Higher Order Derivatives of Lyapunov Functions for Stability of Systems with Inputs

Shenyu Liu Daniel Liberzon

***Abstract*— In this paper we study the alternative method for determining the stability of dynamical systems by inspecting the higher order derivatives of a Lyapunov function. The system can be time invariant or time varying; in both cases we define the higher order derivatives when there are inputs. We then claim and prove that if there exists a linear combination of those higher order derivatives with non-negative coefficients (except that the coefficient of the 0-th order term needs to be positive) which is negative semi-definite, then the system is globally uniformly asymptotically stable. The proof involves repeated applications of comparison principle for first order differential relations. We also show that a system with inputs whose auxiliary system admits a Lyapunov function satisfying the aforementioned conditions is input-to-state stable.**

I. Introduction

For general nonlinear systems, asymptotic stability is typically shown through Lyapunov's direct method (see, e.g., [1]), which involves constructing a positive definite Lyapunov function V whose time derivative $\dot{V}$ along solutions is negative definite. Because of the opposite sign definite constraints and the fact that $\dot{V}$ is coupled to V via the system's dynamics, although the classical Lyapunov results are theoretically elegant, they have lots of difficulties in the application. In most cases, $\dot{V}$ might be positive somewhere and V as a function of time when it is evaluated along a solution of the system becomes non-monotonic. Nevertheless, there is still a chance that the system is asymptotically stable as long as V converges to 0 asymptotically with bounded overshoots.

The motivation for the study of non-monotonic V is behind the simple idea that "if over any long enough period V decreases more than it increases, then V is asymptotically decreasing". One way to ensure that V does not increase too much is to restrict the length of time it can continuously increase, as studied in [2],[3]. This also leads to our previous work on "almost" Lyapunov functions with small $\dot{V} \geq 0$ regions in [4],[5]. Another way is to force "$\ddot{V}$", the rate of the change of $\dot{V}$, to be negative enough so that $\dot{V}$ will become negative again soon enough. This idea leads to the study of higher order derivatives of V, which takes its roots in the paper [6] by Butz, where a linear combination of higher order derivatives of V up to order 3 was studied. The collection of higher order derivatives is also called vector Lyapunov function in [7],[8], which can be used to analysis the stability of a system. A similar idea of higher order derivatives of V for analyzing discrete time systems is studied in [9] and then this technique is practiced for fuzzy systems in [10]. Much later after Butz's work, in [11] a general result was concluded for time varying systems that under some mild assumptions, as long as there exists a negative definite linear combination of the higher order derivatives of V and the coefficients form a Hurwitz polynomial, then the system is globally uniformly asymptotically stable. The result is derived by repeatedly applying the comparison principle for first order differential relations. In [12] the same authors argued that in fact Hurwitzness is not needed; as long as the coefficients are non-negative, the same results can be concluded.

All the aforementioned literature only deals with systems without inputs. In this work we extend the concept of higher order derivatives of Lyapunov functions to systems with inputs. For such systems, input-to-state stability is an important and widely used concept for characterizing a system's response to inputs and is the target property that we would like our systems to have. However, higher order derivatives of Lyapunov functions are not well defined because of the presence of inputs. Inspired by the work in [11], we realize that a generalized, upper-bounding higher order derivatives can be used in our case and this construction is valid whether the system is time invariant or time varying. As a result, by repeated applications of comparison principle for first order differential relations, we are able to show that whenever V along a solution goes beyond a positive value, it has to drop back to the same value in finite time. The system can then be shown to be globally uniformly asymptotically stable. Deploying the equivalences of stability definitions between a system with inputs and its auxiliary system as in [13], we are eventually able to conclude a sufficient condition based on higher order derivatives of V for showing input-to-state stability.

This paper is divided into 6 sections. Section II provides the necessary preliminaries for the later part of the paper, with our main theorem results stated in the end. Section III is the detailed proof for our main theorem. Section IV gives two examples where our techniques can be used for showing the stability of systems. Section V is the discussion and eventually Section VI concludes this paper.

II. Preliminaries and results

A. *Stability definitions and sign definite functions*

Consider a nonlinear time varying system with inputs

$$\dot{x} = f(t, x, u) \tag{1}$$

S. Liu and D. Liberzon are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, U.S.A. This work was supported by the NSF grant CMMI-1662708 and the AFOSR grant FA9550-17-1-0236.

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous in t and locally Lipschitz in x and u. $U \subseteq \mathbb{R}^m$ is the input value set and the input function $u(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to U) =: \mathcal{M}_U$; that is, $u(\cdot)$ is a locally essentially bounded function. For a specific initial condition x_0 at time t_0 and input u, denote the solution state of (1) at time t by $x(t, x_0, u)$.

A system (1) is said to be *input-to-state stable* (ISS) [14] if there exist $\gamma \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$[1] such that for all $x_0 \in \mathbb{R}, u \in \mathcal{M}_U$ and $t \geq t_0$,

$$|x(t, x_0, u)| \leq \beta(|x_0|, t - t_0) + \gamma(\|u\|_{[t_0,t]}). \tag{2}$$

When the input has negligible effects on the stablity of the system, we also say that the system (1) is *globally uniformly asymptotically stable* (GUAS) if there exists $\beta \in \mathcal{KL}$ such that

$$|x(t, x_0, u)| \leq \beta(|x_0|, t - t_0) \tag{3}$$

for all $x_0 \in \mathbb{R}, u \in \mathcal{M}_U$ and $t \geq t_0$. GUAS coincides with the same classical definition of GUAS for systems without inputs

$$\dot{x} = f(t, x) \tag{4}$$

by taking $U = \{0\}$. ISS and GUAS are connected via auxiliary system as discussed in the celebrated work [13]:

Lemma 1 *The system* (1) *is ISS if and only if its auxiliary system*

$$\dot{x} = f(t, x, \rho(|x|)d) =: f'(t, x, d), \quad |d| \leq 1 \tag{5}$$

is GUAS for some $\rho \in \mathcal{K}_\infty$.

