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# On Stability of Nonlinear Slowly Time-Varying and Switched Systems

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*Abstract*— **In this paper, we apply a total variation approach to bridge the stability criteria for nonlinear time-varying systems and nonlinear switched systems. In particular, we derive a set of stability conditions applying to both nonlinear time-varying systems and nonlinear switched systems. We show that the derived stability conditions, when applied to nonlinear time-varying systems and nonlinear switched systems, recover the existing stability results in the literature. We also show that the derived stability conditions can be applied to qualitatively recover a unified stability criterion for slowly time-varying linear systems and switched linear systems proposed in our recent work.**

## I. Introduction

The stability criteria for slowly time-varying systems and switched systems are fundamental, and they have drawn extensive research interest over the past decades. For a slowly time-varying (linear or nonlinear) system, a group of stability results in [1]–[6] state that the system is globally asymptotically stable if the system is stable at each frozen time[1], and the system parameters vary slowly enough. Similarly, a group of stability results in [7]–[10] state that a switched (linear or nonlinear) system is globally asymptotically stable if each subsystem is stable, and the system switches slowly enough among its subsystems. Although the two groups of results are similar and a switched system can be viewed as a time-varying system with piecewise constant system parameters, the stability conditions for switched systems cannot be directly recovered from the stability conditions for time-varying systems. The gap blocking this transition is caused by the common approach to characterize the variation of system parameters in a time-varying system. To be more specific, the common approach (for example, conditions involving the integral of the norm of system parameters' time derivatives) requires that the systems parameters are differentiable (or Lipschitz continuous) in time, and thus it cannot be directly applied to the piecewise constant case.

In our recent work [11], [12], a set of stability conditions was derived for slowly time-varying linear systems. Differently from the prior works, the variation of system parameters is characterized by their *total variation* over the time interval. The concept of total variation can be applied to both differentiable functions and piecewise constant functions. Benefitting from this property, it was shown that the stability conditions derived in [11], [12] can be applied to recover both the stability conditions derived in [1]–[3], for slowly time-varying linear systems, and the stability conditions derived in [8]–[10], for switched linear systems[2].

The work in [11], [12] brings up awareness of the fact that the studies of stability conditions for slowly time-varying systems and switched systems should not be disjoint, and it provides an approach to translate the results from one field to the other. Thus, it motivates us to apply the total variation approach to further bridge the gap between stability conditions for *nonlinear* time-varying systems and switched systems. For nonlinear time-varying systems, a group of stability results have been derived in [4]–[6]. The stability results in [4]–[6] impose slightly different conditions on system dynamics, yet they all require that the variation of system parameters, characterized by the time integral of the norm of the time derivative of the parameters, should be upper bounded by an affine function of the length of the time interval. More results on this line of research can be found in [13] and the references therein. For nonlinear switched systems, the stability criteria proposed in [8]–[10] can also be applied to the nonlinear case. We are interested in unifying the above two groups of results.

In this paper, we first recall the concept of total variation (Section II). Then, we derive a set of stability conditions for nonlinear time-varying systems (main results, Section III), which generalize the stability results in [4] to incorporate piecewise differentiable system parameters. Next, we apply the derived stability conditions to nonlinear switched systems and show that they match the stability conditions in [10] for the nonlinear case (Section IV). Then, we apply the main results to linear time-varying systems. We show that with one minor additional assumption, one can recover (qualitatively) the unified stability criteria for slowly time-varying and switched linear systems proposed in [11], [12] (Section V). Finally, we draw concluding remarks and discuss future directions for research (Section VI).

## II. Preliminaries: total variation

The concept of total variation for real-valued and vector-valued functions has been well documented in the literature [14]. This concept has been generalized to the matrix case in the recent work [12]. We recall the definition of total variation here. We denote by $P := \{t_i | i = 0, \dots, k\}$ a partition of interval $[a, b]$, where $a = t_0 < t_1 < \cdots < t_k = b$. Let $\mathbb{P}$ be the set of all partitions of $[a, b]$. Given a vector-valued function $u(\cdot) : \mathbb{R} \to \mathbb{R}^n$, its total variation over $[a, b]$

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[1] A time-varying system is stable at a frozen time means the system with parameters fixed at that frozen time is stable.

