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Small-gain theorems of LaSalle type for hybrid systems

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***Abstract*— We study stability of hybrid systems described as feedback interconnections of smaller subsystems, within a Lyapunov-based ISS small-gain analysis framework. We focus on constructing a weak (nonstrictly decreasing) Lyapunov function for the overall hybrid system from weak ISS-Lyapunov functions for the subsystems in the interconnection. Asymptotic stability of the hybrid system is then concluded by applying results of LaSalle type. The utility of this approach is illustrated on feedback systems arising in event-triggered control and quantized control.**

I. INTRODUCTION

It was observed in [1] that hybrid systems can be naturally viewed as feedback interconnections of smaller subsystems, as exemplified by—but not limited to—the idea that a hybrid system is a feedback interconnection of its continuous and discrete dynamics. The utility of this observation is that it renders applicable to hybrid systems the small-gain analysis tools, which are well established in systems theory (see, e.g., [2, Chapter 10]) and have proved very helpful in analysis—as well as design—of several classes of systems (such as control systems with saturation [3] and networked control systems [4]). The specific instance of a small-gain theorem most relevant here applies to nonlinear state-space systems and relies on the notion of *input-to-state stability (ISS)* [5]. This ISS small-gain theorem states that a feedback interconnection of two ISS systems is asymptotically stable if a composition of their ISS gain functions is smaller than identity.

One important advantage of ISS small-gain theorems is that they enable explicit construction of Lyapunov functions for the overall system starting with ISS-Lyapunov functions for the individual subsystems. Such constructions were studied for continuous-time systems in [6] and for discrete-time systems in [7]. For hybrid systems, Lyapunov-based small-gain theorems of this kind were investigated in [8] and [9]. The present paper takes this line of research in a somewhat different direction; namely, we focus on constructing *weak* (i.e., nonstrictly decreasing) Lyapunov functions which enable stability analysis with the help of results of Barbashin-Krasovskii-LaSalle type. The individual ISS-Lyapunov functions that we start with are also “weak” in the sense that they are allowed to decrease nonstrictly along the continuous dynamics for one subsystem and the discrete dynamics for the other subsystem, respectively; thus the subsystems are not required to be ISS. A preliminary variant of such a construction for the case of a hybrid system decomposed into its continuous and discrete dynamics appeared in [8]; here we develop a much more general result and study its implications (see Section III).

Compared to stability theorems based on strictly decreasing Lyapunov functions, stability results of LaSalle type suffer from several limitations. In particular, they do not allow the presence of external disturbances and are not helpful for characterizing robustness of stability under perturbations of the system. Nevertheless, in the setting of hybrid systems we find that working with nonstrictly decreasing Lyapunov functions is a very natural approach that is capable of yielding useful results. This approach is natural because it is often convenient to look for Lyapunov functions that, for example, decrease along the continuous flow while remaining constant during discrete events, or vice versa. To support the claim that this approach is useful, we employ it to analyze two classes of control synthesis algorithms of current interest: event-triggered control and quantized feedback control (see Section IV).

II. PRELIMINARIES

A. Derivatives and comparison functions

For a continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ we will write the directional derivative of V along v as $V'(x;v) := \langle \nabla V(x), v\rangle$, where ∇V denotes the gradient of V. Our construction of weak Lyapunov functions for interconnected systems involves taking a maximum of two C^1 functions, which results in a function that is typically not C^1 but just locally Lipschitz. We extend the above directional derivative notation to Lipschitz functions by interpreting it as the *Clarke derivative*:

$$V'(x;v) = V^\circ(x;v) := \limsup_{h\to 0^+,\, y\to x} (V(y+hv)-V(y))/h.$$

For C^1 functions, the two derivative concepts coincide. The following is a consequence of [10, Prop. 2.1.2 and 2.3.12].

Lemma 1 *Consider two functions $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}$ with well-defined Clarke derivatives for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Introduce three sets $A := \{x : V_1(x) > V_2(x)\}$, $B := \{x : V_1(x) < V_2(x)\}$, $\Gamma := \{x : V_1(x) = V_2(x)\}$. Then, for any $v \in \mathbb{R}^n$, the function $U(x) := \max\{V_1(x), V_2(x)\}$ satisfies $U'(x;v) = V_1'(x;v)\ \forall x \in A$, $U'(x;v) = V_2'(x;v)\ \forall x \in B$, and $U'(x;v) \le \max\{V_1'(x;v), V_2'(x;v)\}\ \forall x \in \Gamma$.*

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The following lemma was proved in [6].

