Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Quasi-ISS Reduced-Order Observers and Quantized Output Feedback

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***Abstract*— We formulate and study the problem of designing nonlinear observers whose error dynamics are input-to-state stable (ISS) with respect to additive output disturbances as long as the plant's input and state remain bounded. We present a reduced-order observer design which achieves this quasi-ISS property when there exists a suitable state-independent error Lyapunov function. We show that our construction applies to several classes of nonlinear systems previously studied in the observer design literature. As an application of this robust observer concept, we prove that quantized output feedback stabilization is achievable when the system possesses a quasi-ISS reduced-order observer and a state feedback law that yields ISS with respect to measurement errors. A worked example is included.**

I. Introduction

The basic problem addressed in this paper is the design of nonlinear observers that possess robustness to additive disturbances affecting the output measurements. In addition to asking that the state estimation error converge to 0 in the absence of such disturbances, we want it to still converge to 0 if a disturbance is present but converges to 0, and to remain bounded if the disturbance is bounded. For nonlinear systems, a natural way to formulate this robustness property is in terms of input-to-state stability (ISS) of the state estimation error with respect to the output disturbance, within the framework introduced in [17].

Somewhat surprisingly, it appears that this observer design problem—which is quite basic and easy to formulate—has not been systematically studied in the literature. The two exceptions we are aware of are the papers [16] and [12]. The former paper proposed a novel observer design technique for a class of nonlinear systems based on passivation of the error dynamics. It then showed that, when an additional condition is satisfied, the observer can be redesigned to provide an ISS-like property with respect to the output measurement disturbance, with the ISS gain depending on the size of the input and the output. The more recent paper [12] considers the problem of stabilizing a nonlinear system by quantized output feedback control. In this setting, the output quantization error plays the role of a measurement disturbance. It is shown in [12] that if an observer with the ISS property mentioned above exists and, moreover, a controller providing ISS with respect to the state estimation error is available, then quantized output feedback stabilization is achievable.

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In this paper, we join the independent efforts that have led to the work reported in [16] and [12] and present new results both on the ISS nonlinear observer design problem itself and on its quantized output feedback control application. Unlike in these earlier works, the focus of this paper is on reduced-order observers. Similarly to [16], we allow the ISS gain to depend on the supremum norms of the input and the state up to the current time. This property, which we call "quasi-ISS," is precisely defined in Section II. Our main result is Theorem 1 in Section III. It establishes the quasi-ISS property of a reduced-order observer design based on a coordinate transformation under the assumption that there exists a state-independent error Lyapunov function. We also include a worked example. In Section IV we further study this state-independent error Lyapunov function assumption, with the goal of identifying useful classes of systems for which it holds and hence a quasi-ISS observer can be constructed. We show that several system classes previously studied in the nonlinear observer design literature (specifically, in [1], [5], [8], [10], [19]) fall into this category. In Section V we turn our attention to the quantized output feedback stabilization problem formulated in [12]. We prove that with a quasi-ISS reduced-order observer in place of a truly ISS full-order observer as in [12], a stabilization result analogous to the one from [12] can be obtained (although the proof is quite different). Finally, Section VI offers a quick summary and outlook.

II. Preliminaries

We consider a general nonlinear system ("plant")

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, f is locally Lipschitz, and h is continuously differentiable with locally Lipschitz derivative (called a C^1_L function in this paper). In addition, it is assumed that $f(0,0)=0$ and $h(0)=0$.

A *state observer* for the plant is a pair consisting of a dynamical system and a static map

$$\begin{aligned} \dot{\hat{\xi}} &= F(\bar{y},\hat{\xi},u) \\ \hat{x} &= H(\bar{y},\hat{\xi},u) \end{aligned} \tag{2}$$

where $\hat{\xi} \in \mathbb{R}^l$ is the observer state, $\hat{x} \in \mathbb{R}^n$ is the estimate of the plant state x, and $\bar{y}$ is the measurement of y that may be corrupted by a disturbance d:

$$\bar{y} := y + d. \tag{3}$$

For (2) to be an asymptotic observer, it is necessary for the estimate $\hat{x}$ to converge to the plant state x (i.e., $\hat{x}(t) \to x(t)$ as $t \to \infty$)[1] when $d \equiv 0$. As explained in the introduction, we want the observer to have a stronger property characterizing robustness to nonzero d. Before we formalize this property, we introduce the notation

$$\tilde{x} := \hat{x} - x$$

for the state estimation error. We use the shorthand notation

$$a \vee b := \max\{a, b\}.$$

(Alternatively, the sum could be used instead of the maximum in the following definition to arrive at an equivalent property, but the formulation in terms of the maximum is more convenient for our purposes.)

