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Brief paper

Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability[☆]

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ABSTRACT

We study switched nonlinear differential algebraic equations (DAEs) with respect to existence and nature of solutions as well as stability. We utilize piecewise-smooth distributions introduced in earlier work for linear switched DAEs to establish a solution framework for switched nonlinear DAEs. In particular, we allow induced jumps in the solutions. To study stability, we first generalize Lyapunov's direct method to non-switched DAEs and afterwards obtain Lyapunov criteria for asymptotic stability of switched DAEs. Developing appropriate generalizations of the concepts of a common Lyapunov function and multiple Lyapunov functions for DAEs, we derive sufficient conditions for asymptotic stability under arbitrary switching and under sufficiently slow average dwell-time switching, respectively.

1. Introduction

We consider switched nonlinear differential algebraic equations (DAEs) of the form

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, \ldots, \mathrm{p}\}$, $\mathrm{p} \in \mathbb{N}$, is the switching signal and $E_p : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $p \in \{1, \ldots, \mathrm{p}\}$, are smooth functions. In particular, we assume that each subsystem is a DAE in *quasi-linear form* (Reich, 1990)

$$E(x)\dot{x} = f(x). \tag{2}$$

Equations of this kind occur, for example, when modeling (nonlinear) electrical circuits (Chua & Rohrer, 1965) or coupled mechanical systems (Schiehlen, 1990). Classical linear DAEs (i.e. without switching) of the form $E\dot{x} = Ax$, with matrices $E, A \in \mathbb{R}^{n\times n}$, which are also known as *singular systems* or *descriptor systems*, naturally appear when modeling electrical circuits because Kirchhoff's circuit laws add algebraic equations to the differential equations stemming from capacitors and inductances. For more details and further motivation for studying (non-switched) DAEs the reader is referred to Kunkel and Mehrmann (2006). Adding, for example, (ideal) switches to an electrical circuit or allowing for sudden structural changes in mechanical systems yields a switched DAE as in (1). When studying the zero dynamics of an ordinary differential equation (ODE) one arrives at a DAE because of the additional algebraic constraint $0 = y = h(x)$, where $h : \mathbb{R}^n \to \mathbb{R}^m$ is the output function. In particular, using a switched controller to stabilize the zero dynamics (as was done in Nešić, Skafidas, Mareels, and Evans (1999)) yields a switched DAE (1) even if one starts with an ODE.

The switching signal in (1) is time-dependent and not state-dependent. Although state-dependent switching has high relevance for applications, we focus our attention in this paper only on time-dependent switching. Some reasons for this are the following: (i) we view the switching signal as an exogenous signal, which is a natural approach for studying electrical circuits with (physical) switches or sudden component faults in electrical and mechanical systems, (ii) the distributional solution framework utilized in this paper does not allow for accumulation of switching times (Zeno behavior) which in general can occur for state-dependent switching.

The aim of this paper is a stability analysis of (1) with the help of Lyapunov functions. For this we first need to establish a Lyapunov theory for non-switched DAEs in quasi-linear form (2)

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and secondly we need to define a suitable solution framework for the switched DAE (1).

The use of Lyapunov functions is a powerful tool to study stability of nonlinear differential equations. However, it is not immediately clear how Lyapunov functions can be defined for *implicit* differential equations such as (2). Of course, it is possible to define a Lyapunov function in a very general setting just by the property that it decreases along solutions, but we believe that only a definition for a Lyapunov function which does not refer to the individual solutions makes Lyapunov functions so useful. The main problem is that, given a function $x \mapsto V(x)$, its derivative along solutions $\dot{V}(x) = \nabla V(x)\dot{x}$ cannot be expressed directly in terms of the right-hand side $f(x)$, because $\dot{x}$ is not explicitly given. We resolve this problem and generalize the well known Lyapunov's Direct Method to implicit differential equations of the form (2). In the linear case $E\dot{x} = Ax$ there have been generalizations of Lyapunov's Direct Method (e.g. in Owens and Debeljkovic (1985) and Takaba, Morihira, and Katayama (1995)) but no general definition of a Lyapunov function was given.

One major problem of studying switched DAEs of the form (1) is the presence of jumps in the solutions induced by the presence of so-called *consistency spaces*. A special case is the problem of inconsistent initial values which has been studied extensively (see e.g. Cobb, 1982; Frasca, Çamlıbel, Goknar, Iannelli, & Vasca, 2010; Liu, Yan, & Teo, 1995; Verghese, Levy, & Kailath, 1981) and the references in the latter. We are using the piecewise-smooth distributional framework from Trenn (2009a,b) to define solutions of the switched DAE (1). In this framework $\dot{x}$ is well defined even when x contains jumps, in which case $\dot{x}$ contains *Dirac impulses*. It should be noted that a general distributional solution framework (i.e. not considering the smaller space of piecewise-smooth distributions) will not work, because (i) the nonlinear function evaluations $E(x)$ and $f(x)$ are not defined for distributions and (ii) the product $E(x)\dot{x}$ is not defined even when $E(x)$ is a piecewise-smooth function.

All results presented here apply of course also to the linear switched DAE

$$E_\sigma \dot{x} = A_\sigma x, \tag{3}$$

where $E_p, A_p \in \mathbb{R}^{n\times n}$ for $p \in \{1, \dots, \mathrm{p}\}$. In this case some of the results simplify significantly and we will formulate corollaries to highlight the linear case. We have studied stability of the linear switched DAE (3) already in Liberzon and Trenn (2009). However, our nonlinear results presented here applied to the linear switched DAE (3) still generalize these results. In particular, the notion of a Lyapunov function as well as the dwell-time stability results are significantly generalized.

Although the two research fields 'DAEs' and 'switched systems' are now relatively mature (see e.g. the textbooks Kunkel & Mehrmann, 2006; Liberzon, 2003) the combination of both has not been studied much even in the linear case. The existing literature available on switched DAEs (Geerts & Schumacher, 1996a,b; Meng, 2006; Meng & Zhang, 2006; Raouf & Michalska, 2010; Wunderlich, 2008; Zhai, Kou, Imae, & Kobayashi, 2006) does not consider stability problems in a nonlinear setup. Furthermore, the fundamental problem that one needs distributional solutions for a switched linear DAE (3) and at the same time Eq. (3) cannot be evaluated for distributional x is not resolved there.

It might be possible to reformulate the switched DAE (1) as a hybrid system in the framework of Goebel, Sanfelice, and Teel (2009) by writing (1) as $\dot{x} \in E_\sigma(x)^{-1}\big(f_\sigma(x)\big)$; however, by doing so, we lose the special structure of (1). In particular, the jumps of the states are implicitly given by (1) and no additional jump map needs to be considered. This is a major difference between switched DAEs and switched ODEs with reset maps.

A system class which has similarities with switched DAEs (1) is that of *complementarity systems* (see, e.g., Acary, Brogliato, & Goeleven, 2008; Çamlıbel, Heemels, van der Schaft, & Schumacher, 2003; Heemels, Schumacher, & Weiland, 2000; van der Schaft & Schumacher, 1996). The main similarity is the existence of different modes which are described by differential–algebraic equations. Roughly speaking, the different modes in the complementarity framework stem directly from the complementarity condition (certain variables must be zero) and a mode change is triggered by violation of positivity of certain variables. In particular, the switches between the different modes are state-dependent; hence the solution theory is rather different. Another difference of the complementarity framework is the existence of two different types of variables: the state variable (whose derivative appears explicitly in the system description) and complementarity variables which have to fulfill the complementarity conditions. This distinction is not made in our approach: in one mode a certain state-variable could be governed by a differential equation, in another mode this variable could be governed by a simple algebraic equation. A further comparison of the linear switched DAE (3) with the linear complementarity framework from Heemels et al. (2000) reveals that the consistency projectors are (modulo a restriction to the state variable) identical in both frameworks but the different modes in Heemels et al. (2000) have the same E-matrix which simplifies the analysis significantly.

