unmodeled dynamics are bounded by $\bar{\delta}$ and the initial state of the unmodeled dynamic is bounded by E, then $W(t) \leqslant \overline{W}$ $\forall t \geqslant 0$. It follows that all the states are bounded. We then have (4.95) with $\bar{\gamma}_1 := \bar{c}_0\gamma_1, \bar{\gamma}_2 := \bar{c}_0\gamma_2$ where $\bar{c}_0 := \max\{\frac{\hat{\lambda}-\lambda}{c_1 a_0}, \frac{\hat{\lambda}-\lambda}{c_2 a_0}\}$. $\square$

4.3.6 Stability of the closed-loop control system

We say that the unmodeled dynamics of z are bounded by δ_Δ if (it has bounded input/output-to-state gain):

$$|g_\Delta(z,x,u,t)|^2 \leqslant \delta_\Delta \gamma_g^0(|z(0)|)e^{-\lambda_\Delta t} + \delta_\Delta \gamma_g^1(\|x\|_{[0,t)}) + \delta_\Delta \gamma_g^2(\|u\|_{[0,t)}) \tag{4.90a}$$

$$|h_\Delta(z,x,u,t)|^2 \leqslant \delta_\Delta \gamma_h^0(|z(0)|)e^{-\lambda_\Delta t} + \delta_\Delta \gamma_h^1(\|x\|_{[0,t)}) + \delta_\Delta \gamma_h^2(\|u\|_{[0,t)}) \tag{4.90b}$$

for some numbers $\delta_\Delta, \lambda_\Delta > 0$ and functions $\gamma_g^i, \gamma_h^i \in \mathcal{K}_\infty, i = 0, 1, 2$. Define

$$\delta_P := \max\{\|\delta_A(t)\|_{[0,\infty)}, \|\delta_B(t)\|_{[0,\infty)}, \|\delta_C(t)\|_{[0,\infty)} + \sup_{p,q\in\mathcal{P}} \|C_p - C_q\|\}. \tag{4.91}$$

Theorem 4.5 *Consider the supervisory adaptive control scheme described in Section 4.3.1. For every $M, \bar{v}, \bar{w} > 0$, there exist an event profile $\bar{\alpha}$ and a number $\bar{\delta} \geqslant 0$ such that if $\delta_P \leqslant \bar{\delta}$, $\|v\|_\infty \leqslant \bar{v}$, and $\|w\|_\infty \leqslant \bar{w}$, for all unmodeled dynamics such that* $\max\{\delta_\Delta, \delta_P\} \leqslant \bar{\delta}$ *and all the initial states $|(x(0), z(0))| < M$, all the closed-loop signals are bounded and*

$$|x(t)|^2 \leqslant \bar{\gamma}(\max\{|x(0)|, |z(0)|\})e^{-\lambda t} + \alpha \quad \forall t \geqslant 0$$

for some $\bar{\gamma} \in \mathcal{K}_\infty$ and $\alpha > 0$ for every switching signal s not faster than $\bar{\alpha}$. Further, for every $\varepsilon_x > 0$, we can make $|x(t)| < \varepsilon_x$ $\forall t \geqslant T$ for a large enough T if $\bar{v}, \bar{w}$, and $\bar{\delta}$ are small enough.

Proof *Bounds of the unmodeled dynamics*

Note that $u = K_{\sigma(t)}\hat{x}_{\sigma(t)}$ so $|u(t)| \leqslant c|x_{\mathbb{CE}}(t)| = c|z(t)|$ and $\|u\|_{[0,t)} \leqslant c\|z\|_{[0,t)}$ for some $c > 0$. Also, $x = \tilde{x}_{s(t)} + \hat{x}_{s(t)}$, so $|x(t)| \leqslant |\tilde{x}_{s(t)}(t)| + |\hat{x}_{s(t)}(t)|$ and $\|x\|_{[0,t)} \leqslant \|\zeta\|_{[0,t)} + a\|z\|_{[0,t)}$ for some constant $a > 0$. From (4.90a), using the separation property of class $\mathcal{K}_\infty$ functions (that for every $\gamma \in \mathcal{K}_\infty$, $\exists \gamma_1, \gamma_2 \in \mathcal{K}_\infty : \gamma(r+t) \leqslant \gamma_1(r) +$

$\gamma_2(t)\ \forall r,t)$, we have

$$|\Delta_1(t)|^2 = |g_\Delta(z,x,u,t)|^2 \leqslant \delta_g\gamma_g^0(|z(0)|)e^{-\lambda_\Delta t} + \delta_g\tilde{\gamma}_g^1(\|\zeta\|_{[0,t)}) + \delta_g\tilde{\gamma}_g^2(\|z\|_{[0,t)}) \quad (4.92)$$

for some $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{K}_\infty$ and $\delta_g > 0$ such that $\delta_g \to 0$ as $\delta_\Delta \to 0$. If $\lambda_\Delta > \lambda$, taking $e^{\lambda t}$-weighted $\mathcal{L}_2$ of both sides of (4.90b) and using the separation property of class $\mathcal{K}_\infty$ functions, we have

