On Stability of Nonlinear Slowly Time-Varying and Switched Systems

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A system with time-varying parameters is known to be stable when\(^1\)

- The system with parameters fixed at each frozen time is **stable**.
- The system parameters vary **slowly enough**.

\(^1\)Desoer 69; Ilchmann et al. 87; Lawrence and Rugh 90; Khalil and Kokotovic 91; Ioannou and Sun 96; Khalil 02
A switched system is known to be stable when\(^2\)

- Each subsystem is **stable**.
- The system switches **slowly enough** among its subsystems.

\(^2\)Morse 96; Hespanha and Morse 99; Zhao et al. 12; Kundu and Chatterjee 15
Motivation and Challenge

- **Motivation**: obtaining **unified** stability criteria for slowly time-varying and switched systems.

- **Gap**: system parameters were assumed (in literature) to be
  - Continuous / differentiable for slowly time-varying systems.
  - Piecewise constant for switched systems.
Solution

- Relax regularity assumption on system parameters as piecewise differentiable (with discontinuities).

- Apply the concept of total variation to characterize the variation of system parameters.
Solution

- Prior work\textsuperscript{3}: Obtained unified stability criteria for slowly time-varying and switched linear systems.

- In this talk, we consider the nonlinear case.

Preliminaries: Total Variation

- The **total variation** of a vector-valued function $u(\cdot)$ over $[a, b]$ is defined by

\[
\int_a^b \|du\| := \sup_{P \in \mathbb{P}} \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|
\]

where

- $P = \{t_i | i = 0, \ldots, k\}$ is a partition of $[a, b]$.
- $\mathbb{P}$ is the set of all partitions of $[a, b]$. 
Total Variation under Regularity Conditions

Lemma 1 (see, e.g., Gao et al., 2018)

Under suitable regularity conditions on $u(\cdot)$, the total variation is given by

$$\int_a^b \| du \| = \sum_{i=0}^m \int_{d_i}^{d_{i+1}} \| \dot{u}(t) \| dt + \sum_{i=1}^{m+1} \| u(d_i) - u(d_i^-) \|$$

where $d_i$ are discontinuities of $u(\cdot)$

- Total variation is capable of capturing both differentiable functions and piecewise constant functions.
Nonlinear Time-varying / Switched Systems

\[ \dot{x} = f(x, u(t)) \]

- \( x \in \mathbb{R}^n \) is the state.
- \( u(t) \in \Gamma \subset \mathbb{R}^m \) is the time-varying parameter.
- \( f(\cdot, \cdot) \) is locally Lipschitz over \( \mathbb{R}^n \times \Gamma \).
- \( f(0, u) = 0 \) for all \( u \in \Gamma \).
- \( \Gamma \) is compact and convex.
Main Result: Theorem 1

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The nonlinear time-varying system is globally exponentially stable if

- There exist a candidate Lyapunov function $V(x, u)$ and positive constants $c_1, c_2, c_3, c_4$ such that for all $x \in \mathbb{R}^n$ and $u \in \Gamma$,

\[
c_1 \|x\|^2 \leq V(x, u) \leq c_2 \|x\|^2
\]

\[
\frac{\partial V}{\partial x} f(x, u) \leq -c_3 \|x\|^2 \quad \left\| \frac{\partial V}{\partial u} \right\| \leq c_4 \|x\|^2
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$$\frac{\partial V}{\partial x} f(x, u) \leq -c_3\|x\|^2 \quad \|\frac{\partial V}{\partial u}\| \leq c_4\|x\|^2$$

- There exist positive constants $\mu$ and $\alpha$ with $\mu < c_1c_3/c_2c_4$ such that for any $[t_1, t_2]$, $u(\cdot)$ satisfies

$$\int_{t_1}^{t_2} \|du\| \leq \mu(t_2 - t_1) + \alpha$$
Sketch of the Proof

Without loss of generality, we assume that $u(\cdot)$ has only one discontinuity over $[t_1, t_2]$ at $t_d$. 
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- From
  \[
  \dot{V} = \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial u} \dot{u} \leq -c_3 \|x\|^2 + c_4 \|x\|^2 \|\dot{u}\|
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we show that

$$V(t_d^-) \leq V(t_1) \exp \left( -\frac{c_3}{c_2} (t_d - t_1) + \frac{c_4}{c_1} \int_{t_1}^{t_d} \|\dot{u}\| dt \right)$$