Lemma 2 *Let*[1] $\chi_1, \chi_2 \in \mathcal{K}_\infty$ *satisfy* $\chi_1 \circ \chi_2(r) < r$ *for all* $r > 0$. *Then, there exists a* $\mathcal{K}_\infty$ *function* ρ *such that*
- $\chi_1(r) < \rho(r) < \chi_2^{-1}(r)$ *for all* $r > 0$*;*
- $\rho(r)$ *is* C^1 *on* $(0, \infty)$ *and* $\rho'(r) > 0$ *for all* $r \in (0, \infty)$.

B. Hybrid system model

Motivated by hybrid system models proposed in [11], we consider hybrid systems with inputs that are described by a combination of continuous flow and discrete jumps, of the following form (see also [12]):

$$\dot{x} = f(x, w), \qquad (x, w) \in C \tag{1}$$
$$x^+ = g(x, w), \qquad (x, w) \in D \tag{2}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, C, D are sets closed in $\mathbb{R}^n \times \mathbb{R}^m$ and such that $C \cup D \subseteq \mathbb{R}^n \times \mathbb{R}^m$, and f and g are continuous maps from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. The solutions of the hybrid system are defined on so-called hybrid time domains. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if $E = \cup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A *hybrid signal* is a function defined on a hybrid time domain. A *hybrid input* is a hybrid signal $w : \operatorname{dom} w \to \mathbb{R}^m$ such that $w(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A *hybrid arc* is a hybrid signal $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ and a hybrid input $w : \operatorname{dom} w \to \mathbb{R}^m$ are a solution pair to the hybrid model (1), (2) if: $\operatorname{dom} x = \operatorname{dom} w$; for all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ such that $(t, j) \in \operatorname{dom} x$ we have $(x(t, j), w(t, j)) \in C$ and $\dot{x}(t, j) = f(x(t, j), w(t, j))$; and for $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$ we have $(x(t, j), w(t, j)) \in D$ and $x(t, j+1) = g(x(t, j), w(t, j))$. Here, $x(t, j)$ represents the state of the hybrid system after t time units and j jumps. Under appropriate regularity conditions on C, D, f, g the hybrid system possesses solutions, which may be non-unique (see [11]). Similarly, one can define solutions for hybrid systems for the case of no inputs [13]. A solution is called *complete* if its domain is unbounded. The hybrid system model considered in [11], [13] actually replaces the differential equation in (1) and the difference equation in (2) by inclusions, with f and g becoming set-valued maps satisfying suitable regularity assumptions; it is straightforward to adapt the results that follow to this more general setting.

In this paper we consider the situation where the hybrid system takes the form (1), (2) without the inputs, i.e.,

$$\dot{x} = f(x), \qquad x \in C \tag{3}$$
$$x^+ = g(x), \qquad x \in D \tag{4}$$

[1]A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is zero at zero and strictly increasing. It is of *class* $\mathcal{K}_\infty$ if it is unbounded; note that $\mathcal{K}_\infty$ functions are globally invertible. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$.

and is then decomposed as follows:

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2), \qquad x \in C \tag{5}$$
$$x_1^+ = f_1(x_1, x_2), \quad x_2^+ = g_2(x_1, x_2), \qquad x \in D \tag{6}$$

where $x := (x_1, x_2)$ which is a shorthand notation we use for $(x_1^T, x_2^T)^T$, $x_i \in \mathbb{R}^{n_i}$, $f := (f_1, f_2)$, $g := (g_1, g_2)$, and $n := n_1 + n_2$. We regard this system as a feedback connection of two hybrid subsystems, one having state x_1 and input x_2 and the other having state x_2 and input x_1. Decomposing the hybrid system (1), (2) in this way is very natural and not restrictive; for example, we can always realize it as a feedback interconnection of its continuous and discrete dynamics.

C. (Pre-)asymptotic stability

We now define the desired asymptotic property of solutions that we want to achieve for the system (3), (4), which we will be able to establish as an eventual consequence of our weak Lyapunov function construction. This stability property is a variant of the asymptotic stability notion, which is standard in the nonlinear systems literature; for hybrid system models considered here, it is discussed in [13]. For simplicity, we limit ourselves here to global stability properties with respect to an equilibrium at the origin. The hybrid system (3), (4) is *globally pre-asymptotically stable (pre-GAS)* if all its solutions satisfy

$$|x(t, j)| \leq \beta(|x(0, 0)|, t + j) \tag{7}$$

for all $(t, j) \in \operatorname{dom} x$, where β is a function of class $\mathcal{KL}$. If a system is pre-GAS then all complete solutions converge to 0. Completeness is not part of the stability definition, and needs to be checked separately. As shown in [13, Theorem S3], for hybrid systems with local existence of solutions, establishing completeness of solutions amounts to ruling out the possibility of finite escape time (during flow) and of jumping out of $C \cup D$; the former can be done using well-known results on ODEs (and in fact (7) precludes finite escape time), while the latter is automatic when $C \cup D = \mathbb{R}^n$. If all solutions are complete, then the prefix "pre-" is dropped.

