Limited-Information Control of Hybrid Systems via Reachable Set Propagation

Daniel Liberzon*
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, IL 61821, U.S.A.
liberzon@uiuc.edu

ABSTRACT

This paper deals with control of hybrid systems based on limited information about their state. Specifically, measurements being passed from the system to the controller are sampled and quantized, resulting in finite data-rate communication. The main ingredient of our solution to this control problem is a novel method for propagating over-approximations of reachable sets for hybrid systems through sampling intervals, during which the discrete mode is unknown. In addition, slow-switching conditions of the (average) dwell-time type and multiple Lyapunov functions play a central role in the analysis.

Categories and Subject Descriptors

H.1.1 [**Models and Principles**]: Systems and Information Theory—*general systems theory, information theory*

Keywords

Hybrid systems, quantized control, reachable sets

1. INTRODUCTION

The topic of this paper is control of hybrid systems based on limited information about their state. More precisely, by "limited information" here we mean that measurements being passed from the system to the controller are sampled and quantized using a finite alphabet, resulting in finite data-rate communication. Our aim is to bring together two research areas—hybrid systems and control with limited information—which have both enjoyed a lot of activity in the last two decades and made great impact on applications, but synergy between which has been lacking.

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Hybrid systems are ubiquitous in realistic system models, because of their ability to capture the presence of two types of dynamical behavior within the system: continuous flow and discrete transitions. Amidst the large body of research on hybrid and switched systems, particularly relevant here is the work on stability analysis and stabilization of such systems, covered in the books [36, 19], the survey [32], and the many references therein. Among the specific technical tools typically employed to study these problems, common and multiple Lyapunov functions and slow-switching assumptions are prominently featured. There is also recent work that combines these Lyapunov-based approaches with tools from program analysis like ranking functions and abstraction [28, 5].

Feedback control problems with limited information have been an active research area for some time now, as surveyed in [25] (several specifically relevant works will be cited below). Information flow in a feedback control loop is an important consideration in many application-related scenarios. One notable example is *networked control systems*, where the controller is not collocated with the plant and both control and measurement signals are transmitted through a network. Even though in modern applications a lot of communication bandwidth is usually available, there are also multiple resources competing for this bandwidth (due to many control loops sharing a network cable or wireless medium). In the networked control systems literature, Lyapunov analysis and data-rate/transmission-interval bounds are commonly employed tools [27, 26]. In many other applications, one has constraints on the sensors dictated by cost concerns or physical limitations, or constraints on information transmission dictated by security considerations. Besides multiple practical motivations, the questions of how much information is really needed to solve a given control problem, or what interesting control tasks can be performed with a given amount of information, are quite fundamental from the theoretical point of view.

Control problems with limited information do not seem to have received much attention so far in the context of hybrid systems. (One exception we are aware of is some existing work on quantized Markov jump linear systems [24, 39, 22], but these systems are quite different from the models we consider, and the information structure considered in these references implies that the discrete mode is always known to the controller, which would remove most of the difficulties present in our problem formulation. On the other hand, control of hybrid systems with unknown discrete state was also considered in [37] but there the continuous state was not quantized.) In view of the commonality of the technical tools employed for the analysis of hybrid systems and for control design with limited information, evident from the previous discussion, we contend that a marriage of these two research areas is quite natural. In particular, multiple Lyapunov functions and slow-switching assumptions of the (average) dwell-time type will play a crucial role in our work.

In order to understand how much information is needed—and how this information should be used—to control a given system, we must understand how the uncertainty about the system's state evolves over time along its dynamics. In more precise terms, we need to be able to characterize propagation of reachable sets or their suitable over-approximations during the sampling interval (for a known control input). The reason is that at each sampling time, the quantizer singles out a bounded set which contains the continuous state and the controller determines the control signal to be applied to the system over the next sampling interval; no further information about the state is available during this interval, and at the next sampling time a bounded set containing the state must be computed to generate the next quantized measurement. The system will be stabilized if the factor by which the state estimation error is reduced at the sampling times is larger than the factor by which it grows between the sampling times. Thus propagation of reachable set bounds is a crucial ingredient in the available results on quantized sampled-data control of non-hybrid systems (such as our earlier work [18] which provides the basis for some of the ideas presented here), and the bulk of the effort required to handle the hybrid system scenario will be concentrated in implementing this step and analyzing its consequences.

If the discrete state were precisely known to the controller at each time, then the problem of reachable set propagation would be just a sequence of corresponding problems for the individual continuous modes, and as such would pose very little extra difficulty. (This would essentially correspond to the situation considered, in a discrete-time stochastic setting, in [24].) On the other hand, if the discrete state were completely unknown, then the set of possible trajectories of the hybrid system would be too large to hope for a reasonable (not overly conservative) solution. To strike a balance between these two situations, we assume here that we have a partial knowledge of the discrete state of the hybrid system; specifically, we assume that the discrete state is known at each sampling time, whereas its behavior between switching times is unknown except for an upper bound on the number of discrete transitions. Under these assumptions we implement a novel method for computing over-approximations of reachable sets at the end of each sampling interval. If in addition the allowed data rate is large enough, then we are able to design a provably correct communication and control strategy.

