

CONTROL of SWITCHED SYSTEMS with LIMITED INFORMATION

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PROBLEM FORMULATION

Switched system: $\dot{x} = A_\sigma x + B_\sigma u$

$\{(A_p, B_p) : p \in \mathcal{P}\}$ are (stabilizable) modes,

\mathcal{P} is a (finite) index set, $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is a switching signal
(can be state-dependent, realizing discrete state in hybrid system)

Information structure:

Sampling: state x is measured at times $t_k = k\tau_s$, $k = 0, 1, \dots$

Quantization: each $x(t_k)$ is encoded by an integer from 0 to N^n and sent to the controller, along with $\sigma(t_k) \in \mathcal{P}$ ($n = \dim x$)

Data rate: $\frac{\log_2(N^n + 1) + \log_2 |\mathcal{P}|}{\tau_s}$

Objective: design an encoding & control strategy s.t. $x(t) \rightarrow 0$
based on this limited information about $x(\cdot)$ and $\sigma(\cdot)$

MOTIVATION

Switching:

- ubiquitous in realistic system models
- lots of research on stability & stabilization under switching
- tools used: common & multiple Lyapunov functions, slow switching assumptions

Quantization:

- coarse sensing (low cost, limited power, hard-to-reach areas)
- limited communication (shared network resources, security)
- theoretical interest (how much info is needed for a control task)
- tools used: Lyapunov analysis, data-rate / MATI bounds

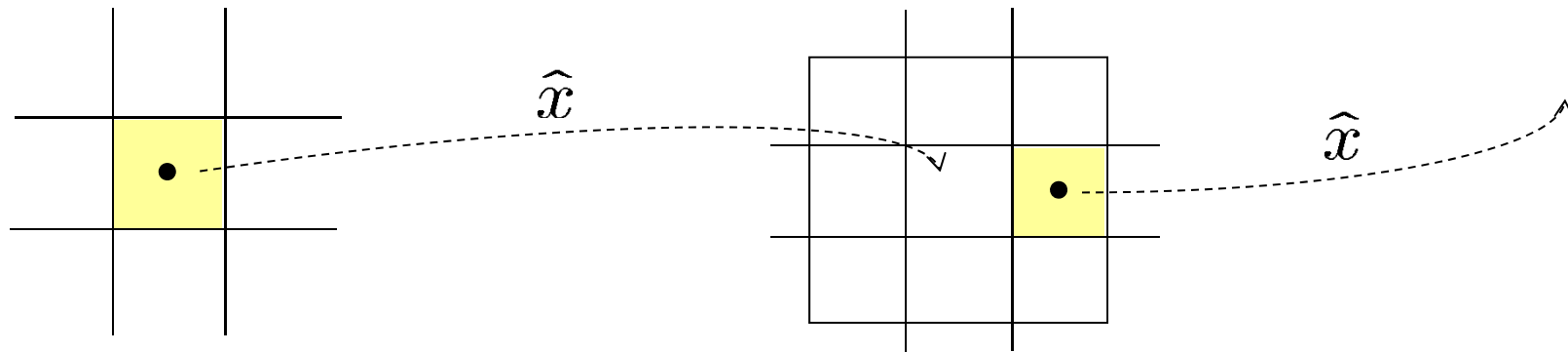
Commonality of tools is encouraging

Almost no prior work on quantized control of switched systems (except quantized MJLS [Nair et. al. 2003, Dullerud et. al. 2009])

NON-SWITCHED CASE

Quantized control of a single LTI system:

[Baillieul, Brockett-L, Hespanha, Matveev-Savkin, Nair-Evans, Tatikonda]



System can be stabilized if

$$\underbrace{\text{error reduction factor at } t_k}_{N(=3)} > \underbrace{\text{growth factor on } [t_k, t_{k+1}]}_{\text{e.g. } \|e^{A\tau_s}\|_\infty}$$

Control: $u = K\hat{x}$

Crucial step: obtaining a **reachable set over-approximation**
at next sampling instant

How to do this for switched systems?

