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Cyclic pursuit without coordinates: convergence to regular polygon formations

Maxim Arnold* Yuliy Baryshnikov*,† Daniel Liberzon†

Abstract— **We study a multi-agent cyclic pursuit model where each of the identical agents moves like a Dubins car and maintains a fixed heading angle with respect to the next agent. We establish that stationary shapes for this system are regular polygons. We derive a sufficient condition for local convergence to such regular polygon formations, which takes the form of an inequality connecting the angles of the regular polygon with the heading angle of the agents. A block-circulant structure of the system's linearization matrix in suitable coordinates facilitates and elucidates our analysis. Our results are complementary to the conditions for rendezvous obtained in earlier work [Yu et al., IEEE Trans. Autom. Contr., Feb. 2012].**

I. INTRODUCTION

This paper is a continuation of the work described in [1] where a particular multi-agent pursuit model was considered. Amidst the large and growing body of literature on distributed coordination and control (partially surveyed in [1]), a distinguishing feature of the model considered in [1] is the minimalistic nature of the sensing and control mechanisms. Specifically, each agent (which moves like a Dubins car) has a limited sensor only able to detect whether or not the target agent that it is supposed to pursue is contained within some sector of its windshield, and implements a simple quantized control law which ensures that this containment condition remains satisfied. Using the perimeter of the polygon formed by the agents as a Lyapunov function, it was shown in [1] that under suitable assumptions the agents converge to a single point (rendezvous) if the aforementioned windshield sector angle is sufficiently small, in the absence of any metric information about their relative locations.

Here we are interested in the complementary case where the windshield sector angle is too large and consequently the agents diverge. Our previous simulation studies[1] indicated that in this case, the agents tend to form regular polygons. The present paper is devoted to a theoretical justification of this empirically observed phenomenon. We focus on the case where identical agents are arranged in a cycle (cyclic pursuit) and each agent maintains the one that it pursues exactly on the boundary of its windshield; this situation—sometimes called the "constant bearing" case, because of the constant relative heading angle for all agents—is consistent with the control strategy presented in [1] and represents its sliding mode regime. We identify regular polygons as stationary shapes and, after applying a coordinate transformation and a time rescaling, develop a sufficient condition for their local attractivity. This is achieved through eigenvalue analysis of the system's linearization matrix, which has a convenient block-circulant structure in the new coordinates. The obtained condition takes the form of an inequality involving trigonometric functions of the angles of the regular polygon and the heading angle of the agents. Another, simpler but more conservative stability condition is also derived.

Convergence to regular polygon formations has been investigated before. Richardson [2] studied direct cyclic pursuit, in which each agent is modeled as an integrator and moves directly towards the next agent; he showed that regular polygons represent stable stationary shapes. A series of papers by Behroozi and Gagnon (see [3] and the references therein) examined a complementary situation where agents move away from one another. The work by Marshall et al. [4] is probably the closest to ours in terms of techniques, as they also use properties of block-circulant matrices to show convergence of cyclic pursuit formations of Dubins car agents to regular polygons. However, the control model in [4] assumes exact knowledge of the heading angle and sets the angular speed proportional to the heading error, while here we just maintain constant heading error through implementing the control law from [1] having access to only coarsely quantized angular measurements. As a consequence, in [4] regular polygons are true equilibria, whereas here a regular polygon shape is preserved but the size of the formation grows; this makes the details of the analysis quite different. Subsequent work [5] by the same authors extended their results from the case of fixed forward speed (also considered here) to the case where a controller gain determines the agents' speeds. Related work by Pavone and Frazzoli [6] assumes a common constant offset angle, as we do, but uses the distance between agents for control. Sinha and Ghose [7] characterized equilibrium formations for more heterogeneous agents with different controller gains, but did not study their stability. We also mention two very recent relevant works: Galloway et al. [8] gave a detailed geometric analysis of shape dynamics equilibria (but not their stability) for several pursuit strategies including a constant-bearing one, and Sharma et al. [9] used eigenvalue computations to show convergence to regular shapes for formations of agents modeled as double integrators. In both of these papers, unlike

*Department of Mathematics, University of Illinois at Urbana-Champaign, U.S.A. MA and YB supported by AFOSR MURI grant FA9550-10-1-0567.

†Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, U.S.A.. DL supported by NSF grants CNS-1217811 and ECCS-1231196.

[1]The simulation program that was developed to accompany [1] is fully accessible as a Java applet through Java-enabled web browsers at http://msl.cs.uiuc.edu/∼jyu18/pe/rendezvous.html

here, the control law assumes exact knowledge of inter-agent distance.

We proceed as follows. Relevant details of the system model are described in Section II. In Section III we show that stationary shapes are regular polygons. Section IV introduces a convenient coordinate transformation for the system. Section V derives stability conditions and illustrates them with some examples. Section VI concludes the paper.

II. System equations

We consider a collection of n agents a_j, $j = 1, \ldots, n$ in the plane, arranged cyclically so that a_{j+n} is identified with a_j. We denote by l_j the distance between the agents a_{j-1} and a_j; by ϕ_j the oriented angle from the horizontal axis (with positive direction) to the ray $\overrightarrow{a_j a_{j+1}}$; by α_j the oriented angle from the ray $\overrightarrow{a_j a_{j+1}}$ to the velocity vector of a_j; and by ψ_j the angle $a_{j-1}a_ja_{j+1}$ (see Figure 1). We note that the values of the angles ψ_j are jointly constrained; see Section V for more details.

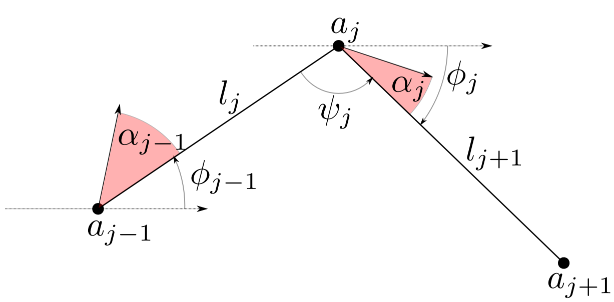

Fig. 1. Agents' positions and orientation angles

We assume that all agents move with the unit forward speed. It is then straightforward to show that the angles ϕ_j satisfy the differential equations

$$\dot{\phi}_j = -\frac{1}{l_{j+1}}(\sin\alpha_j + \sin(\psi_{j+1} + \alpha_{j+1}))$$

The angles ψ_j of the polygon describing the formation satisfy $\psi_j = \pi + \phi_j - \phi_{j-1}$ and therefore

$$\begin{aligned}\dot{\psi}_j = &-\frac{1}{l_{j+1}}(\sin\alpha_j + \sin(\psi_{j+1} + \alpha_{j+1})) \\ &+ \frac{1}{l_j}(\sin\alpha_{j-1} + \sin(\psi_j + \alpha_j))\end{aligned} \quad (1)$$

For the distances l_j one can similarly derive the equations

$$\dot{l}_j = -(\cos\alpha_{j-1} + \cos(\psi_j + \alpha_j)) \quad (2)$$

(cf. [4], [1]). Finally, the angles α_j satisfy

$$\dot{\alpha}_j = f(\alpha_j) - \dot{\phi}_j$$

where the function f describes an angular control law being implemented. In this paper, we assume that this control law forces all α_j to be constant and equal:

$$\alpha_j \equiv \alpha \qquad \forall j \quad (3)$$

In the framework of [1] this corresponds to the sliding mode regime where each agent a_j maintains the agent that it pursues, a_{j+1}, exactly on the boundary of its angular field of view (which is the same for all agents). This is indeed a reasonable abstraction of the more detailed behavior described in [1], and is consistent with the equations of motion considered there (we refer the reader to [1], particularly to Proposition 16 in that paper, for more details on this). We stress that this control law assumes no knowledge of distances between the agents and requires only very coarse information about their relative heading. A small additional assumption we are making here compared to [1] is that all angles have the same sign, i.e., the next agent is always on the same side of the previous agent's windshield.

