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State-norm estimators for switched nonlinear systems under average dwell-time

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Abstract— **In this paper, we consider the concept of state-norm estimators for switched nonlinear systems under average dwell-time switching signals. State-norm estimators are closely related to the concept of input/output-to-state stability (IOSS). We show that if the average dwell-time is large enough, a non-switched state-norm estimator for a switched system exists in the case where each of its constituent subsystems is IOSS. Furthermore, we show that a switched state-norm estimator, consisting of two subsystems, exists for a switched system in the case where only some of its constituent subsystems are IOSS and others are not, provided that the average dwell-time is large enough and the activation time of the non-IOSS subsystems is not too large. In both cases, the stated sufficient conditions are also sufficient for the switched system to be IOSS. For the case where some subsystems are not IOSS, we also show that the switched state-norm estimator can be constructed in such a way that its switching times are independent of the switching times of the switched system it is designed for.**

I. Introduction

Switched systems consist of a family of dynamical subsystems, out of which one at a time is active as specified by a switching signal. In recent years, various properties of switched systems, especially stability issues, were extensively studied in literature (see e.g. [1] and the references therein). A well-known fact concerning switched systems is that in general, properties of the single subsystems are not necessarily inherited by the switched system. For example, a switched system consisting of linear exponentially stable subsystems might become unstable ([1]) if certain switching laws are applied. For proving stability properties in the setup of constrained switching, the concept of average dwell-time switching signals, introduced in [2], has proved very useful; this concept will also be used in this paper.

It is well known that state estimation is a very important issue in control theory. In many applications, the full system state is not available, but only certain outputs can be measured. Yet, for control purposes, often the full system state x is needed, a problem which is tackled through the design of observers which yield an estimate $\hat{x}$ of the system state x out of the observations of past inputs and outputs. However, for continuous-time nonlinear systems, and even more for switched nonlinear systems, the design of such observers is a challenging task far from being solved completely. On the other hand, for some control purposes, it may suffice to gain an estimate of the magnitude, i.e., the norm $\|x\|$, of the system state x (see [3], [4] and the references therein). The notion of such a state-norm estimator was introduced in [3] for continuous-time nonlinear systems; furthermore, the close relation of state-norm estimators to the input/output-to-state (IOSS) property was pointed out. Namely, in [3] and [4] it was shown that for continuous-time nonlinear systems, the existence of an appropriately defined state-norm estimator is equivalent to the system being IOSS (and also to the existence of an IOSS-Lyapunov function for the system). In [5], it was also shown how an estimate of the norm $\|x\|$ can be exploited in constructing an observer, which in turn can be used for output feedback design to globally stabilize the system [6]. The concept of state-norm estimators was extended to switched systems in [7] for the case of arbitrary switching and a common IOSS-Lyapunov function for the switched system.

In this paper, we derive sufficient conditions for the existence of state-norm estimators for switched systems in the setting of multiple IOSS-Lyapunov functions and constrained switching. We consider both the cases where all of the constituent subsystems are IOSS as well as where some of the subsystems are unstable. In both cases, we show how a state-norm estimator for the switched system can be constructed. If all subsystems are IOSS, we can obtain a non-switched state-norm estimator, whereas in the case where also some unstable subsystems are present, a switched state-norm estimator can be constructed, consisting of one stable and one unstable mode. It turns out that in the latter case, the switched state-norm estimator can be constructed in such a way that it exhibits the nice property that its switching times are independent of the switching times of the switched system it is designed for. As in the non-switched case, results on state-norm estimators in the described setup for switched systems are closely related to the IOSS concept. Of course, due to our setup, here we cannot establish an equivalence relationship between the existence of a state-norm estimator and the existence of a (common) IOSS-Lyapunov function as in [4] and [7]; but it turns out that under the same sufficient conditions as stated in [8] in order to ensure that a switched system is IOSS, also a state-norm estimator for this switched system exists.

The remainder of this paper is structured as follows. In Section II, we introduce notations and definitions needed later on. Sections III and IV contain the main results of the

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paper including constructive proofs illustrating how state-norm estimators can be obtained, both for the cases where all of the subsystems are IOSS (Section III) as well as where some of the subsystems lack this property (Section IV). In Section V, some conclusions are given.

II. Preliminaries

Consider a family of systems

$$\begin{aligned} \dot{x} &= f_p(x,u) \\ y &= h_p(x) \end{aligned} \qquad p \in \mathcal{P} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^l$ and $\mathcal{P}$ is an index set. For every $p \in \mathcal{P}$, $f_p(\cdot,\cdot)$ and $h_p(\cdot)$ are locally Lipschitz and $f_p(0,0) = h_p(0) = 0$. A *switched system*

$$\begin{aligned} \dot{x} &= f_\sigma(x,u) \\ y &= h_\sigma(x) \end{aligned} \tag{2}$$

is generated by the family of systems (1) and a switching signal $\sigma(\cdot)$, where $\sigma : [0,\infty) \to \mathcal{P}$ is a piecewise constant, right continuous function which specifies at each time t the index of the active system.

According to [2] we say that a switching signal has *average dwell-time* τ_a if there exist numbers $N_0, \tau_a > 0$ such that

$$\forall T \geq t \geq 0 : \qquad N_\sigma(T,t) \leq N_0 + \frac{T-t}{\tau_a}, \tag{3}$$

where $N_\sigma(T,t)$ is the number of switches occurring in the interval $(t,T]$.

Denote the switching times in the interval $(0,t]$ by $\tau_1, \tau_2, ..., \tau_{N_\sigma(t,0)}$ (by convention, $\tau_0 := 0$) and the index of the system that is active in the interval $[\tau_i, \tau_{i+1})$ by p_i.