REACHABLE SET ALGORITHMS

Many computational (on-line) methods for hybrid systems

- Puri–Varaiya–Borkar (1996): approximation by piecewise-constant differential inclusions; unions of polyhedra
- Henzinger–Preußig–Stursberg–et. al. (1998, 1999): approximation by rectangular automata; tools: *HyTech*, also *PHAVer* by Frehse (2005)
- Asarin–Dang–Maler (2000, 2002): linear dynamics; rectangular polyhedra; tool: *d/dt*
- Mitchell–Tomlin–et. al. (2000, 2003): nonlinear dynamics; level sets of value functions for HJB equations
- Kurzhanski–Varaiya (2002, 2005): affine open-loop dynamics; ellipsoids
- Chutinan–Krogh (2003): nonlinear dynamics; polyhedra; tool: *CheckMate*
- Girard–Le Guernic–et. al. (2005, 2008, 2009, 2011): linear dynamics; zonotopes and support functions; tool: *SpaceEx*

OUR APPROACH

We develop a method for propagating reachable set over-approximations for switched systems which is:

- Analytical (off-line)
- Leads to an **a priori data-rate bound** for stabilization (may be more conservative than on-line methods)
- Works with linear dynamics and hypercubes (with moving center)
- Tailored to switched systems (time-dependent switching) but can be adopted/refined for hybrid systems

SLOW-SWITCHING and DATA-RATE ASSUMPTIONS

1) \exists dwell time τ_d (lower bound on time between switches)

2) \exists average dwell time (ADT) τ_a s.t.

$$\text{number of switches on } (s, t] \leq N_0 + \frac{t - s}{\tau_a} \quad \forall t > s \geq 0$$

3) $\tau_a > \tau_d \geq \tau_s$ (sampling period)

Implies: ≤ 1 switch on each sampling interval $(t_k, t_{k+1}]$

We'll see how large τ_a should be for stability

Define $\Lambda_p := \|e^{A_p \tau_s}\|_\infty, \quad p \in \mathcal{P}$

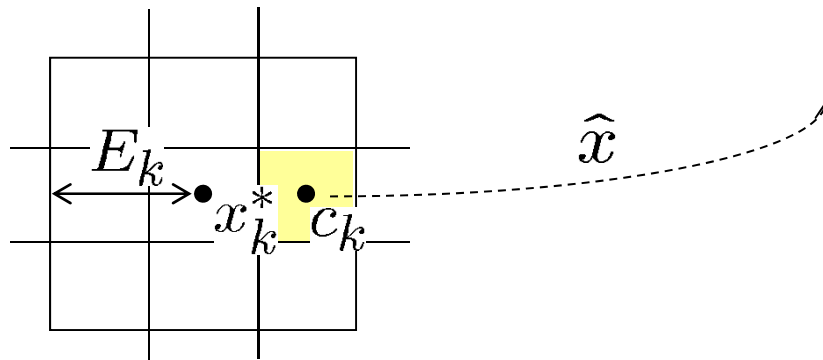
4) $\Lambda_p < N \quad \forall p$

(usual data-rate bound for individual modes)

ENCODING and CONTROL STRATEGY

Goal: generate, on the decoder/controller side, a sequence of points $x_k^* \in \mathbb{R}^n$ and numbers $E_k > 0$ s.t.

$$\|x(t_k) - x_k^*\| \leq E_k \quad \forall k \quad (\text{always } \infty\text{-norm})$$



Let $p := \sigma(t_k)$

Pick K_p s.t. $A_p + B_p K_p$ is Hurwitz

Define state estimate $\hat{x}(\cdot)$ on $[t_k, t_{k+1})$ by

$$\dot{\hat{x}} = (A_p + B_p K_p) \hat{x}, \quad \hat{x}(t_k) = c_k$$

Define control $u(\cdot)$ on $[t_k, t_{k+1})$ by

$$u(t) = K_p \hat{x}(t)$$

GENERATING STATE BOUNDS

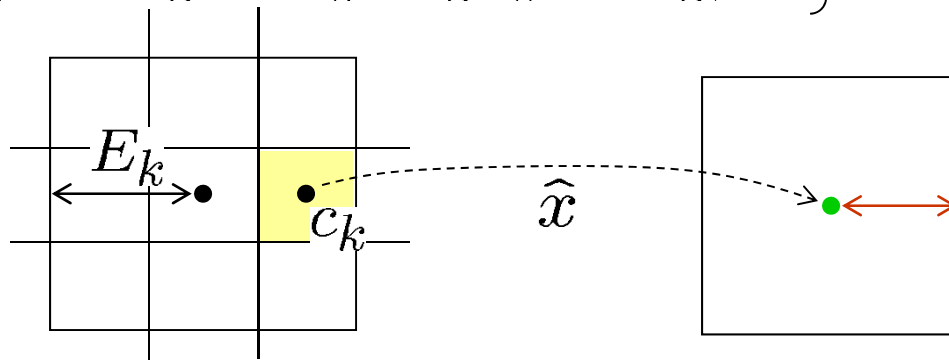
Choosing a sequence E_0, E_1, E_2, \dots that grows faster than system dynamics, for some k_0 we will have $\|x(t_{k_0})\| \leq E_{k_0}$