[2] The results derived in [8]–[10] are for general switched systems; they were applied to the linear case in [11], [12].

is defined as

$$\int_a^b \|du\| := \sup_{P \in \mathbb{P}} \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|.$$

The norm in the expression above is the Euclidean norm. In particular, if $u(\cdot)$ satisfies some regularity conditions (Condition 1), its total variation over an interval has an explicit expression (Lemma 1). We use $u(t^-)$ to denote the left limit of $u(\cdot)$ at $t$, and we introduce the following set of conditions.

*Condition 1 ([12], Assumption 2):* A vector-valued function $u(\cdot)$ satisfies

(i) $u(\cdot)$ is continuous from the right everywhere on $[a, b)$ and has left limits everywhere on $(a, b]$.
(ii) $u(\cdot)$ has a finite number of discontinuities over $(a, b)$, denoted by $\{d_1, d_2, \ldots, d_m\}$, and $a := d_0 < d_1 < \cdots < d_m < d_{m+1} =: b$.
(iii) $u(\cdot)$ is continuously differentiable on $(d_i, d_{i+1})$, and $\dot{u}(\cdot)$ is Riemann integrable on $[d_i, d_{i+1}]$, for all $i \in \{0, 1, \ldots, m\}$.
(iv) $\|\dot{u}(\cdot)\|$ is Riemann integrable on $[d_i, d_{i+1}]$, for all $i \in \{0, 1, \ldots, m\}$.

*Lemma 1 ([12], Lemma 1):* Given a vector-valued function $u(\cdot)$ satisfying Condition 1, its total variation over $[a, b]$ has the following expression:

$$\int_a^b \|du\| = \sum_{i=0}^{m} \int_{d_i}^{d_{i+1}} \|\dot{u}(t)\| dt + \sum_{i=1}^{m+1} \left\| u(d_i) - u(d_i^-) \right\|.$$

In [12], Condition 1 and Lemma 1 were stated for a matrix-valued function, yet a vector-valued function can be viewed as a special case of a matrix-valued function.

## III. Stability of Nonlinear Slowly Time-Varying Systems

In this section, we derive a set of stability conditions for nonlinear time-varying systems, which generalize the existing stability results in [4, Section 9.6].

Consider a nonlinear time-varying system

$$\dot{x} = f(x, u(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $u(t) \in \Gamma \subset \mathbb{R}^m$ is the time-dependent system parameter. Assume that $f(x, u)$ is locally Lipschitz over $\mathbb{R}^n \times \Gamma$. Furthermore, given any $u \in \Gamma$, the equation

$$f(x, u) = 0$$

is assumed to have a solution $x = 0$. Then, system (1) has an equilibrium point at $x = 0$, which is invariant over $u \in \Gamma$.

*Remark 1:* As a starting point, we consider in this paper the time-varying system with a fixed equilibrium point. However, it is more general to assume that the system equation with time-varying system parameters admits an equilibrium point that varies in time on a manifold. This is not pursued in this paper.

*Theorem 1:* The nonlinear time-varying system (1) is globally exponentially stable[3] if the following conditions are satisfied.

(i) The set of all possible system parameters, $\Gamma$, is compact and convex.
(ii) There exist a candidate Lyapunov function $V(x, u)$, continuously differentiable in $x$ and $u$, and positive constants $c_1, c_2, c_3, c_4$ such that for all $x \in \mathbb{R}^n$ and $u \in \Gamma$,

$$c_1 \|x\|^2 \le V(x, u) \le c_2 \|x\|^2, \tag{2}$$

$$\frac{\partial V}{\partial x} f(x, u) \le -c_3 \|x\|^2, \tag{3}$$

$$\left\| \frac{\partial V}{\partial u} \right\| \le c_4 \|x\|^2. \tag{4}$$

(iii) There exist positive constants $\mu$ and $\alpha$, with $\mu < c_1 c_3 / c_2 c_4$, such that for any bounded time interval $[t_1, t_2]$, $u(\cdot)$ satisfies Condition 1, and its total variation satisfies

$$\int_{t_1}^{t_2} \|du\| \le \mu (t_2 - t_1) + \alpha. \tag{5}$$

*Remark 2:* The inequality (3) implies that for all $u \in \Gamma$, the equation $f(x, u) = 0$ has a unique solution, i.e., $x = 0$.

*Remark 3:* The exact choice of the system parameter $u(t)$, in particular its dimension, affects the total variation $\int_{t_1}^{t_2} \|du\|$, and thus it affects condition (iii) in Theorem 1. For example, one could add a virtual dimension to $u(t)$, such that the total variation becomes larger and condition (iii) becomes more restrictive. Pursuing this issue further is out of the scope of this paper.