The switched system (2) is *input/output-to-state stable (IOSS)* [3] if there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ [1] and $\beta \in \mathcal{KL}$ [2] such that for all $x_0 \in \mathbb{R}^n$ and each input $u(\cdot)$, the corresponding solution satisfies

$$|x(t)| \leq \beta(|x_0|,t) + \gamma_1(\|u\|_{[0,t]}) + \gamma_2(\|y\|_{[0,t]}) \tag{4}$$

for all $t \geq 0$, where $\|\cdot\|_J$ denotes the supremum norm on an interval J. If no outputs are considered and equation (4) holds for $\gamma_2 \equiv 0$, then the system is said to be input-to-state stable (ISS).

In the following, the notion of a state-norm estimator will formally be introduced, which will be done in consistency with [3].

Definition 1: Consider a system

$$\dot{z} = g(z,u,y) \tag{5}$$

whose inputs are the input u and the output y of the switched system (2). Denote by $z(t)$ the solution trajectory of (5) starting at z_0 at time $t = 0$. We say that (5) is a *state-norm estimator* for the switched system (2) if the following properties hold:

1) The system (5) is ISS with respect to (u,y).
2) There exist functions $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that, for arbitrary initial states x_0 for (2) and z_0 for (5) and each input $u(\cdot)$,

$$|x(t)| \leq \beta(|x_0| + |z_0|, t) + \gamma(|z(t)|) \tag{6}$$

for all $t \geq 0$. □

Definition 1 ensures that the norm of the switched system state at time t, $|x(t)|$, can be bounded above by the norm of the state-norm estimator at time t, $|z(t)|$, modulo a decaying term of the initial conditions of the switched system and the state-norm estimator. In this sense, (5) "estimates" the norm of the switched system (2), and thus it is called a state-norm estimator.

Remark 1: In [3], it was shown that for continuous-time systems, the existence of a state-norm estimator according to Definition 1 implies that the system is IOSS. This is still the case if we consider switched systems, as the proof works in the exact same way as for continuous-time, non-switched systems. Thus, in the following, we concentrate on establishing conditions under which a state-norm estimator exists, and on how such a state-norm estimator can be constructed.

III. State-norm estimators: all subsystems IOSS

In this section, the first main result will be stated and proven, i.e., how and under what conditions a state-norm estimator for a switched system can be constructed in the case where all of the constituent subsystems are IOSS.

Theorem 1: Consider the family of systems (1). Suppose there exist functions $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathcal{R}^n \to \mathcal{R}$ and constants $\lambda_s > 0$, $\mu \geq 1$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ we have

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \tag{7}$$

$$\frac{\partial V_p}{\partial x} f_p(x,u) \leq -\lambda_s V_p(x) + \gamma_1(|u|) + \gamma_2(|h_p(x)|) \tag{8}$$

$$V_p(x) \leq \mu V_q(x). \tag{9}$$

If σ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s}, \tag{10}$$

then there exists a (nonswitched) state-norm estimator for the switched system (2). A possible choice for such a state-norm estimator is

$$\dot{z}(t) = -\lambda_s^* z(t) + \gamma_1(|u(t)|) + \gamma_2(|y(t)|), \quad z_0 \geq 0 \tag{11}$$

for some $\lambda_s^* \in (0, \lambda_s)$.

Remark 2: The conditions in Theorem 1, i.e., (7) – (10), are the same conditions under which it was proven in

[1] A function α: $[0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

[2] A function β: $[0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$, and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Theorem 1 in [8] that the switched system (2) is IOSS. In [8], condition (8) was stated in a slightly different way, namely

$$|x| \geq \varphi_1(|u|) + \varphi_2(|h_p(x)|) \Rightarrow \frac{\partial V_p}{\partial x} f_p(x,u) \leq -\bar{\lambda}_s V_p(x) \quad (12)$$

for some $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ and $\bar{\lambda}_s > 0$. However, these two different formulations are equivalent ([4],[9]); here, we use (8) instead of (12) as this formulation is better suited for the proof later on. Furthermore, as stated in Remark 1, the existence of a state-norm estimator for a system implies that it is IOSS. Thus Theorem 1 can be seen as an alternative way of establishing IOSS for the switched system (2), which yields the nice "intermediate" result of obtaining a state-norm estimator for the considered switched system.

Remark 3: Similar conditions to those in Theorem 1 are quite common in the literature, when average dwell-time switching signals are considered. Conditions (7) and (8) are necessary and sufficient conditions for the subsystems to be IOSS [4], and the function V_p is called an exponential decay IOSS-Lyapunov function for the p-th subsystem [4]. Finally, condition (9) ensures that the IOSS-Lyapunov function for the different subsystems are somehow compatible; e.g., it doesn't hold if one of IOSS-Lyapunov functions is quadratic and a different one quartic.

Remark 4: In Theorem 1 (and also the following Theorems), for technical reasons in the proofs, we restrict the initial condition of the state-norm estimator to be nonnegative, whereas in Definition 1 we allow (in consistency with [3]) the initial condition of the state-norm estimator to be arbitrary. However, as we design the state-norm estimator and thus can choose any initial condition we want, this is not a major restriction.

Proof of Theorem 1: Consider as a candidate for a state-norm estimator the system (11) with $\lambda_s^* \in (0, \lambda_s)$. In the following, we have to verify that (11) satisfies the two properties of Definition 1, namely that it is ISS with respect to the inputs (u, y) and that (6) holds. It is easy to see that (11) is ISS with respect to the inputs (u, y), as it is a linear, exponentially stable system driven by these inputs. Thus it remains to show that (6) holds.