III. Stationary shapes

The agents a_j, $j = 1, \ldots, n$ connected by the line segments $a_j a_{j+1}$ (with $a_{n+1} := a_1$) form a polygon in the plane. Following [2], by a *shape* we understand an equivalence class of polygons with respect to scaling and rigid motions (translations and rotations).[2] We call a shape *stationary* if it is invariant under the system dynamics. A necessary condition for stationary shape is that the angles are preserved:

$$\dot{\psi}_j \equiv 0 \qquad \forall j \quad (4)$$

Then the right-hand side of (2) is constant which gives

$$l_j(t) = l_j(0) - t\big(\cos\alpha + \cos(\psi_j + \alpha)\big) \quad (5)$$

and from (1) it follows that

$$\begin{aligned}&\frac{\sin(\psi_{j+1} + \alpha) + \sin\alpha}{l_{j+1}(0) - t(\cos(\psi_{j+1} + \alpha) + \cos\alpha)} \\ &\quad = \frac{\sin(\psi_j + \alpha) + \sin\alpha}{l_j(0) - t(\cos(\psi_j + \alpha) + \cos\alpha)} \qquad \forall t\end{aligned} \quad (6)$$

Consider first the scenario when $\sin(\psi_j + \alpha) + \sin\alpha = 0$ for all j. Then for each j we must have either $\psi_j = 2\pi - 2\alpha$ or $\psi_j = \pi$. In the latter case we see from (5) that the edge length l_j remains constant. For the shape to be stationary, all edge lengths must then be constant, implying that $\psi_j = \pi$ for all j which is not possible. This leaves only the possibility that $\psi_j = 2\pi - 2\alpha$ for all j. On the other hand, if we assume that $\sin(\psi_j + \alpha) + \sin\alpha \neq 0$ for all j, then we can rewrite (6) as

$$\begin{aligned}&\frac{l_{j+1}(0) - t(\cos(\psi_{j+1} + \alpha) + \cos\alpha)}{\sin(\psi_{j+1} + \alpha) + \sin\alpha} \\ &\quad = \frac{l_j(0) - t(\cos(\psi_j + \alpha) + \cos\alpha)}{\sin(\psi_j + \alpha) + \sin\alpha} \qquad \forall t\end{aligned}$$

and, differentiating with respect to t, arrive at

$$\frac{\cos(\psi_{j+1} + \alpha) + \cos\alpha}{\sin(\psi_{j+1} + \alpha) + \sin\alpha} = \frac{\cos(\psi_j + \alpha) + \cos\alpha}{\sin(\psi_j + \alpha) + \sin\alpha}$$

[2]Reflections are not relevant here because our formation cannot change its orientation.

Using the trigonometric identities $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$ and $\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$ we obtain

$$\cot\Big(\frac{\psi_{j+1}}{2}+\alpha\Big) = \cot\Big(\frac{\psi_j}{2}+\alpha\Big)$$

which implies $\psi_{j+1} = \psi_j$. Therefore, any stationary shape for the system (1)–(2) should have equal angles between the edges:

$$\psi_j \equiv \psi \qquad \forall j \tag{7}$$

The equation (2) then prescribes the same time derivative for all edge lengths, resulting in

$$l_j(t) = l_j(0) - t\big(\cos\alpha + \cos(\psi+\alpha)\big) \tag{8}$$

It is easy to check that for sufficiently small α we have $\cos\alpha + \cos(\psi+\alpha) > 0$ and so the edge lengths $l_j(t)$ are decreasing. For a convex regular n-gon with angles $\psi = \pi(1-2/n)$ this is true if $\alpha < \pi/n$. We know from [1] that when $\alpha < \pi/n$, the perimeter of the polygon is a Lyapunov function for the system, hence the positions of all agents converge to a single point. (In fact, in [1] this property is shown assuming neither (3) nor (7) but just $\alpha_j < \pi/n$ for all j.) In this paper we are interested in the opposite situation where

$$\cos\alpha + \cos(\psi+\alpha) < 0 \tag{9}$$

which we henceforth assume to hold. In this scenario the edge lengths $l_j(t)$ are increasing, with the pairwise ratios between them all tending to 1. (We ignore the easy case when (9) turns to an equality and hence the edges remain constant.) It is now clear that for a shape to be stationary, the edge lengths must be initially equal so that they remain equal for all time, i.e., we must have $l_1(0) = \cdots = l_n(0) =: l_0$ and

$$l_j(t) = l_0 - t\big(\cos\alpha + \cos(\psi+\alpha)\big) \qquad \forall j \tag{10}$$

This means of course that the only possible stationary shapes are regular polygons. It is easy to see from (1) and (2) that regular polygons are indeed stationary shapes. We summarize this conclusion in a lemma.

