Robust Control with Compensation of Disturbances for Systems with Quantized Output[1]

Igor B. Furtat*. Alexander L. Fradkov*. Daniel Liberzon**

**Institute for Problems of Mechanical Engineering Russian Academy of Sciences, 61 Bolshoy ave V.O., St.-Petersburg, 199178, Russia (Tel: +7-812-321-47-66; e-mail: cainenash@mail.ru).*
**ITMO University, 49 Kronverkskiy ave, Saint Petersburg, 197101, Russia.*
***Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W. Main Street, Urbana, IL 61801, USA.*

Abstract: The paper deals with the robust output feedback discrete control of continuous-time linear plants with arbitrary relative degree under parametric uncertainties and external bounded disturbances with quantized output signal. The parallel reference model (auxiliary loop) to the plant is used for obtaining the uncertainties acting on the plant. The proposed algorithm guarantees that the output of the plant tracks the reference output with the required accuracy.

1. INTRODUCTION

In recent years much attention has been given to the investigation of constraints on a communication channel included in a feedback control loop. Signal quantization (via quantizer or encoder) is usually considered as a source of independent discrete random noise which additively acts on the system. In Widrow (1961), Gray and Neuhoff (1988) this assumption allows to simplify the investigation of systems with quantization, in particularly, for control of plants described by linear models. However, this assumption may be too rough if the value of the quantization step is commensurate with the range of signal variation (Delchamps (1989), Baillieul (2002)).

The quantization level in a discrete closed-loop system can cause oscillatory processes similar to oscillations in continuous nonlinear systems. In addition to the analysis problem there are many results on control synthesis. Some of these results are related to minimizing errors caused by quantization in the control loop. Usually these problems are posed in terms of optimization of an integral performance index (loss function). The earliest works (Tou (1963), Lewis and Tou (1965), Larson (1967)) are dedicated to solution of this problem. The paper of Larson (1967) is devoted to the synthesis of an optimal control system for discrete linear plants with quantization of the input signal. From Larson (1967) we have for input signal the one-dimensional density distribution. The optimization criterion for the encoder is the expected value of some cost function. The solution of this optimization problem is based on using standard methods of mathematical programming.

In Fischer (1982) the algorithm for optimal quantization is derived from the solution of an optimization problem for a closed-loop system with a linear-quadratic criterion with Gaussian noise (LQG-problem). Similar results are presented by Curry (1969) where the optimal stabilization problem of a linear stochastic discrete plant with a quadratic cost function is considered. In Curry (1969) the measuring device is described by a nonlinear static characteristic (which may be a nonlinear characteristic of the encoder at the output of the plant). Also in Curry (1969) an innovation (a mismatch between the plant output and the conditional mean value produced by a Kalman filter) is received by the encoder. The papers of Goodman and Gersho (1974), Zhang and Lockhart (1995), Zierhofer (2000), Aldajani and Sayed (2001), Venayagamoorthy and Zha (2007), Golding and Schultheiss (1967) are devoted to design of adaptive quantizers where the range of signal conversion is changed automatically. In Goodman and Gersho (1974) the width of the band quantization is changed at the encoding of each signal. Note that these results are only devoted to the problem of transmitting signals. In subsequent works (see for example Gomez-Estern, Canudas de Wit, Rubio and Fornes (2007), Andrievsky, Fradkov and Peaucelle (2007), Zheng, Duni and Rao (2007)) the use of adaptive quantizers in control systems and estimation systems is considered.

The paper of Liberzon (2003) is concerned with global asymptotic stabilization of continuous-time systems subject to quantization. A hybrid control strategy (Brockett and Liberzon (2000)) relies on the possibility of making discrete on-line adjustments of quantizer parameters. Sharon and Liberzon (2012) considered the problem of achieving input-to-state stability with respect to external disturbances for control systems with quantized measurements. Quantizers considered in that paper take finitely many values and have an adjustable center and zoom.

[1] This work was supported by grants of Russian Foundation for Basic Researcher No. 13-08-01014, 12-08-01183, Federal Program "Cadres" (agreement No. 8855) and was partially financially supported by Government of Russian Federation, Grant 074-U01.