For a function $P(x) : \mathbb{R}^n \to \mathbb{R}$, we say P is *positive definite* ($P \succ 0$) if $P(0) = 0$ and $P(x) > 0$ when $x \neq 0$. We say P is *positive semi-definite* ($P \succeq 0$) if $P(0) = 0$ and $P(x) \geq 0$ when $x \neq 0$. P is said to be *negative definite* ($P \prec 0$) or *negative semi-definite* ($P \preceq 0$), if $-P$ is positive definite or positive semi-definite, respectively. We say P is *sign indefinite* if P is neither positive semi-definite nor negative semi-definite. For a function $V(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$, we abuse the same terminologies and say $V(t, x)$ being positive/negative (semi)-definite or sign indefinite if this property holds for the function $V(\cdot, t)$ for all $t \geq t_0$. In addition, we write $P \succ Q$ if $P - Q \succ 0$ and similar notations for the other sign definite relations.

B. Construction of higher order derivatives of V

We start from the discussion on the system without inputs. It is well-known in [1] that the system (4) is GUAS if there exists a differentiable function $V(t, x)$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad \forall t \geq t_0, x \in \mathbb{R}^n \tag{6}$$

[1] $\gamma \in \mathcal{K}_\infty$ if $\gamma(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous, strictly increasing function and $\gamma(0) = 0, \lim_{s\to\infty} \gamma(s) = \infty$. $\beta \in \mathcal{KL}$ if $\beta(s, t) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function such that $\beta(\cdot, t)$ is increasing, $\beta(s, \cdot)$ is decreasing and $\beta(0, t) = 0$ for all $t \geq t_0$, $\lim_{t\to\infty} \beta(s, t) = 0$ for all $s \geq 0$.

and

$$\dot{V}(t, x) := \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) \preceq -\alpha_3(|x|). \tag{7}$$

The second condition (7) can be replaced by the following one

$$\dot{V}(t, x) + aV(t, x) \preceq 0 \tag{8}$$

for some $a > 0$ (so $\alpha_3 = a\alpha_1$) and still show GUAS. When extending this idea to systems with inputs (1) and higher order derivatives of V, there are two technical issues that need to be resolved. First, if we use the same definition in (7) for "$\dot{V}$", it will become input dependent and hence undesired for our study because our GUAS property is uniform with respect to input. A good way to get rid of the dependence on u is to take the supremum of those "$\dot{V}$" with respect to u:

$$V_1(t, x) := \frac{\partial V(t, x)}{\partial t} + \sup_{u \in U} \left(\frac{\partial V(t, x)}{\partial x} f(t, x, u) \right).$$

This V_1 seems to be a good candidate for our first order derivative of V for the system (1). The second technical issue is that we cannot directly differentiate V twice along a solution in order to get an expression for the second order derivative because it involves the derivative of u with respect to t, which may not even exist as u is not assumed to be differentiable. On the other hand, we cannot differentiate V_1 to get the second order derivative either because differentiability of V_1 is unclear due the sup function used in its definition. Note that we only needs upper bounds for the derivatives; they do not need to be tight and thus a solution to this technical issue is by finding a smooth upper bound $v_1 \succeq V_1$. This v_1 will be the actual first order derivative of V used for our analysis. An advantage of using smooth functions is that they can be used to generate the subsequent higher order derivatives. In other words, we define $v_0(t, x) := V(t, x)$, and for all $i = 1, 2, \cdots$, iteratively define and construct

$$\begin{aligned} &V_i(t, x) := \frac{\partial v_{i-1}(t, x)}{\partial t} + \sup_{u \in U} \left(\frac{\partial v_{i-1}(t, x)}{\partial x} f(t, x, u) \right), \\ &v_i(t, x) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \text{ s.t. } v_i \succeq V_i. \end{aligned} \tag{9}$$

We call those v_i functions the higher order derivatives of V if they exist. We say the higher order derivatives of V up to order m are *globally decrescent* if there exists $\phi' \in \mathcal{K}_\infty$ such that

$$v_i(t, x) \leq \phi'(|x|) \quad \forall t \geq t_0, x \in \mathbb{R}^n, i = 0, \cdots m.$$

Because of the assumption (6) on V, it is equivalent to write the above requirements in a compact form:

$$v_i \preceq \phi(V) \quad \forall i = 0, \cdots m \tag{10}$$

for some $\phi \in \mathcal{K}_\infty$. Now suppose for some $m \in \mathbb{N}$, there exist

$$a_0 > 0, \ a_i \geq 0 \quad \forall i = 1, \cdots, m \tag{11}$$

such that

$$\sum_{i=0}^{m} a_i v_i \preceq 0, \tag{12}$$

we want to conclude stability properties for the system. They are summarized as our main theorem in the next subsection.