The structure of the paper is as follows. In Section 2 we study the non-switched DAE (2) and generalize Lyapunov's Direct Method to the DAE case in Theorem 2.7. This result is based on a presumably new definition of a Lyapunov function for the DAE (2) as formulated in Definition 2.5. In Section 3 the distributional solution framework for switched DAEs of the form (1) is introduced. We formulate Assumption A4 which under certain regularity assumptions on the subsystems guarantees existence and uniqueness of solutions of the switched DAE (1), see Theorem 3.3. In Section 3.2 we consider the linear case and observe with Corollary 3.9 that the linear equivalent of Assumption A4 ensures the existence of impulse-free solutions of the linear switched DAE (3). Finally, in Section 4 we generalize the well-known results that the existence of a "common Lyapunov function" implies asymptotic stability under arbitrary switching; the novel element is that this Lyapunov function must take into account the consistency projectors as formulated in Theorem 4.1. We also prove a result on stability under average dwell time in the spirit of Hespanha and Morse (1999) for switched nonlinear DAEs (1) in Theorem 4.2.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the natural numbers, integers, real and complex numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, the kernel (null space) of M is $\ker M$, the image (range, column space) of M is $\operatorname{im} M$, and the transpose of M is $M^\top \in \mathbb{R}^{m\times n}$. For a matrix $M \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subset \mathbb{R}^n$, the image of $\mathcal{S}$ under M is $M\mathcal{S} := \{Mx \in \mathbb{R}^n \,|\, x \in \mathcal{S}\}$ and the pre-image of $\mathcal{S}$ under M is $M^{-1}\mathcal{S} := \{x \in \mathbb{R}^n \,|\, \exists y \in \mathcal{S} : Mx = y\}$. The identity matrix is denoted by I. For a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ the left-sided evaluation $\lim_{\varepsilon\searrow 0} f(t-\varepsilon)$ at $t \in \mathbb{R}$ is denoted by $f(t-)$. The space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ is denoted by $\mathcal{C}^1$, the space of piecewise-smooth functions is denoted by $\mathcal{C}^\infty_{\mathrm{pw}}$, the space of distributions is denoted by $\mathbb{D}$, the space of piecewise-smooth distributions is denoted by $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and $\delta_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \subseteq \mathbb{D}$ denotes the Dirac impulse at $t \in \mathbb{R}$; for details see the Appendix. The set of switching signals considered here is

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \{1, \dots, \mathrm{p}\} \,\middle|\, \begin{array}{l}\sigma \text{ is right continuous with a}\\ \text{locally finite number of jumps}\end{array} \right\}$$

where $\mathrm{p} \in \mathbb{N}$ is the number of subsystems.

2. Non-switched DAEs and Lyapunov functions

2.1. Classical solutions and consistency spaces

Consider for now the (non-switched) nonlinear DAE (2). A (classical, local) *solution* of (2) is any differentiable function $x : \mathfrak{I} \to \mathbb{R}^n$ which fulfills (2) on some interval $\mathfrak{I} \subseteq \mathbb{R}$. Due to the time-invariant nature of (2) we can always assume that $\mathfrak{I} = [0, T)$ for some $T \in (0, \infty]$.

Definition 2.1 (*Consistency Space*)**.** The *consistency space* of (2) is given by

$$\mathfrak{C}_{E,f} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } x : [0,T) \to \mathbb{R}^n \\ \text{with } x(0) = x_0 \end{array} \right\}.$$

Each $x_0 \in \mathfrak{C}_{E,f}$ is called a *consistent initial value*.

Time-invariance of (2) implies that all solutions x of (2) evolve within $\mathfrak{C}_{E,f}$, i.e. $x(t) \in \mathfrak{C}_{E,f}$ for all $t \in [0, T)$. In general, it is not easy to characterize the solution behavior of (2) (for details see, e.g., Rabier & Rheinboldt, 1994; Reich, 1990; Schlacher & Zehetleitner, 2004). Here we just assume that the solution behavior is not drastically different from the regular linear case.

Assumption. The nonlinear DAE (2) satisfies:

A1. $f(0) = 0$, in particular $0 \in \mathfrak{C}_{E,f}$.
A2. $\mathfrak{C}_{E,f}$ is a closed manifold (possibly with boundary) in $\mathbb{R}^n$.
A3. For each $x_0 \in \mathfrak{C}_{E,f}$ there exists a unique solution $x : [0, \infty) \to \mathbb{R}^n$ with $x(0) = x^0$ and $x \in (\mathcal{C}^1 \cap \mathcal{C}^\infty_{\text{pw}})^n$.

Remark 2.2 (*On A3*)**.** First note that we exclude systems which exhibit finite escape time. Secondly, the assumption that the differentiable solution is also piecewise-smooth is just a technical assumption which will be needed later for studying switched DAEs.

For the linear case

$$E\dot{x} = Ax \tag{4}$$

with $E, A \in \mathbb{R}^{n\times n}$ Assumptions A1 and A2 are fulfilled trivially (by linearity the consistency space is a linear subspace, see also the forthcoming Theorem 2.3), and A3 is fulfilled if and only if the matrix pair (E, A) is *regular*, i.e. the polynomial $\det(Es - A) \in \mathbb{R}[s]$ is not the zero polynomial (for details see, e.g., the textbook Kunkel & Mehrmann, 2006). Furthermore, regularity of the matrix pair (E, A) is equivalent to the existence of invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that a coordinate transformation of the codomain and domain by S and T yields the *quasi-Weierstrass form*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{5}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $n_1 \in \mathbb{N}$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 = n - n_1$, is *nilpotent*, i.e. $N^{n_2} = 0$. We call (5) the *quasi*-Weierstrass form (following Berger, Ilchmann, & Trenn, in press) because we do not assume that J and N are in Jordan canonical form as is the case for the Weierstrass canonical form (Gantmacher, 1959; Weierstraß, 1868). The smallest number $\nu \in \mathbb{N}$ such that $N^\nu = 0$ is called the *index* of the corresponding linear DAE $E\dot{x} = Ax$. It is not difficult to see that the consistency space $\mathfrak{C}_{E,A}$ is spanned by the first n_1 columns of T. A convenient way to calculate the matrices S and T is the usage of the *Wong sequences* of subspaces (named after Wong, 1974).[2]

Theorem 2.3 (*Armentano, 1986*[3])**.** *Consider a regular matrix pair (E, A) with index ν and define the associated Wong sequences by, $i \in \mathbb{N}$,*

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad \mathcal{V}^* := \bigcap_i \mathcal{V}_i,$$

$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \qquad \mathcal{W}^* := \bigcup_i \mathcal{W}_i.$$

The Wong sequences are nested and become stationary after exactly ν steps. For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^* = \mathcal{V}_\nu$ and $\operatorname{im} W = \mathcal{W}^* = \mathcal{W}_\nu$ the matrices $T := [V\, W]$ and $S := [EV\, AW]^{-1}$ are invertible and put (E, A) into the quasi-Weierstrass form (5). In particular

$$\mathfrak{C}_{E,A} = \mathcal{V}^*.$$

Remark 2.4 (*Linear Index-One Case*)**.** From the quasi-Weierstrass form (5) it can be deduced that the (classical) solutions of (4) do not depend on N, or, in other words, the solutions remain the same when N is set to be the zero matrix. Assuming that N is the zero matrix is by definition equivalent to assuming that the matrix pair (E, A) is *index-one*. The importance of N only shows up when studying switched DAEs, where a non-zero N might produce impulses in the solutions (we will study impulse-free solutions in more detail in Section 3.2). An easy way to exclude impulsive behaviors is an index-one assumption for all subsystems, i.e. assuming that in each quasi-Weierstrass form (5) the nilpotent matrix is the zero matrix. However this assumption excludes a large class of interesting switched DAEs. For example, if all subsystems have the same consistency space, then all solutions of the corresponding switched systems will have neither jumps nor impulses, independently of whether or not the subsystems are index-one. In Section 3 we propose Assumption A4, whose linear equivalent (13) ensures impulse-free solutions and is implied by the above two stricter conditions (index-one or same consistency spaces).

2.2. Stability and Lyapunov functions

We call the DAE (2) *asymptotically stable* when all solutions converge to zero as $t \to \infty$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each consistent initial value $x^0 \in \mathfrak{C}_{E,f}$ with $|x^0| < \delta$ the corresponding solution $x : [0, \infty) \to \mathfrak{C}_{E,f}$ fulfills $|x(t)| < \varepsilon$ for all $t \geq 0$. The only difference with the classical definition of asymptotic stability is the restriction to consistent initial values. Later, in the switched case, we have to reconsider this restriction, because due to the switching it is not guaranteed that the initial value at a switching instant is consistent.