The special interest of the present paper is control of a plant under parametric uncertainties and disturbances. Guo, Zhang, Zhao (2011) study the adaptive tracking control for systems with quantized output observations and one unknown parameter. A projection algorithm is proposed for parameter identification, based on which an adaptive control law is designed via the certainty equivalence principle. By use of the conditional expectation of the quantized observation with respect to the estimates, it is shown that the identification algorithm is both almost surely and mean square convergent, the closed-loop system is stable, and the adaptive tracking control is asymptotically optimal. The result by Konaka, Suzuki, Okuma (2002) deals with a control problem in which the continuous plant is controlled by a discrete logic-based controller, while the control requirements are specified for continuous variables. A new indirect adaptive control strategy for a line-following control of a two-wheeled vehicle with quantized input and output is proposed. The vehicle is supposed to have a low-resolution sensor and actuator. It is shown that some unknown parameters of the system can be estimated from quantized input and output by making use of partial information on the system. The paper by Zheng and Yang (2012) is concerned with the quantized output feedback stabilization problem for a class of uncertain systems with nonsmooth nonlinearities in the actuator device via sliding mode control schemes. It is assumed that system signals are quantized before being transmitted through communication channels. A dynamical compensator is developed to estimate unmeasurable system states. Then a sliding surface, in the augmented space using the system output and the estimated state, is proposed, and an adaptive sliding mode control scheme with a static adjustment law of the quantization parameter is established.

In the above results the problem of feedback quantized output control for dynamical plants with any relative degree under parametric uncertainties and uncontrollable disturbances is not studied. The present paper is dedicated to solving this problem. The parallel reference model (auxiliary loop) to the plant is used for obtaining the uncertainties acting on the plant. A robust algorithm was first proposed by Tsykunov (2007) for control of a continuous-time plant under parametric uncertainties and external disturbance. The idea of this method consists in implementing an auxiliary loop with desired parameters parallel to the plant. The difference between the output of the plant and the output of the auxiliary loop gives a function which depends on parametric and external disturbances. This function gives the control law that guarantees required accuracy of the control system. The proposed algorithm provides the tracking by the output of the plant of the reference output with the required accuracy. In Parsheva and Tsykunov (2009), Furtat (2010) this robust algorithm is generalized for discrete control of a continuous-time plant. In this paper we propose a generalization of results by Parsheva and Tsykunov (2009) for discrete control of continuous-time linear plants under parametric uncertainties and external bounded disturbance with quantization of an output signal. It is assumed that only scalar input and output of the plant are accessible for measurement. The proposed algorithm guarantees that the output of the plant tracks the reference output with the required accuracy.

2. PROBLEM STATEMENT

Consider the equation of a plant

$$\begin{aligned} Q(p)y(t) &= kR(p)u(t) + f(t), \\ p^{i-1}y(0) &= y_{0i}, \quad i = 1, \dots, n, \end{aligned} \tag{1}$$

where $y(t) \in R$ is an output being quantized, $u(t) \in R$ is an input, $f(t) \in R$ is an uncontrollable bounded disturbance, $Q(p) = p^n + q_{n-1}p^{n-1} + \dots + q_1 p + q_0$ and $R(p) = p^m + r_{m-1}p^{m-1} + \dots + r_1 p + r_0$ are linear differential operators with unknown coefficients, $\gamma = n - m \geq 1$, γ is a relative degree, $k > 0$, y_{0i} are unknown initial conditions, $p = d / dt$.

We are interested in the situation where only quantized measurements $q(y(t))$ of the output are available. We also assume that there exist positive real number $\bar{y}$ and quantization function $\bar{q}(y(t))$ such that the following condition holds

$$q(y(t)) = \begin{cases} \bar{q}(y(t)), \text{ if } |y(t)| \leq \bar{y}, \\ \bar{y}\text{sign}(y(t)), \text{ if } |y(t)| > \bar{y}. \end{cases} \tag{2}$$

Also assume that there exist a positive real number $\bar{\delta}$ such that if $|y(t)| \leq \bar{y}$ then

$$|\bar{q}(y(t)) - y(t)| \leq \bar{\delta} \,. \tag{3}$$

Equation (3) causes the quantizer error. Assume that we are given a smooth reference signal $y_m(t)$ and a sequence of sampling times t_k. The problem is to design a discrete control law such that the following condition holds:

$$|q(y(t_k)) - y_m(t_k)| < \delta \text{ for } t > T, \tag{4}$$

where t_k is a sampling time, $t_k \leq t \leq t_{k-1}$, $\delta > 0$ is a small enough number, $T > 0$ is a transient time.