Before proving Theorem 1, we would like to clarify the novelty of this theorem. The set of stability conditions proposed in [4, Section 9.6] requires item (ii) in Theorem 1, yet it replaces item (iii) in Theorem 1 by the stronger condition that $u(\cdot)$ is continuously differentiable, and

$$\int_{t_1}^{t_2} \|\dot{u}(t)\| dt \le \mu (t_2 - t_1) + \alpha,$$

where $\mu < c_1 c_3 / c_2 c_4$ and $\alpha > 0$. By introducing the concept of total variation, Theorem 1 generalizes the result in [4, Section 9.6] to the case where $u(\cdot)$ is piecewise differentiable with discontinuities at the non-differentiable points. This generalization enables us to further derive a set of stability conditions for switched systems, which will be discussed later in Section IV.

One may find the assumption described by (4) not as standard as those described by (2) and (3). The following lemma (Lemma 2) from [4] justifies that if the system satisfies some general regularity conditions, and if the system with fixed parameters is globally exponentially stable, then there exists a candidate Lyapunov function satisfying (2)-(4)

[3]The definition of global exponential stability can be found in [4, Definition 4.5].

in Theorem 1. The proof of Lemma 2 applies the converse Lyapunov theorem, for exponential stability.

*Lemma 2 ([4], Lemma 9.8):* Consider the system (1) and suppose that the Jacobian matrices $\partial f(x,u)/\partial x$ and $\partial f(x,u)/\partial u$ satisfy

$$\left\|\frac{\partial f(x,u)}{\partial x}\right\| \le L_1,\ \left\|\frac{\partial f(x,u)}{\partial u}\right\| \le L_2\|x\|,\ \forall\, x\in\mathbb{R}^n,\ u\in\Gamma$$

for some $L_1, L_2 \ge 0$. Furthermore, assume there exist positive constants $k$ and $\gamma$ such that the trajectories of the system, with constant $u(t)\equiv u\in\Gamma$, satisfy

$$\|x(t)\| \le k\|x(0)\|e^{-\gamma t},\quad \forall\, \|x(0)\|\in\mathbb{R}^n,\ t\ge 0.$$

Then, there exists a function $V(x,u)$ defined over $\mathbb{R}^n\times\Gamma$ satisfying (2)-(4).

We are now in a position to prove Theorem 1.

*Proof:* (Proof of Theorem 1) Let an arbitrary time interval $[t_1,t_2]$ be given. Without loss of generality, we assume that $u(\cdot)$ has only one discontinuity over $[t_1,t_2]$, denoted by $t_d$. By Condition 1, $u(\cdot)$ is differentiable and $\|\dot u(\cdot)\|$ is integrable over the sub-intervals $(t_1,t_d)$ and $(t_d,t_2)$. Furthermore, $u(\cdot)$ is right continuous and has left limit at $t_d$. Denote by $V(t)$ the Lyapunov function evaluated along $x(t)$ and $u(t)$, namely, $V(t)=V(x(t),u(t))$. Since $u(\cdot)$ has a discontinuity at $t_d$, $V(\cdot)$ also has a discontinuity at $t_d$.

Consider the interval $[t_1,t_d)$. By (2)–(4) in Theorem 1, we have

$$\begin{aligned}\dot V &= \frac{\partial V}{\partial x}f(x,u)+\frac{\partial V}{\partial u}\dot u\\ &\le -c_3\|x\|^2+c_4\|x\|^2\,\|\dot u\|\\ &\le \Big(-\frac{c_3}{c_2}+\frac{c_4}{c_1}\|\dot u\|\Big)V.\end{aligned}$$

By the Gronwall-Bellman inequality, we obtain

$$\begin{aligned}V(t_d^-) &\le V(t_1)\exp\left(\int_{t_1}^{t_d} -\frac{c_3}{c_2}+\frac{c_4}{c_1}\|\dot u\|dt\right)\\ &= V(t_1)\exp\left(-\frac{c_3}{c_2}(t_d-t_1)+\frac{c_4}{c_1}\int_{t_1}^{t_d}\|\dot u\|dt\right).\end{aligned}\tag{6}$$