Definition 1 (Quasi-ISS observer) *We say that the system* (2) *is a* quasi-ISS observer *for the plant* (1) *if there exists a function $\tilde{\beta} \in \mathcal{KL}$ and, for each $K > 0$, there exists a function $\tilde{\gamma}_K \in \mathcal{K}_\infty$ such that*

$$|\tilde{x}(t)| \le \tilde{\beta}(|\tilde{x}(0)|, t) \vee \tilde{\gamma}_K(\|d\|_{[0,t]}) \tag{4}$$

whenever $\|u\|_{[0,t]} \le K$ and $\|x\|_{[0,t]} \le K$.

The quasi-ISS observer property means that as long as the plant's control u and state x remain uniformly bounded, the dynamics of the state estimation error $\tilde{x}$ are ISS with respect to the disturbance d. The following example is intended to motivate why for nonlinear systems it is somewhat natural to ask for the boundedness of u and x.

Example 1: Consider the plant $\dot{x} = -x + x^2 u$ with $y = x$. Obviously, $\dot{\hat{x}} = -\hat{x} + y^2 u$ is an asymptotic observer. However, with the perturbed measurement $\bar{y} = y + d$, the error dynamics become $\dot{\tilde{x}} = -\tilde{x} + 2xud + ud^2$. This system is ISS from d to $\tilde{x}$ when $u(t)$ and $x(t)$ are bounded, and the ISS gain function depends on the bounds on u and x.

III. Quasi-ISS (Reduced-order) Observers

We assume that there exists a global coordinate change[2] $z = \Phi(x)$ such that the system (1) is globally diffeomorphic to a system with linear output of the form

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_1(z_1, z_2, u) \\ f_2(z_1, z_2, u) \end{bmatrix} = f(z, u) \\ y &= z_1 \end{aligned} \tag{5}$$

where $z_1 \in \mathbb{R}^p$ and $z_2 \in \mathbb{R}^{n-p}$. (By abuse of notation, we use the same f for (5) as in (1).)

Since Φ and its inverse are C^1, there exist class-$\mathcal{K}$ functions p_K and q_K, parameterized by the constant K, such that $|\Phi(\hat{x}) - \Phi(x)| \le p_K(|\hat{x} - x|)$ and $|\Phi^{-1}(\hat{z}) - \Phi^{-1}(z)| \le q_K(|\hat{z} - z|)$ under the condition that $|x| \le K$ (and thus, $|z| = |\Phi(x)| \le \bar{K}$ with some $\bar{K}$). This means, in particular, that once we have a quasi-ISS observer for (5), a quasi-ISS observer for (1) is obtained under the coordinate change Φ^{-1} with $\hat{x} := \Phi^{-1}(\hat{z})$. Therefore, we are interested in the construction of a quasi-ISS observer for (5) in this section. Such a quasi-ISS observer, of order $n - p$, will now be presented. In fact, the construction is based on the reduced-order observer design of [7], [15] under the following assumption.

Assumption 1 (Reduced-order error Lyapunov function) *There exist a C_L^1 function $l : \mathbb{R}^p \to \mathbb{R}^{n-p}$, a C^1 function $V : \mathbb{R}^{n-p} \to \mathbb{R}$, and class-$\mathcal{K}_\infty$ functions α_i, $i = 1, \ldots, 4$, such that, for all e, z and u,*

$$\begin{aligned} &\text{(a)}\ \alpha_1(|e|) \le V(e) \le \alpha_2(|e|), \ \left|\frac{\partial V}{\partial e}(e)\right| \le \alpha_4(|e|), \\ &\text{(b)}\ \frac{\partial V}{\partial e}(e)\Big([f_2(z_1, e + z_2, u) + \frac{\partial l}{\partial z_1}(z_1) f_1(z_1, e + z_2, u)] \\ &\qquad - [f_2(z_1, z_2, u) + \frac{\partial l}{\partial z_1}(z_1) f_1(z_1, z_2, u)]\Big) \le -\alpha_3(|e|), \end{aligned} \tag{6}$$

and there exists a class-$\mathcal{K}_\infty$ function α such that

$$\alpha(s)\alpha_4(s) \le \alpha_3(s). \tag{7}$$

Theorem 1 *Under Assumption 1, the system*

$$\begin{aligned} \dot{\hat{\xi}} &= f_2(\bar{y}, \hat{\xi} - l(\bar{y}), u) + \frac{\partial l}{\partial z_1}(\bar{y}) f_1(\bar{y}, \hat{\xi} - l(\bar{y}), u) \\ \hat{z}_1 &= \bar{y} \\ \hat{z}_2 &= \hat{\xi} - l(\bar{y}) \end{aligned} \tag{8}$$

where $\hat{\xi} \in \mathbb{R}^{n-p}$ is the observer state, and $\hat{z}_1$ and $\hat{z}_2$ are the estimates of z_1 and z_2, respectively, becomes a quasi-ISS reduced-order observer for the system (5).