Lemma 1 *For the system* (1)–(2) *under the assumptions* (3) *and* (9)*, a shape is stationary if and only if it consists of polygons whose angles ψ_j are all equal to some angle ψ and whose edge lengths l_j are all equal, i.e., regular polygons.*

Remark 1 Let us notice that in addition to the most intuitive convex regular n-gon there are many others. The simplest example comes from a pentagon and a pentagram (see Figure 2, left). Since the condition of Lemma 1 states only the equality of all angles, it does not prohibit other star-shaped regular n-gons. Interestingly, computer simulations show that at least some of them are indeed observed. A common scenario leading to such a stationary shape is the following: a configuration forms several nested loops tangent to each other (see Figure 2, right).

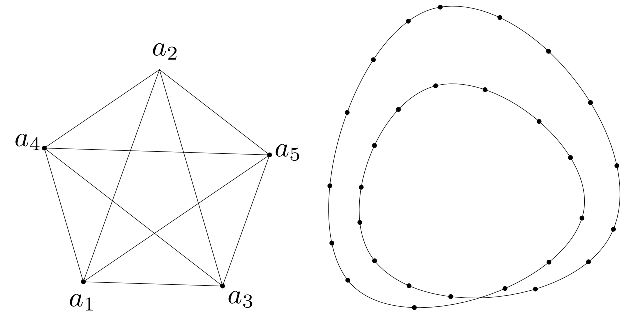

Fig. 2. Illustrating different shape possibilities

We emphasize that regular polygons are not true equilibria of our system. Rather, they correspond to shapes that are preserved by the system trajectories (7), (10). It is convenient to convert these trajectories into equilibria with the help of a coordinate transformation introduced next. (An analogous transformation was considered in [8].)

IV. Time-space coordinate transformation

Let

$$P(t) := \sum_{j=1}^{n} l_j(t)$$

denote the time-varying perimeter of our polygon, and consider the rescaled time

$$\tau := \int_0^t \frac{ds}{P(s)}$$

so that $d\tau/dt = P^{-1}(t)$ and hence

$$d_\tau := d/d\tau = P(t)d/dt$$

In this rescaled time the equation (1) takes the form

$$\begin{aligned} d_\tau\psi_j = &\frac{1}{\rho_j}(\sin\alpha + \sin(\psi_j+\alpha)) \\ &- \frac{1}{\rho_{j+1}}(\sin\alpha + \sin(\psi_{j+1}+\alpha)) \end{aligned} \tag{11}$$

where

$$\rho_j(t) := \frac{l_j(t)}{P(t)}$$

For ρ_j we have from (2)

$$\begin{aligned} d_\tau\rho_j = P\dot{\rho}_j = \dot{l}_j - \rho_j\sum_{j'=1}^{n}\dot{l}_{j'} = -(\cos\alpha + \cos(\psi_j+\alpha)) \\ + \rho_j\sum_{j'=1}^{n}(\cos\alpha + \cos(\psi_{j'}+\alpha)) \end{aligned} \tag{12}$$

The stationary shapes

$$\psi_1 = \cdots = \psi_n \equiv \psi, \quad l_1 = \cdots = l_n \tag{13}$$

identified in Lemma 1 correspond to the stationary points (equilibria)

$$\psi_1 = \cdots = \psi_n \equiv \psi, \quad \rho_1 = \cdots = \rho_n = 1/n \tag{14}$$

of the transformed system (11)–(12) in the coordinates (ψ_j, ρ_j) and with the rescaled time τ. This observation enables us to reduce the study of convergence to the stationary shapes for the original system trajectories to the analysis of asymptotic stability of the equilibria (14) for the transformed system. This analysis will be completed in the next section by showing that all eigenvalues of the linearization matrix of (11)–(12) around the equilibria (14) have negative real parts, which will imply that these equilibria are locally exponentially stable. The desired convergence property for the original system will then follow from the next lemma.