We apply a similar argument on the interval $[t_d,t_2]$, and we obtain

$$V(t_2)\le V(t_d)\exp\left(-\frac{c_3}{c_2}(t_2-t_d)+\frac{c_4}{c_1}\int_{t_d}^{t_2}\|\dot u\|dt\right).\tag{7}$$

Consider any $x_1,x_2\in\mathbb{R}^n$ and $u_1,u_2\in\Gamma$. Since $\Gamma$ is a convex set, and $V(x,u)$ is differentiable in $x$ and $u$, by the Mean Value Theorem, there exists $\lambda\in[0,1]$ such that[4]

$$\begin{aligned}&V(x_2,u_2)-V(x_1,u_1)\\ =\ &\frac{\partial V}{\partial x}(\lambda x_1+(1-\lambda)x_2,\lambda u_1+(1-\lambda)u_2)\cdot(x_2-x_1)\\ &+\frac{\partial V}{\partial u}(\lambda x_1+(1-\lambda)x_2,\lambda u_1+(1-\lambda)u_2)\cdot(u_2-u_1).\end{aligned}$$

[4]Denote by $\mathbf{z_1}=[x_1,u_1]$ a vector whose elements are $x_1$ and $u_1$. Similarly, let $\mathbf{z_2}=[x_2,u_2]$. Then, the Mean Value Theorem states that there exists $\lambda\in[0,1]$ such that $V(\mathbf{z_2})-V(\mathbf{z_1})=\nabla V(\lambda\mathbf{z_1}+(1-\lambda)\mathbf{z_2})\cdot(\mathbf{z_2}-\mathbf{z_1})$, where $\nabla V=[\frac{\partial V}{\partial x},\frac{\partial V}{\partial u}]^T$.

Taking $x_1=x_2=x$, we have

$$\begin{aligned}&V(x,u_2)-V(x,u_1)\\ =\ &\frac{\partial V}{\partial u}(x,\lambda u_1+(1-\lambda)u_2)\cdot(u_2-u_1).\end{aligned}$$

By (4),

$$V(x,u_2)-V(x,u_1)\le c_4\|x\|^2\|u_2-u_1\|.\tag{8}$$

Taking $g(y)=\exp(y-1)-y,\ y\in\mathbb{R}$, and then we have

$$\begin{aligned}g'(y) &= \exp(y-1)-1,\\ g''(y) &= \exp(y-1).\end{aligned}$$

It can be checked that $g'(y)=0$ if and only if $y=1$. In addition, $g''(y)>0,\ \forall\, y\in\mathbb{R}$. Hence, $g(y)$ attains global minimum at $y=1$, and $g(1)=0$. This implies that

$$y\le\exp(y-1),\quad \forall\, y\in\mathbb{R}.$$

Hence,

$$\begin{aligned}\frac{V(x,u_2)}{V(x,u_1)} &\le \exp\left(\frac{V(x,u_2)}{V(x,u_1)}-1\right)\\ &= \exp\left(\frac{V(x,u_2)-V(x,u_1)}{V(x,u_1)}\right).\end{aligned}$$

Plugging (2) and (8) into the inequality above, we further have

$$\begin{aligned}\frac{V(x,u_2)}{V(x,u_1)} &\le \exp\left(\frac{c_4\|x\|^2\|u_2-u_1\|}{c_1\|x\|^2}\right)\\ &= \exp\left(\frac{c_4}{c_1}\|u_2-u_1\|\right)\end{aligned}\tag{9}$$

for all $u_1,u_2\in\Gamma,\ x\in\mathbb{R}^n$. In particular, taking $x=x(t_d)$, $u_1=u(t_d^-)$ and $u_2=u(t_d)$, we obtain

$$V(t_d)\le V(t_d^-)\exp\left(\frac{c_4}{c_1}\|u(t_d)-u(t_d^-)\|\right).\tag{10}$$

Note that $u(t_d^-)\in\Gamma$ since $\Gamma$ is a compact (and thus closed) set.

Combining (5)-(7) and (10) together, we obtain

$$\begin{aligned}V(t_2) &\le V(t_1)\exp\left(-\frac{c_3}{c_2}(t_2-t_1)+\frac{c_4}{c_1}\int_{t_1}^{t_2}\|du\|\right)\\ &\le V(t_1)\exp\left(-(\frac{c_3}{c_2}-\frac{c_4}{c_1}\mu)(t_2-t_1)+\frac{c_4}{c_1}\alpha\right).\end{aligned}$$

Since $\mu<c_1c_3/c_2c_4$, $V(\cdot)$ decays exponentially fast with rate $c_3/c_2-(c_4/c_1)\mu$. By (2), we conclude that $x(\cdot)$ also decays exponentially fast. ■

## IV. Stability of Switched Nonlinear Systems

In this section, we apply the main result obtained in Section III to derive a set of stability conditions for switched systems, and we show that the derived results match the existing stability conditions in [10].