C. Main results

Theorem 1 *Given a system* (1) *and a positive definite function* $V(t,x)$ *satisfying* (6)*, generate the higher order derivatives* v_i *via* (9) *by* f *and* V*. If the global decrescent condition* (10) *is satisfied and* (12) *holds for some* $m \in \mathbb{N}$ *with* a_i*'s satisfying* (11)*, then the system* (1) *is GUAS.*

Inspired by Lemma 1, we also conclude a result on showing ISS of the system (1) via higher order derivatives of V:

Corollary 1 *Given a system* (1) *and a positive definite function* $V(t,x)$ *satisfying* (6)*, generate the higher order derivatives* v_i *via* (9) *by* f' *and* V *where* f' *is defined in* (5) *with some* $\rho \in \mathcal{K}_\infty$*. If all the hypotheses in Theorem 1 are satisfied, then the system* (1) *is ISS.*

In the case when both V and f are smooth enough and there are no inputs, v_i's reduce to the usual higher order derivatives of V. When the equality in (12) is achieved everywhere, it becomes a linear differential equation

$$a_m v^{(m)} + \cdots + a_1 \dot{v} + a_0 = 0, \tag{13}$$

which is associated with a characteristic polynomial

$$a_m s^m + \cdots + a_1 s + a_0 = 0. \tag{14}$$

Recall that (11) is only a necessary condition for the above polynomial to be Hurwitz. This means that it is possible that there exist some a_i's satisfying (11) but a solution $v(t)$ of (13) diverges as t increases. Since $v(t)$ also satisfies the differential relation (12), at a first glance it contradicts the result in Theorem 1 that the system is GUAS. However, we argue that this v cannot be a positive definite Lyapunov function and hence Theorem 1 is not conflicted. Note that (14) has no non-negative real roots when the coefficients satisfy (11). It then must have positive complex roots if the solution of (13) diverges. Hence $v(t)$ will oscillate stronger and stronger in order to diverge so $v(t)$ will become negative for some t large enough. In other words,

Corollary 2 *If the characteristic polynomial* (14) *of the linear differential equation* (13) *is not Hurwitz but the coefficients* a_i*'s satisfy the condition* (11)*, then the solution of* (13) *with any initial condition* $v(0) > 0$ *has to be negative for some* $t > 0$*.*

III. Proof of main theorem

Without loss of generality we can always assume v_m is the highest order term in (12) with non-zero coefficient a_m. By scaling we can also assume that $a_m = 1$. Consider a solution $x(t, x_0, u)$ with arbitrary initial condition $x_0 \in \mathbb{R}^n$ and $u \in M_U$. Simplify the notation with $x(t) = x(t, x_0, u)$, representing the state of the system at time t. By the construction (9) we see that for all $t \geq t_0, i \in \mathbb{N}$,

$$\dot{v}_{i-1}(t,x(t)) = \frac{\partial v_{i-1}(t,x(t))}{\partial t} + \frac{\partial v_{i-1}(t,x(t))}{\partial x} \cdot f(t,x(t),u(t)) \leq v_i(t,x(t)) \tag{15}$$

which can be written as $\dot{v}_{i-1} \preceq v_i$ in short. Interesting results can be developed based on this sequence of first order differential relations.

Lemma 2 *Let* $x(t)$ *be an arbitrary solution of system* (1) *with* $t_0 = 0$*. Under assumptions* (11) *and* (12)*, for any* $b > 0$ *if* $v_0(t,x(t)) \geq b$ *for all* $t \in [0,T]$ *for some* $T \geq 0$*, then*

$$v_0(t,x(t)) \leq -b\sum_{j=1}^{m} a_{m-j}\frac{t^j}{j!} + \sum_{j=0}^{m-1}\sum_{i=0}^{j}\frac{t^j}{j!}a_{m+i-j}v_i(0,x_0) \tag{16}$$

for all $t \in [0,T]$*.*

Proof: This proof is inspired by the work in [11]. We claim that for any $k = 0, 1, \cdots m-1$,

$$\sum_{i=1}^{m-k} a_{i+k} v_{i-1}(t,x(t)) \leq -b\sum_{j=1}^{k+1} a_{k+1-j}\frac{t^j}{j!} + \sum_{j=0}^{k}\sum_{i=0}^{m-k-1+j}\frac{t^j}{j!}a_{i+k-j+1}v_i(0,x_0) \tag{17}$$

Recall we have safely assumed $a_m = 1$ so (16) is simply the incidence when $k = m-1$. We use mathematical induction to prove the claim (17). First, for $k = 0$, we need to show that

$$\sum_{i=1}^{m} a_i v_{i-1}(t,x(t)) \leq -ba_0 t + \sum_{i=0}^{m-1} a_{i+1} v_i(0,x_0) \tag{18}$$

To show that, we plug (15) into (12):

$$a_0 v_0(t,x(t)) + \sum_{i=1}^{m} a_i \dot{v}_{i-1}(t,x(t)) \leq \sum_{i=0}^{m} a_i v_i(t,x(t)) \leq 0$$

Shift the v_0 term to the right, integrate both sides from 0 to t and recall that $a_0 > 0, v_0(\tau, x(\tau)) \geq b$,

$$\sum_{i=1}^{m} a_i \Big(v_{i-1}(t,x(t)) - v_{i-1}(0,x_0)\Big) \leq -a_0\int_0^t v_0(\tau,x(\tau))d\tau \leq -ba_0 t$$

Shift the initial terms $v_{i-1}(0,x_0)$ to the right and increase their indices by 1 and we have proven (18).