Assumptions.

1. Coefficients $q_{n-1}, \dots, q_1, q_0, r_{m-1}, \dots, r_1, r_0, k$ belong to a known bounded set Ξ.

2. The plant (1) is minimum phase.

3. Only signals $q(y(t_k))$, $y_m(t_k)$ and $u(t_k)$ are available for measurement in a control system.

3. ROBUST ALGORITHM

According to Furtat (2010), Furtat (2011), Furtat, Fradkov, and Tsykunov (2011), Furtat, Fradkov, and Tsykunov (2013), Furtat (2013), represent the operators $R(p)$ and $Q(p)$ in the form

$$R(p) = R_0(p) + \Delta R(p) \,, \; Q(p) = Q_0(p) + \Delta Q(p) \,. \tag{5}$$

Here $R_0(\lambda)$ and $Q_0(\lambda)$ are Hurwitz polynomials of degrees n and m respectively, $\deg \Delta Q(p) < n$, $\deg \Delta R(p) < m$. The polynomials $R_0(\lambda)$ and $Q_0(\lambda)$ are chosen such that $Q_0(\lambda) / R_0(\lambda) = Q_m(\lambda)$, where $Q_m(\lambda)$ is a Hurwitz polynomial of degree $n - m$. Taking into account (1) and (5), the equation for the tracking error $e(t) = q(y(t)) - y_m(t)$ can be rewritten in the form

$$Q_m(p)e(t) = ku(t) + \rho(t), \tag{6}$$

where

$$\rho(t) = k\frac{\Delta R(p)}{R_0(p)}u(t) + \frac{1}{R_0(p)}f(t) - \frac{\Delta Q(p)}{R_0(p)}y(t) + Q_m(p)\big(q(y(t)) - y(t)\big) + Q_m(p)y_m(t) + \tau(t),$$

where $\tau(t)$ is an exponentially decaying function which depends on initial conditions of (1).

Introduce the control law

$$u(t) = \alpha v(t_k), \quad t_{k-1} \le t \le t_k, \tag{7}$$

where $\alpha > 0$ is a design parameter, $v(t_k)$ is a new control at time t_k.

From (6) we see that function $\rho(t)$ contains a parametric uncertainty and external disturbance of the plant (1) as well as quantization error. Therefore, according to Tsykunov (2007), Parsheva and Tsykunov (2009) consider the auxiliary loop

$$\begin{aligned} &Q_m(p)\bar{e}(t) = \beta v(t_k), \\ &p^{i-1}\bar{e}(t) = 0,\, i = 1, \ldots, \gamma, \end{aligned} \tag{8}$$

where $\beta > 0$ is a designed parameter, $\bar{e}(t)$ is an output of auxiliary loop (8). The auxiliary loop is a parallel model which characterizes the desired behaviour of transients in a closed-loop system. Therefore, taking into account (6), (7), and (8), the equation for the error function $\sigma(t) = e(t) - \bar{e}(t)$ can be written in the form

$$Q_m(p)\sigma(t) = \varphi(t), \tag{9}$$

Where $\varphi(t) = (k\alpha - \beta)v(t_k) + \rho(t)$ is a new disturbance function. The signal $\sigma(t)$ contains information about parametric and external disturbances of plant (1). According to the Problem Statement the control system is discrete. Therefore, the value $\varphi(t_k)$ satisfies the relation

$$\varphi(t_k) = q_m^{\mathrm{T}}\xi(t_k), \tag{10}$$

where q_m is a vector composed of coefficients of the operator $Q_m(p)$, $\xi(t_k) = \left[\sigma(t_k), \dot{\sigma}(t_k), \ldots, \sigma^{(\gamma)}(t_k)\right]^{\mathrm{T}}$, $\sigma^{(i)}(t_k)$ is the i-th derivative of the signal $\sigma(t)$ taken at time t_k.