III. Construction of weak Lyapunov functions

In this section we present the main result of the paper, which specifies how to construct a weak Lyapunov function by using appropriate weak ISS-Lyapunov functions for subsystems in a feedback interconnection when an appropriate small gain condition is satisfied. Implications of this result for system trajectories are then discussed.

A. Main result

Consider the hybrid system (3), (4) decomposed as (5), (6).

Assumption III.1 *There exist* C^1 *functions* $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ *such that the following hold:*
1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ *such that for all* $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$ *we have* $\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|)$.

2) There exist functions $\chi_i \in \mathcal{K}_\infty$, $i = 1, 2$, a positive definite function $\alpha_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a function $R : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ such that for all $x \in C$ we have

$$V_1(x_1) \geq \chi_1(V_2(x_2)) \;\Rightarrow\; V_1'(x_1; f_1(x)) \leq -\alpha_1(V_1(x_1)), \quad (8)$$

$$V_2(x_2) \geq \chi_2(V_1(x_1)) \;\Rightarrow\; V_2'(x_2; f_2(x)) \leq -R(x_2). \quad (9)$$

3) There exist a positive definite function $\lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\lambda_2(s) < s \; \forall s > 0$ and a function $Y : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ such that for all $x \in D$ we have

$$V_1(g_1(x)) \leq \max\{V_1(x_1) - Y(x_1), \chi_1(V_2(x_2))\}, \quad (10)$$

$$V_2(g_2(x)) \leq \max\{\lambda_2(V_2(x_2)), \chi_2(V_1(x_1))\} \quad (11)$$

where χ_i are the same as in item 2.
4) The following small-gain condition holds:

$$\chi_1 \circ \chi_2(s) < s \qquad \forall s > 0. \quad (12)$$

The next result asserts the existence of a weak Lyapunov function nondecreasing along trajectories of the overall hybrid system, suitable for an application of a Barbashin-Krasovskii-LaSalle-type theorem as we show afterwards.

Theorem 1 Consider the hybrid system (5), (6). Suppose that Assumption III.1 holds. Let $\rho \in \mathcal{K}_\infty$ be generated via Lemma 2 using χ_1, χ_2. Let

$$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}. \quad (13)$$

Then, there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, a positive definite function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\sigma(s) < s \; \forall s > 0$, and a positive semi-definite function $S : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following hold:
1) For all $x \in \mathbb{R}^n$ we have $\psi_1(|x|) \leq V(x) \leq \psi_2(|x|)$.
2) For all $x \in C$ we have

$$V'(x; f(x)) \leq \max\{-\alpha_1(V(x)), -S(x_2)\}. \quad (14)$$

3) For all $x \in D$ we have

$$V(g(x)) \leq \max\{V(x) - Y(x_1), \sigma(V(x))\}. \quad (15)$$

Proof: Since ρ is generated using χ_1, χ_2 via Lemma 2, we have

$$\chi_1(r) < \rho(r) \text{ and } \chi_2(r) < \rho^{-1}(r) \qquad \forall r > 0. \quad (16)$$

Let $q(r) := \rho'(r)$. The proof of item 1 is straightforward and it is omitted. We now establish item 2. Let $S(\cdot) := q(V_2(\cdot)) \cdot R(\cdot)$ and $\sigma(\cdot) := \max\{\chi_1 \circ \rho^{-1}(\cdot), \rho \circ \lambda_2 \circ \rho^{-1}(\cdot), \rho \circ \chi_2(\cdot)\}$. By construction of ρ (see (16)) we easily see that $\sigma(s) < s$ for all $s > 0$. We introduce three subsets of $\mathbb{R}^n$ and investigate the behavior of V on each one intersected with C (and later with D). Define

$$\begin{aligned} A &:= \{(x_1, x_2) : V_1(x_1) < \rho(V_2(x_2))\}, \\ B &:= \{(x_1, x_2) : V_1(x_1) > \rho(V_2(x_2))\}, \\ \Gamma &:= \{(x_1, x_2) : V_1(x_1) = \rho(V_2(x_2))\}. \end{aligned}$$

Consider first $x \in A \cap C$. Here $V(x) = \rho(V_2(x_2))$ and so (9) applies by virtue of (16), hence $V'(x; f(x)) = q(V_2(x_2)) \cdot V_2'(x_2; f_2(x)) \leq -q(V_2(x_2))R(x_2) = -S(x_2)$. Next, consider $x \in B \cap C$ so that $V(x) = V_1(x_1)$. Now (8) applies and we have

$$\begin{aligned} V'(x; f(x)) &= V_1'(x_1; f_2(x)) \\ &\leq -\alpha_1(V_1(x_1)) = -\alpha_1(V(x)). \end{aligned} \quad (17)$$