Proof: Define $\xi := z_2 + l(z_1)$. Then, the plant (5) is globally converted into

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, \xi - l(z_1), u) \\ \dot{\xi} &= f_2(z_1, \xi - l(z_1), u) + \frac{\partial l}{\partial z_1}(z_1) f_1(z_1, \xi - l(z_1), u) \\ &=: F(z_1, \xi, u) \\ y &= z_1 \end{aligned}$$

in which F is defined for convenience. With this F, the reduced-observer (8) is simply written as $\dot{\hat{\xi}} = F(\bar{y}, \hat{\xi}, u)$.

Let $e := \hat{\xi} - \xi$. Then, from Assumption 1, we obtain that

$$\begin{aligned} \dot{V} = \frac{\partial V}{\partial e}(e)\Big\{ & f_2(z_1 + d, \hat{\xi} - l(z_1 + d), u) \\ & + \frac{\partial l}{\partial z_1}(z_1 + d) f_1(z_1 + d, \hat{\xi} - l(z_1 + d), u) \\ & - f_2(z_1, \xi - l(z_1), u) - \frac{\partial l}{\partial z_1}(z_1) f_1(z_1, \xi - l(z_1), u)\Big\} \end{aligned}$$

[1] The convergence property is meaningful only when the trajectory x of the plant does not have a finite escape time.

[2] The coordinate transformation is given by $\Phi(x) = [h(x)^T, \phi(x)^T]^T$ with a C_L^1 function $\phi : \mathbb{R}^n \to \mathbb{R}^{n-p}$ that makes $\Phi(x)$ a globally one-to-one map and $\frac{\partial \Phi}{\partial x}(x)$ nonsingular for all x.

$$
\begin{aligned}
&= \frac{\partial V}{\partial e}(e)\Big\{ f_2(z_1+d, \hat{\xi} - l(z_1+d), u) \\
&\quad + \frac{\partial l}{\partial z_1}(z_1+d) f_1(z_1+d, \hat{\xi} - l(z_1+d), u) \\
&\quad - f_2(z_1+d, \xi - l(z_1+d), u) \\
&\quad - \frac{\partial l}{\partial z_1}(z_1+d) f_1(z_1+d, \xi - l(z_1+d), u) \Big\} \\
&\quad + \frac{\partial V}{\partial e}(e)\left\{ F(\bar{y}, \xi, u) - F(y, \xi, u) \right\}.
\end{aligned}
$$

Since Assumption 1(b) holds for any z_1, it follows that

$$
\begin{aligned}
\dot{V} &\le -\alpha_3(|e|) + \frac{\partial V}{\partial e}(e)\left\{ F(\bar{y}, \xi, u) - F(y, \xi, u) \right\} \\
&\le -\alpha_3(|e|) + \alpha_4(|e|)\gamma(z_1, \xi, u)\rho(|d|)
\end{aligned}
$$

where γ is a continuous positive function and ρ is a class-$\mathcal{K}$ function such that

$$
|F(z_1+d, \xi, u) - F(z_1, \xi, u)| \le \gamma(z_1, \xi, u)\rho(|d|)
$$

whose existence has been proven in [3], [4]. Therefore, it follows from (7) that, for an arbitrary $\varepsilon \in (0,1)$, we have

$$
\begin{aligned}
|e| \ge \alpha^{-1}\left((1-\varepsilon)^{-1}\gamma(z_1, z_2 + l(z_1), u)\rho(|d|) \right) \\
\implies \quad \dot{V} \le -\varepsilon\alpha_3(|e|). \quad (9)
\end{aligned}
$$