If the vector $\xi(t_k) = \left[\sigma(t_k), \dot{\sigma}(t_k), \ldots, \sigma^{(\gamma)}(t_k)\right]^{\mathrm{T}}$ were available for measurement, then the control could be defined by

$$v(t_k) = -\frac{1}{\beta}q_m^{\mathrm{T}}\xi(t_k), \quad t_{k-1} \le t \le t_k. \tag{11}$$

It follows from (10) and (11) that

$$v(t_k) = -\frac{1}{\beta}\varphi(t_k), \quad t_{k-1} \le t \le t_k. \tag{12}$$

Solving the system consisting of equations (12) and $\varphi(t_k) = (k\alpha - \beta)v(t_k) + \rho(t_k)$ with respect to $v(t_k)$, we get

$$v(t_k) = -\frac{1}{ka}\rho(t_k), \quad t_{k-1} \le t \le t_k. \tag{13}$$

Substituting (13) and (7) to (6), we obtain $Q_m(p)e(t) = \rho(t) - \rho(t_k)$. According to Edwards and Neville (2002), Parsheva and Tsykunov (2009) there exist $\Delta t_0 > 0$ and $\upsilon > 0$ such that for any sample rate $\Delta t < \Delta t_0$ the inequality $|\rho(t) - \rho(t_k)| < \sqrt{\upsilon}$ hold, where $\Delta t = t_k - t_{k-1}$, $\upsilon > 0$.

However, it follows from Problem Statement that derivatives of the function $\sigma(t)$ are not available for measurement. Therefore, consider the estimate of the signal $\varphi(t_k)$ of the form

$$\tilde{\varphi}(t_k) = q_m^{\mathrm{T}}\tilde{\xi}(t_k), \tag{14}$$

where $\tilde{\xi}(t_k) = \left[\sigma(t_k), \sigma_1(t_k), \ldots, \sigma_\gamma(t_k)\right]^{\mathrm{T}}$, $\sigma_i(t_k)$ is an estimate of i-th derivative of the signal $\sigma(t_k)$ taken at time t_k. The vector $\tilde{\xi}(t_k)$ is obtained from the observer

$$\tilde{\xi}(t_k) = G\tilde{\xi}(t_{k-1}) + b\sigma(t_k). \tag{15}$$

Here

$$G = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -\frac{1}{\Delta t} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{1}{\Delta t^\gamma} & \cdots & -\frac{1}{\Delta t} & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ \frac{1}{\Delta t} \\ \vdots \\ \frac{1}{\Delta t^\gamma} \end{bmatrix}.$$

Equation (15) is written by using the right hand differences

$$\begin{aligned} \sigma(t_k) &= \sigma(t_k), \\ \sigma_1(t_k) &= \frac{\sigma(t_k) - \sigma(t_{k-1})}{\Delta t}, \\ &\vdots \\ \sigma_\gamma(t_k) &= \frac{\sigma_{\gamma-1}(t_k) - \sigma_{\gamma-1}(t_{k-1})}{\Delta t}. \end{aligned} \tag{16}$$

Rewrite relations (16) in the matrix form

$$\tilde{\xi}(t_k) = G_0\big(\tilde{\xi}(t_k) - \tilde{\xi}(t_{k-1})\big) + b_0\sigma(t_k), \tag{17}$$

where $G_0 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\Delta t} & 0 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{\Delta t} & 0 \end{bmatrix}$, $b_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Solving (17) with respect to $\tilde{\xi}(t_k)$, we obtain (15).

Taking into account (15), write control $v(t_k)$ as

$$v(t_k) = -\frac{1}{\beta} q_m^{\mathrm{T}} \tilde{\xi}(t_k), \quad t_{k-1} \le t \le t_k. \tag{18}$$

For implementing the auxiliary loop (8) in discrete form, first, transform (8) to a state space form

$$\dot{\bar{\varepsilon}}(t) = A_m \bar{\varepsilon}(t) + \beta B_m v(t_k),\ \bar{e}(t) = L\bar{\varepsilon}(t), \tag{19}$$

where $\bar{\varepsilon}(t) \in R^{\gamma}$ is a state vector, the matrices A_m, B_m and $L = [1\ 0\ ...\ 0]$ are obtained according to transformation from (8) to (19). Further, transform equations (19) to the discrete form

$$\bar{\varepsilon}(t_{k+1}) = \bar{A}\bar{\varepsilon}(t_k) + \beta\bar{B}v(t_k),\ \bar{e}(t_k) = L\bar{\varepsilon}(t_k), \tag{20}$$

where $\bar{A} = e^{A_m \Delta t}$, $\bar{B} = \int_t^{t+\Delta t} e^{A_m(t-s)} B_m ds$.