Note that as $\gamma_1(|u|) + \gamma_2(|y|) \geq 0$, we have

$$\dot{z}(t) \geq -\lambda_s^* z(t)$$

and thus, as $z_0 \geq 0$,

$$z(t) \geq e^{-\lambda_s^* t} z_0 \geq 0 \quad (13)$$

for all $t \geq 0$. Furthermore, for all $0 \leq \tau_i \leq t$ we get

$$z(\tau_i) \leq e^{\lambda_s^*(t-\tau_i)} z(t). \quad (14)$$

Now consider the function $W(t) := V_{\sigma(t)}(x(t)) - z(t)$. Using (8), (11), and (13), we obtain that in any interval $[\tau_i, \tau_{i+1})$,

$$\dot{W} = \dot{V}_{p_i} - \dot{z} \leq -\lambda_s V_{p_i} + \lambda_s^* z \leq -\lambda_s V_{p_i} + \lambda_s z = -\lambda_s W$$

and thus

$$\begin{aligned} W(\tau_{i+1}) &= V_{\sigma(\tau_{i+1})}(\tau_{i+1}) - z(\tau_{i+1}) \\ &\leq \mu V_{\sigma(\tau_i)}(\tau_{i+1}^-) - z(\tau_{i+1}) \\ &= \mu W(\tau_{i+1}^-) + (\mu-1) z(\tau_{i+1}) \\ &\leq \mu W(\tau_i) e^{-\lambda_s(\tau_{i+1}-\tau_i)} + (\mu-1) z(\tau_{i+1}) \end{aligned} \quad (15)$$

Iterating (15) from $i = 0$ to $i = N_\sigma(t, 0)$ and using (14), we arrive at

$$\begin{aligned} W(t) &= \mu^{N_\sigma(t,0)} \Big(e^{-\lambda_s t} W(0) \\ &\quad + (\mu-1) \sum_{k=1}^{N_\sigma(t,0)} \mu^{-k} e^{-\lambda_s(t-\tau_k)} z(\tau_k) \Big) \\ &\leq e^{N_\sigma(t,0)\ln\mu - \lambda_s t} W(0) \\ &\quad + (\mu-1) z(t) \sum_{k=1}^{N_\sigma(t,0)} e^{(N_\sigma(t,0)-k)\ln\mu - (\lambda_s - \lambda_s^*)(t-\tau_k)} \end{aligned} \quad (16)$$

Since $N_\sigma(t,0) - k = N_\sigma(t, \tau_k)$, we get, using (3),

$$\begin{aligned} &(N_\sigma(t,0) - k)\ln\mu - (\lambda_s - \lambda_s^*)(t - \tau_k) \\ &\leq N_\sigma(t,\tau_k)\ln\mu - (\lambda_s - \lambda_s^*)(t-\tau_k) \\ &\leq \left(N_0 + \frac{t-\tau_k}{\tau_a} \right) \ln\mu - (\lambda_s - \lambda_s^*)(t-\tau_k) \\ &\leq N_0 \ln\mu - \lambda(t-\tau_k) \end{aligned} \quad (17)$$

for some $\lambda \in (0, \lambda_s - \lambda_s^*)$ if the average dwell time τ_a satisfies the bound

$$\tau_a > \frac{\ln\mu}{\lambda_s - \lambda_s^*}. \quad (18)$$

The average dwell-time property (3) furthermore implies that

$$t - \tau_k \geq (N_\sigma(t,0) - k - N_0)\tau_a. \quad (19)$$

Combining (17) and (19) we arrive at

$$\begin{aligned} &\sum_{k=1}^{N_\sigma(t,0)} e^{(N_\sigma(t,0)-k)\ln\mu - (\lambda_s-\lambda_s^*)(t-\tau_k)} \\ &\leq e^{N_0(\ln\mu + \lambda\tau_a)} \sum_{k=1}^{N_\sigma(t,0)} e^{-\lambda\tau_a(N_\sigma(t,0)-k)} =: a_1. \end{aligned}$$

Applying the index shift $i := N_\sigma(t,0) - k$ we obtain

$$\begin{aligned} a_1 &= e^{N_0(\ln\mu+\lambda\tau_a)} \sum_{i=0}^{N_\sigma(t,0)-1} e^{-\lambda\tau_a i} \\ &\leq e^{N_0(\ln\mu+\lambda\tau_a)} \sum_{i=0}^{\infty} e^{-\lambda\tau_a i} \\ &= e^{N_0(\ln\mu+\lambda\tau_a)} \frac{1}{1-e^{-\lambda\tau_a}} =: a_2. \end{aligned} \quad (20)$$

Thus, by virtue of (16), we get

$$\begin{aligned} W(t) &\leq e^{N_\sigma(t,0)\ln\mu - \lambda_s t} W(0) + (\mu-1) a_2 z(t) \\ &= e^{(N_0 + \frac{t}{\tau_a})\ln\mu - \lambda_s t} W(0) + (\mu-1) a_2 z(t) \\ &\leq \mu^{N_0} e^{-\lambda' t} W(0) + (\mu-1) a_2 z(t) \end{aligned}$$

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for some $\lambda' \in (\lambda_s^*, \lambda_s)$ if τ_a satisfies (18). This leads to

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq (1 + a_2(\mu - 1))z(t) \\ &\quad + \mu^{N_0} e^{-\lambda' t}(V_{\sigma(0)}(x_0) - z_0) \\ &\leq (1 + a_2(\mu - 1))|z(t)| \\ &\quad + 2\mu^{N_0} e^{-\lambda' t}\alpha_2(|x_0| + |z_0|), \end{aligned}$$

if we assume without loss of generality that $\alpha_2(r) \geq r$ for all $r \geq 0$. Using (7), we finally arrive at

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(2(1 + a_2(\mu - 1))|z(t)|) \\ &\quad + \alpha_1^{-1}(4\mu^{N_0} e^{-\lambda' t}\alpha_2(|x_0| + |z_0|)) \\ &=: \gamma(|z(t)|) + \beta(|x_0| + |z_0|, t), \end{aligned} \tag{21}$$

which means that our state-norm estimator candidate (11) satisfies the condition (6).

Concluding the proof, we note that as we can choose λ_s^* arbitrarily close to 0, we can choose it small enough such that for any average dwell time τ_a satisfying (10), condition (18) is also satisfied and thus a state-norm estimator for the switched system (2) exists. □

Remark 5: If a state-norm estimator is constructed as proposed in Theorem 1, a degree of freedom in the design is the choice of λ_s^*. The only restriction is that condition (18) has to be satisfied, which, as stated in the last sentence of the proof, is always possible and gives an upper bound for the values λ_s^* can take. Choosing λ_s^* as large as possible would be desirable as the state-norm estimator (11) then has a better convergence rate. However, if λ_s^* is chosen close to its largest possible value, i.e., such that (18) is only barely satisfied, then (17) is only valid for λ very close to zero. According to (20), this leads to a large value for a_2, which in turn implies that the gain γ in (21), with which $|x|$ can be bounded in terms of $|z|$, becomes also large, which is not desirable. Thus a tradeoff for a good choice of λ_s^* has to be found.