$$\|(h_\Delta(z,x,u,\tau))_{0,t}\|_{2,\lambda}^2 \leqslant \delta_h\gamma_h^0(|z(0)|)\frac{e^{-\lambda t}}{\lambda_\Delta - \lambda} + \delta_h\frac{1}{\lambda}\tilde{\gamma}_h^1(\|\zeta\|_{[0,t)}) + \delta_h\frac{1}{\lambda}\tilde{\gamma}_h^2(\|z\|_{[0,t)})$$

for some $\tilde{\gamma}_{\Delta_2}^1, \tilde{\gamma}_{\Delta_2}^2 \in \mathcal{K}_\infty$ and $\delta_h > 0$ such that $\delta_h \to 0$ as $\delta_\Delta \to 0$. The foregoing inequality and the definition of Δ_2 as in (4.38e) yield

$$|\Delta_2(t)|^2 \leqslant \gamma\bar{c}_1\frac{1}{\lambda}\|z\|_{[0,t)}^2 + \delta_h\gamma_h^0(|z(0)|) + \delta_h\frac{1}{\lambda}\tilde{\gamma}_h^1(\|\zeta\|_{[0,t)}) + \delta_h\frac{1}{\lambda}\tilde{\gamma}_h^2(\|z\|_{[0,t)}). \quad (4.93)$$

We rewrite (4.92) and (4.93) into the forms as in (4.72):

$$|\Delta_1(t)|^2 \leqslant \delta\gamma_{\Delta_1}^0(|z(0)|) + \delta\gamma_{\Delta_1}^1(\|\zeta\|_{[0,t)}) + \delta\gamma_{\Delta_1}^2(\|x_{\mathbb{CE}}\|_{[0,t)}), \quad (4.94a)$$

$$|\Delta_2(t)|^2 \leqslant \delta\gamma_{\Delta_2}^0(|z(0)|) + \delta\gamma_{\Delta_2}^1(\|\zeta\|_{[0,t)}) + \delta\gamma_{\Delta_2}^2(\|x_{\mathbb{CE}}\|_{[0,t)}) \quad (4.94b)$$

for some $\gamma_{\Delta_i}^j \in \mathcal{K}_\infty$, $i = 1,2$, $j = 0,1,2$ and $\delta > 0$ such that $\delta \to 0$ if $\delta_\Delta \to 0$ and $\bar{c}_1 \to 0$.

Stability

The equations (4.32a), (4.33), (4.34a), (4.34b), (4.36), and (4.37) describe an interconnected switched system and the equation $\dot{z} = f_\Delta(z,x,u,t) = \tilde{f}_\Delta(z,\zeta,z,t)$ (noting that $x(t) = z(t) + \hat{x}_{s(t)}(t)\ \forall t$ and $u = K_{\sigma(t)}\hat{x}_{\sigma(t)}$) and the inequality (4.94) describes the unmodeled dynamics for the interconnected switched system as in the framework presented in Section 4.3.5.

From the definition of δ_P as in (4.91), $\bar{c}_1 \to 0$ if $\delta_P \to 0$ where $\bar{c}_1$ is as in (4.38). Hence, $\delta \to 0$ if $\max\{\delta_\Delta, \delta_P\} \to 0$ where δ is as in (4.94). We also have that

$\bar{\delta}_1 \to 0, \delta_2 \to 0$ if $\delta_P \to 0$. Thus, for every $\delta_0 > 0$, there exists $\bar{\delta} > 0$ such that if $\max\{\delta_\Delta, \delta_P\} \leqslant \bar{\delta}$, then $\max\{\bar{\delta}_1, \bar{\delta}_2, \delta\} \leqslant \delta_0$.

The stability result for interconnected switched systems in Theorem 4.4 states that for every $\bar{\delta}_a, \bar{\delta}_b, X_0, Z_0, E > 0$, there exists a number $\delta_0 > 0$ such that if $\max\{\bar{\delta}_1, \bar{\delta}_2, \delta\} \leqslant \delta_0$ and $|z(0)| < E$, then all the closed-loop signals are bounded and

$$\max\{|\tilde{x}_{s(t)}(t)|^2, |x_{\mathbb{CE}}(t)|^2\} \leqslant \bar{\gamma}_1(W_{\bar{\delta}_a,\bar{\delta}_b}(|\tilde{x}_{s(0)}(0)|, |x_{\mathbb{CE}}(0)|))e^{-\lambda} + \bar{\gamma}_2(\alpha_3) \quad \forall t \quad (4.95)$$

for all $|\tilde{x}_{s(t)}(0)|^2 \leqslant X_0, |x_{\mathbb{CE}}(0)|^2 \leqslant Z_0$ and every s not faster than the profile $\alpha_{\rho,W_0,\delta_d,\overline{N}}$ for some $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathcal{K}_\infty$. Let $M := \min\{\sqrt{2X_0^2 + 2Z_0^2}, E\}$, $\bar{\gamma}(r) := \bar{\gamma}_1(W_{\bar{\delta}_a,\bar{\delta}_b}(r, r))$, and $\alpha = \bar{\gamma}_2(\alpha_3)$. Since $\bar{\gamma}_1 \in \mathcal{K}_\infty$, we have $\bar{\gamma} \in \mathcal{K}_\infty$. Using $|x(t)|^2 \leqslant 2(|\tilde{x}_{s(t)}(t)|^2 + |x_{\mathbb{CE}}(t)|^2)$, it follows that

$$|x(t)|^2 \leqslant \bar{\gamma}(\max\{|x(0)|, |z(0)|\})e^{-\lambda t} + \alpha \quad \forall t \geqslant 0 \quad (4.96)$$

for some $\bar{\gamma} \in \mathcal{K}_\infty$ if all the initial state is bounded by M.