Our plan of attack on the problem described above is as follows. As a first step, we consider *switched systems* which can be viewed as high-level abstractions of hybrid systems. Instead of having a precise model of the discrete dynamics governing the transitions of the discrete state, in a switched system the continuous dynamics are given explicitly (by ordinary differential equations) while discrete dynamics are captured more abstractly by a class of possible switching signals which specify the discrete state as a function of time. In other words, we basically first consider systems with time-dependent switching (switched systems) instead of systems with state-dependent switching (hybrid systems). This system class allows us to focus our attention on formulating an appropriate "slow-switching" assumption; the relevant definitions are given in Section 2. We then develop a novel method for propagating over-approximations of reachable sets for such switched systems; this method is described in Sections 3 and 4. With these two ingredients—a slow-switching condition and an algorithm for reachable set propagation—we are able to define and validate a control strategy based on sampled and quantized measurements of the continuous state; the analysis is sketched in Section 5 followed by a short simulation example in Section 6. After this, we build a bridge to hybrid systems by returning to a detailed description of the discrete dynamics in Section 7.

2. PROBLEM FORMULATION

2.1 Switched system

The system to be controlled is the switched linear control system

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $\{(A_p, B_p) : p \in \mathcal{P}\}$ is a collection of matrix pairs defining the individual control systems ("modes") of the switched system, $\mathcal{P}$ is a finite index set, and $\sigma : [0, \infty) \to \mathcal{P}$ is a right-continuous, piecewise constant function called the *switching signal* which specifies the active mode at each time. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation away from the discontinuities of σ (in particular, we assume for now that there are no state jumps, but state jumps can also be handled as explained in Section 7.3). The switching signal σ is fixed but not known to the controller a priori. The discontinuities of σ are called "switching times" or simply "switches" and we let $N_\sigma(t, s)$ stand for their number on a semi-open interval $(s, t]$:

$$N_\sigma(t, s) := \text{number of switches on } (s, t].$$

Our first basic assumption is that the switching is not too fast, in the following sense.

Assumption 1 (Slow Switching)

1. There exists a number $\tau_d > 0$ (called a *dwell time*) such that any two switches are separated by at least τ_d, i.e., $N_\sigma(t, s) \leq 1$ when $t - s \leq \tau_d$;
2. There exist numbers $\tau_a > \tau_d$ (called an *average dwell time*) and $N_0 \geq 1$ such that

$$N_\sigma(t, s) \leq N_0 + \frac{t - s}{\tau_a} \qquad \forall\, t > s \geq 0.$$

The concept of average dwell time was introduced in [13] and has since then become standard; it includes dwell time as a special case (for $N_0 = 1$). Note that if the constraint $\tau_a > \tau_d$ were violated, the average dwell-time condition (item 2) would be implied by the dwell-time condition (item 1). Switching signals satisfying Assumption 1 were considered in [38], where they were called "hybrid dwell-time" signals.

Our second basic assumption is stabilizability of all individual modes.

Assumption 2 (Stabilizability) For each $p \in \mathcal{P}$ the pair (A_p, B_p) is stabilizable, i.e., there exists a state feedback gain matrix K_p such that $A_p + B_p K_p$ is Hurwitz (all eigenvalues have negative real parts).

In the sequel, we assume that a family of such stabilizing gain matrices K_p, $p \in \mathcal{P}$ has been selected and fixed. We understand that (at least some of) the open-loop matrices A_p, $p \in \mathcal{P}$ are not Hurwitz. Note, however, that even if all the individual modes are stabilized by state feedback (or stable without feedback), stability of the switched system is not guaranteed in general.

2.2 Information structure

The task of the controller is to generate a control input $u(\cdot)$ based on limited information about the state $x(\cdot)$ and about the switching signal $\sigma(\cdot)$. The information to be communicated to the controller is subject to the following two constraints.

Sampling: State measurements are taken at times $t_k := k\tau_s$, $k = 0, 1, 2, \dots$, where $\tau_s > 0$ is a fixed *sampling period*.

Quantization: Each state measurement $x(t_k)$ is encoded by an integer from 0 to N^n, where N is an odd positive integer, and sent to the controller. In addition, the value of $\sigma(t_k) \in \mathcal{P}$ is also sent to the controller.

As a consequence, data is transmitted to the controller at the rate of $(\log_2(N^n + 1) + \log_2 |\mathcal{P}|)/\tau_s$ bits per time unit, where $|\mathcal{P}|$ is the number of elements in $\mathcal{P}$. We assume the data transmission to be noise-free and delay-free. We take the sampling period τ_s to be no larger than the dwell time from Assumption 1 (item 1):

$$\tau_s \le \tau_d. \tag{2}$$

This guarantees that at most one switch occurs within each sampling interval.[1] Since the average dwell time τ_a in Assumption 1 (item 2) is larger than τ_d, we know that switches actually occur less often than once every sampling period. The reason for taking the integer N to be odd is to ensure that the control strategy described later preserves the equilibrium at the origin.

Throughout the paper, we work with the ∞-norm $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ on $\mathbb{R}^n$ and the corresponding induced matrix norm $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |A_{ij}|$ on $\mathbb{R}^{n\times n}$, both of which we denote simply by $\|\cdot\|$. To formulate our final basic assumption, we define

$$\Lambda_p := \|e^{A_p \tau_s}\|, \qquad p \in \mathcal{P}. \tag{3}$$

Assumption 3 (Data Rate) $\Lambda_p < N$ for all $p \in \mathcal{P}$.

We can view the above inequality as a data-rate bound because it requires N to be sufficiently large relative to τ_s, thereby imposing (indirectly) a lower bound on the available data rate. A very similar data-rate bound but for the case of a single mode appears in [18], where it is shown to be sufficient for stabilizing a non-switched linear system. That bound is slightly conservative compared to known bounds that characterize the minimal data rate necessary for stabilization (see, e.g., [34], [15]). However, the control scheme of [18] can be refined by tailoring it better to the structure of the system matrix A, and then the data rate that it requires will approach the minimal data rate (see also the discussion in [31, Section V]). Therefore, it is fair to say that Assumption 3 does not introduce a significant conservatism beyond requiring that the data rate be sufficient to stabilize each individual mode of the switched system (1).