Definition 2.5 (*Lyapunov Function*)**.** Consider the DAE (2) satisfying A1–A3. Any continuously differentiable non-negative function $V : \mathfrak{C}_{E,f} \to \mathbb{R}_{\geq 0}$ fulfilling the following properties is called a *Lyapunov function* for (2):

L1. V is positive definite, i.e. $V(x) = 0 \Leftrightarrow x = 0$, and for all $x \in \mathfrak{C}_{E,f}$ each sublevel set $V^{-1}[0, V(x)] \subseteq \mathfrak{C}_{E,f}$ is bounded (hence compact by A2),
L2. there exists a continuous $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $\nabla V(x)z = F(x, E(x)z)$ for all $x \in \mathfrak{C}_{E,f}$, $z \in T_x\mathfrak{C}_{E,f}$, where $T_x\mathfrak{C}_{E,f}$ is the tangent space of $\mathfrak{C}_{E,f}$ at x,
L3. defining $\dot{V}(x) := F\big(x, f(x)\big)$ we have $\dot{V}(x) < 0$ for all $x \in \mathfrak{C}_{E,f} \setminus \{0\}$.

[2] These sequences can be traced back to Dieudonné (1946); however, he only implicitly considers the second Wong sequence via a duality argument. Although some authors use these sequences (Aplevich, 1991; Kuijper, 1994; van der Schaft & Schumacher, 1996), the connection between them and the quasi-Weierstrass form, as established by Theorem 2.3, seems not to be very well known.

[3] See also Berger et al. (in press).

Note that in the linear case (4) the tangent space $T_x\mathfrak{C}_{E,A}$ is identical to the consistency space $\mathfrak{C}_{E,A}$ for all $x \in \mathfrak{C}_{E,A}$, hence L2 simplifies in this case. Furthermore, for any non-trivial solution x of (2) with a Lyapunov function V it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \nabla V\big(x(t)\big)\dot{x}(t) \overset{\textbf{L2}}{=} F\big(x(t), E(x(t))\dot{x}(t)\big)$$
$$= F\big(x(t), f(x(t))\big) = \dot{V}\big(x(t)\big) \overset{\textbf{L3}}{<} 0, \tag{6}$$

hence V is *decreasing along solutions*.

Remark 2.6 (*Weaker Version of L2*). In L2 one could also work with $\nabla V(x)z \leq F(x, E(x)z)$ instead of $\nabla V(x)z = F(x, E(x)z)$. However, the definition in L3 for $\dot{V}$ would then be misleading, because $F(x, f(x))$ would only be an upper bound of $\dot{V}$. In order to keep the spirit of the classical concept of a Lyapunov function we chose to use L2 but all results here hold true also for the weaker version. Furthermore, L2 could be formulated with $z \in E(x)^{-1}(f(x))$ instead of $z \in T_x\mathfrak{C}_{E,f}$ because z is a placeholder for $\dot{x}$ when applied later and therefore all relevant z must be solutions of $E(x)z = f(x)$. Depending on the specific problem it might be easier or more difficult to characterize $E(x)^{-1}(f(x))$ instead of $T_x\mathfrak{C}_{E,f}$.

Theorem 2.7 (*Lyapunov's Direct Method*). *Consider the DAE* (2) *satisfying* A1–A3. *If there exists a Lyapunov function for* (2) *then* (2) *is (globally) asymptotically stable.*

Proof (*Stability*). For $\varepsilon > 0$ consider the set $B_\varepsilon := \{x \in \mathfrak{C}_{E,f} \mid |x| = \varepsilon\}$ which is empty or compact by Assumption A2. If $B_\varepsilon = \emptyset$ then each solution starting within the set enclosed by B_ε cannot leave this set, hence stability follows in this case. Otherwise, let $b := \min_{x\in B_\varepsilon} V(x)$ where positive definiteness of V implies $b > 0$. Continuity of V and $V(0) = 0$ guarantees the existence of $\delta > 0$ such that $V(x) < b$ for all $|x| < \delta$, in particular $\delta < \varepsilon$. From (6) it follows that $t \mapsto V\big(x(t)\big)$ is decreasing for any solution x of (2), hence any solution x with $|x(0)| < \delta$ fulfills $V(x(t)) < b$ for all $t \geq 0$. Seeking a contradiction, assume there exists $t > 0$ such that $|x(t)| \geq \varepsilon$, then, by continuity of x together with $|x(0)| < \delta < \varepsilon$, there exists $t_1 \in (0, t)$ such that $|x(t_1)| = \varepsilon$ which leads to $b \leq V(x(t_1)) < b$.

Convergence to zero

Step 1: $V\big(x(t)\big) \to 0$ as $t \to \infty$.

Let $x : [0, \infty) \to \mathfrak{C}_{E,f}$ be any non-trivial solution, then the non-negative function $t \mapsto v(t) := V\big(x(t)\big) \geq 0$ is strictly decreasing by (6). Therefore, $\underline{v} = \lim_{t\to\infty} v(t)$ is well defined. Seeking a contradiction, assume $\underline{v} > 0$. Then $v(t) \in [\underline{v}, v(0)]$ for all $t \geq 0$. By L1 and continuity of V, $\mathcal{K} := V^{-1}[\underline{v}, v(0)]$ is a compact set, hence $\mathcal{M} := \dot{V}(\mathcal{K}) \subseteq \mathbb{R}$ is also compact (since $\dot{V}$ is continuous) and $0 \notin \mathcal{M}$. This implies that $m := -\max \mathcal{M} > 0$ and, in particular, $v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}V\big(x(t)\big) = \dot{V}\big(x(t)\big) \leq -m < 0$ for all $t \geq 0$. Hence $v(t) \leq v(0) - mt$ for all $t \geq 0$, which contradicts $v(t) \geq 0$ for all $t \geq 0$, hence $\underline{v} = 0$ must hold.

Step 2: $x(t) \to 0$ as $t \to \infty$.

Seeking a contradiction, assume $x(t) \not\to 0$, then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ such that $|x(t_n)| > \varepsilon$. By L1 and (6), each solution x evolves within the compact set $V^{-1}[0, V(x(0))]$, hence there exists a convergent subsequence of $x(t_n)$ with limit $x^* \neq 0$. By continuity and positive definiteness of V we arrive at the contradiction $0 = \lim_{t\to\infty} V(x(t)) = V(x^*) > 0$. □

Remark 2.8 (*The Linear Case*). In the linear, regular case it is well-known (Owens & Debeljkovic, 1985) that $E\dot{x} = Ax$ is asymptotically stable if, and only if, there exists a solution $(P, Q) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ of the generalized Lyapunov equation[4]

$$A^\top PE + E^\top PA = -Q, \tag{7}$$

where $P = P^\top$ is positive definite and $Q = Q^\top$ is positive definite on $\mathfrak{C}_{E,A}$. In fact, it is easy to see that then $V(x) = (Ex)^\top PEx$ is a Lyapunov function in the sense of Definition 2.5 with

$$\nabla V(x)z = (Ex)^\top PEz + (Ez)^\top PEx =: F(x, Ez)$$

and

$$\dot{V}(x) = x^\top(E^\top PA + A^\top PE)x = -x^\top Qx < 0 \quad \text{on } \mathfrak{C}_{E,A}.$$

If the linear system $E\dot{x} = Ax$ is index-one, i.e. $N = 0$ in the quasi-Weierstrass form (5), it is shown in Ishihara and Terra (2002) and Takaba et al. (1995) that asymptotic stability is also equivalent to the existence of a solution $P \in \mathbb{R}^{n\times n}$ of

$$P^\top A + A^\top P = -Q, \qquad P^\top E = E^\top P \geq 0,$$

for any positive definite $Q \in \mathbb{R}^{n\times n}$. The corresponding "asymmetric" Lyapunov function[5] is given by $V(x) = (Ex)^\top Px$, with $\nabla V(x)z = (Ex)^\top Pz + (Ez)^\top Px = x^\top P^\top Ez + (Ez)^\top Px =: F(x, Ez)$ and $\dot{V}(x) = x^\top(P^\top A + A^\top P)x = -x^\top Qx < 0$.

We conclude this section with an example which illustrates the application of Theorem 2.7.