Second, suppose (17) holds for some $k = 0, 1, \ldots, m-2$. We show that (17) also holds for the incidence $k+1$. To do that, we plug (15) into (17):

$$
\begin{aligned}
&a_{1+k}v_0(t,x(t)) + \sum_{i=2}^{m-k} a_{i+k}\dot{v}_{i-2}(t,x(t)) \\
&\quad \le \sum_{i=1}^{m-k} a_{i+k}v_{i-1}(t,x(t)) \\
&\quad \le -b\sum_{j=1}^{k+1} a_{k+1-j}\frac{t^j}{j!} + \sum_{j=0}^{k}\sum_{i=0}^{m-k-1+j}\frac{t^j}{j!}a_{i+k-j+1}v_i(0,x_0)
\end{aligned}
$$

Shift the v_0 term to the right, integrate both sides from 0 to t and recall that $a_{1+k} \ge 0, v_0(\tau, x(\tau)) \ge b$,

$$
\begin{aligned}
&\sum_{i=2}^{m-k} a_{i+k}\Big(v_{i-2}(t,x(t)) - v_{i-2}(0,x_0)\Big) \\
&\quad \le -a_{1+k}\int_0^t v_0(\tau,x(\tau))d\tau - b\sum_{j=1}^{k+1} a_{k+1-j}\int_0^t \frac{\tau^j}{j!}d\tau \\
&\qquad + \sum_{j=0}^{k}\sum_{i=0}^{m-k-1+j}\int_0^t \frac{\tau^j}{j!}d\tau a_{i+k-j+1}v_i(0,x_0) \\
&\quad \le -a_{1+k}bt - b\sum_{j=1}^{k+1} a_{k+1-j}\frac{t^{j+1}}{(j+1)!} \\
&\qquad + \sum_{j=0}^{k}\sum_{i=0}^{m-k-1+j}\frac{t^{j+1}}{(j+1)!}a_{i+k-j+1}v_i(0,x_0).
\end{aligned}
$$

Notice that the first term $-a_{1+k}bt$ can be combined into the first summation with an index $j = 0$. In addition, shift the initial terms $v_{i-2}(0,x_0)$ to the right and we have

$$
\begin{aligned}
\sum_{i=2}^{m-k} a_{i+k}v_{i-2}(t,x(t)) &\le -b\sum_{j=0}^{k+1} a_{k+1-j}\frac{t^{j+1}}{(j+1)!} \\
&+ \sum_{j=0}^{k}\sum_{i=0}^{m-k-1+j}\frac{t^{j+1}}{(j+1)!}a_{i+k-j+1}v_i(0,x_0) \\
&+ \sum_{i=2}^{m-k} a_{i+k}v_{i-2}(0,x_0)
\end{aligned}
$$

Rearrange the summation indices; namely, let the summation on the left side start with $i = 1$, the first summation on the right start with $j = 1$, the outer summation of the second term start with $j = 1$ and the last summation start with $i = 0$, we have

$$
\begin{aligned}
\sum_{i=1}^{m-k-1} a_{i+k+1}v_{i-1}(t,x(t)) &\le -b\sum_{j=1}^{k+2} a_{k-j+2}\frac{t^j}{j!} \\
&+ \sum_{j=1}^{k+1}\sum_{i=0}^{m-k+j-2}\frac{t^j}{j!}a_{i+k-j+2}v_i(0,x_0) \\
&+ \sum_{i=0}^{m-k-2} a_{i+k+2}v_i(0,x_0)
\end{aligned}
$$

Notice that the last term can be combined into the nested summations with an index $j = 0$. As a result, we have

$$
\begin{aligned}
&\sum_{i=1}^{m-(k+1)} a_{i+(k+1)}v_{i-1}(t,x(t)) \\
&\qquad \le -b\sum_{j=1}^{(k+1)+1} a_{(k+1)-j+1}\frac{t^j}{j!} \\
&\qquad + \sum_{j=0}^{k+1}\sum_{i=0}^{m-(k+1)-1+j}\frac{t^j}{j!}a_{i+(k+1)-j+1}v_i(0,x_0).
\end{aligned}
$$

Compared with (17), the above inequality is exactly the incidence of $k+1$ and hence we have proven the lemma. ■

Lemma 3 *Suppose all the hypotheses in Lemma 2 hold. For any $\delta > 0, \epsilon > 0$, there exist a function $\bar{v}(\delta) \in \mathcal{K}_\infty$, a set $D = \{(\delta,\epsilon) \in \mathbb{R}^2 : \delta > 0, 0 < \epsilon \le \bar{v}(\delta)\}$ and a function $\bar{T}(\delta,\epsilon) : D \to \mathbb{R}_{\ge 0}$ with the following properties:*

1) *$\bar{T}(\delta,\epsilon)$ is increasing in δ when ϵ is fixed, and decreasing in ϵ when δ is fixed,*
2) *$\bar{T}(\delta,\bar{v}(\delta)) = 0$ and $\lim_{\epsilon\to 0^+}\bar{T}(\delta,\epsilon) = \infty$ for all $\delta > 0$*

such that if $v_0(0,x_0) = \delta$,

1) *$v_0(t,x(t)) \le \bar{v}(\delta)$ for all $t \ge 0$;*
2) *$v_0(t,x(t)) \le \epsilon$ for all $t \ge \bar{T}(\delta,\epsilon)$.*

Proof: We pick $b \in (0,\delta]$ and Let $t \ge 0$ be the maximal time such that $v_0(t,x(t)) \ge \delta$ for all $t \in [0,T]$. Then by Lemma 2 we have (16) for all $t \in [0,T]$. Split the second summation term in (16) into two parts so that one of them involves v_0 terms only:

$$
\begin{aligned}
&v_0(t,x(t)) \le -b\sum_{j=1}^{m} a_{m-j}\frac{t^j}{j!} \\
&+ \sum_{j=0}^{m-1} a_{m-j}\frac{t^j}{j!}v_0(0,x_0) + \sum_{j=1}^{m-1}\sum_{i=1}^{j}\frac{t^j}{j!}a_{m+i-j}v_i(0,x_0)
\end{aligned} \tag{19}
$$