No unmodeled dynamics, no disturbances, and no noise

Consider the case where there are no unmodeled dynamics, disturbances, and noise, i.e., $\bar{v} = 0, \bar{w} = 0$, $\bar{\delta}_1 = \bar{\delta}_2 = \delta = 0$. The constants $\alpha_1, \alpha_2, \alpha_3$ depends on ε as follows. For notational brevity, we write $a \approx b$ if $a = kb$ for some constant k that does not depend on b. From (4.38) and (4.48), we have $c_3 = m - c_4 \ln \varepsilon$, $c_4 = m/\ln(1+h)$, and so $\gamma_0 = \mu^{c_3} = \mu^m \varepsilon^{-c_4 \ln \mu} = \mu^m \varepsilon^{-\kappa}$, i.e., $\gamma_0 \approx \varepsilon^{-\kappa}$. From the definitions of γ_1, γ_2 as in the proof of Lemma 4.3, $\gamma_1, \gamma_2 \approx \gamma_0$, and hence, $\gamma_1, \gamma_2 \approx \varepsilon^{-\kappa}$. It follows that $\gamma_3 \approx \varepsilon^{\kappa}$. From the definition of $\bar{\nu}$ in (4.63d) and (4.48), we have $\bar{\nu} = d = \varepsilon$. Then $\bar{\nu}^{\kappa+1} \approx \varepsilon^{\kappa+1}$, and so $\gamma_3 \bar{\nu}^{\kappa+1} \approx \varepsilon$. From (4.63c), with $\bar{v} = 0$, we get $\alpha_3 \approx \varepsilon$ which means $\alpha_3 \to 0$ as $\varepsilon \to 0$. Equation (4.69) implies that there exists $T > 0$ such that $\{|x(t)|^2, |z(t)|^2\} \leqslant \gamma_2(\alpha_3) + \epsilon \; \forall t > T$ for a large enough T. Thus, we can get $|x(t)|$ be as small as desired by reducing ε close to 0. One possible way of reducing ε

is as follows. Choose $\varepsilon(0) = \varepsilon_0$ small enough so that $\bar{\gamma}_2(\varepsilon_0) < \min\{X_0, Z_0\}^2/4$. Let $T_0 := \ln \frac{\min\{X_0,Z_0\}^2/4-\bar{\gamma}_2(\varepsilon_0)}{\bar{\gamma}(M)}$. Then $|x(t)| \leqslant \min\{X_0, Z_0\}/2$ for all $t \geqslant T_0$ in view of (**??**) and (4.96). Choose $\varepsilon_1 < \varepsilon_0$ such that $\bar{\gamma}_2(\varepsilon_1) < \min\{X_0, Z_0\}^2/16$. At the time T_0, set ε to this value ε_1. Note that $\alpha_1(T_0) \approx \{\gamma_1(T_0), \gamma_2(T_0)\} \approx \varepsilon_1^{-\kappa}$ but $W(T_0) \approx \bar{\gamma}_2(\varepsilon_0) \approx \varepsilon$ (from the definition of γ_2 as in the proof of Lemma 3.2 and $\bar{\gamma}_2 = c_0\gamma_2$), and so, we have $\alpha_1 W^\kappa(T_0) + \alpha_2 \leqslant \overline{W}$. The equation (4.96) is also applicable starting from the time T_0 so if we let $T_1 := \ln \frac{\min\{X_0,Z_0\}^2/16-\bar{\gamma}_2(\varepsilon_1)}{\bar{\gamma}(M)}$, then $|x(t)| \leqslant \min\{X_0, Z_0\}/4$ for all $t \geqslant T_0 + T_1$. Repeating the procedure, at step k, we have $|x(t)| \leqslant \min\{X_0, Z_0\}/2^k$ for all $t \geqslant T_0 + \cdots + T_{k-1}$. As $k \to \infty$, $|x(t)| \to 0$.