Example 2.9. Consider the nonlinear DAE

$$\begin{bmatrix} \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_1 \sin x_3 - x_2 \cos x_3 \\ x_1 \cos x_3 - x_2 \sin x_3 \\ x_3 - x_1^2 - x_2^2 \end{pmatrix}, \tag{8}$$

which fulfills our Assumptions A1, A2 and A3. The consistency space is given by the equations $x_3 = x_1^2 + x_2^2$ and $x_1 \cos x_3 = x_2 \sin x_3$; the projection to the x_1–x_2-plane is illustrated in Fig. 1. Note that the consistency space can be parameterized by

$$\mathfrak{C}_{E,f} = \left\{(\theta \sin\theta^2, \theta\cos\theta^2, \theta^2)^\top \,|\, \theta \in \mathbb{R}\right\}.$$

The corresponding tangent space is given by, for $x \neq 0$,

$$T_x\mathfrak{C}_{E,f} = \operatorname{span}\left\{(x_1 + 2x_2x_3, x_2 - 2x_1x_3, 2x_3)^\top\right\} \tag{9}$$

and $T_0\mathfrak{C}_{E,f} = \operatorname{span}\left\{(0, 1, 0)^\top\right\}$. We propose the following Lyapunov function candidate:

$$V(x) = x_3.$$

For all $x \in \mathfrak{C}_{E,f}$ it follows that $x_3 = x_1^2 + x_2^2$, hence V fulfills L1. Aiming for a function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying L2, i.e. for $x \in \mathfrak{C}_{E,f}$ and $z \in T_x\mathfrak{C}_{E,f}$,

$$F(x, E(x)z) = z_3 = \nabla V(x)z, \tag{10}$$

we choose, for $x \in \mathfrak{C}_{E,f}$ and $v \in E(x)^{-1}(T_x\mathfrak{C}_{E,f})$,

$$F(x, v) := \frac{2x_3 v_1}{x_1 \sin x_3 + x_2 \cos x_3}.$$

---

[4] Actually, in Owens and Debeljkovic (1985) only the *complex-valued* case is studied; however, by considering the real part of the generalized Lyapunov equation (7) we also obtain real-valued matrix pairs (P, Q) with the desired properties.

[5] We thank Emilia Fridman for making us aware of this Lyapunov function construction.

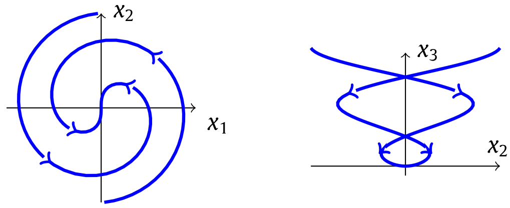

Fig. 1. Consistency space of Example 2.9 in the x_1–x_2-plane (left) and in the x_2–x_3-plane (right); the dynamics within the consistency space are shown by the arrows.

Then by using (9) as well as $x_1 \cos x_3 = x_2 \sin x_3$ we indeed obtain (10). Finally,

$$\dot V(x) = F(x, f(x)) = F\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} -x_1 \sin x_3 - x_2 \cos x_3 \\ x_1 \cos x_3 - x_2 \sin x_3 \\ x_3 - x_1^2 - x_2^2 \end{pmatrix} \right)$$
$$= \frac{2x_3(-x_2 \sin x_3 - x_2 \cos x_3)}{x_1 \sin x_3 + x_2 \cos x_3} = -2x_3,$$

hence L3 is fulfilled and V is a Lyapunov function for (8) and Theorem 2.7 shows that (8) is globally asymptotically stable.

3. Solutions of switched DAEs

3.1. The general nonlinear case

Recall the switched nonlinear DAE (1)

$$E_\sigma(x)\dot x = f_\sigma(x),$$

where each subsystem $E_p(x)\dot x = f_p(x)$, $p = \{1, \dots, \mathrm{p}\}$, fulfills Assumptions A1–A3 and $\sigma \in \Sigma$ is the switching signal. As an underlying solution framework for (1), we will use the space $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ of *piecewise smooth distributions* which was introduced in Trenn (2009a,b) for studying linear switched DAEs. For a short summary of the basic definition and the main properties of piecewise-smooth distributions see the Appendix.

Definition 3.1 (*Solution of* (1)). A *solution* of (1) on some interval $\mathfrak{I} \subseteq \mathbb{R}$ is any *piecewise-smooth function* $x \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ such that (1) restricted to $\mathfrak{I}$ holds as an equation of piecewise-smooth distributions, i.e.

$$(E_\sigma(x)(x_{\mathbb{D}})')_{\mathfrak{I}} = (f_\sigma(x)_{\mathbb{D}})_{\mathfrak{I}}.$$

The product $E(x)(x_{\mathbb{D}})'$ in Definition 3.1 is well defined, since by assumption $t \mapsto E(x(t))$ is piecewise smooth and $(x_{\mathbb{D}})'$ is a piecewise-smooth distribution. Note that this definition of a solution does not allow for Dirac impulses in the solution. There are two reasons for this: (i) it is not clear how a nonlinear function of a Dirac impulse should be defined in general and (ii) for a stability analysis the existence of Dirac impulses in the solution can be interpreted as an undesired unstable solution. However, in Section 3.2 we will also study solutions with impulses for *linear* switched DAEs. The following assumption is essential for existence and uniqueness of solutions of the switched DAE (1).

Assumption. The switched DAE (1) and the corresponding consistency spaces $\mathfrak{C}_p := \mathfrak{C}_{E_p,f_p}$, $p \in \{1, \dots, \mathrm{p}\}$, satisfy

A4. $\forall p, q \in \{1, \dots, \mathrm{p}\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists$ unique $x_0^+ \in \mathfrak{C}_q : x_0^+ - x_0^- \in \ker E_q(x_0^+)$.

Assumption A4 makes it possible to define *nonlinear consistency projectors* Π_q, $q \in \{1, \dots, \mathrm{p}\}$:

$$\Pi_q : \bigcup_p \mathfrak{C}_p \to \mathfrak{C}_q, \qquad x_0^- \mapsto x_0^+,$$

where x_0^+ is the unique value given by Assumption A4. In particular, $\Pi_q(x) = x$ for all $x \in \mathfrak{C}_q$.

Remark 3.2 (*Motivation of Assumption A4*). For a motivation of Assumption A4 consider the situation where the system switches from subsystem $p \in \{1, \dots, \mathrm{p}\}$ to subsystem $q \in \{1, \dots, \mathrm{p}\}$ at some switching time $t \in \mathbb{R}$. Any solution x (in the sense of Definition 3.1) fulfills $x(t-) \in \mathfrak{C}_p$ and $x(t+) \in \mathfrak{C}_q$ and the impulsive part of $\dot x$ at t is given by $\dot x[t] = (x(t+) - x(t-))\delta_t$, where δ_t denotes the Dirac impulse at t. Since the right-hand side $f_\sigma(x)_{\mathbb{D}}$ does not contain impulses it follows that $E_\sigma(x)\dot x[t] = 0$ must hold. This directly implies the existence part of Assumption A4. Furthermore, uniqueness of x (for a given past) follows when $x(t+)$ is uniquely given by $x(t-)$, hence Assumption A4 is a necessary condition for existence and uniqueness of a solution x of (1) in the sense of Definition 3.1. In the ODE case, i.e. $E_\sigma(x) \equiv I$, Assumption A4 is trivially fulfilled with $x_0^+ := x_0^-$. In the linear case an easy check for Assumption A4 is possible, see Section 3.2.

The following theorem shows that Assumption A4 is also sufficient for existence and uniqueness of solutions of (1).

Theorem 3.3 (*Existence and Uniqueness*). *Consider the switched nonlinear DAE* (1) *satisfying* A4 *and* A1–A3 *for each subsystem. Then for every switching signal* $\sigma \in \Sigma$ *and every* $x_0 \in \mathfrak{C}_{\sigma(0-)}$ *there exists a unique solution* $x \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ *of* (1) *on* $[0, \infty)$ *with* $x(0-) = x_0$. *Furthermore, for all* $t \in [0, \infty)$ *and all solutions* x *of* (1),

$$x(t) = \Pi_{\sigma(t)}(x(t-)),$$

where Π_p, $p \in \{1, \dots, \mathrm{p}\}$, *are the consistency projectors induced by* A4. *In particular, on each interval which does not contain a switching time,* x *is a classical solution of the corresponding subsystem.*

Proof. *Step* 1: Existence of a solution.

