NONLINEAR CONTROL with LIMITED INFORMATION

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Plenary talk, 2nd Indian Control Conference, Hyderabad, Jan 5, 2015
INFORMATION FLOW in CONTROL SYSTEMS

Plant

Controller
INFORMATION FLOW in CONTROL SYSTEMS

- Coarse sensing
- Limited communication capacity
  - many control loops share network cable or wireless medium
  - microsystems with many sensors/actuators on one chip
- Need to minimize information transmission (security)
- Event-driven actuators
- Theoretical interest
BACKGROUND

Previous work:

[Brockett, Delchamps, Elia, Mitter, Nair, Savkin, Tatikonda, Wong,…]

• Deterministic & stochastic models
• Tools from information theory
• Mostly for linear plant dynamics

Our goals:

• Handle nonlinear dynamics
• Unified framework for
  • quantization
  • time delays
  • disturbances
OUR APPROACH

( Goal: treat nonlinear systems; handle quantization, delays, etc. )

• Model these effects via **deterministic** error signals, \( e \)
• Design a control law ignoring these errors, \( u = k(x) \)
• “Certainty equivalence”: apply control, \( u = k(x + e) \)
  combined with estimation to reduce \( e \) to zero

Caveat:
This doesn’t work in general, need robustness from controller

Technical tools:
• Input-to-state stability (ISS)   • Small-gain theorems
• Lyapunov functions             • Hybrid systems
$z \in \mathcal{R}^k \xrightarrow{\text{Encoder}} i \in \{1, \ldots, N\} \xrightarrow{\text{Decoder}} q(z) \in \mathcal{Q}$

$\mathcal{R}^k$ is partitioned into quantization regions
QUANTIZATION and ISS

\[ \dot{x} = f(x, u) \]
QUANTIZATION and ISS

\[ \dot{x} = f(x, k(x)) \quad \text{– assume glob. asymp. stable (GAS)} \]
QUANTIZATION and ISS

\[ \dot{x} = f(x, k(q(x))) \]

no longer GAS
\[ \dot{x} = f(x, k(q(x))) \]
\[ = f(x, k(x + e)) \]

quantization error

Assume \[ \exists V : \]
\[ |x| \geq \rho(|e|) \]

\[ \downarrow \]
\[ \frac{\partial V}{\partial x} f(x, k(x+e)) < 0 \]

class \( \mathcal{K}_\infty \)
\[ \dot{x} = f(x, k(q(x))) = f(x, k(x + e)) \]

Quantization error

Assume \( \exists V : \)

\[ |x| \geq \rho(|e|) \]

\[ \frac{\partial V}{\partial x} f(x, k(x + e)) < 0 \]

Solutions that start in \( \mathcal{R}_1 \) enter \( \mathcal{R}_2 \) and remain there

This is input-to-state stability (ISS) w.r.t. measurement errors

In time domain: \( |x(t)| \leq \beta(|x_0|, t) + \gamma(\|e\|_{[0,t]}) \)
LINEAR SYSTEMS

\[ \dot{x} = Ax + Bu \]

\[ \exists \text{ feedback gain } K \text{ & Lyapunov function } V = x^T P x : \]

\[ (A + BK)^T P + P(A + BK) = -I \]

Quantized control law: \[ u = Kq(x) = K(x + e) \]

Closed-loop: \[ \dot{x} = (A + BK)x + BK e \]

(automatically ISS w.r.t. \( e \))

\[ \dot{V} < 0 \text{ if } |x| > 2\|PBK\|\|e\| \]

\[ 2\|PBK\| \Delta \]
DYNAMIC QUANTIZATION
DYNAMIC QUANTIZATION

\[ q(x/\mu), \quad \mu \text{ – zooming variable} \]

Hybrid quantized control: \( \mu \) is discrete state
DYNAMIC QUANTIZATION

\( q(x/\mu) \), \( \mu \) – zooming variable

Hybrid quantized control: \( \mu \) is discrete state
DYNAMIC QUANTIZATION

\[ q(x/\mu), \quad \mu \text{ – zooming variable} \]

Hybrid quantized control: \( \mu \) is discrete state

Zoom out to overcome saturation
DYNAMIC QUANTIZATION

\[ q(x/\mu), \quad \mu \text{ – zooming variable} \]

Hybrid quantized control: \( \mu \) is discrete state

After ultimate bound is achieved,
recompute partition for smaller region

Can recover global asymptotic stability

Proof: ISS from \( x \) to \( \mu \) + ISS from \( \mu \) to \( x \) + small-gain condition

[L–Nešić, ’05, ’06, L–Nešić–Teel ’14]
QUANTIZATION and DELAY

Architecture-independent approach

Delays possibly large

Based on the work of Teel
QUANTIZATION and DELAY

\[ \dot{x} = f(x, q(k(x(t - \tau)))) \]
\[ = f(x, k(x) + \theta + e) \]

where

\[ \theta(t) := k(x(t - \tau)) - k(x(t)) \]
\[ e(t) := q(k(x(t - \tau))) - k(x(t - \tau)) \]

Can write

\[ \theta(t) = - \int_{t-\tau}^{t} \frac{d}{ds} k(x(s)) \, ds \]

hence

\[ |\theta(t)| \leq \tau \gamma \left( \| (x, e) \|_{[t-2\tau, t]} \right) \]
Assuming ISS w.r.t. actuator errors:

\[ |x| \geq \rho(|\theta + e|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x) + \theta + e) < 0 \]

In time domain:

\[ |x(t)| \leq \beta(|x_0|, t) + \gamma_\theta \|\theta\|_{[0,t]} + \gamma_e \|e\|_{[0,t]} \]
\[ \leq \beta(|x_0|, t) + \gamma_1 (\tau \gamma_2 (\|x\|_{[t-2\tau,t]})) + \gamma_3 (\|e\|_{[t-2\tau,t]}) \]