IV. State-norm estimators: some subsystems not IOSS

In the following, we will consider the case where some of the subsystems of the family (1) are not IOSS, i.e., (8) doesn't hold for all $p \in \mathcal{P}$, but only for a subset $\mathcal{P}_s$ of $\mathcal{P}$.

Let $\mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_u$ such that $\mathcal{P}_s \cap \mathcal{P}_u = \emptyset$. Denote by $T^u(t,\tau)$ the total activation time of the systems in $\mathcal{P}_u$ and by $T^s(t,\tau)$ the total activation time of the systems in $\mathcal{P}_s$ during the time interval $[\tau, t)$, where $0 \leq \tau \leq t$. Clearly, $T^s(t,\tau) = t - \tau - T^u(t,\tau)$.

A. Known switching times

In the following Theorem, we show that given certain conditions hold, a state-norm estimator can be constructed if the exact switching times between an IOSS and a non-IOSS subsystem of (2) are known..

Theorem 2: Consider the family of systems (1). Let the conditions (7) and (9) hold for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$. Furthermore, suppose there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and constants $\lambda_s, \lambda_u > 0$ such that

$$\frac{\partial V_p}{\partial x} f_p(x,u) \leq -\lambda_s V_p(x) + \gamma_1(|u|) + \gamma_2(|h_p(x)|) \quad \forall p \in \mathcal{P}_s \tag{22}$$

$$\frac{\partial V_p}{\partial x} f_p(x,u) \leq \lambda_u V_p(x) + \gamma_1(|u|) + \gamma_2(|h_p(x)|) \quad \forall p \in \mathcal{P}_u \tag{23}$$

for all $x \in \mathbb{R}^n$.

If there exist constants $\tau_0, \rho \geq 0$ such that

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u} \tag{24}$$

$$\forall\ t \geq \tau \geq 0: \quad T^u(t,\tau) \leq \tau_0 + \rho(t - \tau), \tag{25}$$

and if $\sigma(\cdot)$ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}, \tag{26}$$

then there exists a switched state-norm estimator $\dot{z} = g_{s(t)}(z,u,y)$ for the switched system (2), consisting of two subsystems, where $s : [0,\infty) \to \{0,1\}$ is a switching signal whose switching times are those switching times of σ where a switch from a system in $\mathcal{P}_s$ to a system in $\mathcal{P}_u$ or vice versa occurs. A possible choice for the two subsystems is

$$\begin{aligned} \dot{z} &= g_0(z,u,y) = -\lambda_s^* z(t) + \gamma_1(|u(t)|) + \gamma_2(|y(t)|) \\ \dot{z} &= g_1(z,u,y) = \lambda_u^* z(t) + \gamma_1(|u(t)|) + \gamma_2(|y(t)|) \end{aligned} \tag{27}$$

with an appropriate choice of $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$.

Remark 6: Similarly to Remark 2, also for the case where some of the subsystems are not IOSS it holds that the conditions in Theorem 2 are the same conditions (modulo a slightly different but equivalent formulation of (22)–(23) as pointed out in Remark 2) under which it was proven in Theorem 2 in [8] that the switched system (2) is IOSS. Furthermore, the idea to restrict the fraction of time during which the unstable subsystems are active (through (24)–(25)) was e.g. also used in [10] and [11], where stability of switched systems consisting of both stable and unstable subsystems was considered.

We will now proceed with the proof of Theorem 2. Note that this proof follows the lines of the proof of Theorem 1; however, as some of the subsystems are not IOSS, some of the intermediate steps get more involved.

Proof of Theorem 2: Consider as a candidate for a switched state-norm estimator the system

$$\dot{z} = g_{s(t)}(z,u,y), \qquad z_0 \geq 0, \tag{28}$$

where the switching signal $s(t)$ is defined by

$$s(t) = \begin{cases} 0 & \text{if } \sigma(t) \in \mathcal{P}_s \\ 1 & \text{if } \sigma(t) \in \mathcal{P}_u \end{cases} \tag{29}$$

and g_i, $i \in \{0,1\}$ is the family of two systems (27) with $\lambda_s^* \in (0,\lambda_s)$ and $\lambda_u^* \geq \lambda_u$.

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This means that the system (28) consists of a subsystem $\dot{z} = g_0$, which is ISS with respect to the inputs (u, y) and which is active whenever one of the IOSS subsystems of (2) is active, and an unstable subsystem $\dot{z} = g_1$, which is active whenever one of the unstable subsystems of (2) is active. Thus the switching times of s coincide with those switching times of σ, where a switch from a system in $\mathcal{P}_s$ to a system in $\mathcal{P}_u$ or vice versa occurs, and the activation time of g_1 in any interval $[\tau, t)$, denoted by $T^u_z(t, \tau)$, is equal to the activation time $T^u(t, \tau)$ of the unstable systems of the switched system (2) in this interval.

In the following, we have to verify that our state-norm estimator candidate (28) satisfies the two properties of Definition 1, namely that it is ISS with respect to the inputs (u, y) and that (6) holds.

As stated in Remark 6, in [8] it was shown that the first property is satisfied, i.e., the system (28) is ISS with respect to the inputs (u, y), if the conditions of Theorem 2 are satisfied for the state-norm estimator candidate (28). Choosing e.g. $V_0(z) = V_1(z) =: V(z) = \frac{1}{2}z^2$, it is straightforward to verify that this is the case if $T^u_z(t, \tau)$ satisfies (25) with $\rho < \frac{\lambda^*_s}{\lambda^*_s + \lambda^*_u}$, but with no further condition on the average dwell time τ^z_a of the switching signal s, as (26) yields $\tau^z_a > 0$, or differently stated, $V(z)$ is a common ISS-Lyapunov function for the system (28).