By global decrescent condition (10) we have $\phi \in \mathcal{K}_\infty$ such that for all $i = 1,\cdots m-1$,

$$
v_i(0,x_0) \le \phi(V(0,x_0)) = \phi(v_0(0,x_0)) = \phi(\delta).
$$

Substitute the above uniform bounds on v_i's into (19),

$$
\begin{aligned}
&v_0(t,x(t)) \le -b\sum_{j=1}^{m} a_{m-j}\frac{t^j}{j!} \\
&\qquad + \delta\sum_{j=0}^{m-1} a_{m-j}\frac{t^j}{j!} + \phi(\delta)\sum_{j=1}^{m-1}\sum_{i=1}^{j}\frac{t^j}{j!}a_{m+i-j}
\end{aligned} \tag{20}
$$

To find the $\bar{v}$ function, we consider the case $b = \delta$. Thus we have

$$
\begin{aligned}
v_0(t,x(t)) \le -\delta a_0\frac{t^m}{m!} + \delta + \phi(\delta)\sum_{j=1}^{m-1}\sum_{i=1}^{j}\frac{t^j}{j!}a_{m+i-j} \\
=: p_\delta(t)
\end{aligned} \tag{21}
$$

Notice that $p_\delta(t)$ is an m-th degree polynomial in t, whose coefficients depend on δ. In addition, the coefficient of the highest degree term is negative so $p_\delta(t)$ is bounded from above for all $t \geq t_0$. By simple computation, $p_\delta(0) = \delta$, $\frac{d}{dt}p_\delta(0) = \phi(\delta) > 0$ so it can be concluded that the maximum value of $p_\delta(t)$ is achieved somewhere at $t^* > 0$ and $p_\delta(t^*) > \delta$; in addition since $p_0(t) \equiv 0$ so the maximum value of $p_\delta(t)$ approaches to 0 as δ decreases to 0. Consequently it can be concluded that there exists $\bar{v} \in \mathcal{K}_\infty$ so that $p_\delta(t) \leq \bar{v}(\delta)$ for all $t \geq t_0$. We claim this $\bar{v}$ is the desired upper bound for $v_0(t, x(t))$. Indeed if this is not true, then it means there exists $s^* > 0$ such that $v_0(s^*, x(s^*)) > \bar{v}(\delta)$. As $v_0(t_0, x_0) = \delta$, there exists $s_0 \in [0, s^*)$ such that $v_0(s_0, x(s_0)) = \delta$ and $v_0(t, x(t)) \geq \delta$ for all $t \in [s_0, s^*]$. We shift s_0 to be the initial time 0 and the assumptions in Lemma 2 are satisfied with $T = s^*$. Note that semi-definite relation (12) and the global decrescent condition (10) still hold for all $t \geq s_0$ by same analysis we will still have the inequality (21). Hence we must have $v_0(s^*, x(s^*)) \leq p_\delta(s^* - s_0) \leq \bar{v}(\delta)$, which is a contradiction.

To find the $\bar{T}$ function, we let $b = \bar{v}^{-1}(\epsilon)$. In this way because $\epsilon \leq \bar{v}(\delta)$, indeed we have $b \leq \delta$, which does not conflict with our previous choice of $b \in (0, \delta]$. Thus (20) becomes:

$$
\begin{aligned}
v_0(t, x(t)) &\leq -\bar{v}^{-1}(\epsilon) \sum_{j=1}^{m} a_{m-j} \frac{t^j}{j!} + \delta \sum_{j=0}^{m-1} a_{m-j} \frac{t^j}{j!} \\
&\quad + \phi(\delta) \sum_{j=1}^{m-1 \sum_{i=1}^{j}} \frac{t^j}{j!} a_{m+i-j} \\
&= -\bar{v}^{-1}(\epsilon) a_0 \frac{t^m}{m!} + (\delta - \bar{v}^{-1}(\epsilon)) \sum_{j=1}^{m-1} a_{m-j} \frac{t^j}{j!} \\
&\quad + \delta + \phi(\delta) \sum_{j=1}^{m-1} \sum_{i=1}^{j} a_{m+i-j} \frac{t^j}{j!} \\
&= -\bar{v}^{-1}(\epsilon) a_0 \frac{t^m}{m!} + \delta \\
&\quad + \sum_{j=1}^{m-1} \frac{t^j}{j!} \left(\phi(\delta) \sum_{i=1}^{j} a_{m+i-j} + (\delta - \bar{v}^{-1}(\epsilon)) a_{m-j} \right) \\
&=: p_{\delta,\epsilon}(t)
\end{aligned}
$$

Again $p_{\delta,\epsilon}(t)$ is an m-th degree polynomial in t. Define $\bar{T}(\delta, \epsilon) := \arg\min_{t \geq t_0} \{p_{\delta,\epsilon}(t) \leq \bar{v}^{-1}(\epsilon)\}$, which is finite since the highest degree term in the polynomial $p_{\delta,\epsilon}(t)$ has negative coefficient and thus decreases to $-\infty$ when t increases, and is positive when $p_{\delta,\epsilon}(0) = \delta > \bar{v}^{-1}(\epsilon)$. We claim this is the $\bar{T}$ function that we are looking for.