[1]This assumption is made for simplicity. It could be relaxed to allow multiple switches, up to a fixed number, per sampling interval. This would make our formulas more complicated but would not cause conceptual difficulties.

2.3 Main objective

The control objective is to asymptotically stabilize the system defined in Section 2.1 while respecting the information constraints described in Section 2.2. More concretely, we want to provide a constructive proof of the following result.

Theorem 1 (Main Result) Consider the switched linear system (1) and let Assumptions 1–3 and the inequality (2) hold. If the average dwell time τ_a is large enough, then there exists an encoding and control strategy that yields the following two properties:

Exponential convergence: There exist a number $\lambda > 0$ and a function $g : [0, \infty) \to (0, \infty)$ such that for every initial condition x_0 and every time $t \ge 0$ we have

$$\|x(t)\| \le e^{-\lambda t} g(\|x_0\|). \tag{4}$$

Lyapunov stability: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t)\| < \varepsilon \quad \forall\, t \ge 0. \tag{5}$$

A precise lower bound on the average dwell time τ_a is derived in the course of the proof (see the formula (24) in Section 5). As for the function g in the exponential convergence property, $g(r)$ does not go to 0 as $r \to 0$ and, in general, g grows faster than any linear function at infinity (see the formula (25) in Section 5 and the discussion at the end of Section 4.3). For this reason, Lyapunov stability needs to be established separately, and the two properties (exponential convergence and Lyapunov stability) combined still do not give the standard global exponential stability, but rather just global asymptotic stability with an exponential convergence rate.

The control strategy that we will develop to prove Theorem 1 is a dynamic one: it involves an additional state denoted by $\hat{x}$. Theorem 1 only discusses the behavior of the state x, which is the main quantity of interest, but it can be deduced from the proof that the controller state $\hat{x}$ satisfies analogous bounds. We will also see that $\hat{x}$ is potentially discontinuous at the sampling times t_k (which are not synchronized with the switching times of the original system); in other words, our controller is a hybrid one.

3. BASIC ENCODING AND CONTROL SCHEME

In this section we outline our encoding and control strategy, assuming for now that the state x satisfies known bounds at the sampling times. The problem of generating such state bounds is solved in the next section.

First, suppose that at some sampling time t_{k_0} we have

$$\|x(t_{k_0})\| \le E_{k_0}$$

where $E_{k_0} > 0$ is a number known to the controller. (In Section 4.3 we will show how such a bound can be generated for an arbitrary initial state x_0, by using a "zooming-out" procedure.) At the first such sampling time our controller is initialized. The encoder works by partitioning the hypercube $\{x \in \mathbb{R}^n : \|x\| \leq E_{k_0}\}$ into N^n equal hypercubic boxes, N per each dimension, and numbering them from 1 to N^n in some specific way. It then records the number of the box that contains[2] $x(t_{k_0})$ and sends it to the controller, along with the value of $\sigma(t_{k_0})$. We assume that the controller knows the box numbering system used by the encoder, so it can decode the box number. It lets $c_{k_0} \in \mathbb{R}^n$ be the center of the box containing $x(t_{k_0})$. We then have

$$\|x(t_{k_0}) - c_{k_0}\| \leq \frac{E_{k_0}}{N}.$$

For $t \in [t_{k_0}, t_{k_0+1})$, the control is set to

$$u(t) = K_{\sigma(t_{k_0})}\hat{x}(t)$$

where $\hat{x}$ is defined to be the solution of

$$\dot{\hat{x}} = (A_{\sigma(t_{k_0})} + B_{\sigma(t_{k_0})}K_{\sigma(t_{k_0})})\hat{x} = A_{\sigma(t_{k_0})}\hat{x} + B_{\sigma(t_{k_0})}u$$

with the boundary condition

$$\hat{x}(t_{k_0}) = c_{k_0}.$$

At a general sampling time t_k, $k \geq k_0 + 1$, suppose that a point $x_k^* \in \mathbb{R}^n$ and a number $E_k > 0$ are known such that

$$\|x(t_k) - x_k^*\| \leq E_k. \tag{6}$$

Of course the encoder has precise knowledge of x; the quantities x_k^* and E_k have to be obtainable on the decoder/controller side, based on the knowledge of the system matrices (but not the switching signal) and previously received measurements. We explain later how such x_k^* and E_k can be generated. The encoder also computes x_k^* and E_k in the same way, to ensure that the encoder and the decoder are synchronized. The encoding is then done as follows. Partition the hypercube $\{x \in \mathbb{R}^n : \|x - x_k^*\| \leq E_k\}$ into N^n equal hypercubic boxes, N per each dimension. Send the number of the box to the controller, along with the value of $\sigma(t_k)$. On the decoder/controller side, let c_k be the center of the box containing $x(t_k)$. This gives

$$\|x(t_k) - c_k\| \leq \frac{E_k}{N} \tag{7}$$

and also

$$\|c_k - x_k^*\| \leq \frac{N-1}{N}E_k. \tag{8}$$

Note that the formula (8) is also valid for $k = k_0$ if we set $x_{k_0}^* := 0$, a convention that we follow in the sequel. For $t \in [t_k, t_{k+1})$ define the control, along the same lines as before, by

$$u(t) = K_{\sigma(t_k)}\hat{x}(t)$$

where $\hat{x}$ is the solution of

$$\dot{\hat{x}} = (A_{\sigma(t_k)} + B_{\sigma(t_k)}K_{\sigma(t_k)})\hat{x} = A_{\sigma(t_k)}\hat{x} + B_{\sigma(t_k)}u \tag{9}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{10}$$

[2]In case $x(t_{k_0})$ lies on the boundary of several boxes, either one of these boxes can be chosen.