Small boundedness for small unmodeled dynamics, disturbances, and noise

Denote by $\underline{\bar{\nu}}$ and $\underline{\alpha_3}$ the values of $\bar{\nu}$ and α_3 in the case without unmodeled dynamics, disturbances, and noise. In the case with unmodeled dynamics, disturbances, and noise, we have $d := \varepsilon + 4\frac{\gamma}{\lambda}\bar{w}^2$ and hence, $\bar{\nu} := \varepsilon + 4\frac{\gamma}{\lambda}\bar{w}^2 + \bar{\delta}_z + \bar{\Delta}_2 + \frac{\bar{c}\hat{\gamma}_1}{(\hat{\lambda}-\lambda)\lambda}\bar{\Delta}_1 + \frac{\bar{c}\hat{\gamma}_2}{\hat{\lambda}-\lambda}\bar{v}^2$ in view of (4.81d). Sine $\bar{\delta}_z \to 0$ as $\bar{\delta}_2 \to 0$, $\bar{\Delta}_1 \to 0$ as $\delta \to 0$, and $\bar{\Delta}_2 \to 0$ as $\{\delta, \delta_P\} \to 0$, we have $\bar{\nu} \to \underline{\bar{\nu}}$ as $\{\bar{v}, \bar{w}, \bar{\delta}_2, \delta, \delta_P\} \to 0$. Also, from (4.81c), $\alpha_3 \to \underline{\alpha_3}$ as $\bar{v} \to 0$ and $\bar{\delta}_1 \to 0$. Thus, we can asymptotically recover the case without unmodeled dynamics, disturbances, and noise as $\{\bar{v}, \bar{w}, \bar{\delta}_2, \delta, \delta_P\} \to 0$, which means we can make $|x(t)|$ be as small as desired for all $t \geqslant T$ for some large enough T if $\bar{v}, \bar{w}, \bar{\delta}_2, \delta$, and δ_P are small enough. $\square$

4.3.7 Example

Example 4.3 Consider the following uncertain system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ a_1(t) & a_2(t) \end{bmatrix} x + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned}$$

where $a_1(t)$, $a_2(t)$, and $b(t)$ are unknown (this system with known constant $b(t)$, $a_1 = -2\sin(t)$, $a_2 = 6$ is termed a "fast" system in example 7.26 in [85]). We know the uncertainty set Ω which these parameters belong to, $(a_1(t), a_2(t), b(t)) \in \Omega =: [-5,5] \times [-10,10] \times [-3,3]$. We design a supervisory adaptive controller for the plant as outlined in Section 4.3.1, using the nominal values $\{-4,-2,2,4\}$ for $a_1(t)$, $\{-8,-4,4,8\}$ for $a_2(t)$, and $\{-2,2\}$ for $b(t)$. Controllers place closed-loop poles at $-10,-5$ and observer poles at $-30,-20$. The design constants are $h = 0.01$,$\lambda = 0.4$,$\epsilon = 10^{-6}$, $\gamma = 10^4$.

We simulate the supervisory adaptive control in the presence of unmodeled dynamics z, disturbances w, and noises v as follows:

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ a_1(t) & a_2(t) \end{bmatrix} x + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u + 0.01z + 0.01u + w \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + v \\
\dot{z} &= -0.8z + u,
\end{aligned}$$

where $a_1(t) = 1 + 2\sin(4t)$, $a_2(t) = -6$, $b(t) = (1 + 0.5\sin(2t))\cos(0.5t)$, w uniformly randomly distributed between $[-0.1, 0.1]$, and v uniformly randomly distributed between $[-0.5, 0.5]$. The result is plotted in Fig. 4.8. This control problem is challenging for other adaptive control schemes due to the presence of unmodeled dynamics, disturbances, and noises but especially because the sign of $b(t)$ is not known. However, Fig. 4.8 shows that the closed-loop states are bounded and close to zero except when there are abrupt changes in the parameter b.

4.4 Model Reference Supervisory Adaptive Control

Our results on stability of interconnected switched systems can also be applied to the output tracking problem for uncertain time-varying SISO systems. We describe