Finally, consider $x \in \Gamma \cap C$. Using Lemma 1 and noting that Γ is contained in the closure of both A and B, we can use the same inequalities as above to obtain $V'(x; f(x)) \leq \max\{V_1'(x_1; f_1(x)), q(V_2(x_2)) \cdot V_2'(x_2; f_2(x))\} \leq \max\{-\alpha_1(V(x)), -q(V_2(x_2))R(x_2)\} = \max\{-\alpha_1(V(x)), -S(x_2)\}$. Therefore, (14) holds.

We now establish item 3. Using the definition of V in (13) and item 3 of Assumption III.1, we have for all $x \in D$:

$$\begin{aligned} V(g(x)) &= \max\{V_1(g_1(x)), \rho(V_2(g_2(x)))\} \\ &\leq \max\{V_1(x_1) - Y(x_1), \chi_1(V_2(x_2)), \\ &\quad \rho \circ \lambda_2(V_2(x_2)), \rho \circ \chi_2(V_1(x_1))\} \\ &\leq \max\big\{V(x) - Y(x_1), \chi_1 \circ \rho^{-1}(V(x)), \\ &\quad \rho \circ \lambda_2 \circ \rho^{-1}(V(x)), \rho \circ \chi_2(V(x))\big\} \\ &\leq \max\{V(x) - Y(x_1), \sigma(V(x))\}. \end{aligned}$$

Therefore, (15) is verified. □

Remark 1 We can draw stronger conclusions if either $R(\cdot)$ or $Y(\cdot)$ or both are positive definite functions rather than merely nonnegative. If both $R(\cdot)$ and $Y(\cdot)$ are positive definite, then V becomes a strong Lyapunov function (i.e., strictly decreasing away from 0). □

Remark 2 The following "natural decomposition" of the hybrid system (3), (4) is often of interest:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \quad \dot{x}_2 = 0, && x \in C \\ x_1^+ &= x_1, \quad x_2^+ = g_2(x_1, x_2), && x \in D \end{aligned} \quad (18)$$

where we can call x_1 and x_2 the continuous and discrete state variables, respectively. Note that x_1 does not change during the jumps while x_2 does not change during the flow. Suppose that there exist C^1 functions V_1 and V_2 and functions $\psi_{ij}, \chi_i, \alpha_1, \lambda_2$ such that (8), (11) and items 1 and 4 of Assumption III.1 hold (all functions are from the same classes as in Assumption III.1). Let V be defined via (13). Then, it is easy to see that (9) holds with $R(\cdot) \equiv 0$ and (10) holds with $Y(\cdot) \equiv 0$, hence Theorem 1 applies. □

B. Consequences for trajectories

We can translate the properties of the weak Lyapunov function V established in Theorem 1 into a stability property of the system trajectories by using Theorem 23 of [13], which is a version of the Barbashin-Krasovskii-LaSalle theorem for hybrid systems. That result and the conclusion of Theorem 1 imply that the system (5), (6) is pre-GAS if V does not stay constant and positive along any complete solution. Interestingly, by a further analysis but without assuming anything beyond the hypotheses of Theorem 1, we can show that V cannot stay constant and positive along any solution satisfying the following mild condition. Let us say that a hybrid time domain (or a hybrid arc defined on this domain) has *flow followed by a jump* if its projection onto the t-axis contains two intervals $[t_k, t_{k+1}]$ and $[t_{k+1}, t_{k+2}]$ with $t_k < t_{k+1}$. This name is justified by the fact that a flow occurs on $[t_k, t_{k+1}]$, followed by a jump at $t = t_{k+1}$.

Proposition 1 *Let the hypotheses of Theorem 1 hold, and let V be defined via (13). Then, V does not stay constant and positive along any solution of* (5), (6) *that has flow followed by a jump.*

Proof: We continue to use the notation and calculations of the proof of Theorem 1. First, note that V actually decreases strictly on $B \cap C$ (during flow) and on $A \cap D$ (during jumps), i.e., we have

$$V'(x; f(x)) < 0 \qquad \forall x \in B \cap C, \tag{19}$$

$$V(g(x)) < V(x) \qquad \forall x \in A \cap D. \tag{20}$$

The first of these properties is an immediate consequence of (17). To see why the second one is true, consider $x \in A \cap D$ so that $V(x) = \rho(V_2(x_2)) > V_1(x_1)$. We have $V(g(x)) = \max\{V_1(g_1(x)), \rho(V_2(g_2(x)))\}$. By (10) and (16), $V_1(g_1(x)) \le \max\{V_1(x_1) - Y(x_1), \chi_1(V_2(x_2))\} < \rho(V_2(x_2)) = V(x)$. On the other hand, by (11) and (16) again, $\rho(V_2(g_2(x))) \le \max\{\rho \circ \lambda_2(V_2(x_2)), \rho \circ \chi_2(V_1(x_1))\} \le \max\{\rho \circ \lambda_2 \circ \rho^{-1}(V(x)), \rho \circ \chi_2(V(x))\} < V(x)$. Hence, (20) is established.