The above procedure is to be repeated for each subsequent value of k. Note that $\hat{x}$ is, in general, discontinuous (only right-continuous) at the sampling times, and we will use the notation $\hat{x}(t_k^-) := \lim_{t \nearrow t_k} \hat{x}(t)$. In the earlier work [18], x_k^* was obtained directly from $\hat{x}$ via $x_k^* := \hat{x}(t_k^-)$. On sampling intervals containing a switch this construction no longer works, and the task of defining x_k^* as well as E_k becomes more challenging.

4. STATE BOUNDS: REACHABLE SET OVER-APPROXIMATIONS

Proceeding inductively, we start with known x_k^* and E_k satisfying (6), where $k \geq k_0$, and show how to find x_{k+1}^* and E_{k+1} such that

$$\|x(t_{k+1}) - x_{k+1}^*\| \leq E_{k+1}. \tag{11}$$

Generation of E_{k_0} is addressed at the end of the section.

4.1 Sampling interval with no switch

We first consider the simpler case when $\sigma(t_k) = \sigma(t_{k+1}) = p \in \mathcal{P}$. By (2) we know that no switch has occurred on $(t_k, t_{k+1}]$, since two switches would have been impossible. So, we know that on the whole interval $[t_k, t_{k+1}]$ mode p is active. We can then proceed as in [18]. It is clear from (1) and (9) that the error $e := x - \hat{x}$ satisfies $\dot{e} = A_p e$ on $[t_k, t_{k+1})$, and we know from (10) and (7) that $\|e(t_k)\| \leq E_k/N$, hence

$$\|e(t_{k+1}^-)\| \leq \Lambda_p \frac{E_k}{N} =: E_{k+1} \tag{12}$$

where Λ_p was defined in (3). It remains to let

$$x_{k+1}^* := \hat{x}(t_{k+1}^-) = e^{(A_p + B_p K_p)\tau_s}c_k \tag{13}$$

and recall that x is continuous to see that (11) indeed holds.

4.2 Sampling interval with a switch

Suppose now that $\sigma(t_k) = p$ and $\sigma(t_{k+1}) = q \neq p$. Then again by (2) the controller knows that exactly one switch, from mode p to mode q, has occurred somewhere on the interval $(t_k, t_{k+1}]$, but it does not know exactly where. This case is more challenging.

Let the (unknown) time of the switch from p to q be $t_k + \bar{t}$, where $\bar{t} \in (0, \tau_s]$.

4.2.1 Analysis before the switch

On $[t_k, t_k + \bar{t})$ mode p is active, and we can derive as before that

$$\|x(t_k + \bar{t}) - \hat{x}(t_k + \bar{t})\| \leq \|e^{A_p \bar{t}}\| \frac{E_k}{N}.$$

But $\hat{x}(t_k + \bar{t})$ is unknown, so we need to describe a set that contains it. Choose an arbitrary $t' \in [0, \tau_s]$ (which may vary with k). By (9) and (10) we have[3]

$$\hat{x}(t_k + t') = e^{(A_p + B_p K_p)t'}c_k \tag{14}$$

and

$$\hat{x}(t_k + \bar{t}) = e^{(A_p + B_p K_p)(\bar{t} - t')}\hat{x}(t_k + t')$$

[3]In case either $t_k + t'$ or $t_k + \bar{t}$ equals t_{k+1}, the value of $\hat{x}$ at that time should be replaced by the left limit $\hat{x}(t_k + t'^-)$ or $\hat{x}(t_k + \bar{t}^-)$, respectively.

hence

$$\begin{aligned}&\|\hat{x}(t_k+\bar{t})-\hat{x}(t_k+t')\|\\&\qquad\le\|e^{(A_p+B_pK_p)(\bar{t}-t')}-I\|\|\hat{x}(t_k+t')\|\\&\qquad\le\|e^{(A_p+B_pK_p)(\bar{t}-t')}-I\|\|e^{(A_p+B_pK_p)t'}\|\|c_k\|.\end{aligned}$$

We also have from (8) that

$$\|c_k\|\le\|x_k^*\|+\frac{N-1}{N}E_k. \tag{15}$$

By the triangle inequality, we obtain

$$\begin{aligned}&\|x(t_k+\bar{t})-\hat{x}(t_k+t')\|\\&\le\|e^{(A_p+B_pK_p)(\bar{t}-t')}-I\|\|e^{(A_p+B_pK_p)t'}\|\\&\times\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)+\|e^{A_p\bar{t}}\|\frac{E_k}{N}=:D_{k+1}(\bar{t}).\end{aligned}$$

4.2.2 *Analysis after the switch*

On the interval $[t_k+\bar{t},t_{k+1})$, the closed-loop dynamics are

$$\begin{pmatrix}\dot{x}\\\dot{\hat{x}}\end{pmatrix}=\begin{pmatrix}A_q & B_qK_p\\0 & A_p+B_pK_p\end{pmatrix}\begin{pmatrix}x\\\hat{x}\end{pmatrix}. \tag{16}$$

