On Stability of Nonlinear Slowly Time-Varying and Switched Systems

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Literature: Stability of Slowly Time-varying Systems

A system with time-varying parameters is known to be stable when[1]

- The system with parameters fixed at each frozen time is stable.
- The system parameters vary slowly enough.

[1] Desoer 69; Ilchmann et al. 87; Lawrence and Rugh 90; Khalil and Kokotovic 91; Ioannou and Sun 96; Khalil 02

Literature: Stability of Switched Systems

A switched system is known to be stable when[2]

- Each subsystem is stable.
- The system switches slowly enough among its subsystems.

[2] Morse 96; Hespanha and Morse 99; Zhao et al. 12; Kundu and Chatterjee 15

Motivation and Challenge

- Motivation: obtaining unified stability criteria for slowly time-varying and switched systems.

- Gap: system parameters were assumed (in literature) to be
  - Continuous / differentiable for slowly time-varying systems.
  - Piecewise constant for switched systems

Solution

- Relax regularity assumption on system parameters as piecewise differentiable (with discontinuities).
- Apply the concept of total variation to characterize the variation of system parameters.

Solution

- Prior work[3]: Obtained unified stability criteria for slowly time-varying and switched linear systems.

- In this talk, we consider the nonlinear case.

[3] X. Gao, D. Liberzon, J. Liu, and T. Başar. Unified stability criteria for slowly time-varying and switched linear systems. *Automatica*, 96:110-120, 2018.

Preliminaries: Total Variation

- The total variation of a vector-valued function $u(\cdot)$ over $[a, b]$ is defined by

$$\int_a^b \|du\| := \sup_{P \in \mathbb{P}} \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|$$

  where

  - $P = \{t_i | i = 0, \ldots, k\}$ is a partition of $[a, b]$.
  - $\mathbb{P}$ is the set of all partitions of $[a, b]$.

Total Variation under Regularity Conditions

Lemma 1 (see, e.g., Gao et al., 2018)

Under suitable regularity conditions on $u(\cdot)$, the total variation is given by

$$\int_a^b \|du\| = \sum_{i=0}^{m} \int_{d_i}^{d_{i+1}} \|\dot{u}(t)\| dt + \sum_{i=1}^{m+1} \left\| u(d_i) - u(d_i^-) \right\|$$

where d_i are discontinuities of $u(\cdot)$

- Total variation is capable of capturing both differentiable functions and piecewise constant functions.

Nonlinear Time-varying / Switched Systems

$$\dot{x} = f(x, u(t))$$

- $x \in \mathbb{R}^n$ is the state.
- $u(t) \in \Gamma \subset \mathbb{R}^m$ is the time-varying parameter.
- $f(\cdot, \cdot)$ is locally Lipschitz over $\mathbb{R}^n \times \Gamma$.
- $f(0, u) = 0$ for all $u \in \Gamma$.
- Γ is compact and convex.

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- There exist a candidate Lyapunov function $V(x, u)$ and positive constants c_1, c_2, c_3, c_4 such that for all $x \in \mathbb{R}^n$ and $u \in \Gamma$,

$$c_1\|x\|^2 \leq V(x, u) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -c_3\|x\|^2 \qquad \left\|\frac{\partial V}{\partial u}\right\| \leq c_4\|x\|^2$$

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- There exist positive constants μ and α with $\mu < c_1 c_3 / c_2 c_4$ such that for any $[t_1, t_2]$, $u(\cdot)$ satisfies

$$\int_{t_1}^{t_2} \|du\| \leq \mu(t_2 - t_1) + \alpha$$

Sketch of the Proof

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- From

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we show that
$$\begin{aligned} V(t_d^-) &\le V(t_1)\exp\left(-\frac{c_3}{c_2}(t_d - t_1) + \frac{c_4}{c_1}\int_{t_1}^{t_d} \|\dot{u}\| dt\right) \\ V(t_2) &\le V(t_d)\exp\left(-\frac{c_3}{c_2}(t_2 - t_d) + \frac{c_4}{c_1}\int_{t_d}^{t_2} \|\dot{u}\| dt\right) \end{aligned}$$

Sketch of the Proof

- In addition, we show (using MVT and some additional calculations)

$$V(t_d) \ \leq \ V(t_d^-) \exp\left(\frac{c_4}{c_1} \, \|u(t_d) - u(t_d^-)\| \right)$$

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- Combining the three inequalities, we have

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- Since

$$\int_{t_1}^{t_2}\|du\| \leq \mu(t_2-t_1)+\alpha$$

and $\mu < c_1c_3/c_2c_4$, $V(\cdot)$ decays exponentially fast over $[t_1, t_2]$.