Let a set of systems

$$\dot x=f(x,u_p),\ p\in\mathcal{P}$$

be given, where $x \in \mathbb{R}^n$ is the system state, $u_p$ is the system dependent parameter, and $\mathcal{P}$ is the index set, which is finite or countably infinite. The system parameter $u_p$ is picked from a set $\Gamma \subset \mathbb{R}^m$, which could be an uncountable set. Assume that $f(\cdot,\cdot)$ is locally Lipchitz over $\mathbb{R}^n \times \Gamma$. In addition, assume that for any $u \in \Gamma$, the equation $f(x,u) = 0$ has a solution at $x = 0$.

Now consider a switched system

$$\dot{x} = \bar{f}_{\sigma(t)}(x), \tag{11}$$

where $\sigma(\cdot) : [0,\infty) \to \mathcal{P}$ is the switching signal and

$$\bar{f}_{\sigma(t)}(x) := f(x, u_{\sigma(t)}).$$

By convention, it is assumed that $\sigma(\cdot)$ is piecewise constant and is continuous from right everywhere. We introduce $N_\sigma^{pq}(t, t+T),\ p,q \in \mathcal{P}$ as the number of switches from subsystem $p$ and subsystem $q$ over the time interval $[t, t+T]$. By Theorem 1, we have the following corollary.

*Corollary 1:* The switched system (11) is globally exponentially stable if the following conditions hold.

(i) $\Gamma$ is a compact and convex set.

(ii) There exists a Lyapunov function $V(x,u)$, continuously differentiable over $\mathbb{R}^n \times \Gamma$, which satisfies (2)-(4) in Theorem 1 with constants $c_1, c_2, c_3, c_4$.

(iii) There exist positive constants $\mu$ and $\alpha$, with $\mu < c_1c_3/c_2c_4$, such that

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p - u_q\| \leq \mu T + \alpha,\ \forall\, t, T \geq 0. \tag{12}$$

*Proof:* By the piecewise constant property of $\sigma(\cdot)$, it can be readily seen that $u_{\sigma(\cdot)}$ is also piecewise constant. Hence, the total variation of $u_{\sigma(\cdot)}$ over $[t, t+T]$ can be expressed by

$$\int_t^{t+T} \|du_\sigma\| = \sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p - u_q\|.$$

Then (12) is equivalent to (5), and thus global exponential stability can be established by Theorem 1. ■

By [10, Theorem 5], the switched system (11) is globally exponentially stable if the following conditions are satisfied[5]: for each $p \in \mathcal{P}$, there exists a Lyapunov function $V_p(x)$, continuously differentiable in $x$, satisfying

(I) There exist positive constants $c_1$ and $c_2$ such that

$$c_1\|x\| \leq V_p(x) \leq c_2\|x\|, \quad \forall\, p \in \mathcal{P},\ x \in \mathbb{R}^n.$$

(II) There exists a positive constant $\gamma$ such that

$$\frac{\partial V_p}{\partial x} f(x, u_p) \leq -\gamma V_p(x), \quad \forall\, p \in \mathcal{P},\ x \in \mathbb{R}^n.$$

(III) For each pair of $p, q \in \mathcal{P}$, there exists a positive constant $\nu_{pq}$ such that

$$V_p(x) \leq \nu_{pq} V_q(x), \quad \forall\, x \in \mathbb{R}^n.$$

(IV) There exists a positive constant $\eta$ such that

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\ln \nu_{pq} < \gamma T + \eta, \quad \forall\, t, T \geq 0.$$

We now compare the above stability result with Corollary 1. Assume that items (i) and (ii) in Corollary 1 hold. By (2), condition (I) in [10, Theorem 5] is satisfied with the Lyapunov functions

$$V_p(x) := V(x, u_p),\ p \in \mathcal{P}.$$

By (2) and (3), we have

$$\frac{\partial V_p}{\partial x} f(x,u_p) \leq -c_3\|x\|^2 \leq -\frac{c_3}{c_2} V_p(x),\ \ \forall\, p \in \mathcal{P},\ x \in \mathbb{R}^n.$$