To show the first property of $\bar{T}(\delta, \epsilon)$ in Lemma 3, We see that when ϵ is fixed, $\delta_1 > \delta_2$ implies $p_{\delta_1,\epsilon}(t) - \bar{v}^{-1}(\epsilon) > p_{\delta_2,\epsilon}(t) - \bar{v}^{-1}(\epsilon)$ for all $t \geq t_0$ and hence $\bar{T}(\delta_1, \epsilon) > \bar{T}(\delta_2, \epsilon)$; when δ is fixed, $\epsilon_1 > \epsilon_2$ implies $p_{\delta,\epsilon_1}(t) - \bar{v}^{-1}(\epsilon_1) < p_{\delta,\epsilon_2}(t) - \bar{v}^{-1}(\epsilon_2)$ for all $t > 0$ and hence $\bar{T}(\delta, \epsilon_1) < \bar{T}(\delta, \epsilon_2)$. To show the second property, we see that $p_{\delta,\bar{v}(\delta)}(0) = \delta = \bar{v}^{-1}(\bar{v}(\delta))$ so $\bar{T}(\delta, \bar{v}(\delta)) = 0$. In addition, $p_{\delta,0}(t) = \delta > 0$ for all $t \geq t_0$ and because $p_{\delta,\epsilon}(t)$ is continuous in ϵ, we must have $\lim_{\epsilon \to 0^+} \bar{T}(\delta, \epsilon) = \infty$.

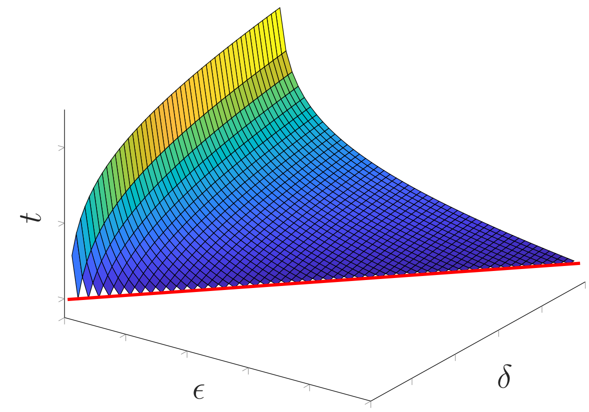

Fig. 1. A graphical view of $\bar{T}$ function. The red curve is $\epsilon = \bar{v}(\delta)$ and the surface is the value of $\bar{T}(\delta, \epsilon)$. Swapping ϵ and t we can get a class $\mathcal{KL}$ function which bounds $V(t, \delta)$.

Eventually, to show $v_0(t, x(t)) \leq \epsilon$ for all $t \geq \bar{T}(\delta, \epsilon)$, recall $v_0(t, x(t)) \geq b = \bar{v}^{-1}(\epsilon)$ and it is bounded from above by $p_{\delta,\epsilon}(t)$ for $t \in [0, T]$. By the definition of $\bar{T}$ we must have $T \leq \bar{T}$. In other words there exists $\bar{t} \leq \bar{T}$ such that $v(\bar{t}, x(\bar{t})) = \bar{v}^{-1}(\epsilon)$. Hence by the first conclusion on $\bar{v}$ we have $v_0(t, x(t)) \leq \epsilon$ for all $t \geq \bar{T}(\delta, \epsilon) \geq \bar{t}$. ■

Proof: [Proof of Theorem 1:] Briefly speaking, the first property in Lemma 3 implies global stability and the second property implies uniform attractivity and hence the system is GUAS. We present an alternative proof here via the construction of a class $\mathcal{KL}$ function as required by (3).

Without loss of generality and by majorization, we can always assume $\bar{T}$ form Lemma 3 is a continuous function over D while preserving its properties. A graphical view of $\bar{T}$ function is illustrated in the Figure 1. For each $\delta > 0$, define $W_\delta(\epsilon) = \bar{T}(\delta, \epsilon)$. By the properties of $\bar{T}$ in Lemma 3, we see that its inverse function $W_\delta^{-1} : [0, \infty) \to (0, \bar{v}(\delta)]$ exists and is a decreasing function such that $W_\delta^{-1}(0) = \bar{v}(\delta)$, $\lim_{t\to\infty} W_\delta^{-1}(t) = 0$. In addition, from the second conclusion on $v_0(t, x(t))$ after $\bar{T}$ we see that when $v_0(t_0, x_0) = \delta$, $v_0(t, x(t)) \leq W_\delta^{-1}(t - t_0)$.

Define

$$
\bar{\beta}(\delta, t) := \begin{cases} W_\delta^{-1}(t) & \delta > 0 \\ 0 & \delta = 0 \end{cases}
$$

We claim that $\bar{\beta} \in \mathcal{KL}$. We are left to show $\bar{\beta}(\delta, t)$ is increasing in δ and it is continuous at $\delta = 0$. Let $\delta_1 > \delta_2 > 0$ and $t \geq t_0$. Since t is in the range of the function $\bar{T}(\delta, \epsilon)$ for any $\delta > 0$, $t = \bar{T}(\delta_1, \epsilon_1) = \bar{T}(\delta_2, \epsilon_2)$ for some ϵ_1, ϵ_2. Then we must have $\epsilon_1 > \epsilon_2$ because $\bar{T}(\delta, \epsilon)$ is increasing in δ and decreasing in ϵ. In other words,

$$
\bar{\beta}(\delta_1, t) = W_{\delta_1}^{-1}(t) = \epsilon_1 > \epsilon_2 = W_{\delta_2}^{-1}(t) = \bar{\beta}(\delta_2, t)
$$

So $\bar{\beta}(\delta, t)$ is increasing in δ. In addition, we have $\bar{\beta}(\delta, t) \leq \bar{\beta}(\delta, 0) = \bar{v}(\delta) \to 0$ as $\delta \to 0$ so $\lim_{\delta\to 0} \bar{\beta}(\delta, t) = 0 = \bar{\beta}(0, t)$ and the function is continuous at $\delta = 0$.