Now, suppose that there is a trajectory with the indicated property along which V stays equal to a positive constant. Consider the interval $[t_k, t_{k+1}]$ that comes from the definition of "flow followed by a jump," so that $x(t,k) \in C$ for $t \in [t_k, t_{k+1})$ and the next jump will happen at $t = t_{k+1}$. We cannot have $x(t,k) \in B \cap C$ for any $t \in [t_k, t_{k+1})$, because then (19) applied with $x = x(t,k)$ would force $V(x(\cdot,k))$ to decrease during flow. Hence, $x(t,k) \in A \cup \Gamma\ \forall t \in [t_k, t_{k+1})$. We thus have $V(x(t,k)) = \rho(V_2(x(t,k))$ on this interval, and $V_2(x(\cdot,k))$ must remain constant during flow. As for $V_1(x(\cdot,k))$, by (8) and (16) it decreases during flow when $x(\cdot,k)$ is in A sufficiently close to Γ or in $\Gamma \backslash \{0\}$. However, $x(\cdot,k)$ cannot reach Γ while remaining in $A \cup \Gamma$ if V_1 decreases and V_2 stays constant. Thus $x(t_{k+1},k)$, which is the state right before the next jump, must be in A, hence in $A \cap D$. But then V will decrease during this jump by virtue of (20), and we reach a contradiction. $\square$.

We can now state the following as a direct corollary of [13, Theorem 23] and our Theorem 1 and Proposition 1.

Corollary 1 *Suppose that the hybrid system* (5), (6) *fulfills Assumption III.1, and let V be defined via (13). If V does not stay constant and positive along any complete solution of* (5), (6) *that does not have flow followed by a jump, then the system is pre-GAS.*

In other words, by Proposition 1 solutions that have flow followed by a jump do not require any further attention, and it is just the other complete solutions that we need to analyze separately. More precisely, all complete hybrid solutions can be classified into the following three types: (i) (eventually) continuous solutions, i.e., solutions which (possibly after jumping finitely many times) only flow; (ii) (eventually) discrete solutions, i.e., solutions which (possibly after flowing for some finite time) only jump; and (iii) solutions that continue to have both flow and jumps for arbitrarily large times. While it may be difficult to check whether a given solution is of type (iii), this is not necessary. Indeed, every solution of type (iii) has flow followed by a jump, and Proposition 1 rules out the possibility of our weak Lyapunov function V being constant and positive along such solutions. The possibility of V being constant and positive along solutions of types (i) and (ii) is something that needs to be analyzed separately. To this end, it is useful to observe that if V is constant along a solution that is eventually continuous, then there is a purely continuous solution along which V is constant, and the same is true for eventually discrete vs. purely discrete solutions. So, only purely continuous and purely discrete solutions need to be checked. In practice, these classes of solutions are not very rich and we expect to be able to quickly rule out either the existence of such solutions or the possibility of V staying constant and positive along them. We will see examples of such reasoning in Section IV, where it will go through thanks to additional structure relating the flow and jump sets to the gain functions. What we mean by this is that in the general setting considered so far, the flow and jump sets C and D are completely separate from the gain functions χ_1 and χ_2, while in the design examples treated in Sections IV-A and IV-B there is a close relation between them.

IV. Two applications

A. Emulation with event-triggered sampling

In this section we revisit results in [14]. Consider a continuous-time plant $\dot{x} = f(x,u)$ for which a state feedback controller $u = k(x)$ was designed to globally asymptotically stabilize the closed-loop system. Suppose that we want to implement the controller in a sampled-data fashion so that we take samples of $x(\cdot)$ at times $t_k, k \in N$ and let $u(t) = k(x(t_k)), t \in [t_k, t_{k+1})$. The sampling times t_k will be designed in an event-driven fashion. To this end, introduce an auxiliary variable $e(t) := x(t_k) - x(t)$ and assume that there exist C^1 functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ and $\psi_{ij}, \chi_1, \alpha_1 \in \mathcal{K}_\infty$, $i,j = 1,2$ such that for all x, e we have

$$\psi_{11}(|x|) \le V_1(x) \le \psi_{21}(|x|),\ \ \psi_{12}(|e|) \le V_2(e) \le \psi_{22}(|e|), \tag{21}$$

$$V_1(x) \ge \chi_1(V_2(e)) \ \Rightarrow \\ \langle \nabla V_1, f(x, k(x+e)) \rangle \le -\alpha_1(V_1(x)). \tag{22}$$