Therefore, condition (II) is satisfied with $\gamma = c_3/c_2$. By (9) in the proof of Theorem 1, we have

$$\frac{V_p(x)}{V_q(x)} \leq \exp\left(\frac{c_4}{c_1}\|u_p - u_q\|\right),\ \ \forall\, p,q \in \mathcal{P},\ x \in \mathbb{R}^n.$$

Hence, condition (III) is satisfied with $\nu_{pq} = \exp(c_4/c_1 \cdot \|u_p - u_q\|)$. By plugging in the above expressions for $\gamma$ and $\nu_{pq}$, condition (IV) can be expressed by

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\frac{c_4}{c_1}\|u_p - u_q\| < \frac{c_3}{c_2}T + \eta,$$

which is equivalent to

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p - u_q\| < \frac{c_1c_3}{c_2c_4}T + \frac{c_1}{c_4}\eta.$$

The above inequality is further equivalent to

$$\sum_{p,q\in\mathcal{P},p\neq q} N_\sigma^{pq}(t,t+T)\|u_p - u_q\| \leq \mu T + \alpha,$$

with

$$\mu < \frac{c_1c_3}{c_2c_4}, \quad \alpha = \frac{c_1}{c_4}\eta.$$

Hence, condition (IV) is exactly the same as item (iii) in Corollary 1. In view of this, we conclude that Corollary 1 recovers the result [10, Theorem 5] under the same assumptions on the Lyapunov function, i.e., items (i) and (ii).

## V. Stability of Slowly Time-Varying Linear Systems

In this section, we apply Theorem 1 to slowly time-varying linear systems. We show that Theorem 1, when applied to the linear case, qualitatively recovers the stability criteria proposed in our earlier works [11], [12]. To present and prove this statement, we need to introduce some notations and definitions. We use $\|\cdot\|$ and $\|\cdot\|_F$ to denote the standard induced 2-norm and the Frobenius norm, respectively, for matrices. Given a matrix-valued function $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$, its total variation over $[a,b]$ is defined as

$$\int_a^b \|dA\| := \sup_{p\in\mathbb{P}} \sum_{i=1}^k \|A(t_i) - A(t_{i-1})\|,$$

[5]This result was recovered in [12, Theorem 5] for the purpose of comparing it with stability conditions for switched linear system. Here, we compare it with Corollary 1 which deals with nonlinear switched systems.

where $p$ is a partition of $[a, b]$, and $\mathbb{P}$ is the set of all partitions of $[a, b]$. Similarly, the total variation of $A(\cdot)$ over $[a, b]$, under the Frobenius norm, is defined as

$$\int_a^b \|dA\|_F := \sup_{p\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\|_F.$$

A well-known property of matrix norms (see [15, eqn. 2.3.7]) states that given any matrix $A' \in \mathbb{R}^{n\times n}$,

$$\|A'\| \leq \|A'\|_F. \tag{13}$$

Then, we have

$$\int_a^b \|dA\| \leq \int_a^b \|dA\|_F. \tag{14}$$

We define a mapping $U(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ such that for any $A' \in \mathbb{R}^{n\times n}$,

$$U(A') := [a'_{11}, \ldots, a'_{1n}, a'_{21}, \ldots, a'_{2n}, \ldots, a'_{n1}, \ldots, a'_{nn}]^T,$$

where

$$A' = \begin{bmatrix} a'_{11} & \ldots & a'_{1n} \\ \vdots & \ddots & \vdots \\ a'_{n1} & \ldots & a'_{nn} \end{bmatrix}.$$

It is clear that $U(\cdot)$ is a bijection, and we denote by $U^{-1}(\cdot)$ the inverse mapping of $U(\cdot)$. Furthermore, by the definitions of the Euclidean norm for vectors and the Frobenius norm for matrices, we have

$$\|U(A')\| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} {a'_{ij}}^2} = \|A'\|_F. \tag{15}$$

Given a matrix-valued function $A(\cdot)$, we denote by

$$u_A(\cdot) := U(A(\cdot)) \tag{16}$$

the composition between $U(\cdot)$ and $A(\cdot)$. By (15), we have

$$\int_a^b \|du_A\| = \int_a^b \|dA\|_F.$$

To apply Theorem 1 to slowly time-varying linear systems, we need to rely on the following lemma. The lemma states that when system (1) is linear in $x$, a Lyapunov function satisfying (2)–(4) in Theorem 1 exists, and the constants $c_1, c_2, c_3, c_4$ admit explicit expressions.