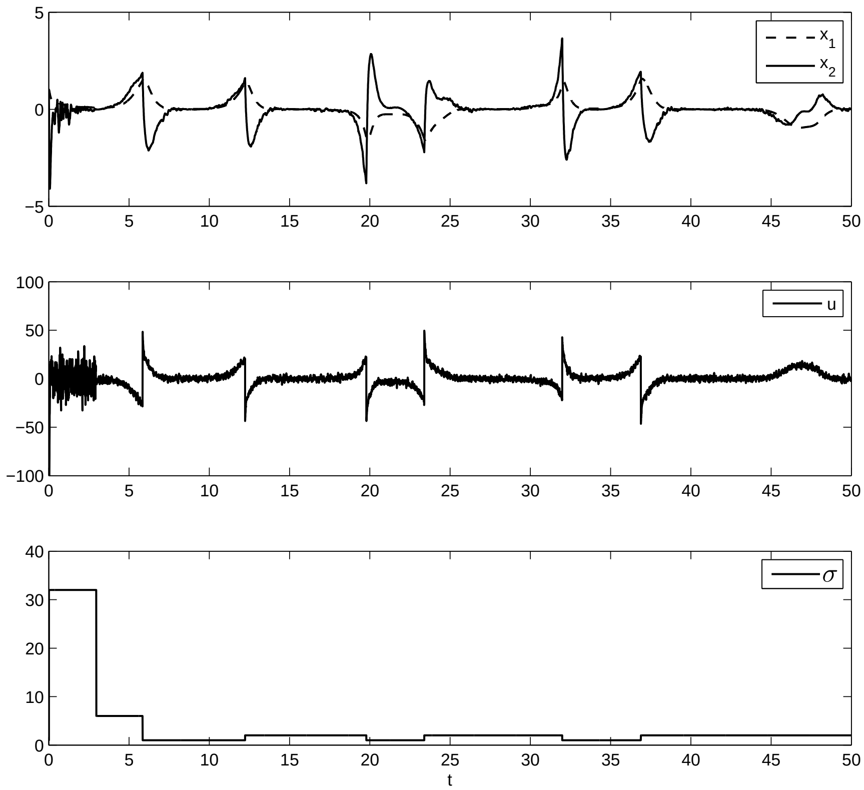

Figure 4.8: supervisory adaptive stabilization

here a supervisory model reference adaptive control (SMRAC) scheme for uncertain time-varying SISO plants. While much research has applied supervisory adaptive control to stabilization and set-point tracking problems, supervisory adaptive control has not been applied to model reference output tracking. We provide here details of this application of supervisory adaptive control.

4.4.1 Model reference controller for a known LTI plant

We describe how to construct a particular model reference controller (MRC) for a LTI SISO system (see [80]). Consider a plant in transfer function form

$$\frac{y}{u} = g\frac{\sum_{i=0}^{n-m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} =: g\frac{b(s)}{a(s)}, \tag{4.97}$$

where $b_{n-m} = a_n = 1$. For an integer r, define $\bar{\mathbf{1}}_r := [1\ 0 \dots 0]^T \in \mathbb{R}^r$ and $\underline{\mathbf{1}}_r := [0 \dots 0\ 1]^T \in \mathbb{R}^r$. Let

$$\frac{a(s)}{b(s)} = \beta(s) - \frac{\alpha(s)}{b(s)}, \tag{4.98}$$

where $\alpha(s) = \sum_{i=0}^{n-m-1} \alpha_i s^i$ and $\beta(s) = \sum_{i=0}^{m} \beta_i s^i$, $\beta_m = 1$. We construct the following state-space realization of (4.97):

$$\begin{aligned} \begin{pmatrix} \dot{w} \\ \dot{v} \end{pmatrix} &= \begin{bmatrix} A_1 & \underline{\mathbf{1}}_{n-m}\bar{\mathbf{1}}_m^T \\ \underline{\mathbf{1}}_m \alpha & A_2 \end{bmatrix} \begin{pmatrix} w \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ g\underline{\mathbf{1}}_m \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & \bar{\mathbf{1}}_m^T \end{bmatrix} \begin{pmatrix} w \\ v \end{pmatrix} \end{aligned} \tag{4.99}$$

where $A_1 = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ -b_0 & -b_1 & \dots & -b_{n-m-1} \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ -\beta_0 & -\beta_1 & \dots & -\beta_{m-1} \end{bmatrix}$, and $\alpha = [\alpha_0\ \dots\ \alpha_{n-m-1}]$.

We want the output y to track the output of a reference model

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m u_m, \quad x_m(0) = x_m^0 \\ y_m &= C_m x_m \end{aligned} \tag{4.100}$$

where $x_m \in \mathbb{R}^{n_m}, n_m \geqslant m$. Let $P_m(s) = C_m(sI - A_m)^{-1}B_m$. Let $q(s) = \sum_{i=0}^{m} q_i s^i$, $q_m = 1$, be an m-order polynomial that has all roots in the left-half complex plane. Since $n_m \geqslant m$, there exist $k_1 \in \mathbb{R}^{1\times n_m}$ and $k_2 \in \mathbb{R}$ such that

$$\frac{P_m(s)}{\bar{\mathbf{1}}_m^T(sI - \bar{A}_2)^{-1}\underline{\mathbf{1}}_m} = k_2 + k_1(sI - A_m)^{-1}B_m, \tag{4.101}$$

where $\bar{A}_2 = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ -q_0 & -q_1 & \dots & -q_{m-1} \end{bmatrix}$. A tracking controller is given by

$$u = \frac{1}{g}(-\alpha w + (q - \beta)v + k_1 x_m + k_2 u_m) \tag{4.102}$$

where $q = [-q_0 \ \dots \ -q_{m-1}]$ and $\beta = [-\beta_0 \ \dots \ -\beta_{m-1}]$.

4.4.2 Supervisory adaptive tracking

Consider an uncertain time-varying plant described in the linear time-varying (LTV) differential equation model

$$\frac{y}{u} = g(t)\frac{\sum_{i=0}^{n-m} b_i(t)s^i}{\sum_{i=0}^{n} a_i(t)s^i} =: g\frac{b(s,t)}{a(s,t)}, \tag{4.103}$$

where some (or all) of the coefficients $a_i(t), b_i(t)$ are not known. We group all unknown coefficients into a vector $\nu(t)$ (also called the unknown parameter). We know the uncertainty set Ω to which the unknown parameter belongs. The uncertain plant (4.103) with model uncertainty is

$$\frac{y}{u} = \big(g\frac{b(s,t)}{a(s,t)} + \Delta_1\big)(1 + \Delta_2), \tag{4.104}$$

where Δ_1, Δ_2 are unmodeled dynamics.