Letting

$$z:=\begin{pmatrix}x\\\hat{x}\end{pmatrix},\qquad\bar{A}_{pq}:=\begin{pmatrix}A_q & B_qK_p\\0 & A_p+B_pK_p\end{pmatrix} \tag{17}$$

we can write (16) in the more compact form

$$\dot{z}=\bar{A}_{pq}z. \tag{18}$$

The previous analysis shows that

$$\left\|z(t_k+\bar{t})-\begin{pmatrix}\hat{x}(t_k+t')\\\hat{x}(t_k+t')\end{pmatrix}\right\|\le D_{k+1}(\bar{t})$$

(noting the property $\|(a^T,b^T)^T\|\le\max\{\|a\|,\|b\|\}$ of the ∞-norm). Consider the auxiliary system copy (on $\mathbb{R}^{2n}$)

$$\dot{\bar{z}}=\bar{A}_{pq}\bar{z},\qquad\bar{z}(0)=\begin{pmatrix}\hat{x}(t_k+t')\\\hat{x}(t_k+t')\end{pmatrix}.$$

We have

$$\|z(t_{k+1}^-)-\bar{z}(\tau_s-\bar{t})\|\le\|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|D_{k+1}(\bar{t}).$$

We now need to generate a bound for the unknown $\bar{z}(\tau_s-\bar{t})$. Similarly to what we did before, pick a $t''\in[0,\tau_s]$. Then $\bar{z}(t'')=e^{\bar{A}_{pq}t''}\bar{z}(0)$ and $\bar{z}(\tau_s-\bar{t})=e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}\bar{z}(t'')$, hence

$$\begin{aligned}&\|\bar{z}(\tau_s-\bar{t})-\bar{z}(t'')\|\le\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\|\bar{z}(t'')\|\\&\le\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\|e^{\bar{A}_{pq}t''}\|\|\bar{z}(0)\|\\&=\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\|e^{\bar{A}_{pq}t''}\|\|\hat{x}(t_k+t')\|\\&\le\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\|\\&\times\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)\end{aligned}$$

where we used (14) and (15) in the last step. By the triangle inequality,

$$\begin{aligned}&\|z(t_{k+1}^-)-\bar{z}(t'')\|\le\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\\&\times\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)\\&+\|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|D_{k+1}(\bar{t})=:E_{k+1}(\bar{t}).\end{aligned}$$

To eliminate the dependence on the unknown $\bar{t}$, we take the maximum over $\bar{t}$ (with t' and t'' fixed as above):

$$\begin{aligned}&E_{k+1}:=\max_{0\le\bar{t}\le\tau_s}E_{k+1}(\bar{t})=\max_{0\le\bar{t}\le\tau_s}\Big\{\|e^{\bar{A}_{pq}(\tau_s-\bar{t}-t'')}-I\|\\&\times\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)\\&+\|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|\Big(\|e^{(A_p+B_pK_p)(\bar{t}-t')}-I\|\|e^{(A_p+B_pK_p)t'}\|\\&\times\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)+\|e^{A_p\bar{t}}\|\frac{E_k}{N}\Big)\Big\}.\end{aligned}$$

We can use the inequalities

$$\|M-I\|\le\|M\|+1,\qquad\|e^{As}\|\le e^{\|A\|s} \tag{19}$$

to obtain a more conservative upper bound which is more useful for computations:

$$\begin{aligned}&E_{k+1}\le\left(e^{\|\bar{A}_{pq}\|\max\{t'',\tau_s-t''\}}+1\right)\|e^{\bar{A}_{pq}t''}\|\|e^{(A_p+B_pK_p)t'}\|\\&\times\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)+e^{\|\bar{A}_{pq}\|\tau_s}\\&\times\Big(\left(e^{\|A_p+B_pK_p\|\max\{t',\tau_s-t'\}}+1\right)\|e^{(A_p+B_pK_p)t'}\|\\&\times\left(\|x_k^*\|+\frac{N-1}{N}E_k\right)+e^{\|A_p\|\tau_s}\frac{E_k}{N}\Big).\end{aligned}$$

Note that setting $t'=t''=0$ simplifies the formulas but does not necessarily minimize E_{k+1}. Finally, x_{k+1}^* is defined by projecting $\bar{z}(t'')$ onto the x-component:

$$\begin{aligned}x_{k+1}^*:&=\begin{pmatrix}I_{n\times n} & 0_{n\times n}\end{pmatrix}\bar{z}(t'')\\&=\begin{pmatrix}I_{n\times n} & 0_{n\times n}\end{pmatrix}e^{\bar{A}_{pq}t''}\begin{pmatrix}\hat{x}(t_k+t')\\\hat{x}(t_k+t')\end{pmatrix}\\&=\begin{pmatrix}I_{n\times n} & 0_{n\times n}\end{pmatrix}e^{\bar{A}_{pq}t''}\begin{pmatrix}I_{n\times n}\\I_{n\times n}\end{pmatrix}e^{(A_p+B_pK_p)t'}c_k.\end{aligned} \tag{20}$$

4.3 Initial state bound E_{k_0}

Initially, set the control to $u\equiv 0$. At time 0, choose an arbitrary $E_0>0$ and partition the hypercube $\{x\in\mathbb{R}^n:\|x\|\le E_0\}$ into N^n equal hypercubic boxes, N per each dimension. If x_0 belongs to one of these boxes, then send the number of the box to the controller. Otherwise send 0 (the “overflow” symbol). Choose an increasing sequence $E_1,E_2,\dots$ that grows fast enough to dominate the rate of growth of the open-loop dynamics. For example, we can pick a small $\varepsilon>0$ and let

$$E_k:=e^{(2+\varepsilon)\max_{p\in\mathcal{P}}\|A_p\|t_k}E_0,\qquad k=1,2,\dots \tag{21}$$

There are other options but for concreteness we assume that the specific “zooming-out” sequence (21) is implemented. Repeat the above encoding procedure at each step. (As long as the quantization symbol is 0, there is no need to send the value of σ to the controller.) Then we claim that there will be a time t_{k_0} such that, for the corresponding value E_{k_0}, the symbol received by the controller will not be 0. At this time, the encoding strategy described in Section 3 can be initialized.