Inductively, assuming $\|x(t_k) - x_k^*\| \leq E_k$ we show how to find x_{k+1}^*, E_{k+1} s.t. $\|x(t_{k+1}) - x_{k+1}^*\| \leq E_{k+1}$

Case 1 (easy): sampling interval with **no switch**

$$\sigma(t_k) = \sigma(t_{k+1}) = p \Rightarrow \sigma(t) = p \quad \forall t \in [t_k, t_{k+1}]$$

$$\left. \begin{aligned} \dot{x} &= A_p x + B_p u \\ \hat{\dot{x}} &= A_p \hat{x} + B_p u \\ \hat{x}(t_k) &= c_k \Rightarrow \|e(t_k)\| \leq E_k/N \end{aligned} \right\} \begin{aligned} &\text{Let } e := x - \hat{x} \\ &\Rightarrow \dot{e} = A_p e \\ &\Rightarrow \|e(t_{k+1})\| \leq \|e^{A_p \tau_s}\| \cdot \|e(t_k)\| \\ &\leq \Lambda_p E_k / N =: E_{k+1} \end{aligned}$$

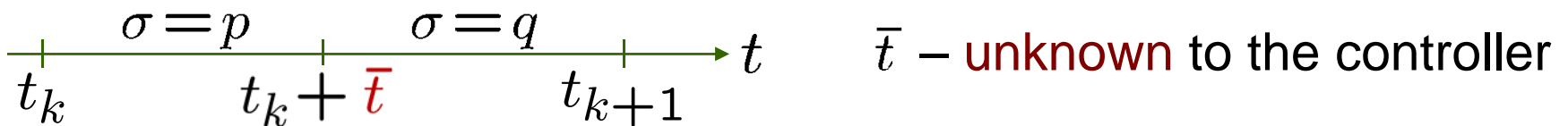


$$x_{k+1}^* := e^{(A_p + B_p K_p) \tau_s} c_k$$

GENERATING STATE BOUNDS

Case 2 (harder): sampling interval **with a switch**

$$\sigma(t_k) = p, \quad \sigma(t_{k+1}) = q \neq p$$

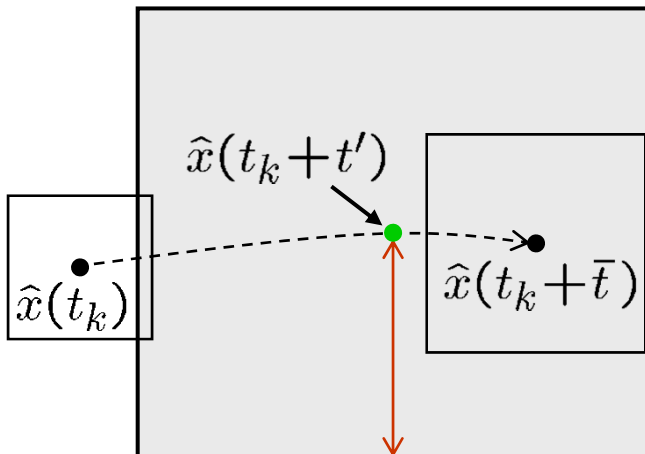


Before the switch: as on previous slide,

$$\|x(t_k + \bar{t}) - \hat{x}(t_k + \bar{t})\| \leq \|e^{A_p \bar{t}}\| E_k / N$$

but this is unknown known

Instead, pick some $t' \in [0, \tau_s]$ and use $\hat{x}(t_k + t')$ as center



(triangle inequality)

Intermediate bound:

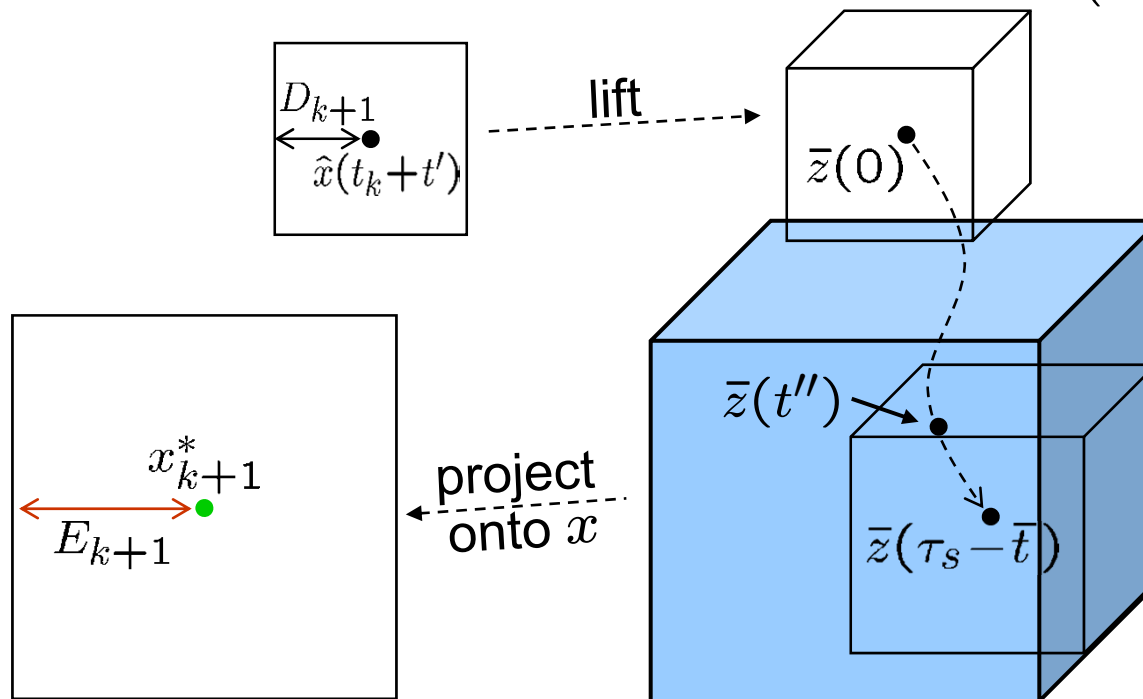
$$\|x(t_k + \bar{t}) - \hat{x}(t_k + t')\| \leq D_{k+1}(\bar{t})$$

GENERATING STATE BOUNDS

After the switch: on $[t_k + \bar{t}, t_{k+1}]$, closed-loop dynamics are

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A_q & B_q K_p \\ 0 & A_p + B_p K_p \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \text{ or } \dot{z} = \bar{A}_{pq} z$$

Auxiliary system in \mathbb{R}^{2n} : $\dot{z} = \bar{A}_{pq} z$, $z(0) = \begin{pmatrix} \hat{x}(t_k + t') \\ \hat{x}(t_k + t') \end{pmatrix}$



Then take maximum over \bar{t} to obtain final bound

STABILITY ANALYSIS: OUTLINE

1) sampling interval with no switch: $\sigma \equiv p$ on $[t_k, t_{k+1}]$

$$x_{k+1}^* = e^{(A_p + B_p K_p)\tau_s} c_k = e^{(A_p + B_p K_p)\tau_s} (x_k^* + (c_k - x_k^*))$$

This is exp. stable DT system w. input $\Delta_k := c_k - x_k^*$

$$\|\Delta_k\| \leq E_k(N-1)/N \quad \text{data-rate assumption}$$

$$\text{and } E_{k+1} = E_k \Lambda_p / N < E_k \Rightarrow E_k \xrightarrow{\text{exp}} 0$$

Thus, the overall “cascade” system is exp. stable

Lyapunov function: $V_p(x, E) := x^T P_p x + \rho_p E^2$

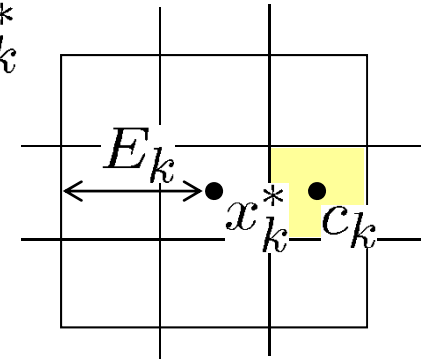
satisfies $V_p(x_{k+1}^*, E_{k+1}) \leq \nu V_p(x_k^*, E_k), \nu < 1$

2) if $[t_k, t_{k+1}]$ contains a switch from p to q , then

$$V_q(x_{k+1}^*, E_{k+1}) \leq \mu V_p(x_k^*, E_k), \mu > 1$$

If ADT satisfies $\tau_a > (1 + \log(\mu) / \log(1/\nu))\tau_s$ then

$$V_{\sigma(t_k)}(x_k^*, E_k) \xrightarrow{\text{exp}} 0 \text{ as } k \rightarrow \infty \Rightarrow \text{same true for } x(t_k)$$