Let $\chi_2 \in \mathcal{K}_\infty$ be arbitrary and satisfy

$$\chi_1 \circ \chi_2(s) = \chi_2 \circ \chi_1(s) < s \qquad \forall s > 0. \tag{23}$$

Our triggering strategy is to update the control whenever $V_2(e) \geq \chi_2(V_1(x))$, which leads to the following closed-loop hybrid system:

$$\begin{aligned} \dot{x} = f(x, k(x+e)),\ \dot{e} = -f(x, k(x+e)),\ (x,e) \in C \\ x^+ = x, \quad e^+ = 0, \qquad (x,e) \in D \end{aligned} \tag{24}$$

where $C := \{(x,e) : V_2(e) \leq \chi_2(V_1(x))\}$ and $D := \{(x,e) : V_2(e) \geq \chi_2(V_1(x))\}$.

Proposition 2 *Suppose that there exist Lyapunov functions V_1, V_2 and functions $\psi_{ij}, \chi_i, \alpha_1 \in \mathcal{K}_\infty$, $i,j = 1,2$ such that (21)–(23) hold. Let V be defined via (13). Then, there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that the following holds for the system (24): $\psi_1(|(x,e)|) \leq V(x,e) \leq \psi_2(|(x,e)|)$,*

$$\begin{aligned} V'\big((x,e); (f(x,k(x+e)), -f(x,k(x+e)))\big) \\ \leq -\alpha_1(V(x,e)) \qquad \forall (x,e) \in C, \end{aligned}$$

and $V(x,0) \leq V(x,e)$ $\forall (x,e) \in D$.

Proof: Note that for all $(x,e) \in D$ we have that the following holds: $V_1(x^+) = V_1(x)$. In view of this and (22), the conditions (8) and (10) of Assumption III.1 hold (with $Y \equiv 0$). Consider an arbitrary $\bar{\chi}_2(s) > \chi_2(s), \forall s > 0$ and such that $\chi_1 \circ \bar{\chi}_2(s) < s$ $\forall s > 0$; such a $\bar{\chi}_2$ always exists since the inequality (23) is strict. Then, we have that for $(x,e) \in C$ the following is *vacuously true*:

$$V_2(e) \geq \bar{\chi}_2(V_1(x)) \Rightarrow \langle \nabla V_2(e), -f(x, k(x+e)) \rangle \leq -R(e)$$

where $R(\cdot)$ can be arbitrary. Moreover, for all e we have $V_2(e^+) = V_2(0) = 0$. Hence, the conditions (9) and (11) of Assumption III.1 hold (with arbitrary λ_2). By construction, the small gain condition (item 4 of Assumption III.1) holds, and the conclusion follows from Theorem 1. □

To apply Corollary 1, according to the remarks following that corollary we need to check complete solutions that are either purely continuous or purely discrete (ignoring of course the trivial solution at the origin). Here we know that after a jump we must flow, since jumps reset e to 0. Thus, the only solutions that we need to analyze are purely continuous ones. However, in view of the ISS condition (22), the definition of C, and the small-gain condition (23), such flow-only behavior is possible only when both x and e converge to 0. Hence, V cannot stay constant and positive along any such solution. Finally, all solutions are complete because the properties of V in Theorem 1 guarantee their boundedness and we have $C \cup D = \mathbb{R}^n$ by construction. We have arrived at the following result.

Corollary 2 *The origin is a globally asymptotically stable equilibrium of the closed-loop hybrid system* (24)*.*

B. Quantized feedback control

This example is in some sense more specialized than the previous one, because we will only work with linear dynamics. On the other hand, this additional structure will permit us to explicitly construct the Lyapunov functions V_1, V_2 (which will be quadratic) and derive expressions for the gain functions χ_1, χ_2 (which will be linear gains), instead of just assuming their existence as we did in the previous example.