For the formulation of our main result introduce the following notations: $P = P^{\mathrm{T}} > 0$ is a solution of Lyapunov equation $A_m^{\mathrm{T}} P + PA_m = -Q$, $Q = Q^{\mathrm{T}} > 0$, $\chi = \frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}$, $R = Q - 2\mu^{-1} PB_m B_m^{\mathrm{T}} P$, $\mu > 0$, $|\varphi(t) - \tilde{\varphi}(t_k)| < \sqrt{\theta}$, $\theta > 0$.

Theorem. Let Assumptions 1-3 hold. Then there exist coefficients $\alpha > 0$, $\beta > 0$ and small enough values $\Delta t_0 > 0$, $\theta > 0$ such that for any sample rate $\Delta t \le \Delta t_0$ the inequality

$$|\varphi(t) - \tilde{\varphi}(t_k)| < \sqrt{\theta}, \quad t_{k-1} \le t \le t_k \tag{21}$$

holds, and for

$$\delta \le \sqrt{\lambda_{\min}^{-1}(P)\left(e^{-\chi T} \varepsilon^{\mathrm{T}}(0) P \varepsilon(0) + \left(1 - e^{-\chi T}\right)\chi^{-1}\mu\theta\right)}, \tag{22}$$

$$\left|p^{i-1}\left(y(0) - y_m(0)\right)\right| < \delta, \quad i = 1, 2, \dots, n, \tag{23}$$

$$|y_m(t_k)| < \bar{y} - \delta, \quad t_{k-1} \le t \le t_k, \tag{24}$$

the objective (4) in control system (7), (15), (18), (20) is achieved for any parameters of plant (1) from the set Ξ.

The Theorem will be proved in the Appendix.

It follows from (22) that the value δ explicitly depends on α, β, $\bar{\delta}$ and Δt. Moreover, the value δ in (4) can be reduced by decreasing the values α, Δt, $\bar{\delta}$ and increasing the value β.

Let us illustrate given results on a numerical example.

4. EXAMPLE

Consider a plant model in the form

$$\left(p^3 + q_1 p^2 + q_2 p + q_3\right) y(t) = \left(r_0 p + r_1\right) u(t) + f(t). \tag{25}$$

The set Ξ is determined by the following inequalities: $-3 \le q_i \le 3$, $i = 1, 2, 3$, $1 \le r_0 \le 2$, $1 \le r_1 \le 12$, $|f(t)| \le 30$. Let $\bar{y} = 1$ in (2).

The reference signal $y_m(t)$ is chosen as follows

$$y_m(t) = 0.2 + 0.5\sin 0.5t + 0.2\sin 1.7t.$$

Choose the equation of the auxiliary loop (8) as

$$\left(p^2 + 5p + 6\right)\bar{e}(t) = \beta v(t_k),\ \bar{e}(0) = \dot{\bar{e}}(0) = 0. \tag{26}$$

Equation (26) with sampling time 0.01 has the following form

$$\begin{aligned} \bar{\varepsilon}(t_{k+1}) &= \begin{pmatrix} 0.9997 & 0.0098 \\ -0.0585 & 0.0951 \end{pmatrix} \bar{\varepsilon}(t_k) \\ &+ \begin{pmatrix} -0.5084 \cdot 10^{-4} \\ 0.0103 \end{pmatrix} \beta v(t_k), \\ \bar{e}(t_k) &= [1\quad 0]\bar{\varepsilon}(t_k),\ \bar{\varepsilon}(0) = [0\ 0]^{\mathrm{T}}. \end{aligned} \tag{27}$$

Introduce the observer equation (15) as follows

$$\begin{aligned} \tilde{\xi}(t_k) &= \begin{pmatrix} 0 & 0 & 0 \\ -10^2 & 0 & 0 \\ -10^4 & -10^2 & 0 \end{pmatrix} \tilde{\xi}(t_{k-1}) + \begin{pmatrix} 1 \\ 10^2 \\ 10^4 \end{pmatrix} \sigma(t_k), \\ \tilde{\xi}(0) &= [0\ 0\ 0]^{\mathrm{T}}. \end{aligned} \tag{28}$$

Let $\alpha = 5$, $\beta = 10$. According to (7) and (18) write the control as

$$u(t) = 5v(t_k),\ v(t_k) = -0.1\left(6\tilde{\xi}_1(t_k) + 5\tilde{\xi}_2(t_k) + \tilde{\xi}_3(t_k)\right), \tag{29}$$

where $\left[\tilde{\xi}_1(t_k), \tilde{\xi}_2(t_k), \tilde{\xi}_3(t_k)\right]^{\mathrm{T}} = \tilde{\xi}(t_k)$.