As before, we partition the uncertainty set into m nonoverlapping subsets and for each subset, pick a nominal value ν_i, $i \in \mathcal{P} = \{1, \dots, \mathrm{m}\}$. Using the realization as in the previous section, the state space representation for each model with a parameter

$\nu_p, p \in \mathcal{P}$ is

$$\dot{x} = A_p x + B_p u \tag{4.105}$$

$$y = Cx \tag{4.106}$$

where $x = (w, v)$, $A_p = \begin{bmatrix} A_{1,p} & \underline{\mathbf{1}}_{n-m}\bar{\mathbf{1}}_m^T \\ \underline{\mathbf{1}}_m\alpha & A_{2,p} \end{bmatrix}$, $B_p := \begin{bmatrix} 0 \\ g_p\underline{\mathbf{1}}_m \end{bmatrix}$, $C = [0 \;\; \bar{\mathbf{1}}_m^T]$, and $A_{1,p}, A_{2,p}, g_p$ are the constants defined as in (4.99) for the system with parameter ν_p.

The multiestimator is still exactly the same as in (4.20). The multicontroller is

$$u_p = \frac{1}{g_p}(K_p \hat{x}_p + k_1 x_m + k_2 u_m), \tag{4.107}$$

where $K_p = [-\alpha_p \;\, (q - \beta_p)]$.

Analysis

The injected system with controller p is

$$\dot{x}_{\mathbb{CE}} = \bar{A}_p x_{\mathbb{CE}} + \bar{B}_p(k_1 x_m + k_2 u_m) + \hat{B}\tilde{y}_p, \tag{4.108}$$

where $x_{\mathbb{CE}} = (\hat{x}_1, \ldots, \hat{x}_m)$, $\bar{A}_p = \begin{bmatrix} A_1 + L_1 C & \ldots & \frac{1}{g_p}B_1 K_p - L_p C & \ldots & 0 \\ 0 & & A_p + \frac{1}{g_p}B_p K_p & \ldots & 0 \\ 0 & & \frac{1}{g_p}B_\mathrm{m} K_p - L_p C & \ldots & A_\mathrm{m} + L_\mathrm{m} C \end{bmatrix}$, $\bar{B}_p := \frac{1}{g_p}\begin{bmatrix} B_1 \\ \vdots \\ B_\mathrm{m} \end{bmatrix}$, $\hat{B} := \begin{bmatrix} L_1 \\ \vdots \\ L_\mathrm{m} \end{bmatrix}$ and $\tilde{y} = y - C\hat{x}_p$. The matrix $\bar{A}_p$ are Hurwitz for all $p \in \mathcal{P}$.

Let ξ be the solution of the switched system with the switching signal σ and the subsystems

$$\dot{\xi} = \bar{A}_p \xi + \bar{B}_p (k_1 x_m + k_2 u_m) \tag{4.109}$$

and $\xi(0) = 0$. Let $e_y^p = \bar{C}_p \xi - C_m x_m$ where $\bar{C}_p := [0_{1\times(p-1)n}\ C\ 0_{1\times(\mathrm{m}-p)n}]$. We have

$$\begin{pmatrix} \dot{\xi} \\ \dot{x}_m \end{pmatrix} = \begin{bmatrix} \bar{A}_p & \bar{B}_p k_1 \\ 0 & A_m \end{bmatrix} \begin{pmatrix} \xi \\ x_m \end{pmatrix} + \begin{bmatrix} \bar{B}_p^1 k_2 \\ B_m \end{bmatrix} u_m. \tag{4.110}$$

By construction (due to the model matching procedure), the transfer function $u_m \mapsto e_y^p$ of (4.110) is zero for all u_m and for all $p \in \mathcal{P}$. That means $e_y^p = [\bar{C}_p\ C_m]\zeta(t)$ for all $t \geqslant t_0$ where ζ is a state of the switched system

$$\dot{\zeta} = \begin{bmatrix} \bar{A}_p & \bar{B}_p k_1 \\ 0 & A_m \end{bmatrix} \zeta =: \hat{A}_p \zeta. \tag{4.111}$$

The matrix $\hat{A}_p$ are Hurwitz for all $p \in \mathcal{P}$.

Consider the switched injected system and let $\bar{x}_{\mathbb{CE}} = x_{\mathbb{CE}} - \xi$. From (4.108) and (4.109), we have

$$\dot{\bar{x}}_{\mathbb{CE}} = \dot{x}_{\mathbb{CE}} - \dot{\xi} = \bar{A}_{\sigma(t)} \bar{x}_{\mathbb{CE}} + \hat{B} \tilde{y}_{\sigma(t)}. \tag{4.112}$$

Let $e = y - y_m$ be the output tracking error. Then $e = C x_{s(t)} - C_m x_m = C(\hat{x}_{s(t)} + \tilde{x}_{s(t)}(t)) - C_m x_m = C \tilde{x}_{s(t)}(t) + e_y^{s(t)}$.