*Lemma 3 ([4], Lemma 9.9):* Consider system (1) with system equation linear in $x$, that is,

$$\dot{x} = f(x, u(t)) = \bar{A}(u(t))x, \tag{17}$$

where $x \in \mathbb{R}^n$, $u(t) = [u_1(t), \ldots, u_m(t)]^T \in \Gamma \subset \mathbb{R}^m$, and $\bar{A}(\cdot) : \mathbb{R}^m \to \mathbb{R}^{n\times n}$. Assume that the following conditions hold.

(i) $\bar{A}(u)$ is continuously differentiable in $u$. Furthermore, there exist positive constants $b_i, i = 1, \ldots, m$, such that

$$\Big\|\frac{\partial \bar{A}(u)}{\partial u_i}\Big\| \leq b_i, \quad \forall\, u \in \Gamma,\ i = 1, \ldots, m.$$

(ii) There exists a positive constant $L$ such that

$$\|\bar{A}(u)\| \leq L, \quad \forall\, u \in \Gamma.$$

(iii) $\bar{A}(u)$ is Hurwitz for all $u \in \Gamma$, and there exist positive constants $c$ and $\lambda$ such that

$$\|e^{\bar{A}(u)s}\| \leq ce^{-\lambda s}, \quad \forall\, s \geq 0,\ u \in \Gamma.$$

Let $V(x, u) = x^T P(u)x$ be the candidate Lyapunov function, where $P(u)$ is the solution to the Lyapunov equation

$$P(u)\bar{A}(u) + \bar{A}^T(u)P(u) = -I.$$

Then, $V(x, u)$ satisfies (2)–(4) in Theorem 1 for all $x \in \mathbb{R}$ and $u \in \Gamma$, with

$$c_1 = \frac{1}{2L}, \quad c_2 = \frac{c^2}{2\lambda}, \quad c_3 = 1, \quad c_4 = \frac{c^4}{2\lambda^2}\sqrt{\sum_{i=1}^{m} b_i^2}.$$

Combining Lemma 3 and Theorem 1, we have the following corollary.

*Corollary 2:* Consider a linear time-varying system

$$\dot{x} = A(t)x, \tag{18}$$

where $x \in \mathbb{R}^n$ and $A(t) \in \mathcal{A} \subset \mathbb{R}^{n\times n}$. The system is globally exponentially stable if the following conditions are satisfied.

(i) $\mathcal{A}$ is a compact and convex set.

(ii) $A'$ is Hurwitz for all $A' \in \mathcal{A}$, and there exist positive constants $c$ and $\lambda$ such that

$$\|e^{A's}\| \leq ce^{-\lambda s} \quad \forall\, A' \in \mathcal{A},\ s \geq 0.$$

(iii) For any time interval $[t_1, t_2]$, $u_A(\cdot)$ satisfies Condition 1, and its total variation satisfies

$$\int_{t_1}^{t_2} \|du_A\| \leq \mu(t_2 - t_1) + \alpha,$$

where $u_A(\cdot)$ is defined in (16),

$$\alpha > 0, \quad \mu < \frac{\beta_1}{2n\beta_2^3}, \quad \beta_1 = \frac{1}{2L}, \quad \beta_2 = \frac{c^2}{2\lambda}, \tag{19}$$

and $L := \max_{A'\in\mathcal{A}} \|A'\|$.

*Proof:* The system equation (18) of the linear time-varying system can be written as

$$\dot{x} = A(t)x = U^{-1}\Big(U\big(A(t)\big)\Big)x = U^{-1}(u_A(t))x,$$

which matches (17) by letting $\bar{A}(\cdot) = U^{-1}(\cdot)$, $u(t) = u_A(t)$, and $\Gamma = U(\mathcal{A}) \subset \mathbb{R}^{n^2}$. It can be readily seen that $\partial\bar{A}/\partial u_i,\ i = 1, \ldots, n^2$, is a sparse matrix with one element being one and all other elements being zero. Hence, we have

$$\Big\|\frac{\partial \bar{A}(u)}{\partial u_i}\Big\| \leq \Big\|\frac{\partial \bar{A}(u)}{\partial u_i}\Big\|_F = 1 \quad \forall\, u \in \mathbb{R}^{n^2},\ i = 1, \ldots, n^2,$$