It remains to show that our state-norm estimator candidate (28) satisfies the second property of Definition 1, i.e., that (6) holds.

As $\gamma_1(|u|) + \gamma_2(|y|) \geq 0$, we have

$$\begin{aligned} g_0(z, u, y) &\geq -\lambda^*_s z(t) \\ g_1(z, u, y) &\geq \lambda^*_u z(t) \end{aligned}$$

and thus, as $z_0 \geq 0$,

$$z(t) \geq e^{-\lambda^*_s T^s(t,0) + \lambda^*_u T^u(t,0)} z_0 \geq 0 \tag{30}$$

for all $t \geq 0$. Furthermore, for all $0 \leq \tau_i \leq t$ we get

$$z(\tau_i) \leq e^{\lambda^*_s T^s(t,\tau_i) - \lambda^*_u T^u(t,\tau_i)} z(t). \tag{31}$$

Note that (25) implies that

$$T^s(t, \tau) \geq (1 - \rho)(t - \tau) - \tau_0. \tag{32}$$

Now consider the function $W(t) := V_{\sigma(t)}(x(t)) - z(t)$. Following the lines of the proof of Theorem 1, we get that for any interval $[\tau_i, \tau_{i+1})$,

$$\begin{aligned} W(\tau_{i+1}) &\leq \mu W(\tau_i) e^{-\lambda_s(\tau_{i+1} - \tau_i)} + (\mu - 1) z(\tau_{i+1}) \\ &\qquad \text{if} \quad s(t) = 0 \text{ in } [\tau_i, \tau_{i+1}) \\ W(\tau_{i+1}) &\leq \mu W(\tau_i) e^{\lambda_u(\tau_{i+1} - \tau_i)} + (\mu - 1) z(\tau_{i+1}) \\ &\qquad \text{if} \quad s(t) = 1 \text{ in } [\tau_i, \tau_{i+1}) \end{aligned}$$

Iterating this from $i = 0$ to $i = N_\sigma(t, 0)$ and using (31), (25), and (32) we arrive at

$$\begin{aligned} W(t) &\leq \mu^{N_\sigma(t,0)} e^{-\lambda_s T^s(t,0) + \lambda_u T^u(t,0)} W(0) \\ &\quad + (\mu - 1) z(t) \sum_{k=1}^{N_\sigma(t,0)} \Big(e^{(N_\sigma(t,0) - k) \ln \mu} \\ &\quad \times e^{-(\lambda_s - \lambda^*_s) T^s(t,\tau_k) - (\lambda^*_u - \lambda_u) T^u(t,\tau_k)} \Big) \\ &\leq e^{(\lambda_s + \lambda_u)\tau_0} e^{N_\sigma(t,0) \ln \mu - \left(\lambda_s - \rho(\lambda_s + \lambda_u)\right) t} W(0) \\ &\quad + (\mu - 1) e^{(\lambda_s + \lambda_u - \lambda^*_s - \lambda^*_u)\tau_0} z(t) \\ &\quad \times \sum_{k=1}^{N_\sigma(t,0)} \Big(e^{(N_\sigma(t,0) - k) \ln \mu} \\ &\quad \times e^{-\left(\lambda_s - \lambda^*_s - \rho(\lambda_s + \lambda_u - \lambda^*_s - \lambda^*_u)\right)(t - \tau_k)} \Big) \end{aligned} \tag{33}$$

Using the average dwell-time property (3) and proceeding as in the proof of Theorem 1, we arrive at

$$\begin{aligned} &\sum_{k=1}^{N_\sigma(t,0)} \Big(e^{(N_\sigma(t,0) - k) \ln \mu} \\ &\quad \times e^{-\left(\lambda_s - \rho(\lambda_s + \lambda_u) - [\lambda^*_s - \rho(\lambda^*_s + \lambda^*_u)]\right)(t - \tau_k)} \Big) \\ &\leq e^{N_0(\ln \mu + \lambda \tau_a)} \frac{1}{1 - e^{-\lambda \tau_a}} =: b \end{aligned} \tag{34}$$

for some $\lambda \in \Big(0, \lambda_s - \rho(\lambda_s + \lambda_u) - \big[\lambda^*_s - \rho(\lambda^*_s + \lambda^*_u)\big]\Big)$ if the average dwell time τ_a satisfies the bound

$$\tau_a > \frac{\ln \mu}{\lambda_s - \rho(\lambda_s + \lambda_u) - \big[\lambda^*_s - \rho(\lambda^*_s + \lambda^*_u)\big]}. \tag{35}$$

Note that if we choose $\lambda^*_s \in (0, \lambda_s)$ and $\lambda^*_u \geq \lambda_u$ such that

$$\rho < \frac{\lambda^*_s}{\lambda^*_s + \lambda^*_u} < \frac{\lambda_s}{\lambda_s + \lambda_u},$$

then the above given interval in which λ is contained is well defined, i.e., $\lambda_s - \rho(\lambda_s + \lambda_u) - \big[\lambda^*_s - \rho(\lambda^*_s + \lambda^*_u)\big] > 0$.