Lemma 2 *If* (14) *is a locally exponentially stable equilibrium of the system* (11)–(12)*, then the stationary shapes* (13) *of the system* (1)–(2) *are locally attractive, i.e., all trajectories with initial conditions sufficiently close to the set of points satisfying* (13) *asymptotically converge to this set.*

PROOF. In view of (8) and (9), the perimeter grows linearly with time; a crude upper bound is $P(t) \le P(0) + 2nt$. It is well known that if (14) is a locally exponentially stable equilibrium of the system (11)–(12) then there exists a Lyapunov function[3] $V = V(\psi_{1\cdots n}, \rho_{1\cdots n})$ which is 0 at the equilibrium (14) and, in some neighborhood of this equilibrium, is positive and exponentially decaying along solutions, in the sense that the function $v(\tau) := V(\psi_{1\cdots n}(\tau), \rho_{1\cdots n}(\tau))$ satisfies $d_\tau v \le -\lambda v$ for some $\lambda > 0$. In terms of the original time t and the function $\bar{v}(t) := v(\tau(t)) = V(\psi_{1\cdots n}(\tau(t)), \frac{1}{P(t)} l_{1\cdots n}(\tau(t)))$ and for initial conditions of (1)–(2) sufficiently close to (13), this translates to

$$\dot{\bar{v}} = \frac{1}{P(t)} d_\tau v \le -\frac{\lambda}{P(0) + 2nt} \bar{v}$$

hence

$$\bar{v}(t) \le e^{-\int_0^t \frac{\lambda}{P(0)+2ns} ds} \bar{v}(0)$$

Since the integral diverges, we have $\bar{v}(t) \to 0$ as $t \to \infty$. This means that $\psi_j(t) \to \psi$ and $l_j(t) \to P(t)/n$ for all j, which gives the claim. □

V. STABILITY ANALYSIS

Now we will focus on deriving conditions under which all equilibria of the form (14) for the system (11)–(12) are stable in the first approximation. Ordering the system coordinates as $(\psi_1, \rho_1, \dots, \psi_n, \rho_n)$, we easily see that the corresponding $2n \times 2n$ Jacobian matrix has the block-circulant form (see, e.g., [10])

$$J = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & A_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_{n-1} & A_0 \end{pmatrix} \qquad (15)$$

[3] We use the shorthand notation $x_{1\cdots n} := (x_1, \dots, x_n)$.

where A_m, $m = 1, \dots, n-1$ is the 2×2 matrix of partial derivatives $\dfrac{\partial(d_\tau \psi_j, d_\tau \rho_j)}{\partial(\psi_{j+m}, \rho_{j+m})}$ evaluated at (14). A direct calculation gives

$$A_0 = \begin{pmatrix} nC & -n^2(\sin\alpha + S) \\ (1 - \frac{1}{n})S & n(\cos\alpha + C) \end{pmatrix},$$
$$A_1 = \begin{pmatrix} -Cn & n^2(\sin\alpha + S) \\ -\frac{1}{n}S & 0 \end{pmatrix},$$
$$A_m = \begin{pmatrix} 0 & 0 \\ -\frac{1}{n}S & 0 \end{pmatrix}, \ m = 2, \dots, n-1$$

where we set

$$C := \cos(\psi + \alpha), \quad S := \sin(\psi + \alpha) \qquad (16)$$

for convenience. It is known (see, e.g., [10], [4]) that the k-th pair of eigenvalues of J is the pair of eigenvalues of the matrix

$$\begin{aligned} A_k^* &:= A_0 + A_1\chi_k + A_2\chi_k^2 + \cdots + A_{n-1}\chi_k^{n-1} \\ &= \begin{pmatrix} nC(1-\chi_k) & -n^2(\sin\alpha + S)(1-\chi_k) \\ S - \frac{1}{n}S\sum_{m=0}^{n-1}\chi_k^m & n(\cos\alpha + C) \end{pmatrix} \end{aligned}$$

where

$$\chi_k := e^{2\pi ik/n}, \qquad k = 1, \dots, n$$

are the n-th roots of unity.

Note that the complex number $z := \sum_{m=0}^{n-1} \chi_k^m$ satisfies $e^{2\pi ik/n} z = \chi_k z = z$, which means that either $k = n$ or $z = 0$. We thus distinguish two cases.

CASE 1: $k = n$.