At last, from the earlier analysis we have $v_0(t, x(t)) \leq \bar{\beta}(v_0(t_0, x_0), t-t_0)$ for any $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U, t \geq t_0$. As $v_0 = V$ by definition, combine this result with (6) and we have

$$\begin{aligned}|x(t)| &\leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1} \circ \bar{\beta}(V(t_0, x_0), t-t_0) \\ &\leq \alpha_1^{-1} \circ \bar{\beta}(\alpha_2(|x_0|), t-t_0) =: \beta(|x(0)|, t-t_0).\end{aligned}$$

By construction $\beta \in \mathcal{KL}$ and hence the system (1) is GUAS.

■

IV. Examples

A. Linear system with unaligned V

Consider a 2-dimensional linear system given by

$$\dot{x} = f(x, u) = Ax + u \tag{22}$$

where $A = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}$. It is not hard to check that A is Hurwitz so the system (22) is ISS. This can be verified by picking a proper quadratic Lyapunov function $V := x^\top P x$ where P satisfies the Lyapunov equation:

$$AP + PA^\top = -Q \tag{23}$$

for some positive definite Q. Consider the canonical Lyapunov function $V = |x|^2$ so that P is the identity matrix. By (23) we find $Q = \begin{pmatrix} 0.2 & -1 \\ -1 & 0.2 \end{pmatrix}$, which is not positive definite. Hence such V is not a Lyapunov function for system (22); in other words, even in the case when $u = 0$, we have

$$\dot{V} = -0.2x_1^2 + 2x_1x_2 - 0.2x_2^2 = -0.2(x_1 - 5x_2)^2 + 4.8x_2^2$$

which may be positive when $x_1 = 5x_2 \neq 0$. Nevertheless, in spite of the sign indefiniteness of $\dot{V}$, we look at its higher order derivatives and we still want to show that (22) is ISS. Pick $\rho(s) = \dfrac{s}{20}$ and consider the auxiliary system

$$\dot{x} = f'(x, u) = Ax + \rho(|x|)u$$

with $|u| \leq 1$. As usual $v_0 = V$ and notice that

$$\frac{\partial v_i}{\partial x} f'(x, u) = \nabla v_i \left(Ax + \frac{|x|}{20} u \right) \leq \nabla v_i Ax + \frac{|\nabla v_i||x|}{20}.$$

When v_i is quadratic in x, $|\nabla v_i| \leq R_i|x|$ for some $R_i \geq 0$ and hence according to (9) we can recursively define

$$v_{i+1} := \nabla v_i Ax + \frac{R_i}{20}|x|^2$$

which is also quadratic in x. According to this rule the first few v_i's can be generated:

$$\begin{aligned} v_1 &= -0.1x_1^2 + 2x_1x_2 - 0.1x_2^2, \\ v_2 &= 4.13x_1^2 - 0.6x_1x_2 - 1.87x_2^2, \\ v_3 &= -1.5907x_1^2 - 15.62x_1x_2 + 1.4093x_2^2. \end{aligned}$$

It is observed that (10) is satisfied since all v_i's are quadratic. Let $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$, we have

$$\begin{aligned}\sum_{i=0}^{3} a_i v_i &= -0.2257x_1^2 + 0.08x_1x_2 - 0.2257x_2^2 \\ &= -x^\top \begin{pmatrix} 0.2257 & -0.04 \\ -0.04 & 0.2257 \end{pmatrix} x \prec 0,\end{aligned}$$

hence according to our Corollary 1 the system (22) is ISS.

B. Slowly varying between two stable modes

Consider the following 2-dimensional, time varying system

$$\begin{aligned}\dot{x} = f(t, x, u) &= \sin^2(kt)A_1x + \cos^2(kt)A_2x + u \\ &=: A(k, t)x + u \end{aligned}\tag{24}$$

where

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$$

and k is a sufficiently small positive number, representing a slow enough variation of the system (24) between the two linear sub-systems $\dot{x} = A_1x + u$ and $\dot{x} = A_2x + u$. The two sub-systems are taken from Chapter 2.1 of [15]. Both sub-systems are stable when there are no inputs; however, as discussed in the cited book, the trajectory of a switched system may diverge for some particular sequence of switches between the two sub-systems. Hence the switched system is not stable under arbitrary switches; there is no common Lyapunov functions between the two sub-systems so there exist no time-independent Lyapunov functions for (24). Nevertheless we want to show that the canonical positive definite function $V(x) = |x|^2$ when applied on (24) satisfies (12) hence it proves ISS of the system when k is sufficiently small.

Again pick $\rho(s) = \frac{s}{20}$ and the auxiliary system is $\dot{x} = f'(t, x, d) = A(k, t)x + \frac{|x|}{20}d$. It can be inductively shown that the higher order derivatives of v_i are of quadratic form

$$v_i(t, x) = x^\top M_i(k, t)x$$

because

$$\begin{aligned}&\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x} \cdot f'(t, x, d) \\ &= x^\top \frac{dM_i}{dt}x + x^\top \left(M_i + M_i^\top\right)\left(Ax + \frac{|x|}{20}d\right) \\ &\leq x^\top \frac{dM_i}{dt}x + x^\top \left(M_i + M_i^\top\right)Ax + \frac{1}{20}\|M_i + M_i^\top\||x|^2 \\ &= x^\top \left(\frac{dM_i}{dt} + \left(M_i + M_i^\top\right)A + \frac{1}{20}\|M_i + M_i^\top\|I_{2\times2}\right)x \\ &=: v_{i+1}.\end{aligned}$$