Combining (33) and (34), we get

$$\begin{aligned} W(t) &\leq \mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} W(0) e^{-\lambda' t} \\ &\quad + b(\mu - 1) e^{(\lambda_s + \lambda_u - \lambda^*_s - \lambda^*_u)\tau_0} z(t) \\ &=: \mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} W(0) e^{-\lambda' t} + b_1 z(t) \end{aligned} \tag{36}$$

for some $\lambda' \in (\lambda^*_s - \rho(\lambda^*_s + \lambda^*_u), \lambda_s - \rho(\lambda_s + \lambda_u))$ if the average dwell-time τ_a satisfies the bound (35), which leads to

$$\begin{aligned} V_{\sigma(t^-)}(x(t)) &\leq (1 + b_1) z(t) \\ &\quad + \mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} e^{-\lambda' t} (V_{\sigma(0)}(x_0) - z_0) \\ &\leq (1 + b_1))|z(t)| \\ &\quad + 2\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} e^{-\lambda' t} \alpha_2(|x_0| + |z_0|), \end{aligned}$$

if we assume without loss of generality that $\alpha_2(r) \geq r$ for all $r \geq 0$. Using (7), we finally arrive at

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(2(1 + b_1))|z(t)|) \\ &\quad + \alpha_1^{-1}\Big(4\mu^{N_0} e^{(\lambda_s + \lambda_u)\tau_0} e^{-\lambda' t} \alpha_2(|x_0| + |z_0|)\Big) \\ &=: \gamma(|z(t)|) + \beta(|x_0| + |z_0|, t), \end{aligned} \tag{37}$$

which means that our state-norm estimator candidate (28) satisfies the condition (6).

Concluding the proof, we note that for any average dwell time τ_a satisfying (26) and any ρ satisfying (24), we can choose $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$ such that the conditions (35) and (36) are also satisfied, and thus a switched state-norm estimator for the switched system (2) exists. □

Remark 7: Similar considerations as in Remark 5 apply to the choice of $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$, if a state-norm estimator is constructed as proposed in Theorem 2. Restrictions in this choice are (35) and (36), which, as stated in the last sentence of the proof, always can be satisfied, yielding an upper bound for possible values of λ_s^*, respectively a lower bound for possible values of λ_u^*. Choosing λ_s^* as large as possible and λ_u^* as small as possible would be desirable, as then the state-norm estimator (28) has a better convergence rate. However, by similar considerations as in Remark 5, one can see that this yields a larger constant b in (34) and thus also a larger constant b_1 in (36), which in turn implies that the gain γ in (37), with which $|x|$ can be bounded in terms of $|z|$, becomes also large, which is not desirable. Thus a tradeoff for a good choice of λ_s^* and λ_u^* has to be found.

B. Unknown switching times

The construction of the state-norm estimator in Theorem 2 requires the exact knowledge of the switching times of the considered switched system (2), at least of those switching times, where a switch from a subsystem in $\mathcal{P}_s$ to a system in $\mathcal{P}_u$ or vice versa occurs. This is a very restrictive assumption, as the switching signal would have to be known a priori or switches would somehow have to be detected instantly. Thus, one would like to have some robustness in the construction of the state-norm estimator with respect to the knowledge of the switching times, i.e., that a state-norm estimator can still be constructed if not the exact switching times are known, but only small time intervals in which the switching times lie. Even more desirable would be the case where a state-norm estimator can be constructed with a switching signal that is independent of the switching times of the switched system the state-norm estimator is designed for. Then, the only knowledge needed about the switching signal σ of the switched system would be that it satisfies some average dwell-time condition, but knowledge about the (exact) switching times would not be needed.

In the following, we show that under the same conditions as in Theorem 2, a state-norm estimator can be constructed whose switching times are independent of the switching times of σ. For the proof of this result, we exploit that a state-norm estimator as proposed in Theorem 2, i.e., with (exact) knowledge of the switching times of σ, exists; however, this knowledge is not needed for designing the switching signal s' of the proposed state-norm estimator.

Theorem 3: Let all the conditions of Theorem 2 hold. Then there exists a switched state-norm estimator $\dot{\zeta} = g_{s'(t)}(\zeta, u, y)$ for the switched system (2), consisting of two subsystems, where $s' : [0, \infty) \to \{0, 1\}$ is a switching signal whose switching times are independent of the switching times of σ. As in Theorem 2, a possible choice for the two subsystems of the state-norm estimator is given by (27).

Proof: Consider again the switched state-norm estimator (28) designed in the proof of Theorem 2 and its state $z(t)$. The idea of this proof is that if we can design a candidate state-norm estimator $\dot{\zeta} = g_{s'(t)}(\zeta, u, y)$ such that for all $t \geq 0$

$$|z(t)| \leq c|\zeta(t)| \tag{38}$$

for some constant $c \geq 1$, then the system $\dot{\zeta} = g_{s'(t)}(\zeta, u, y)$ is also a state-norm estimator for the switched system (2).

Consider the following candidate for a state-norm estimator with switching times independent of the switching times of σ:

$$\dot{\zeta} = g_{s'(t)}(\zeta, u, y), \qquad \zeta_0 \geq 0 \tag{39}$$

where g_i, $i \in \{0, 1\}$ is the family of two systems (27) and the switching signal $s'(t)$ is defined by

$$s'(t) = \begin{cases} 0 & \forall t \in [k\tau_a^\zeta, k\tau_a^\zeta + (1-\rho^\zeta)\tau_a^\zeta) \\ 1 & \forall t \in [k\tau_a^\zeta + (1-\rho^\zeta)\tau_a^\zeta, (k+1)\tau_a^\zeta) \end{cases} \tag{40}$$

where $k = 0, 1, 2, \ldots$. The constants $\tau_a^\zeta > 0$ and $\rho^\zeta > 0$ are chosen such that

$$\rho < \rho^\zeta < \frac{\lambda_s^*}{\lambda_s^* + \lambda_u^*} \tag{41}$$

$$\rho^\zeta \tau_a^\zeta \geq \tau_0 + \rho \tau_a^\zeta, \tag{42}$$

where λ_s^* and λ_u^* are the parameters of the state-norm estimator designed in Theorem 2, which means that in any interval of length τ_a^ζ, the period of time during which $\zeta(t)$ is unstable (namely $\rho^\zeta \tau_a^\zeta$ according to (40)) is greater or equal than the maximum unstable time of $z(t)$ ($\tau_0 + \rho\tau_a^\zeta$ according to (25)).