Since $\chi_n = 1$, we have

$$A_n^* = \begin{pmatrix} 0 & 0 \\ 0 & n(\cos\alpha + C) \end{pmatrix}$$

The eigenvalues of this matrix are 0 and $n(\cos\alpha + C)$. The second eigenvalue is negative by virtue of (9) and (16). As for the zero eigenvalue, the eigenvector of the matrix (15) with this eigenvalue is $(-n(\cos\alpha + C), 0, -n(\cos\alpha + C), 0, \dots, -n(\cos\alpha + C), 0)$. Note, however, that we cannot simultaneously increase all angles of a polygon. In fact, the variables (ψ_j, ρ_j) belong not to the full space $\mathbb{R}^{2n}$ but to the disjoint union of "strata" defined by the constraints $\sum_{j=1}^n \psi_j = \pi(n-2d)$, $d = 1, \dots, n-1$. (Each of these strata contains regular polygons with angles $\psi = \pi(1-2d/n)$; see, e.g., [4].) The above eigenvector corresponding to the zero eigenvalue of J is orthogonal to these strata, hence it is not an admissible direction and so the zero eigenvalue does not affect local exponential stability of the stationary points (14).

CASE 2: $k < n$.

Since $1 + \chi_k + \cdots + \chi_k^{n-1} = 0$ in this case, we have

$$A_k^* = \begin{pmatrix} nC(1-\chi_k) & -n^2(\sin\alpha + S)(1-\chi_k) \\ S & n(\cos\alpha + C) \end{pmatrix}$$

We compute that $\mathrm{tr} A_k^* = n((2-\chi_k)\cos(\psi+\alpha) + \cos\alpha)$ and $\det A_k^* = n^2(1-\chi_k)(1+\cos\psi)$. Thus, rescaling the eigenvalues by $1/n$, we obtain the characteristic equation with complex coefficients

$$\lambda^2 + a\lambda + b = 0 \tag{17}$$

where

$$a = (\chi_k - 2)\cos(\psi + \alpha) - \cos\alpha, \quad b = (1 - \chi_k)(1 + \cos\psi)$$

According to the Routh-Hurwitz criterion for polynomials with complex coefficients from [11], both roots of the quadratic equation (17) will have negative real parts if and only if $\mathrm{Re}(a) > 0$ and

$$(\mathrm{Re}(a))^2\mathrm{Re}(b) + \mathrm{Re}(a)\mathrm{Im}(a)\mathrm{Im}(b) - (\mathrm{Im}(b))^2 > 0 \tag{18}$$

The first of these inequalities holds because $|\cos(\psi + \alpha)| > \cos\alpha$ by (9) and $\mathrm{Re}(2 - \chi_k) \geq 1$. It remains to verify (18). To simplify the notation, we let

$$A := \cos\alpha, \quad B := \cos\psi, \quad x + iy := \chi_k \tag{19}$$

Then we have $\mathrm{Re}(a) = (x-2)C - A$, $\mathrm{Im}(a) = yC$, $\mathrm{Re}(b) = (1-x)(1+B)$, $\mathrm{Im}(b) = -y(1+B)$, and (18) becomes

$$\begin{aligned}&\big((x-2)^2C^2 - 2AC(x-2) + A^2\big)(1-x)(1+B)\\ &\quad - \big((x-2)C - A)y^2C(1+B) - y^2(1+B)^2 > 0\end{aligned}$$

Dividing by $1 + B > 0$ (since $\psi < \pi$) and using the fact that $x^2 + y^2 = 1$, we obtain

$$\begin{aligned}&(1-x)(x^2C^2 - 4xC^2 + 4C^2 - 2ACx + 4AC + A^2)\\ &\quad - (1-x^2)(1 + B + xC^2 - 2C^2 - AC) > 0\end{aligned}$$

Dividing by $1 - x > 0$ (since $1 \leq k < n$) and collecting terms, we arrive at the inequality

$$(3C^2 + AC + 1 + B)x - (6C^2 + A^2 + 5AC - 1 - B) < 0 \tag{20}$$

which must hold for $x \in [-1, \cos(2\pi/n)]$. The left-hand side of (20) is a linear function of x, and therefore (20) holds for all x in the interval $[-1, \cos(2\pi/n)]$ if and only if it holds at the endpoints. For $x = -1$ the left-hand side of (20) equals $-(3C + A)^2 \leq 0$, and when $3C + A = 0$ we have $3C^2 + AC + 1 + B > 0$ (because $\psi \neq \pi$) which means that the left-hand side of (20) is increasing in x and it is enough to check its negativity at $x = \cos(2\pi/n)$. Plugging $x = \cos(2\pi/n)$ into (20) gives the inequality

$$(3C^2 + AC + 1 + B)\cos(2\pi/n) < (6C^2 + A^2 + 5AC - 1 - B) \tag{21}$$

We arrive at the following result.