Applications: Nonlinear Time-varying Systems

- If $u(\cdot)$ is continuously differentiable, the third condition in Theorem 1 becomes
$$\int_{t_1}^{t_2} \|\dot{u}(t)\| dt \leq \mu(t_2 - t_1) + \alpha$$
- Theorem 1 under this case reproduces the stability criteria for nonlinear time-varying systems proposed in [Khalil 02].

Applications: Nonlinear Switched Systems

Given a set of subsystems

$$\dot{x} = f(x, u_p),\ p \in \mathcal{P}$$

- $u_p \in \Gamma$.
- $\mathcal{P}$ is the index set.

Consider a switched system

$$\dot{x} = \bar{f}_{\sigma(t)}(x)$$

- $\bar{f}_{\sigma(t)}(x) := f(x, u_{\sigma(t)})$.
- $\sigma(\cdot)$ is the switching signal.

Applications: Nonlinear Switched Systems

Corollary 1

The switched system is globally exponentially stable if

- Γ *is compact and convex.*
- *There exists a condidate Lyapunov function* $V(x, u)$ *with the same properties as described in Theorem 1.*
- *There exist positive constants* μ *and* α*, with* $\mu < c_1 c_3 / c_2 c_4$*, such that for any* $[t, t+T]$*,*

$$\sum_{p,q \in \mathcal{P}, p \neq q} N_\sigma^{pq}(t, t+T) \|u_p - u_q\| \leq \mu T + \alpha$$

Applications: Nonlinear Switched Systems

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p-u_q\| \leq \mu T+\alpha$$

- $N_\sigma^{pq}(t,t+T)$ is the number of switches from subsystem p to subsystem q over $[t,t+T]$.

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p-u_q\| = \int_t^{t+T}\|du_\sigma\|$$

- It can be shown that Corollary 1 matches the stability criteria for nonlinear switched systems proposed in [Kundu and Chatterjee 15].

Applications: Linear Systems

Consider a linear system

$$\dot{x} = f(x, u(t)) = A(t)x$$

where

$$u(t) = [a_{11}(t), \ldots, a_{1n}(t), \ldots, a_{n1}(t), \ldots, a_{nn}(t)]^T \in \Gamma \subset \mathbb{R}^{n^2}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{bmatrix} \in \mathcal{A} \subset \mathbb{R}^{n \times n}$$

Applications: Linear Systems

Corollary 2

The linear system is globally exponentially stable if

- $\mathcal{A}$ *is compact and convex.*
- A' *is Hurwitz for all* $A' \in \mathcal{A}$*, and there exist positive constants* c *and* λ *such that*
$$\|e^{A's}\| \leq ce^{-\lambda s} \quad \forall\, A' \in \mathcal{A},\ s \geq 0$$
- *There exist positive constants* μ *(small enough) and* α *such that for any* $[t, t+T]$,
$$\int_{t_1}^{t_2} \|dA\|_F \leq \mu(t_2 - t_1) + \alpha$$

Applications: Linear Systems

- $\int_{t_1}^{t_2} \|dA\|_F$ is the total variation of $A(\cdot)$ over $[t_1, t_2]$, defined via the Frobenius norm.
- Corollary 2 (qualitatively) matches the unified stability criteria for slowly time-varying and switched linear systems proposed in [Gao et al., 18]; see also [Pait and Kassab 01].

Future Works

- Bridging the stability results for slowly time-varying systems with stable and unstable system dynamics at different frozen times, and the switched systems with stable and unstable subsystems.

- Extension to the case where the time-varying system admits a time-varying equilibrium point on a manifold.