Let $t_0 = 0$ and $t_i > 0$, $i = 1, 2, \dots$ be the ordered switching times of σ after t_0 and let $p_i := \sigma(t_i)$. Inductively and invoking Assumption A3 choose $x^i \in (\mathcal{C}^1 \cap \mathcal{C}^\infty_{\mathrm{pw}})^n$, $i \in \mathbb{N}$, such that x^i is the unique (classical) solution of $E_{p_i}(x^i)\dot x^i = f_{p_i}(x^i)$ on the interval $[t_i, t_{i+1})$ with $x^i(t_i) = \Pi_{p_i}(x^{i-1}(t_i-))$, where $x^{-1}(t_0-) := x_0$. We show that any $x \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ with $x(0-) = x_0$ and $x_{[t_i,t_{i+1})} = x^i{}_{[t_i,t_{i+1})}$ for $i \in \mathbb{N}$ solves the switched DAE (1) on $[0, \infty)$. By definition x solves (1) on each open interval (t_i, t_{i+1}) and it remains to check that

$$(E_\sigma(x)(x_{\mathbb{D}})')[t_i] = (f_\sigma(x)_{\mathbb{D}})[t_i] = 0 \quad \text{for all } i \in \mathbb{N},$$

where $D[t]$ denotes the impulsive part of $D \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ at $t \in \mathbb{R}$ (see Appendix for details). Invoking the properties of piecewise-smooth distributions, it follows that

$$(E_\sigma(x)(x_{\mathbb{D}})')[t_i] = E_{p_i}(x(t_i))\big(x(t_i) - x(t_i-)\big)\delta_{t_i}$$
$$= E_{p_i}\big(\Pi_{p_i}\big(x(t_i-)\big)\big)\big(\Pi_{p_i}\big(x(t_i-)\big) - x(t_i-)\big)\delta_{t_i} = 0,$$

where the last equation follows from Assumption A4.

Hence x is a solution of (1) on $[0, \infty)$.

Step 2: Uniqueness of the solution.

With the notation as in Step 1 it suffices to show that the solution x as constructed above is unique on $[0, t_1)$, uniqueness on $[t_1, \infty)$ follows then inductively. Let $z \in (\mathcal{C}^\infty_{\mathrm{pw}})^n$ be a solution of (1) on $[0, t_1)$ with $z(0-) = x_0$. With a similar argument as in Step 1 it follows that

$$E_{p_0}(z(0))(z(0) - x_0) = 0,$$

hence Assumption A4 ensures $z(0) = \Pi_{p_0}(x_0) = x(0)$. Furthermore, Assumption A4 also implies that $z(t) = z(t-)$ for all $t \in (0, t_1)$, hence z is continuous on $(0, t_1)$ which together with

Assumption A3 implies that $z_{(0,t_1)} = x_{(0,t_1)}$. Hence uniqueness of the solution is shown. □

Remark 3.4 (*Assumption A4 for Each System*)**.** Note that Assumption A4 applied to each single system, i.e. $p = q$, additionally restricts the possible nonlinear DAEs even without switching: in A4 one can always pick $x_0^+ = x_0^-$ if $p = q$ and the asserted uniqueness of x_0^+ implies therefore

$$\forall x_0^+ \in \mathfrak{C}_p : \ker E_p(x_0^+) \cap \left\{ x_0^+ - x_0^- \,\middle|\, x_0^- \in \mathfrak{C}_p \right\} = \{0\}. \tag{11}$$

So in addition to A1–A3 each subsystem must also fulfill (11). In the linear case it can be shown that A3 already implies (11), but in the general case this is not true as the following example shows:

$$x_2\dot{x}_1 = 0,$$
$$\dot{x}_2 = 1.$$

With initial value $x_2(0) = -t_0 \in \mathbb{R}$, we get the unique solution $x_2(t) = t - t_0$ and $\dot{x}_1(t) = 0$ for all $t \neq t^0$. The only classical solution of the latter is $x_1(t) = x_1^0$, where $x_1^0 \in \mathbb{R}$ is the initial value for x_1. Hence A3 holds. However, condition A4 is not fulfilled because (11) does not hold. In fact, when allowing jumps in solutions (as in the case for switched DAEs) uniqueness of solutions is lost, because x can have an arbitrary jump at $t = t_0$ without violating the DAE (in a distributional sense).

Remark 3.5 (*Index-One Systems*)**.** If the nonlinear DAE (2) can be written as (e.g. via a (nonlinear) coordinate transformation)

$$\dot{x}_1 = g(x_1, x_2),$$
$$0 = h(x_1, x_2),$$

where h is such that x_2 can be solved in terms of x_1, then (in analogy to the linear case) (2) is said to have *index one*. In this case, Assumption A4 clearly holds with consistency projector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \overline{h}(x_1) \end{pmatrix}$, where the function $\overline{h}$ is such that $x_2 = \overline{h}(x_1)$ is the unique solution of $h(x_1, x_2) = 0$. However, Assumption A4 is weaker than the index-one assumption because it could hold even when not all subsystems are index-one; see also Remark 2.4.

3.2. The linear case

Consider the linear switched DAE (3) with switching signal $\sigma \in \Sigma$. As already mentioned above, the Assumptions A1–A3 for each subsystem reduce to the regularity condition $\det(E_p s - A_p) \not\equiv 0$ for each subsystem. Under this assumption (in particular without assuming A4) it already follows from Trenn (2009a,b) that existence and uniqueness of solutions of (3) is guaranteed. However, these solutions are then elements of the space of *piecewise-smooth distributions* and will therefore, in general, contain *Dirac impulses* and their derivatives. The following example illustrates this phenomenon.

Example 3.6. Consider (3) with subsystems given by

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Then the switching signal $\sigma(t) = 1$ on $[0, 1)$ and $\sigma(t) = 2$ on $[1, \infty)$ together with the initial condition $x(0) = (1, 0, 0)$ enforces a jump to zero in x_1 at the switching time. At the switching time the second system is already active, in particular $x_2 = \dot{x}_1$ holds, hence x_2 is the derivative of a jump, i.e. $x_2 = -\delta_1$ contains the Dirac impulse at $t = 1$. Furthermore, also the equation $x_3 = \dot{x}_2$ must hold which yields that $x_3 = -\delta_1'$, i.e. x_3 contains the derivative of a Dirac impulse.

Since the presence of impulses in solutions can be seen as an undesired unstable behavior (see the next section), we would like to give an easily checkable condition which ensures that for arbitrary switching all solutions of (3) are impulse-free (but may still exhibit jumps). It will turn out that this condition is equivalent to Assumption A4 but is easier to check in the linear case. For the formulation of this condition, we define the linear consistency projector of a regular matrix pair (E, A).

Definition 3.7 (*Linear Consistency Projector*)**.** Consider a regular matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and, invoking Theorem 2.3, choose invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that (SET, SAT) is in quasi-Weierstrass form (5) with $n_1 \times n_1$ and $n_2 \times n_2$ the corresponding diagonal block sizes. The *linear consistency projector* is then given by

$$\Pi_{E,A} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where I is an $n_1 \times n_1$ identity matrix.

Let $\mathcal{V}^*$ and $\mathcal{W}^*$ be the limits of the Wong sequences as in Theorem 2.3. Then it is easy to see that the definition of $\Pi_{E,A}$ is independent of the choice of T and that it is a linear projection onto $\mathcal{V}^* = \mathfrak{C}_{E,A}$ along $\mathcal{W}^*$, i.e.

$$\Pi_{E,A}^2 = \Pi_{E,A}, \qquad \operatorname{im} \Pi_{E,A} = \mathcal{V}^*, \qquad \ker \Pi_{E,A} = \mathcal{W}^*. \tag{12}$$

With the help of the linear consistency projectors it is now possible to give an easily checkable characterization of Assumption A4.

Theorem 3.8 (*Linear Version of Assumption A4*)**.** *Consider the switched linear DAE (3) with regular matrix pairs (E_p, A_p) and corresponding consistency projectors $\Pi_p, p \in \{1, \ldots, \mathrm{p}\}$ as in Definition 3.7. Then Assumption A4 is equivalent to*

$$\forall p, q \in \{1, \ldots, \mathrm{p}\} : E_q(\Pi_q - I)\Pi_p = 0 \tag{13}$$

and the linear mapping $x_0^- \mapsto x_0^+ := \Pi_q x_0^-$ coincides with the consistency projector associated with Assumption A4.

Proof. Let $p, q \in \{1, \ldots, \mathrm{p}\}$ and $x_0^- \in \mathfrak{C}_p := \mathfrak{C}_{E_p,A_p}$ be arbitrary and fixed in the rest of the proof.