Consider the linear time-invariant system $\dot{x} = Ax + Bu$ where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and A is a non-Hurwitz matrix. We assume that this system is stabilizable, so that there exist matrices $P = P^T > 0$ and K such that

$$(A + BK)^T P + P(A + BK) \leq -I. \tag{25}$$

We denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a symmetric matrix, respectively. By a *quantizer* we mean a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite or countable subset of $\mathbb{R}^n$, for which there exist positive numbers M (the *range* of q, which can be a finite number or ∞ depending on whether $\mathcal{Q}$ is finite or countable) and Δ (the *quantization error bound*) satisfying $|z| \leq M \ \Rightarrow\ |q(z) - z| \leq \Delta$. We assume that $q(x) = 0$ for x in some neighborhood of 0 (in order that the equilibrium at 0 be preserved under quantized control). It is well known that quantization errors in general destroy asymptotic stability, in the sense that the quantized feedback law $u = Kq(x)$ is no longer stabilizing. To overcome this problem, we will use quantized measurements of the form

$$q_\mu(x) := \mu q(x/\mu), \qquad \mu > 0. \tag{26}$$

The quantizer q_μ has range $M\mu$ and quantization error bound $\Delta\mu$. The "zoom" variable μ will be the discrete variable of the hybrid closed-loop system, initialized at some fixed value. The feedback law will be $u = Kq_\mu(x)$. We consider the following scheme for updating μ, which we refer to as the "quantization protocol":

$$\begin{aligned} \dot{\mu} = 0, \qquad (x,\mu) \in C \\ \mu^+ = \Omega\mu, \qquad (x,\mu) \in D \end{aligned}$$

where $C := \{(x,\mu) : |q_\mu(x)| \geq (\Theta + \Delta)\mu\}$, $D := \{(x,\mu) : |q_\mu(x)| \leq (\Theta + \Delta)\mu\}$, $\Omega \in (0,1)$, and $\Theta > \sqrt{\lambda_{\max}(P)}2\|PBK\|\Delta/\sqrt{\lambda_{\min}(P)}$. The overall closed-loop hybrid system then looks like

$$\begin{aligned} \dot{x} = Ax + BKq_\mu(x), \quad \dot{\mu} = 0, \qquad (x,\mu) \in C \\ x^+ = x, \qquad \mu^+ = \Omega\mu, \qquad (x,\mu) \in D \end{aligned} \tag{27}$$

which is the "natural decomposition" as in Remark 2.

The idea behind achieving asymptotic stability is to "zoom in", i.e., decrease μ to 0 in a suitable discrete fashion. To simplify the exposition, we will assume that the condition $|x| \leq M\mu$ always holds, i.e., x always remains within the range of q_μ. This is automatically true if M is infinite, and can be guaranteed by a proper initialization of μ if a bound on the initial state $x(0)$ is available. For finite M and completely unknown $x(0)$, this can be achieved by incorporating an initial "zooming-out" scheme and subsequently ensuring that the condition is never violated (see [15] for details). For a Lyapunov-based small-gain analysis of a quantization scheme that includes zoom-outs, see the recent work [16].

Lemma 3 *Consider the hybrid system* (27)*. Let* $V_1(x) := x^T Px$*, with* P *and* K *from* (25)*. Let* $V_2(\mu) := \mu^2$*. Pick two numbers* ε_1 *and* ε_2 *satisfying* $0 < \varepsilon_1 < \varepsilon_2$ *and*

$$\Theta \geq \sqrt{\lambda_{\max}(P)}2\|PBK\|\Delta(1+\varepsilon_2)/\sqrt{\lambda_{\min}(P)}. \quad (28)$$

Then:
1) For all $(x,\mu) \in C$ *we have*

$$V_1(x) \geq \chi_1 V_2(\mu) \;\Rightarrow\; \langle \nabla V_1(x), Ax + BKq_\mu(x)\rangle \leq -c_1 V_1(x), \quad (29)$$

$$V_2(\mu) \geq \chi_2 V_1(x) \;\Rightarrow\; \langle \nabla V_2(\mu), 0\rangle \leq -c_2 V_2 \quad (30)$$

with $c_1 := \varepsilon_1/((1+\varepsilon_1)\lambda_{\max}(P))$*,* $\chi_1 := 4\lambda_{\max}(P)\|PBK\|^2\Delta^2(1+\varepsilon_1)^2$*,* $\chi_2 := 1/(4\lambda_{\max}(P)\|PBK\|^2\Delta^2(1+\varepsilon_2)^2)$*, and any* $c_2 > 0$*.*
2) For all $(x,\mu) \in D$ *we have* $V_1(x^+) = V_1(x)$ *and* $V_2(\Omega\mu) = \Omega^2 V_2(\mu) < V_2(\mu)$*.*

This lemma, whose proof we omit, yields the following.