Let quantization interval be equal to 0.1. In Fig. 1 the transient of the tracking error $e(t)$ is given by the following parameters in (25): $q_1 = -3$, $q_2 = -3$, $q_3 = -3$, $r_0 = 1$, $r_1 = 1$, $f(t) = 3 + 17\sin t + 10\cos 1.7t$, $y(0) = \dot{y}(0) = \ddot{y}(0) = 0.9$.

Let quantization interval be equal to 0.05. In Fig. 2 the simulation result of the tracking error $e(t)$ is given by the following parameters in (25): $q_1 = 3$, $q_2 = -1$, $q_3 = 0$, $r_0 = 2$, $r_1 = 12$, $f(t) = 1 + 10\sin 1.1t + 4\cos 0.8t$, $y(0) = \dot{y}(0) = \ddot{y}(0) = 0.8$.

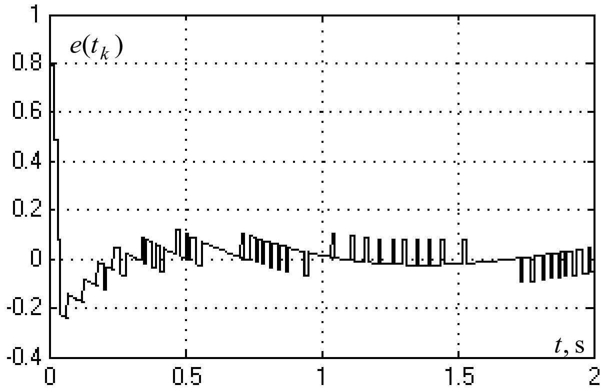

Fig.1. The transients of the tracking error $e(t_k)$.

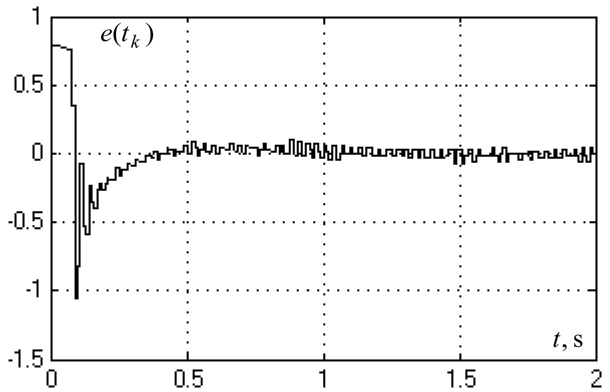

Fig.2. The transients of the tracking error $e(t_k)$.

It follows from Fig. 1 that parametric uncertainties and external disturbances are compensated by control system (27)-(29) with the required accuracy $\delta = 0.11$ achieved after 0.5 s. It follows from Fig. 2 the required accuracy $\delta = 0.06$ is achieved after 0.5 s. Simulation results show that the error $e(t)$ can be reduced by decrease of the value α, Δt, $\bar{\delta}$ and increase of the value β.

5. CONCLUSIONS

In this paper, we have treated the problem of robust output feedback discrete control of continuous-time linear plants under parametric uncertainties and external bounded disturbance with quantized output signal. The parallel reference model (auxiliary loop) which allows obtaining a function containing parametric uncertainties and external disturbances acting on the plant was considered. We proposed an algorithm that provides tracking by the plant output of the reference output with the required accuracy. Relationships between the tracking accuracy and the quantization error, uncertainties of the plant, and parameters of the regulator were derived.

REFERENCES

Aldajani, M.A., and A.H. Sayed (2001). Stability and Performance Analysis of an Adaptive Sigma-Delta Modulator, *IEEE Transaction on Circuits Systems* II, 48(3), 233–244.

Andrievsky, B., A.L. Fradkov, and D. Peaucelle (2007). State Estimation over the Limited-Band Communication Channel for Pitch Motion Control of LAAS Helicopter Benchmark, *Proc. 17th IFAC Symp. Automat. Contr. Aerospace* (*ACA'2007*), Toulouse, France.

Baillieul, J. (2002). Feedback Coding for Information-Based Control: Operating Near the Data Rate Limit, *Proc. 41st IEEE Conf. Decision Control*, ThP02-6, Las Vegas, Nevada, USA, 3229–3236.