Step 1: We show (13) ⇒ A4.

Let $x_0^+ := \Pi_q x_0^- \in \mathfrak{C}_q := \mathfrak{C}_{E_q,A_q}$, then, since $\Pi_p x_0^- = x_0^-$,

$$E_q(x_0^+ - x_0^-) = E_q(\Pi_q \Pi_p x_0^- - \Pi_p x_0^-) = E_q(\Pi_q - I)\Pi_p x_0^- \overset{(13)}{=} 0,$$

hence the existence assertion of Assumption A4 is shown. To show uniqueness of x_0^+, let $z \in \mathfrak{C}_q$ be such that

$$z - x_0^- \in \ker E_q \subseteq \mathcal{W}_q^* = \ker \Pi_q,$$

where $\mathcal{W}_q^*$ is the limit of the corresponding Wong sequence for (E_q, A_q) as in Theorem 2.3. Together with $\Pi_q z = z$ this implies $z = \Pi_q x_0^- = x_0^+$.

Step 2: We show A4 ⇒ (13).

Choose $x_0^+ \in \mathfrak{C}_q$ such that $x_0^+ - x_0^- \in \ker E_q \subseteq \mathcal{W}_q^* = \ker \Pi_q$, hence $x_0^+ = \Pi_q x_0^+ = \Pi_q x_0^-$. Therefore, by $\Pi_p x_0^- = x_0^-$,

$$0 = E_q(x_0^+ - x_0^-) = E_q(\Pi_q \Pi_p x_0^- - \Pi_p x_0^-) = E_q(\Pi_q - I)\Pi_p x_0^-.$$

Since $x_0^- \in \mathfrak{C}_p = \mathcal{V}_p^*$ is arbitrary it follows from $\mathcal{V}_p^* \oplus \mathcal{W}_p^* = \mathbb{R}^n$ together with $\mathcal{W}_p^* = \ker \Pi_p$ that $E_q(\Pi_q - I)\Pi_p = 0$, hence (13) holds. □

Combining Theorems 3.3 and 3.8 yields that for every switched linear DAE (3) with regular matrix pairs (E_p, A_p), $p = 1, \dots, \mathrm{p}$, satisfying (13) there *exists* a solution $x \in (\mathfrak{C}^\infty_{\text{pw}})^n$, unique in this class of functions. By definition, this solution also solves (3) in the distributional framework of Trenn (2009a,b). Since the switched DAE (3) with regular pairs (E_p, A_p), $p = 1, \dots, \mathrm{p}$, has a *unique* distributional solution (for a fixed initial value $x(0-)$) we obtain the following result.

Corollary 3.9 (*Impulse-Free Solutions for* (3)). *Consider the switched DAE* (3) *with arbitrary switching signal* $\sigma \in \Sigma$ *and regular matrix pairs* (E_p, A_p) *with corresponding consistency projectors* $\Pi_p \in \mathbb{R}^{n\times n}$ *given by Definition 3.7. If* (13) *holds, then every distributional solution* $x \in (\mathbb{D}_{\text{pw}\mathfrak{C}^\infty})^n$ *of* (3) *is impulse-free.*

4. Asymptotic stability of switched DAEs

Asymptotic stability for (1), with a fixed switching signal σ, can be defined basically in the same way as for the non-switched case, see Section 2.2; the only difference is that the solutions might have jumps, so we have to decide where to evaluate the initial value. In view of Theorem 3.3, we consider the initial value $x(0-)$. Note that in the linear case Assumption A4 excludes impulses in the solution, which is reasonable for the definition of stability, because an impulse can be interpreted as an infinite peak which remains infinite even when the corresponding solution is scaled so that $|x(0-)|$ gets arbitrarily small.

Theorem 4.1 (*Arbitrary Switching*). *Consider the switched DAE* (1) *satisfying Assumption* A4 *and Assumptions* A1–A3 *for each subsystem with corresponding consistency spaces* $\mathfrak{C}_p := \mathfrak{C}_{E_p,f_p}$ *and consistency projectors* $\Pi_p, p \in \{1, \dots, \mathrm{p}\}$ *induced by* A4. *Assume for each subsystem that there exists a Lyapunov function* $V_p : \mathfrak{C}_p \to \mathbb{R}_{\geq 0}$ *in the sense of Definition 2.5. If*

$$\forall p, q \in \{1, \dots, \mathrm{p}\}\ \forall x \in \mathfrak{C}_p : V_q(\Pi_q(x)) \leq V_p(x), \tag{14}$$

then the switched DAE (1) *is asymptotically stable for any switching signal* $\sigma \in \Sigma$.

Proof. *Step* 1: Definition of a Lyapunov function candidate.

If $x \in \mathfrak{C}_p \cap \mathfrak{C}_q$ for some $p, q \in \{1, \dots, \mathrm{p}\}$ then $x = \Pi_p(x) = \Pi_q(x)$, hence (14) implies $V_p(x) = V_q(x)$. Therefore

$$V : \bigcup_p \mathfrak{C}_p \to \mathbb{R}, \qquad x \mapsto V_p(x) \quad \text{for } x \in \mathfrak{C}_p,$$

is well defined.

Step 2: $V\big(x(t)\big) \to 0$ as $t \to \infty$.

Fix $\sigma \in \Sigma$ and let $x : [0, \infty) \to \mathbb{R}^n$ be a solution of (1) in the sense of Theorem 3.3. Consider an interval $\mathfrak{I} \subseteq \mathbb{R}$ without switching times, then x is also a classical (local) solution of $E_p(x)\dot{x} = f_p(x)$ on $\mathfrak{I}$ where $p := \sigma(\tau)$ for $\tau \in \mathfrak{I}$. From $x(\tau) \in \mathfrak{C}_p$ for all $\tau \in \mathfrak{I}$ it follows that $V(x(\tau)) = V_p(x(\tau))$ and, by Definition 2.5 together with (6),

$$\frac{\mathrm{d}}{\mathrm{d}t} V_p\big(x(\tau)\big) = \dot{V}_p\big(x(\tau)\big) < 0 \quad \forall \tau \in \mathfrak{I}.$$

Let $t \in \mathbb{R}$ be a switching time of σ, then $x(t) = \Pi_{\sigma(t)}\big(x(t-)\big)$ and $x(t-) \in \mathfrak{C}_{\sigma(t-)}$ yield, invoking (14),

$$\begin{aligned} V\big(x(t)\big) &= V_{\sigma(t)}\big(x(t)\big) = V_{\sigma(t)}\big(\Pi_{\sigma(t)}(x(t-))\big) \\ &\leq V_{\sigma(t-)}\big(x(t-)\big) = V\big(x(t-)\big). \end{aligned}$$

Hence $t \mapsto v(t) = V\big(x(t)\big)$ is monotonically decreasing and therefore $\underline{v} := \lim_{t\to\infty} v(t) \geq 0$ is well defined. Seeking a contradiction, assume $\underline{v} > 0$. Analogously to the proof of Theorem 2.7 let $\mathcal{K}_p := V_p^{-1}[\underline{v}, v(0)]$, $\mathcal{M}_p := \dot{V}(\mathcal{K}_p)$ and $m_p := -\max \mathcal{M}_p > 0$. Let $m = \min_p m_p > 0$ then $\frac{\mathrm{d}}{\mathrm{d}t} v(t) < -m < 0$ for all non-switching (hence almost all) times $t \geq 0$, which contradicts $v(t) \geq 0$ and the assertion of Step 2 is shown.

Step 3: $x(t) \to 0$ as $t \to \infty$.

Seeking a contradiction, assume $x(t) \not\to 0$. Then there exist $\varepsilon > 0$ and a sequence $(s_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $s_i \to \infty$ as $i \to \infty$ such that $|x(s_i)| > \varepsilon$ for all $i \in \mathbb{N}$. There is at least one $p \in \{1, \dots, \mathrm{p}\}$ such that the set $\{i \in \mathbb{N} \,|\, \sigma(s_i) = p\}$ has infinitely many elements, therefore, without loss of generality, assume that $\sigma(s_i) = p$ for some p and all $i \in \mathbb{N}$. Since each $x(s_i)$ is contained within the compact set $V_p^{-1}[0, V(x(0))]$, the same argument as in the proof of Theorem 2.7 shows existence of $x^* \neq 0$ such that we arrive at the contradiction $0 = \lim_{t\to\infty} V(x(t)) = \lim_{i\to\infty} V_p(x(s_i)) = V_p(x^*) \neq 0$.