We skip the proof that k_0 is well defined but mention that there exist functions $\eta:[0,\infty)\to\mathbb{Z}_{\ge 0}$ and $\gamma:[0,\infty)\to(0,\infty)$ such that

$$k_0\le\eta(\|x_0\|),\qquad E_{k_0}\le\gamma(\|x_0\|) \tag{22}$$

and

$$\|x(t)\| \le \gamma(\|x_0\|) \qquad \forall\, t \in [0, t_{k_0}]. \tag{23}$$

Both functions depend on the initial choice of E_0. We can pick them so that $\eta(r) = 0$ and $\gamma(r) = E_0$ for all $r \le E_0$. For large values of its argument, $\gamma(\cdot)$ is in general super-linear. In fact, we can calculate that $\gamma(r)$ is of the order of r^2/E_0, and $\eta(r)$ is of the order of $(\max_{p\in\mathcal{P}} \|A_p\|\tau_s)^{-1} \log(r/E_0)$, for large values of r.

5. STABILITY VERIFICATION

We only sketch very briefly the main steps of the stability proof. First, consider a sampling interval with no switch: $\sigma \equiv p$ on $[t_k, t_{k+1}]$. Rewrite (13) as

$$x^*_{k+1} = e^{(A_p+B_pK_p)\tau_s} c_k = e^{(A_p+B_pK_p)\tau_s}(x^*_k + (c_k - x^*_k)).$$

This is an exponentially stable discrete-time system with input $\Delta_k := c_k - x^*_k$. We know from (8) that $\|\Delta_k\| \le E_k(N-1)/N$, whereas (12) and Assumption 3 give us $E_{k+1} = E_k\Lambda_p/N < E_k$. We see that, as long as there are no switches, E_k decays exponentially, hence the overall "cascade" system describing the joint evolution of x^*_k and E_k is exponentially stable. This fact can be formally proved by constructing a Lyapunov function in the form of a sum of a positive definite quadratic form in x^*_k and a positive multiple of E_k^2:

$$V_p(x^*_k, E_k) := (x^*_k)^T P_p x^*_k + \rho_p E_k^2$$

which can be shown to satisfy, on a sampling interval with no switch that we are considering,

$$V_p(x^*_{k+1}, E_{k+1}) \le \nu V_p(x^*_k, E_k)$$

for some number $\nu < 1$ that can be precisely computed.

Next, if a sampling interval $[t_k, t_{k+1}]$ contains a switch from $\sigma = p$ to $\sigma = q$, then the above mode-dependent Lyapunov function satisfies

$$V_q(x^*_{k+1}, E_{k+1}) \le \mu V_p(x^*_k, E_k)$$

for some $\mu > 1$ which again can be computed. It can now be shown that $V_{\sigma(t_k)}(x^*_k, E_k)$ converges to 0 exponentially fast if the average dwell time τ_a introduced in Assumption 1 satisfies the lower bound

$$\tau_a \ge m\tau_s, \qquad m > 1 + \frac{\log \mu}{\log \frac{1}{\nu}}. \tag{24}$$

From this, the same exponential convergence property holds for $x(t_k)$ and, with a bit of extra effort needed to analyze the intersample behavior, can be established for $x(t)$. Specifically, recalling (22) and (23), we can establish the desired exponential convergence property (4) with

$$\lambda := \frac{1}{2\tau_s} \log \frac{1}{\theta}, \qquad \theta := \mu^{1/m}\nu^{(m-1)/m} < 1$$

and

$$g(r) := \bar{c}\Big(\frac{1}{\sqrt{\theta}}\Big)^{1+\eta(r)}\gamma(r). \tag{25}$$

Finally, the proof of Lyapunov stability proceeds along the lines of [18, 20].

6. SIMULATION EXAMPLE

We simulated the above control strategy with the following data: $\mathcal{P} = \{1, 2\}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} 0 & -1 \end{pmatrix}$, $x_0 = (2, 2)^T$, $E_0 = 0.5$, $\tau_s = 0.5$, $N = 5$ (Assumption 3 is satisfied), $\tau_d = 1.05$, $\tau_a = 7.55$, and $N_0 = 5$. Figure 1 plots a typical behavior of the first component x_1 of the continuous state (in solid red) and the corresponding component $\hat{x}_1$ of the state estimate (in dashed green) versus time; switches are marked by blue circles. Observe the initial "zooming-out" phase and the nonsmooth behavior of x when $\hat{x}$ experiences a jump (causing a jump in the control u). The above value of the average dwell time τ_a was picked empirically to be just large enough to provide consistent convergence in simulations. For this example, the theoretical lower bound on the average dwell time from the formula (24) is about 85.5 which is, not surprisingly, quite conservative.