It is straightforward to verify that the activation time of g_1 in any interval $[\tau, t)$, denoted by $T_\zeta^u(t, \tau)$, satisfies

$$T_\zeta^u(t, \tau) \leq \tau_0^\zeta + \rho^\zeta(t - \tau) \tag{43}$$

with $\tau_0^\zeta = \rho^\zeta(1-\rho^\zeta)\tau_a^\zeta$. Using the same argumentation as in the proof of Theorem 2, we see that according to Remark 6, the candidate state-norm estimator (39) is ISS with respect to the inputs (u, y), i.e. the first property of Definition 1 is satisfied.

In the following, we will prove by induction that (38) holds, which implies that the second property of Definition 1 is also satisfied and thus the candidate (39) is a state-norm estimator for the switched system (2). Note that we can write (38) without the absolute values, as $z(t)$ as well as $\zeta(t)$ are greater than or equal to zero for all $t \geq 0$.

As an initial step, note that we can choose z_0 and ζ_0 such that $z_0 \leq \zeta_0$. Now assume that

$$|z(k\tau_a^\zeta)| \leq |\zeta(k\tau_a^\zeta)| \tag{44}$$

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for some integer $k \geq 0$. If we can show that also

$$|z((k+1)\tau_a^\zeta)| \leq |\zeta((k+1)\tau_a^\zeta)| \tag{45}$$

and that

$$|z(t)| \leq c|\zeta(t)| \qquad k\tau_a^\zeta \leq t \leq (k+1)\tau_a^\zeta \tag{46}$$

for some $c \geq 1$, then we can conclude that (38) holds.

Let $t_1 := k\tau_a^\zeta$, $t_2 := k\tau_a^\zeta + (1-\rho^\zeta)\tau_a^\zeta$, and $t_3 := (k+1)\tau_a^\zeta$. Furthermore, let $w(t) := \gamma_1(|u(t)|) + \gamma_2(|y(t)|)$.
First, consider the interval $[t_1, t_2)$. During this interval, $\zeta(t)$ is stable, whereas $z(t)$ might switch between its stable and its unstable mode.

If during some interval $[t', t'')$, where $t_1 \leq t' \leq t'' \leq t_2$, both ζ and z are stable and $z(t') \leq \varepsilon\zeta(t')$ for some $\varepsilon \geq 1$, then

$$\frac{d}{dt}(z-\zeta) = g_0(z,u,y_\sigma) - g_0(\zeta,u,y_\sigma) = -\lambda_s^*(z-\zeta)$$

and thus

$$\begin{aligned} z(t'') - \zeta(t'') &= e^{-\lambda_s^*(t''-t')}(z(t') - \zeta(t')) \\ &\leq e^{-\lambda_s^*(t''-t')}(\varepsilon-1)\zeta(t') \leq (\varepsilon-1)\zeta(t''). \end{aligned}$$

Thus we obtain

$$z(t'') \leq \varepsilon\zeta(t''). \tag{47}$$

If during some interval $[t'', t''')$, where $t_1 \leq t'' \leq t''' \leq t_2$, ζ is stable and z is unstable and $z(t'') \leq \kappa\zeta(t'')$ for some $\kappa \geq 1$, then

$$\begin{aligned} \zeta(t''') &= e^{-\lambda_s^*(t'''-t'')}\left(\zeta(t'') + \int_{t''}^{t'''} e^{\lambda_s^*(s-t'')}w(s)ds\right) \\ &\geq e^{-\lambda_s^*(t'''-t'')}\left(\zeta(t'') + \int_{t''}^{t'''} w(s)ds\right) \end{aligned}$$

and

$$\begin{aligned} z(t''') &= e^{\lambda_u^*(t'''-t'')}\left(z(t'') + \int_{t''}^{t'''} e^{-\lambda_u^*(s-t'')}w(s)ds\right) \\ &\leq e^{\lambda_u^*(t'''-t'')}\left(\kappa\zeta(t'') + \int_{t''}^{t'''} w(s)ds\right) \\ &= e^{(\lambda_u^*+\lambda_s^*)(t'''-t'')}\kappa e^{-\lambda_s^*(t'''-t'')} \\ &\quad \times \left(\zeta(t'') + \frac{1}{\kappa}\int_{t''}^{t'''} w(s)ds\right) \\ &\leq \kappa e^{(\lambda_u^*+\lambda_s^*)(t'''-t'')}\zeta(t''') \end{aligned} \tag{48}$$

Thus, as the interval $[t_1, t_2)$ consists of disjoint subintervals where z is alternating between stable and unstable, and as according to (44) $z(t_1) \leq \zeta(t_1)$, by iterating (47) and (48) we arrive at

$$z(t_2) \leq e^{(\lambda_u^*+\lambda_s^*)T^u(t_2,t_1)}\zeta(t_2). \tag{49}$$

Second, consider the interval $[t_2, t_3)$. During this interval, $\zeta(t)$ is unstable, whereas $z(t)$ might switch between its stable and its unstable mode. Yet, as according to (25), $T^u(t_3, t_1) \leq \tau_0 + \rho(t_3 - t_1) = \tau_0 + \rho\tau_a^\zeta$, the time in the interval $[t_2, t_3)$ where z is unstable is bounded above by

$$T^u(t_3, t_2) \leq \tau_0 + \rho\tau_a^\zeta - T^u(t_2, t_1) \tag{50}$$

and thus

$$\begin{aligned} T^s(t_3, t_2) &= t_3 - t_2 - T^u(t_3, t_2) \\ &\geq \rho^\zeta\tau_a^\zeta - (\tau_0 + \rho\tau_a^\zeta - T^u(t_2, t_1)). \end{aligned} \tag{51}$$