Step 4: Stability of the switched DAE.

We first show that for all $\varepsilon > 0$ there exists $b_\varepsilon > 0$ such that for all $p \in \{1, \dots, \mathrm{p}\}$ and all $x \in \mathfrak{C}_p$

$$V_p(x) < b_\varepsilon \Rightarrow |x| < \varepsilon. \tag{15}$$

Assume the contrary, then there exist $\varepsilon > 0$ and sequences $(p_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $V_{p_n}(x_n) < 1/n$ and $|x_n| \geq \varepsilon$. There exists at least one $p \in \{1, \dots, \mathrm{p}\}$ which occurs infinitely often in the sequence (p_n), so we can, without loss of generality, assume that $p_n = p$ for all $n \in \mathbb{N}$ and some $p \in \{1, \dots, \mathrm{p}\}$. Then, by L1, all x_n are contained in the compact set $V_p^{-1}[0, V_p(x_{n_{\max}})]$ where $n_{\max} := \operatorname{argmax}_n V_p(x_n) < \infty$. This implies that there exists $x^* \in \mathfrak{C}_p$ which is a limit of a subsequence of (x_n) and with $|x^*| \geq \varepsilon$. Hence we arrive at the contradiction $0 = \lim_{n\to\infty} V_p(x_n) = V_p(x^*) \neq 0$ and the claim (15) is shown.

For a given $\varepsilon > 0$ choose $b_\varepsilon > 0$ according to (15). Let $p_0 := \sigma(0-)$, then by continuity of V_{p_0} there exists $\delta > 0$ such that $|x| < \delta$ implies $V_{p_0}(x) < b_\varepsilon$ for all $x \in \mathfrak{C}_{p_0}$. In Step 2 it was shown that $t \mapsto V_{\sigma(t-)}(x(t-))$ is monotonically decreasing, hence $V_{\sigma(t-)}(x(t-)) < b_\varepsilon$ for all $t \geq 0$. Hence (15) yields $|x(t-)| < \varepsilon$ for all $t \geq 0$. □

Condition (14) implies that any two Lyapunov functions V_p and V_q coincide on the intersection $\mathfrak{C}_p \cap \mathfrak{C}_q$, hence Theorem 4.1 is a generalization of the switched ODE case where the existence of a common Lyapunov function is sufficient to ensure stability under arbitrary switching (Liberzon, 2003, Theorem 2.1). However, without condition (14) the existence of a common Lyapunov function is not enough (Liberzon & Trenn, 2009) for asymptotic stability of the switched DAE (1). Under arbitrary switching, solutions will in general exhibit jumps; these jumps are described by the consistency projectors, and these projectors must "fit together" with the Lyapunov functions in the sense of (14) to ensure stability of the switched DAE under arbitrary switching. Finally, with some additional effort it can be shown that the hypotheses of Theorem 4.1 guarantee uniformity of the asymptotic stability with respect to the switching signal.

It is well-known for switched ODEs that by restricting the class of switching signals one can obtain asymptotic stability also in cases where no common Lyapunov function exists. Denote by $N_\sigma(t, T)$ the number of switchings of σ in the interval $[t, T)$ and define the class of *average dwell time* switching signals with average dwell time $\tau_a > 0$ (Hespanha & Morse, 1999)

$$\Sigma_{\tau_a} := \left\{ \sigma \in \Sigma \,\middle|\, \begin{array}{l} \exists N_0 > 0\ \forall t \in \mathbb{R}\ \forall \Delta t > 0 : \\ N_\sigma(t, t + \Delta t) < N_0 + \dfrac{\Delta t}{\tau_a} \end{array} \right\}.$$

The number $N_0 > 0$ is called the *chatter bound* of the switching signal $\sigma \in \Sigma_{\tau_a}$. Note that the subset of average dwell time switching signals with chatter bound $N_0 = 1$ is precisely the class of switching signals with *dwell time* τ_a.

Theorem 4.2 (*Average Dwell Time Switching*). *Consider the switched DAE (1) with corresponding consistency space $\mathfrak{C}_p$ and consistency projectors Π_p, $p \in \{1, \dots, \mathrm{p}\}$. Assume that all subsystems permit Lyapunov functions V_p, $p \in \{1, \dots, \mathrm{p}\}$, which additionally fulfill*

1. *$\exists \lambda > 0 : \dot{V}_p(x) \le -\lambda V_p(x)$ for all $p \in \{1, \dots, \mathrm{p}\}$, $x \in \mathfrak{C}_p$ and*
2. *$\exists \mu \ge 1 : V_q(\Pi_q(x)) \le \mu V_p(x)$ for all $p, q \in \{1, \dots, \mathrm{p}\}$, $x \in \mathfrak{C}_p$.*

Then the switched DAE (1) with switching signal $\sigma \in \Sigma_{\tau_a}$ is asymptotically stable if

$$\tau_a > \frac{\ln \mu}{\lambda}. \tag{16}$$

Proof. With standard arguments (cf. Liberzon, 2003) it follows that the non-negative function $t \mapsto V_{\sigma(t-)}(x(t-))$ is bounded by an exponentially decreasing function and hence converges to zero. Arguments analogous to those in Steps 3 and 4 of the proof of Theorem 4.1 now conclude the proof. □

In the linear case the Lyapunov functions can be chosen according to Remark 2.8; in this case it is possible to express the inequality (16) for the average dwell time directly in terms of eigenvalues of corresponding matrices.

Lemma 4.3 (*The Linear Case*). *Consider the linear switched DAE (3) with the regular matrix pairs (E_p, A_p), $p \in \{1, \dots, \mathrm{p}\}$, with corresponding consistency spaces $\mathfrak{C}_p$, and let (P_p, Q_p) be the solutions of the corresponding generalized Lyapunov equation (7). Choose a matrix O_p with orthonormal columns such that $\operatorname{im} O_p = \operatorname{im} \Pi_p = \mathfrak{C}_p$, where Π_p is the linear consistency projector corresponding to the matrix pair (E_p, A_p). Then, for $p, q \in \{1, \dots, \mathrm{p}\}$,*

$$\forall x \in \mathfrak{C}_p : V_q(\Pi_q x) \le \mu_{p,q} V_p(x),$$

where

$$\mu_{p,q} := \frac{\lambda_{\max}(O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p)}{\lambda_{\min}(O_p^\top E_p^\top P_p E_p O_p)} \ge 0$$

and

$$\forall x \in \mathfrak{C}_p : \dot{V}_p(x) \le -\lambda_p V_p(x),$$

where

$$\lambda_p := \frac{\lambda_{\min}(O_p^\top Q_p O_p)}{\lambda_{\max}(O_p^\top E_p^\top P_p E_p O_p)} > 0$$

and where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively.

Proof. Let $d_p := \dim \mathfrak{C}_p$, i.e. $O_p \in \mathbb{R}^{n \times d_p}$, then $x \in \mathfrak{C}_p$ if, and only if, there exists a unique $z \in \mathbb{R}^{d_p}$ with $x = O_p z$, $O_p^\top x = z$ and $|x| = |z|$. Hence, by choosing z corresponding to $x \in \mathfrak{C}_p$ as above,

$$\begin{aligned} V_p(x) &= z^\top O_p^\top E_p^\top P_p E_p O_p z =: z^\top P_p^z z \\ &\ge \lambda_{\min}(P_p^z)|z|^2 = \lambda_{\min}(P_p^z)|x|^2 \end{aligned}$$

$$V_p(x) \le \lambda_{\max}(P_p^z)|x|^2$$

$$\begin{aligned} V_q(\Pi_q x) &= z^\top O_p^\top \Pi_q^\top E_q^\top P_q E_q \Pi_q O_p z =: z^\top M_{p,q}^z z \\ &\le \lambda_{\max}(M_{p,q}^z)|x|^2 \end{aligned}$$

$$\dot{V}_p(x) = -z^\top O_p^\top Q_p O_p z =: -z^\top Q_p^z z \le -\lambda_{\min}(Q_p^z)|x|^2.$$