Hence we also have the recursive relations

$$M_{i+1} = \frac{dM_i}{dt} + \left(M_i + M_i^\top\right)A + \frac{1}{20}\|M_i + M_i^\top\|I_{2\times2}.$$

In addition, if $M_i(k, t) = P_i(k, t) + o(k)Q_i(k, t)$ such that $o(k)$ converges to 0 as k converges to 0 and $\|Q_i(k, t)\|$ is bounded uniformly with respect to all $k \in \mathbb{R}, t \geq t_0$, then the sign definiteness of $\sum_{i=0}^{m} a_i M_i$ is the same as $\sum_{i=0}^{m} a_i P_i$ when k is sufficiently small. Hence we only need to compute those P_i's. Because we use $V(x) = |x|^2$, $P_0 = M_0 = I_{2\times 2}$ as a start. The other matrices can be generated accordingly:

$$P_1 = \begin{pmatrix} -0.1 & -C \\ -C & -0.1 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} C^2 - 3C + 0.13 & 0.3C \\ 0.3C & C^2 + 3C + 0.13 \end{pmatrix},$$
$$P_3 = \begin{pmatrix} -0.5C^2 + 1.5C + 0.224 & -C^3 + 8.81C \\ -C^3 + 8.81C & -0.5C^2 - 1.5C + 0.224 \end{pmatrix}$$

where $C = \cos(2kt)$. Use the same coefficients as in the previous example; that is, $a_0 = 0.1, a_1 = 8, a_2 = 0.5, a_3 = 1$ and we have

$$\sum_{i=0}^{3} a_i P_i = \begin{pmatrix} -0.411 & -C^3 + 0.96C \\ -C^3 + 0.96C & -0.411 \end{pmatrix}$$

Note $\max_t |\cos^3 2kt - 0.96 \cos 2kt| = 0.256\sqrt{2} < 0.411$ so the above matrix is negative definite. Thus when k is sufficiently small,

$$\sum_{i=0}^{3} a_i v_i \approx \sum_{i=0}^{3} a_i x^\top P_i x \prec 0$$

and the system (24) is ISS by Corollary 1.

V. Discussion

It is appreciated that stability of a system can be shown either by finding a standard Lyapunov function, or via the analysis of higher order derivatives of V as we studied in this work; each method has its own pros and cons. Finding a standard Lyapunov function is not trivial in general; in addition, we point out that for a time varying system, such a standard Lyapunov function may also need to be time varying and hence difficult to find. As shown in our second example, no standard time-independent Lyapunov function exists for this time varying system (24); on the contrary, starting from a simple $V(x) = |x|^2$ and using our techniques on the higher order derivatives, we are able to show that the system is GUES.

On the other hand, while our method for checking stability of the systems via higher order derivatives gives freedom in the choice of the candidate positive definite function V, as a trade off the negative semi-definite linear combination condition is analytically difficult to check. Nevertheless, very often when all the higher order derivatives are polynomials, our negative semi-definite linear combination condition is related to the sum-of-squares (SOS) techniques in semi-definite programming (SDP) (i.e., [16]). It is worth devoting more efforts to the study of numerical SOS SDP implementation and there is a high chance that such problems can be solved efficiently.

The connection between higher order derivatives and the standard Lyapunov function is observed in the work [17]. For an asymptotically stable system with no inputs, if there exists a function $V \in C^\infty(\mathbb{R}^n \to \mathbb{R})$ with $V(0) = 0$ and some coefficients $a_0, a_1, \cdots, a_m$ such that the negative definite linear combination condition $\sum_{i=1}^{m} a_i V^{(i)} \prec 0$ holds where $V^{(i)}$ is the i-th time derivative of V, then

$$W(x) := \sum_{i=1}^{m} a_i V^{(i-1)}(x) \tag{25}$$

is a standard positive definite Lyapunov function with negative definite time derivative. Note that there is no assumption of positive definiteness on V, nor any sign requirements like (11) on the coefficients a_i. Compared with our theorem, the result in [17] seems to be much less conservative. However, we point out that because of the presence of inputs in the system, W constructed via a formula similar to (25) may not be a standard Lyapunov function in our case. To be more precise, because of the inputs, we only have inequality in the relations (15) between higher order derivatives, rather than equality as we have for the case when there are no inputs. Thus as long as there are some negative a_i's, we will not be able to compare $\dot{W}$ with $\sum_{i=1}^{m} a_i V^{(i)}$ and hence the negative definite time derivative of W cannot be concluded.

As a comparison to the classical Lyapunov function theorem, another interesting question to study is whether there also exists a converse theorem with respect to the higher order derivatives? That is, given a positive definite V and a stable system, whether there always exist some non-negative coefficients such that the negative semi-definite linear combination condition of the higher order derivatives of V with these coefficients is satisfied. If starting with any arbitrary V seems too optimistic, we then can consider those "almost" Lyapunov functions whose time derivative is negative everywhere except at small regions in the state space, as studied in our earlier work [5]. It is very likely that such V can be adjusted to be negative definite by adding some higher order derivatives to it. This remains as an interesting future research direction.

VI. Conclusion

In this paper we have studied the alternative method for determining the stability of dynamical systems by inspecting the higher order derivatives of a positive definite function. We have first defined the higher order derivatives for time varying systems with inputs. We then claimed and proved that if there exists a linear combination of those higher order derivatives with non-negative coefficients (except that the coefficient of the 0-th order term needs to be positive) which is negative semi-definite, then the system is GUAS. Consequently if a system whose auxiliary system admits a positive definite function which satisfies the aforementioned conditions, this system is ISS.

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