Proposition 1 *Under the assumptions* (3) *and* (9)*, the stationary points* (14) *are locally exponentially stable equilibria of the system* (11)–(12) *if and only if the inequality* (21) *is satisfied, where* $A := \cos\alpha$*,* $B := \cos\psi$*, and* $C := \cos(\psi + \alpha)$.

In view of Lemma 2, (21) gives us a sufficient condition for local attractivity of the stationary shapes identified in Lemma 1.

To illustrate the above condition, consider again the case of a convex regular n-gon ($n \geq 3$) with angles $\psi = \pi(1 - 2/n)$. We know from [1] that the case when $\alpha < \pi/n$ corresponds to asymptotic convergence to a single point (rendezvous). Interestingly, for $\alpha = \pi/n$ the inequality (21) turns into equality (this can be verified by a straightforward calculation), meaning that the system is critically stable. Moreover, if we consider the left-hand side of (21) as a function of α, then it is not hard to check that its derivative, when evaluated at $\psi = \pi(1 - 2/n)$ and $\alpha = \pi/n$, is negative. This implies that as α is increased beyond π/n, the condition of Proposition 1 is satisfied, at least as long as α does not become too large. In other words, there is no "gap" between the range of values of α where local convergence to stationary shape is guaranteed by Proposition 1 and the range of values of α for which we have rendezvous thanks to the result of [1] (of course, for the former values the polygon is expanding while for the latter values it is contracting).

This situation can be visualized with the help of Figure 3, in which we took $n = 4$ so stationary shapes are squares.[4] For this case, we can see that squares are attractive for $\alpha \in (\pi/4, \pi)$. This is clear from the figure, but can also be confirmed by a direct calculation.

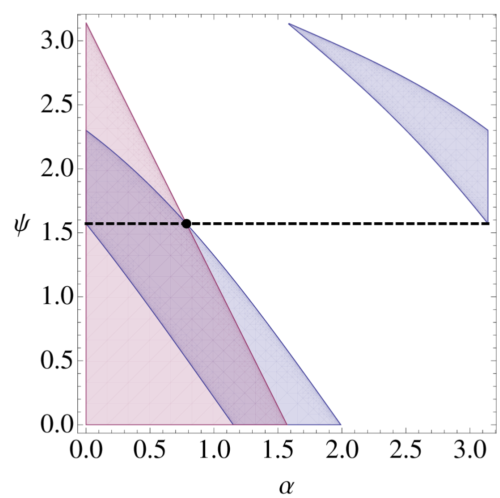

Fig. 3. The case $n = 4$. The region where (9) is violated is shaded in pink; the region where (21) is violated is shaded in blue; the dashed horizontal line corresponds to $\psi = \pi/2$; the marked point on the line corresponds to $\alpha = \pi/4$.

Figure 4 illustrates the case $n = 5$. Besides confirming once again that there is no gap between rendezvous and convergence to stationary shapes, we note that for some values of α both the pentagon ($\psi = 3\pi/5$) and the pentagram ($\psi = \pi/5$) are locally attractive. This means that there must be a trajectory serving as a separatrix between these two equilibria, which is an interesting phenomenon for further study.

If we plug the value $x = 1$ instead of $x = \cos(2\pi/n)$ into (20), we obtain

$$3C^2 + A^2 + 4AC - 2 - 2B > 0 \tag{22}$$

[4] This and the subsequent figures were produced using *Mathematica* [12].