By assumption, the matrices $Q_p^z = Q_p^{z\top} \in \mathbb{C}^{d_p \times d_p}$ and $P_p^z = P_p^{z\top} \in \mathbb{C}^{d_p \times d_p}$ are positive definite, hence $\lambda_{\min}(Q_p^z) > 0$ and $\lambda_{\max}(P_p^z) \ge \lambda_{\min}(P_p^z) > 0$. Therefore,

$$\mu_{p,q} := \frac{\lambda_{\max}(M_{p,q}^z)}{\lambda_{\min}(P_p^z)} \ge 0, \qquad \lambda_p := \frac{\lambda_{\min}(Q_p^z)}{\lambda_{\max}(P_p^z)} > 0$$

are well defined. Note that $\lambda_{\max}(M_{p,q}^z) = 0$ is possible, however $\lambda_{\max}(M_{p,p}^z) = \lambda_{\max}(P_p^z) \ge \lambda_{\min}(P_p^z)$, hence $\mu_{p,p} \ge 1$ and $\max_{p,q} \ln \mu_{p,q} \ge 0$. □

Corollary 4.4 (*Average Dwell Time, Linear Case*). *For the switched linear DAE (3) with asymptotically stable subsystems, let $\mu_{p,q}$ and λ_p, $p, q \in \{1, \dots, \mathrm{p}\}$, be given as in Lemma 4.3. Then the linear switched DAE (3) is asymptotically stable if $\sigma \in \Sigma_{\tau_a}$ with*

$$\tau_a > \frac{\max_{p,q} \ln \mu_{p,q}}{\min_p \lambda_p}.$$

Note that the obtained results cannot in general be expressed in terms of the eigenvalues of the matrices Q_p and P_p (or $E_p^\top P_p E_p$); the consistency projectors and basis transformation must be incorporated as formulated in Lemma 4.3. We show the application of Corollary 4.4 with a simple linear example, which is based on Example 1 from Liberzon and Trenn (2009).

Example 4.5. Let

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

The corresponding consistency spaces and consistency projectors are given by

$$\mathfrak{C}_1 := \mathfrak{C}_{E_1, A_1} = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathfrak{C}_2 := \mathfrak{C}_{E_2, A_2} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

In Liberzon and Trenn (2009) it is shown that the corresponding switched DAE is not asymptotically stable under arbitrary switching. However, we can apply the result of Corollary 4.4. As basis matrices for the consistency space choose $O_1 = \frac{1}{2}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$, $O_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Consider the Lyapunov functions $V_1(x) = \frac{1}{2}x_2^2$ and $V_2(x) = \frac{1}{2}(x_1 + x_2)^2$, corresponding to

$$P_1 = P_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} O_1^\top E_1^\top P_1 E_1 O_1 &= \frac{1}{4}, & O_2^\top E_2^\top P_2 E_2 O_2 &= \frac{1}{2}, \\ O_1^\top \Pi_2^\top E_2^\top P_2^\top E_2 \Pi_2 O_1 &= 1, & O_2^\top \Pi_1^\top E_1^\top P_1^\top E_1 \Pi_1 O_2 &= \frac{1}{2}, \\ O_1^\top Q_1 O_1 &= \frac{1}{2}, & O_2^\top Q_2 O_2 &= 1, \end{aligned}$$

hence $\mu := \max_{p,q} \mu_{p,q} = 2$ and $\lambda := \min_p \lambda_p = 2$. Therefore the corresponding switched DAE is asymptotically stable for all switching signals $\sigma \in \Sigma_{\tau_a}$ with $\tau_a > \frac{\ln 2}{2}$. This bound is actually sharp in this example.

5. Conclusion

We have studied switched nonlinear DAEs with respect to solution and stability theory. For the non-switched nonlinear DAE subsystems we generalized the classical Lyapunov's Direct Method; in particular, we defined a Lyapunov function for quasi-linear DAEs in general terms. Furthermore, we studied existence and uniqueness of solutions of a switched nonlinear DAE, provided the subsystems are regular in a certain sense. Finally, we were able to generalize existing stability results on switched ODEs to switched DAEs.

Appendix. Piecewise smooth distributions

We assume familiarity with the definitions and properties of classical distributions as formalized by Schwartz (1950–1951). We denote the space of *test functions* (i.e., smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support) by $\mathcal{C}_0^\infty$, then the *space of distributions* is the dual space of the space of test functions, i.e.

$$\mathbb{D} := \left\{ D : \mathcal{C}_0^\infty \to \mathbb{R} \,|\, D \text{ is linear and continuous} \right\}.$$

The main two properties of distributions are (i) that they can be interpreted as generalized functions and (ii) that they are arbitrarily often differentiable. To be more precise, let $\mathcal{L}_{1,\mathrm{loc}}$ be the space of locally integrable functions, then the mapping

$$\mathcal{L}_{1,\mathrm{loc}} \to \mathbb{D}, \qquad f \mapsto f_{\mathbb{D}} := \left(\varphi \mapsto \int_{\mathbb{R}} f\varphi \right)$$

is well defined (i.e. $f_{\mathbb{D}}$ is indeed a distribution) and an injective homomorphism. The simplest distribution which is not induced by a function is the *Dirac impulse* given by $\delta(\varphi) := \varphi(0)$, or, in general for $t \in \mathbb{R}$, $\delta_t(\varphi) := \varphi(t)$ for $\varphi \in \mathcal{C}_0^\infty$. The derivative of an arbitrary distribution $D \in \mathbb{D}$ is given by $D'(\varphi) := -D(\varphi')$ for $\varphi \in \mathcal{C}_0^\infty$. Distributions can be *multiplied with smooth functions*:

$$(\alpha D)(\varphi) := D(\alpha\varphi), \quad \alpha \in \mathcal{C}^\infty,\ D \in \mathbb{D},\ \varphi \in \mathcal{C}_0^\infty.$$

Let $\mathcal{C}_{\mathrm{pw}}^\infty$ be the space of piecewise-smooth functions, where $\alpha : \mathbb{R} \to \mathbb{R}$ is called piecewise-smooth when there exist a locally finite ordered set $S = \{s_i \in \mathbb{R} \,|\, i \in \mathbb{Z}\}$ and smooth functions $\alpha_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$, such that $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[s_i,s_{i+1})}$. Here, f_I denotes the restriction (or truncation) of a function $f : \mathbb{R} \to \mathbb{R}$ to the interval $I \subseteq \mathbb{R}$, i.e. $f_I(\tau) = f(\tau)$ for $\tau \in I$ and $f_I(\tau) = 0$ otherwise. The space of *piecewise-smooth distributions* is then given by

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \,\middle|\, \begin{array}{l} f \in \mathcal{C}_{\mathrm{pw}}^\infty,\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall \tau \in T : D_\tau \in \mathrm{span}\{\delta_\tau, \delta_\tau', \delta_\tau'', \ldots\} \end{array} \right\}.$$

The properties of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and corresponding definitions are summarized in the following, where $D = f_{\mathbb{D}} + \sum_{\tau\in T} D_\tau \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and $t \in \mathbb{R}$:

1. Closed under differentiation: $D' \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$.
2. Left- and right-evaluation: $D(t+) := f(t)$, $D(t-) := f(t-)$.
3. Impulsive part: $D[t] := D_t$ if $t \in T$, $D[t] = 0$ otherwise.
4. Restriction to interval: $D_I := (f_I)_{\mathbb{D}} + \sum_{\tau\in T\cap I} D_t$, where $I \subseteq \mathbb{R}$ is some interval.
5. Multiplication with piecewise-smooth function: $\alpha D := \sum_{i\in\mathbb{Z}} \alpha_i D_{[s_i,s_{i+1})}$, where $\alpha = \sum_{i\in\mathbb{Z}} (\alpha_i)_{[s_i,s_{i+1})}$ as above; in particular, $\alpha\delta_t = \alpha(t)\delta_t$.

For more details see Trenn (2009a,b). In the proof of Theorem 3.3 we actually need the fact that for any $\alpha \in \mathcal{C}_{\mathrm{pw}}^\infty$, $(\alpha D)[t] = \alpha D[t]$ and

$$(\alpha_{\mathbb{D}})' = \sum_{i\in\mathbb{Z}} (\alpha_i')_{[s_i,s_{i+1})} + \sum_{i\in\mathbb{Z}} (\alpha_i(s_i) - \alpha_{i-1}(s_i))\delta_{s_i},$$

where $\alpha = \sum_{i\in\mathbb{N}} (\alpha_i)_{[s_i,s_{i+1})}$ as above.

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