Suppose there are l switching times of σ during the interval (t_2, t_3), and assume without loss of generality that these are the switching times $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{r+l}$, where r is some nonnegative integer. Furthermore, for convenience, let $\tau_r := t_2$ and $\tau_{r+l+1} := t_3$. Let $\mathcal{K}_1 := \{k \in \{0, \ldots, l\} : \ s(t) = 0 \text{ for t} \in [\tau_{r+k}, \tau_{r+k+1})\}$ and $\mathcal{K}_2 := \{k \in \{0, \ldots, l\} : s(t) = 1 \text{ for t} \in [\tau_{r+k}, \tau_{r+k+1})\}$. Then

$$\begin{aligned} \zeta(t_3) &= e^{\lambda_u^*(t_3-t_2)}\left(\zeta(t_2) + \int_{t_2}^{t_3} e^{-\lambda_u^*(s-t_2)}w(s)ds\right) \\ &\geq e^{\lambda_u^*(t_3-t_2)}\zeta(t_2) \\ &\quad + \sum_{k\in\mathcal{K}_1} e^{\lambda_u^*(t_3-\tau_{r+k+1})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} w(s)ds \\ &\quad + \sum_{k\in\mathcal{K}_2} e^{\lambda_u^*(t_3-\tau_{r+k})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{-\lambda_u^*(s-\tau_{r+k})}w(s)ds \\ &=: e^{\lambda_u^*(t_3-t_2)}\zeta(t_2) + a_1 + a_2 \end{aligned} \tag{52}$$

and

$$\begin{aligned} z(t_3) &= e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) \\ &\quad + \sum_{k\in\mathcal{K}_1}\Big(e^{-\lambda_s^*T^s(t_3,\tau_{r+k+1})+\lambda_u^*T^u(t_3,\tau_{r+k+1})} \\ &\quad \times e^{-\lambda_s^*(\tau_{r+k+1}-\tau_{r+k})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{\lambda_s^*(s-\tau_{r+k})}w(s)ds\Big) \\ &\quad + \sum_{k\in\mathcal{K}_2}\Big(e^{-\lambda_s^*T^s(t_3,\tau_{r+k+1})+\lambda_u^*T^u(t_3,\tau_{r+k+1})} \\ &\quad \times e^{\lambda_u^*(\tau_{r+k+1}-\tau_{r+k})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{-\lambda_u^*(s-\tau_{r+k})}w(s)ds\Big) \\ &\leq e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) + a_1 + a_2 \end{aligned} \tag{53}$$

Using (49)–(53), we arrive at

$$\begin{aligned} \zeta(t_3) - z(t_3) &\geq e^{\lambda_u^*(t_3-t_2)}\zeta(t_2) \\ &\qquad - e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) \\ &\geq e^{\lambda_u^*(t_3-t_2)}\zeta(t_2) \\ &\qquad - e^{-\lambda_s^*(\rho^\zeta\tau_a^\zeta-(\tau_0+\rho\tau_a^\zeta-T^u(t_2,t_1)))} \\ &\qquad \times e^{\lambda_u^*(\tau_0+\rho\tau_a^\zeta-T^u(t_2,t_1))}z(t_2) \\ &\geq \left(e^{\lambda_u^*\rho^\zeta\tau_a^\zeta} - e^{(\lambda_u^*+\lambda_s^*)(\tau_0+\rho\tau_a^\zeta)-\lambda_s^*\rho^\zeta\tau_a^\zeta}\right)\zeta(t_2) \end{aligned} \tag{54}$$

As $\zeta(t_2) \geq 0$, the right-hand side of (54) is nonnegative if and only if

$$\begin{aligned} \lambda_u^*\rho^\zeta\tau_a^\zeta &\geq (\lambda_u^*+\lambda_s^*)(\tau_0+\rho\tau_a^\zeta) - \lambda_s^*\rho^\zeta\tau_a^\zeta \\ &\Updownarrow \\ \rho^\zeta\tau_a^\zeta &\geq \tau_0 + \rho\tau_a^\zeta \end{aligned}$$

which is true according to our choice of ρ^ζ and τ_a^ζ in (41)–(42).

Thus (45) and (46) are satisfied, and the constant c in (46) can be calculated from (49) and the fact that $T^u(t_2, t_1) \leq \tau_0 + \rho(t_2 - t_1) = \tau_0 + \rho(1-\rho^\zeta)\tau_a^\zeta$ as

$$c = e^{(\lambda_u^* + \lambda_s^*)(\tau_0 + \rho(1-\rho^\zeta)\tau_a^\zeta)}.$$

Note that we calculated c from (49) (i.e., the worst-case ratio $\frac{z}{\zeta}$ can be calculated at time t_2) as during the time interval $[t_2, t_3)$, the ratio $\frac{z}{\zeta}$ decreases. Namely, as during the time interval $[t_2, t_3)$ $\zeta(t)$ is unstable and $z(t)$ switches between its stable and unstable mode,

$$\frac{d}{dt}(z - \zeta) \leq \lambda_u^*(z - \zeta).$$

Thus, if $z(t_2) \leq \varepsilon\zeta(t_2)$ for some ε, we obtain that for any $t \in [t_2, t_3)$,

$$\begin{aligned} z(t) - \zeta(t) &= e^{\lambda_u^*(t-t_2)}(z(t_2) - \zeta(t_2)) \\ &\leq e^{-\lambda_s^*(t-t_2)}(\varepsilon - 1)\zeta(t_2) \leq (\varepsilon - 1)\zeta(t). \end{aligned}$$

and thus

$$z(t) \leq \varepsilon\zeta(t),$$

i.e., the ratio $\frac{z(t)}{\zeta(t)}$ is smaller or equal than the ratio $\frac{z(t_2)}{\zeta(t_2)}$.

Therefore, we conclude that the system (39) is also a state-norm estimator for the switched system (2) and by construction of the switching signal s', its switching times are independent of the switching times of the switching signal σ. □

V. Conclusions

In this paper, we considered the concept of state-norm estimators for switched nonlinear systems in the setting of multiple IOSS-Lyapunov functions and constrained switching. We showed that there exists a non-switched state-norm estimator if all subsystems of the considered switched system are IOSS, whereas a switched state-norm estimator exists for the case where some of the subsystems are not IOSS. For the latter case, we showed that the switching signal of the state-norm estimator can be designed such that its switching times are independent of the switching times of the considered switched system. The close connection of state-norm estimators to the IOSS property was demonstrated by the fact that the conditions under which the existence of a state-norm estimator could be established turned out to be the same conditions for which the considered switched system is IOSS.

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