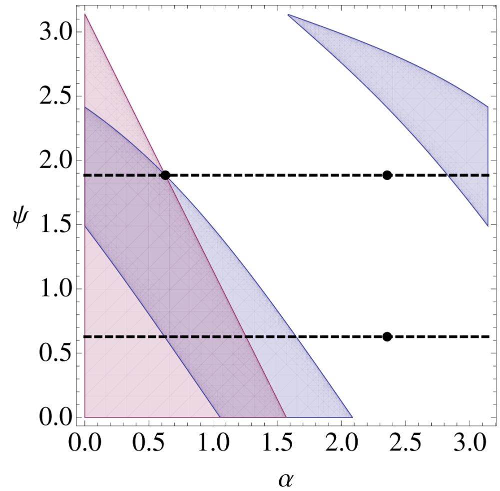

Fig. 4. The case $n = 5$. The meanings of the pink and blue shaded regions are the same as in the previous figure; the dashed horizontal lines correspond to $\psi = \pi/5$ and $\psi = 3\pi/5$; the three marked points on these lines are $(\alpha, \psi) = (\pi/5, 3\pi/5)$, $(3\pi/4, 3\pi/5)$, and $(3\pi/4, \pi/5)$.

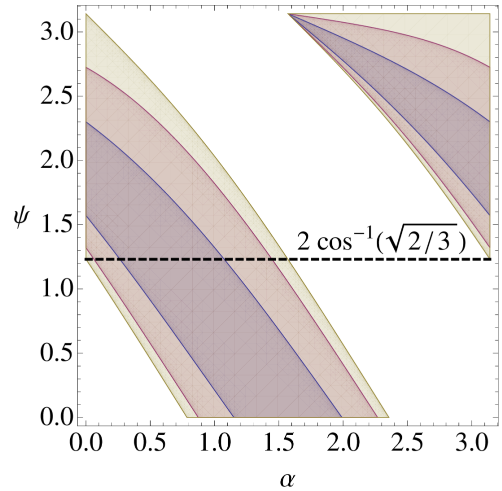

Fig. 5. The regions where (21) is violated for $n = 4$ (shaded blue), where (21) is violated for $n = 10$ (shaded pink), and where (24) is violated (shaded yellow).

This gives us a sufficient condition for local exponential stability which is more conservative than (21) but is simpler and does not depend on n. It is therefore particularly convenient for studying the asymptotics as n gets large. We can further simplify this condition with the help of the following trigonometric identity, which is straightforward to verify:

$$\cos\left(\frac{\gamma}{2}\right)\cos\left(2\beta+\frac{\gamma}{2}\right) = \cos^2\beta - \frac{1}{2} - \frac{1}{2}\cos\gamma + \cos\beta\cos(\beta+\gamma) \tag{23}$$

Splitting the left-hand side of (22) as $A^2 - \frac{1}{2} - \frac{1}{2}B + AC + 3(C^2 - \frac{1}{2} - \frac{1}{2}B + CA)$ and applying (23) first with $\beta = \alpha$ and $\gamma = \psi$, and then with $\beta = \alpha + \psi$ and $\gamma = -\psi$, we see that (22) is equivalent to

$$\cos\left(\frac{\psi}{2}\right)\cos\left(2\alpha + \frac{\psi}{2}\right) + 3\cos\left(\frac{\psi}{2}\right)\cos\left(2\alpha + \frac{3\psi}{2}\right) > 0$$

Dividing by $\cos(\psi/2) > 0$, we are left with the condition

$$\cos\left(2\alpha + \frac{\psi}{2}\right) + 3\cos\left(2\alpha + \frac{3\psi}{2}\right) > 0 \tag{24}$$

which gives the following result.

Proposition 2 *Under the assumptions* (3) *and* (9)*, the stationary points* (14) *are locally exponentially stable equilibria of the system* (11)–(12) *if the inequality* (24) *is satisfied.*

We stress that this condition, although simpler than that of Proposition 1, is only a sufficient one. Figure 5 compares (24) with (21) for different values of n.

For example, for the case $\alpha = \pi$ (as in [3]), with the help of the formula $\cos(3\beta) = 4(\cos\beta)^3 - 3\cos\beta$ the condition (24) reduces to

$$\cos(\psi/2) > \sqrt{2/3}$$

which means that the angle ψ should be sufficiently small. (These angles are below the dashed horizontal line in Figure 5, and we see that indeed we have stability for α close enough to π.) This conclusion appears to be consistent with the findings of [3].

VI. Conclusions and Future Work

For the particular cyclic pursuit scenario considered in this paper, we identified regular polygons as the only stationary shapes and derived sufficient conditions for their local stability. It remains to further investigate the behavior of the system, especially far away from these equilibria. Extensions to non-constant heading angles and to more general graphs (not only cycles), as allowed in [1], are also of interest.

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