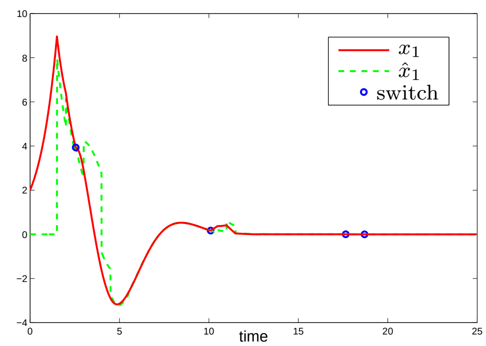

Figure 1: Simulation example

7. HYBRID SYSTEMS

In a hybrid system, the abstract notion of the switching signal σ that we used to define the switched system (1) is replaced (or, we may say, realized) by a discrete dynamics model which generates the sequence of modes, based typically on the evolution of the continuous state. Many specific modeling formalisms for hybrid systems exist in the literature, but a common paradigm which we also have in mind here is that each mode corresponds to a region in the continuous state space (sometimes called the *invariant* for that mode) where the corresponding continuous dynamics are active, and transitions (or switchings) between different modes take place when the continuous state x crosses boundaries (called *switching surfaces*, or *guards*) between these regions. At the times of these discrete transitions, the value of x in general can also jump to a new value according to some *reset map*.

Thus, compared to the switched system model (1), the two main new aspects that must be incorporated are *switching surfaces* and *state jumps*. We will address both these aspects in what follows. However, since we saw that propagating (over-approximations of) reachable sets is a key ingredient of our control strategy, we first discuss some relevant prior work on reachable set computation for hybrid systems in order to put our present developments in a proper context.

7.1 Comparison with existing reachable set algorithms

Without aiming for completeness, we give here an overview of some representative results. We classify them roughly according to the type of dynamics in the considered hybrid system model and the shapes of the sets used for reachable set approximation.

Early work by Puri, Borkar, and Varaiya on differential inclusions [30] approximates a general nonlinear differential inclusion by a piecewise constant one, and computes over-approximations of reachable sets which are unions of polyhedra. Henzinger et. al. [12] and Preußig et. al. [29] approximate hybrid systems by *rectangular automata* (hybrid systems whose regions and flow in each region are defined by constant lower and upper bounds on state and velocity components, respectively) and base reachable set computation on the tool `HyTech`; Frehse later developed a refined tool, `PHAVer` [6], for a similar purpose. Also related to this is reachability analysis using "face lifting" [4]. Asarin et. al. [1, 2] work with linear dynamics and rectangular polyhedra and develop the tool called `d/dt`. They reduce the conservatism due to the so-called "wrapping effect" by combining propagation of exact reachable sets at sampling instants with convex over-approximation during intersample intervals. Similar ideas appeared in the earlier work of Greenstreet and Mitchell [10] who also handle nonlinear models and non-convex polyhedra by using two-dimensional projections. Mitchell et. al. [23, 35] work with general nonlinear dynamics and compute reachable sets as sublevel sets of value functions for differential games, which are solutions of Hamilton-Jacobi PDEs. Kurzhanski and Varaiya [17] work with affine open-loop dynamics and use ellipsoids for reachable set approximation (based on ellipsoidal methods for continuous systems developed in their prior work). They handle discrete transitions by taking the union of reachable sets over possible switching times and covering it with one bounding ellipsoid. Chutinan and Krogh [3] compute optimal polyhedral approximations of continuous flow pipes for general nonlinear dynamics, using the tool `CheckMate`. Stursberg and Krogh [33] work with nonlinear dynamics and "oriented rectangular hulls" relying on principal component analysis. Girard et. al. [8, 9] use a procedure similar to that of Asarin et. al. mentioned earlier, but work with *zonotopes* (affine transforms of hypercubes) which allow more efficient computation for linear dynamics. More recently, this approach was refined with the help of support-function representations [11] and the accompanying tool `SpaceEx` [7] was developed. Another example of very recent work in this area is the result of [16] on computation of ε-reach sets.

There are several similarities between our method and the previous ones just mentioned. Like `d/dt` and related techniques, we also reduce the conservatism due to the "wrapping effect" by making a distinction between sampling and intersample approximations (although we do not present the analysis of our method at sufficient level of detail to fully demonstrate this point here, it should be clear that the bounds derived in Section 4 are valid at sampling times only). Also, similarly to Kurzhanski and Varaiya, we handle discrete transitions by taking the union of reachable sets over possible switching times and covering it with one bounding set, except we work with hypercubes rather than ellipsoids. On the other hand, in spite of the multitude of available methods, these methods were designed for reachability verification and are not directly tailored to control problems of the kind considered here. There are at least two important reasons why we prefer to build on the method from Section 4 rather than just adopt one of the above methods for dealing with hybrid systems:

i) The methods just mentioned are computational (on-line) in nature; by this we mean that approximations of reachable sets are generated in real time as the system evolves. By contrast, the method from Section 4 is analytical (off-line). Indeed, the size of the reachable set bound E_k at each time step, as well as the center point x_k^*, are obtained iteratively from the formulas given in Section 4. In other words, knowing the system data (the matrices A_p and B_p as well as the control gains K_p), we can pre-compute these bounds; there is no need to synchronize their computation with the evolution of the system. Consequently, the corresponding lower bound on the data rate required for stabilization can be obtained *a priori*, which makes more sense in the context of applications where communication strategies are designed separately from on-line control tasks. (On the negative side, this makes the bounds on reachable sets that our method provides more conservative.)

ii) Our method is tailored specifically to linear dynamics and to sets in the shapes of hypercubes. Our choice of hypercubes as bounding sets is very natural from the point of view of quantizer design with rectilinear quantization boxes, such as those arising from simple sensors. (However, in other application contexts it may be possible to work with different set shapes. For example, zonotopes—which are affine transforms of hypercubes—would correspond to pre-processing the continuous state by an affine transformation before passing it to a digital encoder; this generalization appears to be quite promising for more efficient computations.)