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Sketch of the Proof

In addition, we show (using MVT and some additional calculations)

\[ V(t_d) \leq V(t_d^-) \exp \left( \frac{c_4}{c_1} \| u(t_d) - u(t_d^-) \| \right) \]
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- Combining the three inequalities, we have

\[ V(t_2) \leq V(t_1) \exp \left( -\frac{c_3}{c_2} (t_2 - t_1) + \frac{c_4}{c_1} \int_{t_1}^{t_2} \|du\| \right) \]
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- Since
  \[ \int_{t_1}^{t_2} \| du \| \leq \mu (t_2 - t_1) + \alpha \]

and \( \mu < c_1 c_3 / c_2 c_4 \), \( V(\cdot) \) decays exponentially fast over \([t_1, t_2]\).
If $u(\cdot)$ is continuously differentiable, the third condition in Theorem 1 becomes

$$\int_{t_1}^{t_2} \|\dot{u}(t)\| dt \leq \mu(t_2 - t_1) + \alpha$$

Theorem 1 under this case reproduces the stability criteria for nonlinear time-varying systems proposed in [Khalil 02].
Applications: Nonlinear Switched Systems

Given a set of subsystems

\[ \dot{x} = f(x, u_p), \; p \in \mathcal{P} \]

- \( u_p \in \Gamma \).
- \( \mathcal{P} \) is the index set.

Consider a switched system

\[ \dot{x} = \bar{f}_{\sigma(t)}(x) \]

- \( \bar{f}_{\sigma(t)}(x) := f(x, u_{\sigma(t)}) \).
- \( \sigma(\cdot) \) is the switching signal.
Corollary 1

The switched system is globally exponentially stable if

- $\Gamma$ is compact and convex.

- There exists a candidate Lyapunov function $V(x,u)$ with the same properties as described in Theorem 1.

- There exist positive constants $\mu$ and $\alpha$, with $\mu < c_1 c_3 / c_2 c_4$, such that for any $[t, t + T]$,

$$\sum_{p,q \in \mathcal{P}, p \neq q} N_{pq}^{\sigma}(t, t + T) \|u_p - u_q\| \leq \mu T + \alpha$$
Applications: Nonlinear Switched Systems

\[ \sum_{p,q \in \mathcal{P}, p \neq q} N_{pq}^{\sigma}(t, t + T) \| u_p - u_q \| \leq \mu T + \alpha \]

- \( N_{pq}^{\sigma}(t, t + T) \) is the number of switches from subsystem \( p \) to subsystem \( q \) over \([t, t + T] \).

\[ \sum_{p,q \in \mathcal{P}, p \neq q} N_{pq}^{\sigma}(t, t + T) \| u_p - u_q \| = \int_{t}^{t+T} \| du_\sigma \| \]

- It can be shown that Corollary 1 matches the stability criteria for nonlinear switched systems proposed in [Kundu and Chatterjee 15].
Consider a linear system
\[
\dot{x} = f(x, u(t)) = A(t)x
\]
where
\[
u(t) = [a_{11}(t), \ldots, a_{1n}(t), \ldots, a_{n1}(t), \ldots, a_{nn}(t)]^T \in \Gamma \subset \mathbb{R}^{n^2}
\]
\[
A(t) = \begin{bmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{bmatrix} \in \mathcal{A} \subset \mathbb{R}^{n \times n}
\]
Applications: Linear Systems

Corollary 2

The linear system is globally exponentially stable if

- $A$ is compact and convex.
- $A'$ is Hurwitz for all $A' \in A$, and there exist positive constants $c$ and $\lambda$ such that
  \[ \| e^{A's} \| \leq ce^{-\lambda s} \quad \forall A' \in A, \ s \geq 0 \]
- There exist positive constants $\mu$ (small enough) and $\alpha$ such that for any $[t, t + T]$, 
  \[ \int_{t_1}^{t_2} \| dA \|_F \leq \mu(t_2 - t_1) + \alpha \]
Applications: Linear Systems

- $\int_{t_1}^{t_2} \| dA \|_F$ is the total variation of $A(\cdot)$ over $[t_1, t_2]$, defined via the Frobenius norm.

- Corollary 2 (qualitatively) matches the unified stability criteria for slowly time-varying and switched linear systems proposed in [Gao et al., 18]; see also [Pait and Kassab 01].
Future Works

- Bridging the stability results for slowly time-varying systems with stable and unstable system dynamics at different frozen times, and the switched systems with stable and unstable subsystems.

- Extension to the case where the time-varying system admits a time-varying equilibrium point on a manifold.