Brockett, R.W., and D. Liberzon (2000). Quantized feedback stabilization of linear systems, *IEEE Transactions on Automatic Control*, 45, 1279-1289.

Curry, R.E. (1969). A Separation Theorem for Nonlinear Measurements, *IEEE Transactions on Automatic Control*, 14, 561–569.

Delchamps, D.F. (1989). Extracting State Information from a Quantized Output Record, *System Control Letters*, 13, 365–372.

Fischer, T.R. (1982). Optimal Quantized Control, *IEEE Transactions on Automatic Control*, 27(4), 996–998.

Furtat, I.B., and A.M. Tsykunov (2010). Adaptive control of plants of unknown relative degree, *Automation and Remote Control*, 71(6), 1076-1084.

Furtat, I.B. (2010). Continuous-discrete robust control of linear plant, *Mechatronics, Automation, Control*, 4, 10-16 (in Russian).

Furtat, I.B. (2011). Robust synchronization of dynamical networks with compensation of disturbances, *Automation and Remote Control*, 72(12), 2516–2526.

Furtat, I.B., A.L. Fradkov, and A.M. Tsykunov (2011). Robust synchronization of linear networks with compensation of disturbances, *Proc. of the 18th IFAC World Congress on Automatic Control*, Milan, 1255-1260.

Furtat I., A. Fradkov, and A. Tsykunov (2013). Robust synchronization of linear dynamical networks with compensation of disturbances, *International Journal of Robust and Nonlinear Control,* published online Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3024.

Furtat, I.B. (2013). Robust Synchronization of the Structural Uncertainty Nonlinear Network with Delay & Disturbances, *Proc. 11th IFAC International Workshop on Adaptation and Learning in Control and Signal Processing*, Caen, France, 227-232.

Golding, L.S., and P.M. Schultheiss (1967). Study of an Adaptive Quantizer, Proc. IEEE, 55(3), 293–297

Gomez-Estern, F., C. Canudas de Wit, F. Rubio, and J. Forn´es (2007). Adaptive Delta-Modulation Coding for Networked Controlled Systems, *Proc. Amer. Control. Conf.*, N.Y., USA, FrA20.6.

Goodman, D.J., and A. Gersho. (1974). Theory of an Adaptive Quantizer, *IEEE Transaction on Communication*, COM-22 (8), 1037–1045.

Gray, R.M., and D.L. Neuhoff (1988). *Quantization, IEEE Transaction on Information Theory*, 44, 2325-2383.

Guo, J., J.-F., Zhang, and Y. Zhao (2011). Adaptive tracking control with quantized output observations and single unknown parameter. *Proc. 2011 International Conference on Modelling, Identification and Control*, 451–456.

Konaka, E., T. Suzuki, and S. Okuma (2002). Indirect adaptive control of two wheeled vehicle by quantized input and output. *Proc. 2002 International Conference on Control Applications*, vol. 1, 600–605.

Larson, R.E. (1967). Optimum Quantization in Dynamic Systems, *IEEE Transactions on Automatic Control*, 12, 162–168.

Lewis, J.B., and J.T. Tou (1965). Optimum Sampled-Data Systems with Quantized Control Signals, *Transaction on AIEE*, 82(2), 195–201

Liberzon, D. (2003). Hybrid feedback stabilization of systems with quantized signals, *Automatica*, 39, 1543-1554.

Edwards J.T., and J.F. Neville (2002). Boundedness and stability of solutions to difference equations, *Journal of Computational and Applied Mathematics*, 140(1-2), 275–289.

Parsheva, A.I., and A.M. Tsykunov (2009). Digital robust control of linear continuous plant with disturbance compensation, *Vestnik of Astrakhan State Technical University. Series: Control, Computer and Information Sciences*, 2, 61-67 (in Russian).

Sharon, Y., D. Liberzon (2012). Input to state stabilizating controller for systems with coarse quantization, *IEEE Transactions on Automatic Control*, 57(4), 830-844.

Tou, J.T. (1963). Optimum Design of Digital Control Systems, New York: Academic.

Tsykunov, A.M. (2007). Robust control algorithms with compensation of bounded perturbations, *Automation and Remote Control*, 71(7), 1213-1224.

Venayagamoorthy, G.K., and W. Zha (2007). Comparison of Nonuniform Optimal Quantizer Designs for Speech Coding with Adaptive Critics and Particle Swarm, IEEE Transaction on Industry, 43(1), 238–244.