**Small gain:** if \[ \gamma_1 (\tau \gamma_2 (r)) < r \quad \forall r > 0 \]

then we recover ISS w.r.t. \( e \) \quad [Teel '98]
Need: $\gamma_1(\tau \gamma_2(r)) < r$

$\forall \Lambda > \epsilon > 0 \ \exists \tau^* > 0 :$

small gain true $\forall \tau \leq \tau^*$
Need: $\gamma_1(\tau \gamma_2(r)) < r$

$\forall \Lambda > \varepsilon > 0 \ \exists \tau^* > 0 :$

small gain true $\forall \tau \leq \tau^*$
Need: $\gamma_1(\tau \gamma_2(r)) < r$

$\forall \Lambda > \varepsilon > 0 \; \exists \tau^* > 0$:

small gain true \quad $\forall \tau \leq \tau^*$

$\tau \leq \tau^* \Rightarrow$ solutions starting in $R_1$ enter $R_2$ and remain there

Can use “zooming” to improve convergence
EXTERNAL DISTURBANCES [Nešić–L]

State quantization and completely unknown disturbance
EXTERNAL DISTURBANCES

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EXTERNAL DISTURBANCES [Nešić–L]

State quantization and completely unknown disturbance

After zoom-in:

**Issue:** disturbance forces the state outside quantizer range

Must switch repeatedly between zooming-in and zooming-out

**Result:** for linear plant, can achieve ISS w.r.t. disturbance

(ISS gains are nonlinear although plant is linear; cf. [Martins])
NETWORKED CONTROL SYSTEMS

NCS: Transmit only some variables according to time scheduling protocol
Examples: round-robin, TOD (try-once-discard)

QCS: Transmit quantized versions of all variables

NQCS: Unified framework combining time scheduling and quantization

Basic design/analysis steps:
• Design controller ignoring network effects
• Prove discrete protocol stability via Lyapunov function
• Apply small-gain theorem to compute upper bound on maximal allowed transmission interval (MATI)
ACTIVE PROBING for INFORMATION

**Dynamic (time-varying)**

**Dynamic (changes at sampling times)**

very small
Example: \( \dot{x} = f(x, u), \ n = 2, \ N = 9 = 3^n \)

Zoom out to get initial bound
\( \hat{x}(t_0) := 0 \)

Between samplings \( \dot{x} = f(\hat{x}, u) \)
NONLINEAR SYSTEMS

Example: $\dot{x} = f(x, u), \ n = 2, \ N = 9 = 3^n$

Between samplings $\dot{x} = f(\hat{x}, u)$
Let $e := \hat{x} - x$ $\dot{x} = f(x, u)$ \Rightarrow $\dot{e} = f(\hat{x}, u) - f(x, u)$

$\|f(\hat{x}, u) - f(x, u)\|_\infty \leq L\|e\|_\infty$ on a suitable compact region (dependent on $x_0$)

The norm $\|e\|_\infty$:

- grows at most by the factor $\Lambda := e^{L\tau}$ in one period
- is divided by 3 at the sampling time
NONLINEAR SYSTEMS (continued)

\[ e = \hat{x} - x \]

The norm \( ||e||_\infty \):

- grows at most by the factor \( \Lambda := e^{\mathcal{L}_\tau} \) in one period
- is divided by 3 at each sampling time

Pick \( \tau \) small enough s.t. \( \Lambda < 3 \Rightarrow e \to 0 \)

\[ u(t) = k(\hat{x}(t)) \]

\[ \dot{x} = f(x, k(\hat{x})) = f(x, k(x + e)) \]

If this is ISS w.r.t. \( e \) as before, then \( x \to 0 \)
ROBUSTNESS of the CONTROLLER

Option 1. \[ \dot{x} = f(x, k(x + e)) \]

ISS w.r.t. \( e \Rightarrow x \rightarrow 0 \)

Same condition as before (restrictive, hard to check)

Option 2. Look at the evolution of \( \hat{x} \)

\[
\begin{cases}
\dot{\hat{x}} = f(\hat{x}, k(\hat{x})) & t \neq \text{sampl. time} \\
\hat{x}(t) = \hat{x}(t^-) + \Delta e(t), & t = \text{sampl. time}
\end{cases}
\]

ISS w.r.t. \( \Delta e \Rightarrow \hat{x} \rightarrow 0 \Rightarrow x \rightarrow 0 \)

\[\exists \text{ checkable sufficient conditions ([Hespanha-L-Teel])}\]
LINEAR SYSTEMS

$$\dot{x} = Ax + Bu$$
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\hat{x} &= A\hat{x} + Bu
\end{align*}
\]

Between sampling times, \( \Rightarrow \dot{e} = Ae \)

- \( \|e\|_\infty \) grows at most by \( \Lambda := e\|A\|_\infty \tau \) in one period
- divided by 3 at each sampling time

global quantity: \( \Lambda < 3 \Rightarrow e \to 0 \)

amount of static info provided by quantizer

\[
\hat{x} = A\hat{x} + BK\hat{x} = (A + BK)x + BK e \Rightarrow x \to 0
\]

[Baillieul, Brockett-L, Hespanha, Nair-Evans, Petersen-Savkin, Tatikonda]
OTHER RESEARCH DIRECTIONS

- Quantized control of switched systems
- Quantized output feedback and observers (with H. Shim)
- Disturbances and coarse quantizers (with Y. Sharon)
- Modeling uncertainty (with L. Vu)
- Performance-based design (with F. Bullo)
- Multi-agent coordination (with S. LaValle and J. Yu)
- Vision-based control (with Y. Ma and Y. Sharon)