where the inequality holds due to (13). Then, condition (i) in Lemma 3 is satisfied. By the compactness of $\mathcal{A}$, it can be checked that condition (ii) in Lemma 3 is satisfied. Furthermore, condition (iii) in Lemma 3 is satisfied due to condition (ii) in Corollary 2. Therefore, we can apply Lemma 3 and obtain a candidate Lyapunov function satisfying (2)–(4), with constants

$$c_1 = \frac{1}{2L}, \quad c_2 = \frac{c^2}{2\lambda}, \quad c_3 = 1, \quad c_4 = \frac{c^4}{2\lambda^2}\sqrt{\sum_{i=1}^{n^2} 1} = \frac{nc^4}{2\lambda^2}. \tag{20}$$

To apply Theorem 1, we need to further justify that the conditions (i) and (iii) in its statement hold. Since $\Gamma = U(\mathcal{A})$, the compactness and convexity of $\mathcal{A}$ implies[6] those of $\Gamma$, and thus condition (i) in Theorem 1 is satisfied. Furthermore, it can be computed that

$$\frac{\beta_1}{2n\beta_2^3} = \frac{2\lambda^3}{nLc^6} = \frac{1}{2L}\cdot\frac{4\lambda^3}{nc^6} = \frac{c_1c_3}{c_2c_4},$$

where $\beta_1$ and $\beta_2$ are as in (19) and $c_1$–$c_4$ are characterized by (20). Hence, condition (iii) in Corollary 2 is equivalent to condition (iii) in Theorem 1.

Combining the arguments above, we conclude that the stability conditions in Theorem 1 are satisfied, and thus the linear time-varying system is globally exponentially stable. ∎

The result in [12, Theorem 3] states that the linear time-varying system is globally exponentially stable if the following conditions are satisfied.

(i) $\mathcal{A}$ is a compact set.

(ii) $A'$ is Hurwitz for all $A' \in \mathcal{A}$, and there exist positive constants $c$ and $\lambda$ such that

$$\|e^{A's}\| \le ce^{-\lambda s} \quad \forall\, A' \in \mathcal{A},\ s \ge 0.$$

(iii) For any time interval $[t_1, t_2]$, the total variation of $A(\cdot)$ satisfies

$$\int_{t_1}^{t_2} \|dA\| \le \mu(t_2 - t_1) + \alpha,$$

where

$$\alpha > 0, \quad \mu < \frac{\beta_1}{2\beta_2^3}, \quad \beta_1 = \frac{1}{2L}, \quad \beta_2 = \frac{c^2}{2\lambda}.$$

Comparing this result with Corollary 2, we observe that the convexity assumption on $\mathcal{A}$ was not made in [12, Theorem 3]. Furthermore,

$$\int_{t_1}^{t_2} \|du_A\| = \int_{t_1}^{t_2} \|dA\|_F \ge \int_{t_1}^{t_2} \|dA\|,$$

where the inequality holds due to (14), and

$$\frac{\beta_1}{2n\beta_2^3} \le \frac{\beta_1}{2\beta_2^3}.$$

Hence, with a minor additional assumption on the convexity of $\mathcal{A}$, Corollary 2 recovers [11, Theorem 3] qualitatively but not quantitatively.

[6]The convexity of $\Gamma$ is easy to check, yet the proof of compactness requires an application of the equivalence of norms, between the induced 2-norm and the Frobenius norm.

## VI. Conclusions and Future Work

In this paper, we have derived a set of stability conditions unifying the stability results for nonlinear time-varying systems and nonlinear switched systems. By applying the derived stability conditions to linear time-varying systems, we have also recovered qualitatively the unified stability criteria for slowly time-varying linear systems and switched linear systems, which were derived in an earlier work initiating this line of research.

Based on the results of this paper, there are two interesting directions for future study: first, it is assumed in this paper that the time-varying system is stable at each frozen time, and the switched system contains only stable subsystems. However, stability of slowly time-varying systems with stable and unstable system dynamics at different frozen times, and stability of switched systems with both stable and unstable subsystems, have been well studied in the literature. It is worth trying to bridge the two groups of results via the total variation approach. Second, in this work, it is assumed that the systems with different parameters admit the same equilibrium point. It is worth generalizing the stability results derived in this paper to the case where the time-varying system with time-varying parameters admits a time-varying equilibrium point on a manifold.

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