Proposition 3 *Assumption III.1 holds for the system* (27) *and, hence, the conclusion of Theorem 1 holds.*

To conclude asymptotic stability, we can apply Corollary 1. If $x(0) = 0$ then, since $\mu(0)$ is constrained to be positive, we will have a solution that only jumps and along which $\mu \to 0$, hence V does not stay constant. It is not difficult to see that every $x(0) \neq 0$ and every $\mu(0) > 0$ give a solution that has flow followed by a jump. Indeed, after finitely many jumps μ becomes small enough so that $(x,\mu) \in C$ and flow must occur, and then due to (29) x will eventually become small enough so that $(x,\mu) \in D$ and a jump must occur. In fact, [17], [15], [18] contain results along these lines (see in particular Lemma IV.3 in [18]). Finally, it is clear that all solutions are complete because the dynamics are linear and $C \cup D = \mathbb{R}^n$. We have shown

Corollary 3 *The origin is a globally*[2] *asymptotically stable equilibrium of the closed-loop hybrid system* (27)*.*

The above quantization protocol has a clear geometric interpretation. We zoom in if the quantized measurements show that $|x| \leq (\Theta + 2\Delta)\mu$, which is guaranteed to happen whenever $|x| \leq \Theta\mu$. The condition (28) means that for each μ, the ball of radius $\Theta\mu$ around the origin contains the level set of V_1 superscribed around the ball of radius $2\|PBK\|\Delta\mu$, outside of which V_1 is known to decay (thus ensuring that the zoom-in will be triggered). Similar constructions appeared in [17], [15] but they did not employ the small-gain argument and were arguably less transparent.

[2]Recall that we required the condition $|x| \leq M\mu$ to hold for all times. When M is finite, this actually restricts the admissible initial conditions $(x(0), \mu(0))$. However, as shown in [15], if M large enough compared to Δ then an initial "zooming-out" scheme can be used to guarantee that the above requirement is fulfilled from some time onwards.

V. Conclusions

For a hybrid system realized as a feedback interconnection of ISS subsystems satisfying a small-gain condition, we presented a construction of a weak Lyapunov function for the overall system starting from weak ISS-Lyapunov functions for the subsystems. We explained how asymptotic stability can be concluded with the help of this weak Lyapunov function and results of Barbashin-Krasovskii-LaSalle type for hybrid systems, and illustrated this approach in two application-motivated control design contexts. The results were presented in a somewhat specialized setting, in order to make them simple to state and easy to understand. Several generalizations, such as allowing differential/difference inclusions instead of equations and capturing stability with respect to general compact sets, will be reported elsewhere.

References

[1] D. Nešić and D. Liberzon, "A small-gain approach to stability analysis of hybrid systems," in *Proc. 44th IEEE Conf. on Decision and Control*, 2005, pp. 5409–5414.
[2] A. Isidori, *Nonlinear Control Systems II*. London: Springer, 1999.
[3] A. R. Teel, "A nonlinear small gain theorem for the analysis of control systems with saturation," *IEEE Trans. Automat. Control*, vol. 41, pp. 1256–1270, 1996.
[4] D. Nešić and A. R. Teel, "Input-output stability properties of networked control systems," *IEEE Trans. Automat. Control*, vol. 49, pp. 1650–1667, 2004.
[5] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, pp. 435–443, 1989.
[6] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, pp. 1211–1215, 1996.
[7] D. S. Laila and D. Nešić, "Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems," in *Proc. 41st IEEE Conf. on Decision and Control*, 2002, pp. 2292–2297.
[8] D. Liberzon and D. Nešić, "Stability analysis of hybrid systems via small-gain theorems," in *Proc. 9th Int. Workshop on Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, J. P. Hespanha and A. Tiwari, Eds. Berlin: Springer, 2006, vol. 3927, pp. 421–435.
[9] D. Nešić and A. R. Teel, "A Lyapunov-based small-gain theorem for hybrid ISS systems," in *Proc. 47th IEEE Conf. on Decision and Control*, 2008, pp. 3380–3385.
[10] F. H. Clarke, *Optimization and Nonsmooth Analysis*. New York: Interscience, 1983, reprinted as vol. 5 of Classics in Applied Mathematics, SIAM, 1990.
[11] R. Goebel and A. R. Teel, "Solution to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, pp. 573–587, 2006.
[12] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Systems Control Lett.*, vol. 58, pp. 47–53, 2009.
[13] R. Goebel, R. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, pp. 28–93, 2009.
[14] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Automat. Control*, vol. 52, pp. 1680–1685, 2007.
[15] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.
[16] A. R. Teel and D. Nešić, "Lyapunov functions for L_2 and input-to-state stability in a class of quantized control systems," in *Proc. 50th IEEE Conf. on Decision and Control*, 2011, pp. 4542–4547.
[17] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Control*, vol. 45, pp. 1279–1289, 2000.
[18] D. Liberzon and D. Nešić, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Trans. Automat. Control*, vol. 52, pp. 767–781, 2007.