7.2 Switching surfaces

With regards to hybrid systems where mode switching occurs on switching surfaces, the first observation is that our Theorem 1 already covers such systems, because our reachable set over-approximation is computed by taking the union over all possible switching times $\bar{t}$ (see Section 4). Indeed, a switched system admits more solutions than a hybrid system (for which it serves as a high-level abstraction), and so our stabilization result conservatively captures the hybrid system solutions. The main issue is to verify that a given hybrid system fulfills the slow-switching assumption (Assumption 1), i.e., that all solutions satisfy the dwell-time and average dwell-time properties specified there. This can be difficult, but is possible in some cases. Notable examples are hybrid systems whose switching surfaces are concentric circles with respect to some norm, or lines through the origin in the plane. (Average dwell time is not directly helpful but in these cases we can compute dwell time, assuming linear dynamics in each mode.) Some more interesting examples where time-dependent properties (of dwell-time type) are established a posteriori for control systems with state-dependent switching can be found in [14], [21]. Thus, translating a hybrid system to a switched system and applying our previous result off-the-shelf via verifying the slow-switching condition can actually be a reasonable route to follow. In fact, since our strategy guarantees containment of the reachable set at each sampling time within a bounding hypercube, we can just run it and verify empirically whether or not the switching is slow enough for convergence. This is what we actually did in the simulation example given in Section 6. This in some sense moves us closer to the on-line computational methods cited above.

A better approach, however, is to improve our reachable set bounds by explicitly incorporating the information available in a hybrid system about where in the continuous state space the switching can occur. Recall that our information structure makes the current mode available to the controller at each sampling time t_k. So, for example, if we know as in Section 4.1 that no switch has occurred on an interval $(t_k, t_{k+1}]$ and $\sigma(t) = p$ there, then the hypercube $\{x \in \mathbb{R}^n : \|x - x^*_{k+1}\| \le E_{k+1}\}$, which contains the reachable set at time $t = t_{k+1}$, can be reduced by intersecting it with the invariant for mode p. In other words, if a guard passes through this hypercube then we keep only the portion lying on that side of the guard on which mode p is active; the point x^*_{k+1} can also be redefined at this step. The resulting reduction in the size of the bounding set can be quite significant, especially if the set $\{x : \|x - x^*_k\| \le E_k\}$ at time $t = t_k$ was close to some of the switching surfaces. (Note, however, that if the reachable set over-approximation at time t_{k+1} must be a hypercube, then some or all of this size reduction might become undone when passing to a bounding hypercube.) Or, consider the situation of Section 4.2 where a sampling interval $(t_k, t_{k+1}]$ contains a switch from $\sigma = p$ to $\sigma = q$ at an unknown time $\bar{t}$. The bounding set before the switch, $\{x : \|x - \hat{x}(t_k + t')\| \le D_{k+1}(\bar{t})\}$ (see Section 4.2(a)) can be reduced in the same way as above by intersecting it with the invariant for mode p. (Since $\bar{t}$ is unknown, we should either treat it as a parameter for this computation or take the maximum over $\bar{t}$ first; (19) is helpful for doing the latter.) Then, when this possibly reduced intermediate bounding set is used to calculate the bounding set after the switch, which we previously defined as $\{x : \|x - x^*_{k+1}\| \le E_{k+1}\}$ (see Section 4.2(b)), we may reduce it again, this time intersecting it with the invariant for mode q. Overall, this can lead to a significant reduction in the size of the reachable set over-approximation compared to the method of Section 4 which does not assume any relation between the continuous state x and the switching signal (but again, working with hypercubes would not allow us to take full advantage of this size reduction). Additionally, the knowledge of switching surfaces can be used to obtain some information about the unknown switching time $\bar{t}$: for example, if at time t_k we are far from any switching surface, then using the system dynamics we can calculate a lower bound on the time that must pass before a switch can occur.

7.3 State jumps

The reachable set propagation method of Section 4 assumes that there are no state jumps at the switching times, i.e., the reset map is the identity. However, it is not very difficult to augment it to nontrivial reset maps. Specifically, if we have a reset map $R_{pq} : \mathbb{R}^n \to \mathbb{R}^n$ which defines the new state $R_{pq}(x)$ to which x jumps at the time of mode transition from p to q, all we need is a knowledge of some affine Lipschitz bound of the form $\|R_{pq}(x_1) - R_{pq}(x_2)\| \le a\|x_1 - x_2\| + b$. Then, we can apply the transformations $c \mapsto R_{pq}(c)$ and $D \mapsto aD + b$ to the reachable set over-approximations of the form $\{x : \|x - c\| \le D\}$ obtained at each time that corresponds to a switch (these times are $t_k + \bar{t}$ on sampling intervals containing a switch, see Section 4.2). We can continue working with hypercubes because after incorporating resets in this way we still obtain hypercubes. We see that accounting for state jumps does not lead to substantial complications in our reachable set algorithm. (The same claim is true for most of the other existing reachable set algorithms from the literature: many of them assume the identity reset map but can be generalized with not much difficulty.) The stability analysis can proceed similarly, with the constants a, b affecting the evolution of the Lyapunov function and leading to a modified average dwell time bound.

8. CONCLUSIONS

We presented a result on sampled-data quantized state feedback stabilization of switched linear systems, which relies on a slow-switching condition and a novel method for propagating over-approximations of reachable sets. We explained how this result can be applied in the setting of hybrid systems, where it can actually be improved. Future work will focus on refining the reachable set bounds (possibly by combining our method with other known reachable set algorithms for hybrid systems) and on addressing more general systems with external disturbances, output measurements, and nonlinear dynamics.

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