The interconnected switched system that arises from the closed-loop control system consists of the following switched systems:

1. The first switched system is the state error dynamics as in (4.30).

2. The second switched system is

$$\dot{z} = \begin{bmatrix} \bar{A}_\sigma & 0 & 0 \\ 0 & \bar{A}_\sigma & \bar{B}_\sigma \\ 0 & 0 & A_m \end{bmatrix} z + \begin{bmatrix} \hat{B} \\ 0 \\ 0 \end{bmatrix} \tilde{y}_\sigma, \tag{4.113}$$

where $z = (x_{\mathbb{CE}}, \zeta)$.

Using the stability result in Section 4.3.5, we conclude that the model reference supervisory adaptive control scheme can asymptotically drive output tracking error to zero when there is no disturbance, provided that the plant varies slowly enough. In the presence of disturbance and unmodeled dynamics, we can still guarantee boundedness of all the closed-loop states if the disturbances and the unmodeled dynamics are small enough.

Theorem 4.6 *Consider the time-varying uncertain system* (4.103)*. Design a model reference supervisory adaptive control scheme as in Section 4.4.2. Suppose that the initial state is bounded by* X_0*. Let* s *be the approximate switching signal for the time-varying plant and* δ *be the approximation error. There exist an average dwell-time vs. chatter bound curve* ϕ_{δ,X_0}*, and an average dwell-time vs. dwell-time curve* ψ_{δ,X_0} *such that for every switching signal* s *having a dwell-time* δ_d*, a chatter bound* $\overline{N}$*, and an average dwell-time* $\bar{\tau}$ *such that* $\bar{\tau} \geqslant \phi_{\delta,X_0}(\overline{N})$ *and* $\bar{\tau} \geqslant \psi_{\delta,X_0}(\delta_d)$*, the tracking error asymptotically goes to zero. In the presence of disturbances and unmodeled dynamics, all the closed-loop states can be kept bounded if the disturbances and the unmodeled dynamics are small enough.*

4.4.3 Example

Example 4.4 The following model reference adaptive control problem is from [80]. Consider the time-varying uncertain plant

$$\begin{aligned} \dot{x} &= a(t)x + g(t)u, \\ y &= x, \end{aligned} \tag{4.114}$$

where $(a(t), g(t))$ belongs to the uncertainty set $[-1,1] \times ([-2,-1] \cup [1,2])$. We want to design a controller to track the output of the reference model $P_m = \frac{1}{s+1}$. As noted in [80], the fact that $g(t)$ can be either positive or negative makes the adaptive control problem challenging for adaptive control techniques that rely on the knowledge of the sign of $g(t)$.

We chose the nominal values of $a(t)$ being $\{-0.7, -0.3, 0.3, 0.7\}$ and the nominal values of $g(t)$ being $\{-1.5, 1.5\}$. A multiestimator is

$$\dot{\hat{x}}_p = a_p \hat{x}_p + g_p u + (-a_p - 500)(\hat{x}_p - y) \tag{4.115}$$

where p is an index, $p \in \{1, \ldots, 8\}$ (each p corresponding to a pair of nominal values of $a(t)$ and $g(t)$). For every p, a tracking controller is designed using the procedure in [80]:

$$u_p = \frac{1}{g_p}((-100 - a_p)\hat{x}_p + x_m + u_m), \tag{4.116}$$

where x_m is the state and u_m is the input of the reference model. The simulation result is plotted in Fig. 4.9 for $g(t) = (1.2 + 0.2\cos(3u/4))\mathrm{sgn}(\cos(u/4))$ and $a(t) = \cos(u/3)$. We add the sinusoidal noise $0.1\sin(10t)$ between 40 s and 60 s and a uniformly distributed random noise between -0.05 and 0.05 from 60 s to 80 s. As we can see, the control plant tracks the reference output very well. Note that the performance of the adaptive control scheme in [80] would be affected in the presence of random noise (see [80, Remark 6]) whereas, here, our control scheme still performs well in that situation. On the other hand, the scheme in [80] can handle faster variation than the supervisory control scheme here and [80] will also likely give a smaller control input.