SIMULATION EXAMPLE

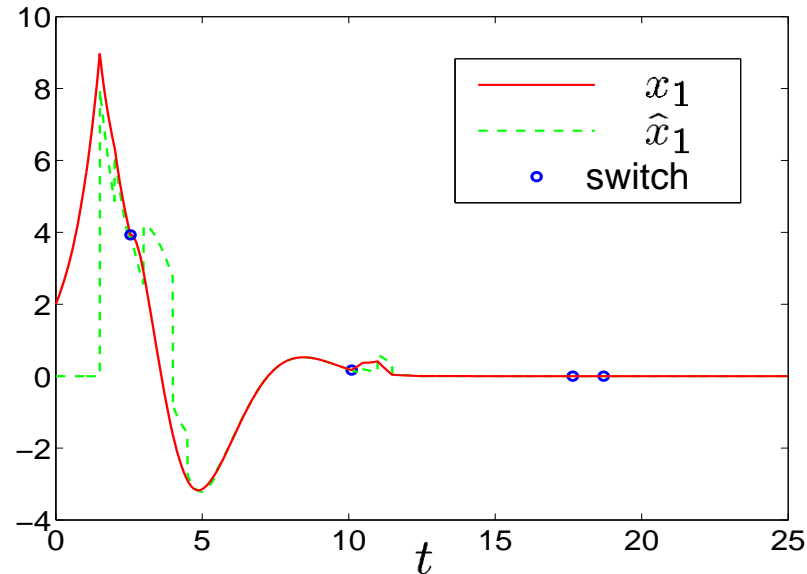
$$\mathcal{P} = \{1, 2\} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -2 & 0 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$E_0 = 0.5 \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

$$\tau_s = 0.5, \quad N = 5 \quad (\text{data-rate assumption holds})$$

$$\tau_d = 1.05, \quad \tau_a = 7.55, \quad N_0 = 5$$



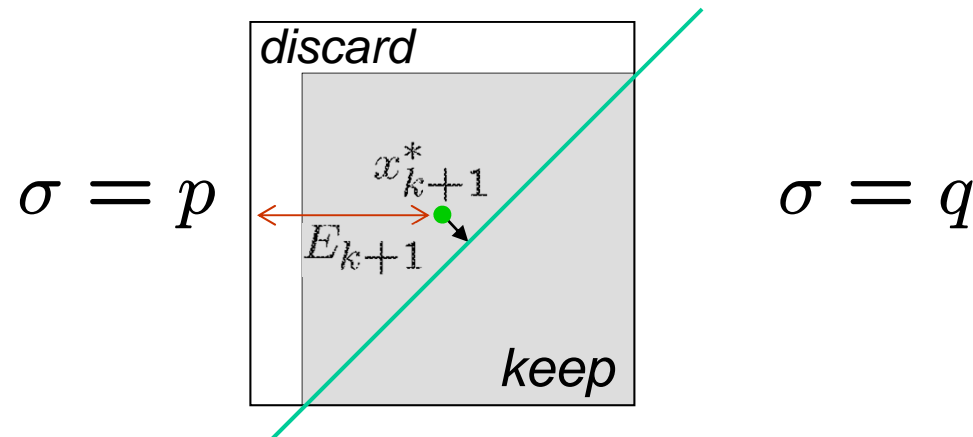
Theoretical lower bound on τ_a is about 50

HYBRID SYSTEMS

Switching triggered by switching surfaces (guards) in state space

- Previous result applies if we can use relative location of switching surfaces to verify slow-switching hypotheses
- Can just run the algorithm and verify convergence on-line
- Can use the extra info to improve reachable set bounds

For example: $\sigma(t_k) = p$, $\sigma(t_{k+1}) = q \neq p$



- State jumps – easy to incorporate

CONCLUSIONS and FUTURE WORK

Contributions:

- Stabilization of switched/hybrid systems with quantization
- Main step: computing over-approximations of reachable sets
- Data-rate bound is the usual one, maximized over modes

Extensions:

- Refining reachable set bounds (set shapes, choice of t' , t'')
- Relaxing slow-switching assumptions ($\tau_d < \tau_s$)
- Less frequent transmissions of discrete mode value

Challenges:

- Output feedback (Wakaiki and Yamamoto, MTNS'14)
- External disturbances (ongoing work with Yang)
- Modeling uncertainty
- Nonlinear dynamics

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$$\dot{x} = A_{\sigma}x + B_{\sigma}u + D_{\sigma}d$$