Widrow, B. (1961). Statistical Analysis of Amplitude-Quantized Sampled-Data Systems, *Transaction on AIEE*, 79(2), 555-567.

Zhang, S.D., and G.B. Lockhart. (1995). Design and Simulation of an Efficient Adaptive Delta Modulation Embedded Coder, *IEE Proc. Vis. Image Signal Proc*., 142(3), 155–160.

Zheng, J., E.R. Duni, and B.D. Rao (2007). Analysis of Multiple-Antenna Systems with Finite-Rate Feedback Using High-Resolution Quantization Theory, *IEEE Transaction on Signal Processing*, 55(4), 1461–1475.

Zheng B.-C., and G.-H. Yang (2012). Quantized output feedback stabilization of uncertain systems with input nonlinearities via sliding mode control. *International Journal of Robust and Nonlinear Control*, Article first published online: 13 AUG 2012, DOI: 10.1002/rnc.2883.

Zierhofer, C.M. (2000). Adaptive Sigma-Delta Modulation with One-Bit Quantization, *IEEE Transaction on Circuits Systems* II, 47(5), 408–415.

APPENDIX A.

Proof of Theorem. Taking into account (7), (10), (12) and (18), rewrite equation (6) in the form

$$Q_m(p)e(t) = \varphi(t) - \tilde{\varphi}(t_k)\,. \tag{30}$$

According to Edwards and Neville (2002), Parsheva and Tsykunov (2009) there exist $\Delta t_0 > 0$ and $\theta > 0$ such that for any sample rate $\Delta t < \Delta t_0$ the inequality $|\varphi(t) - \tilde{\varphi}(t_k)| < \sqrt{\theta}$ holds.

Rewrite equation (27) in a state space form

$$\dot{\varepsilon}(t) = A_m\varepsilon(t) + B_m(\varphi(t) - \tilde{\varphi}(t_k)),\ e(t) = L\varepsilon(t). \tag{31}$$

Choose Lyapunov function $V(t) = V(\varepsilon(t))$ in the form

$$V(t) = \varepsilon^{\mathrm{T}}(t)P\varepsilon(t)\,. \tag{32}$$

Taking the derivative of (32) along the trajectories of (31), we obtain

$$\dot{V}(t) = -\varepsilon^{\mathrm{T}}(t)Q\varepsilon(t) + 2\varepsilon^{\mathrm{T}}(t)PB_m(\varphi(t) - \tilde{\varphi}(t_k)). \tag{33}$$

Consider the following bound

$$\begin{aligned} &2\varepsilon^{\mathrm{T}}(t)PB_m(\varphi(t) - \tilde{\varphi}(t_k)) \\ &\le 2\mu^{-1}\varepsilon^{\mathrm{T}}(t)PB_mB_m^{\mathrm{T}}P\varepsilon(t) + \mu\theta. \end{aligned} \tag{34}$$

Taking into account bound (34) rewrite (33) as follows

$$\dot{V}(t) \le -\varepsilon^{\mathrm{T}}(t)R\varepsilon(t) + \mu\theta\,, \tag{35}$$

Rewrite (35) in the form

$$\dot{V}(t) \le -\chi V(t) + \mu\theta\,, \tag{36}$$

Solving inequality (36) with respect to $V(t)$, we obtain

$$V(t) \le e^{-\chi t}V(0) + \chi^{-1}(1 - e^{-\chi t})\mu\theta\,. \tag{37}$$

It follows from (37) that $\lim_{t\to\infty} V(t) \le \chi^{-1}\mu\theta$. Taking into account (32) and (37), we have

$$|e(t)| \le |\varepsilon(t)| \le \sqrt{\lambda_{\min}^{-1}(P)\left(e^{-\chi t}V(0) + (1 - e^{-\chi t})\chi^{-1}\mu\theta\right)}\,. \tag{38}$$

It follows from (38) that relation (22) holds. Obviously, the value of the right hand side of (22) depends on the values α, β, $\bar{\delta}$ and Δt. Therefore, the value δ in (4) can be reduced by decreasing the value α, Δt, $\bar{\delta}$ and increasing the value β.

Consider inequality (4). Since the upper bound of $y(t)$ is $\bar{y}$, rewrite (4) as

$$|\bar{y} - y_m(t)| < \delta\,. \tag{39}$$

Hence, the conditions (23) and (24) follow from (39).