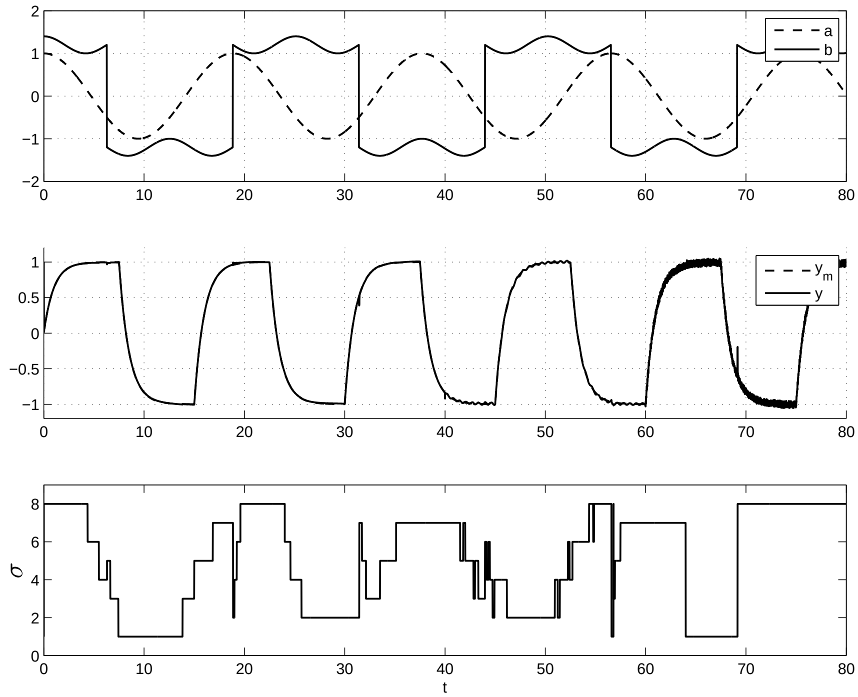

Figure 4.9: Model reference supervisory adaptive control

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

In this dissertation, we formulated a new problem, namely, the invertibility problem for switched systems. This problem seeks a condition on a switched system that guarantees unique reconstruction of the switching signal and the input from an output and a given initial state. In solving the invertibility problem, we introduced the concept of singular pairs, which is peculiar to switched systems and does not have a counterpart in nonswitched systems. We presented a necessary and sufficient condition for invertibility of continuous-time switched linear systems, which says that the individual subsystems should be invertible and there should be no singular pairs with respect to the output set. For switched systems, not necessarily invertible but with invertible subsystems, we provided an algorithm that finds switching signals and inputs that generate a given output with a given initial state. We illustrated our results by detailed examples.

Another contribution of the dissertation was to provide stability results for switched nonlinear systems. We introduced the concept of event profiles to characterize slow switching signals, which is a nonlinear generalization of the concept of average dwell-time (which are equivalent to uniform affine event profiles). Using event profiles, we classified classes of switching signals that guarantee asymptotic stability of autonomous switched nonlinear systems when all subsystems are globally asymptotically stable, even when one cannot find a constant switching gain among the family of Lyapunov functions of the subsystems. For switched nonlinear systems with inputs, when a constant switching gain among the family of ISS-Lyapunov functions of the subsystems exists, we showed that under switching signals with large enough average dwell-time, a switched system is ISS, $e^{\lambda t}$-weighted ISS, and $e^{\lambda t}$-weighted iISS, if the individual subsystems are ISS. When such a constant does not exist, we also provided a slow switching condition in terms of event profiles that guarantee ISS of switched nonlinear systems.

We applied our ISS results for switched nonlinear systems to the problem of adaptively controlling uncertain plants. For nonlinear uncertain plants in the presence of disturbances, using the scale-independent hysteresis switching logic, we showed that the states can be kept bounded for arbitrary initial conditions and bounded disturbances, provided that the injected systems are ISS with respect to the estimation errors and there is a constant switching gain among ISS-Lyapunov functions of the injected systems. We illustrated our results on a plant where it might be difficult to apply traditional adaptive control tools. We relaxed the requirement of a global μ and achieved local boundedness of the plant state in the presence of bounded disturbances; a detailed example of this was also included. For linear time-varying uncertain plants, in the presence of unmodeled dynamics, disturbances, and noise, we showed that the state can be kept bounded provided that the plant varies slowly enough. Further, if there are no unmodeled dynamics, no disturbances, and no noise, the closed-loop state can be made arbitrarily small with appropriate fine-tuning in the supervisory control scheme.

There are several avenues for future research on the topics presented in this dissertation. For the invertibility problem, further investigation of the invertibility issues for discrete-time switched systems as well as a geometric approach for continuous-time switched systems are needed. A geometric result for invertibility will shed more insight into intrinsic properties (that are coordinate independent) and will complement the matrix-oriented approach presented here. For nonswitched systems, invertibility for one time domain implies invertibility for every domain and it is not clear if this is the case for switched systems. The robust invertibility property where the initial condition is not known requires further investigation. An interesting design question is to design the matrices C so that the switched system is invertible. Another direction is to extend the results to switched nonlinear systems. Singular pairs for switched nonlinear systems, though conceptually similar, could have different properties from the linear counterpart and need further investigation. In solving the invertibility problem for switched nonlinear systems, the extension of the structure algorithm for nonswitched nonlinear systems [39] or the geometric approach [86] may be useful.

For stability of switched nonlinear systems, other stability properties such as input-output-to-state stability can also be studied using event profiles. One can also try to find out how the stability results of switched nonlinear systems presented here can be applied to particular applications where the existence of constant switching gains is not guaranteed.

Supervisory control can be further studied to address performance issues such as how to choose the design parameters to achieve certain types of optimality or to improve transient response. Another potential direction is to address the issue of fast-switching plants by finding the design parameters that yield the largest class of hybrid dwell-time signals. Also of future research interest is the case of time-varying plants but with constant unknown parameters, which falls between the case of time-invariant plants with constant unknown parameters and the case of plants with time-varying unknown parameters considered in this dissertation.

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