Now, under the condition that $|z(\tau)| \le K$ and $|u(\tau)| \le K$ for $0 \le \tau \le t$, it can be shown by standard arguments (see, e.g., [17]) that there exist a class-$\mathcal{KL}$ function $\bar{\beta}$ and a class-$\mathcal{K}_\infty$ function $\bar{\gamma}_K$ parameterized by K such that

$$
|e(t)| \le \bar{\beta}(|e(0)|, t) \,\vee\, \bar{\gamma}_K(\|d\|_{[0,t]}). \quad (10)
$$

Recalling (8), we have that

$$
\tilde{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} := \begin{bmatrix} \hat{z}_1 - z_1 \\ \hat{z}_2 - z_2 \end{bmatrix} = \begin{bmatrix} d \\ e - (l(z_1+d) - l(z_1)) \end{bmatrix},
$$

which leads to

$$
|\tilde{z}| \le |d| + |e| + \theta_K(|d|) \quad \text{and} \quad |e| \le |\tilde{z}_2| + \theta_K(|d|) \quad (11)
$$

where θ_K is a class-$\mathcal{K}$ function, parameterized by K, such that $|l(z_1+d) - l(z_1)| \le \theta_K(|d|)$ when $|z_1| \le K$. Combining (10) and (11), it follows that

$$
\begin{aligned}
|\tilde{z}(t)| &\le \bar{\beta}(|e(0)|, t) + \bar{\gamma}_K(\|d\|_{[0,t]}) + \theta_K(|d(t)|) + |d(t)| \\
&\le \bar{\beta}(|\tilde{z}_2(0)| + \theta_K(|d(0)|), t) + \chi_K(\|d\|_{[0,t]}) \quad (12) \\
&\le 2\bar{\beta}(|\tilde{z}_2(0)| + \theta_K(|d(0)|), t) \,\vee\, 2\chi_K(\|d\|_{[0,t]})
\end{aligned}
$$

where $\chi_K(s) := \bar{\gamma}_K(s) + \theta_K(s) + s$. By the identity $\alpha(a + b) \le \alpha(2a) \vee \alpha(2b)$ for class-$\mathcal{K}$ functions, we have that

$$
\begin{aligned}
\bar{\beta}(|\tilde{z}_2(0)| + \theta_K(|d(0)|), t) &\le \bar{\beta}(2|\tilde{z}_2(0)|, t) \vee \bar{\beta}(2\theta_K(|d(0)|), t) \\
&\le \bar{\beta}(2|\tilde{z}_2(0)|, t) \,\vee\, \bar{\beta}(2\theta_K(\|d\|_{[0,t]}), 0) \\
&=: \breve{\beta}(|\tilde{z}_2(0)|, t) \,\vee\, \breve{\theta}_K(\|d\|_{[0,t]}). \quad (13)
\end{aligned}
$$

With (12) and (13), we finally obtain that

$$
\begin{aligned}
|\tilde{z}(t)| &\le 2\breve{\beta}(|\tilde{z}(0)|, t) \,\vee\, 2\breve{\theta}_K(\|d\|_{[0,t]}) \,\vee\, 2\chi_K(\|d\|_{[0,t]}) \\
&=: \tilde{\beta}(|\tilde{z}(0)|, t) \,\vee\, \tilde{\gamma}_K(\|d\|_{[0,t]}),
\end{aligned}
$$

which proves that the quasi-ISS observer property holds. ■

Example 2: Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_1 + 2x_2 + 4x_2^3 + 2u \\
\dot{x}_2 &= x_2^3 + u \\
y &= x_1
\end{aligned} \quad (14)
$$

which is taken from [2]. This system is already in the form (5) (with z replaced by x), and Assumption 1 is satisfied with $V(e) = e^2/2$, $l(x_1) = -(1/4)x_1$, and $\alpha_3(s) = (1/2)s^2$. Indeed, the left-hand side of (6) becomes

$$
\begin{aligned}
e\Big([(e+x_2)^3 + u - \frac{1}{4}(x_1 + 2(e+x_2) + 4(e+x_2)^3 + 2u)] \\
- [x_2^3 + u - \frac{1}{4}(x_1 + 2x_2 + 4x_2^3 + 2u)] \Big) = -\frac{1}{2}e^2
\end{aligned}
$$

which verifies the claim. Therefore, the quasi-ISS reduced-order observer is given by

$$
\begin{aligned}
\dot{\hat{\xi}} &= -\frac{1}{4}\bar{y} - \frac{1}{2}(\hat{\xi} + \frac{1}{4}\bar{y}) + \frac{1}{2}u \\
\hat{x}_1 &= \bar{y} \\
\hat{x}_2 &= \hat{\xi} + \frac{1}{4}\bar{y}.
\end{aligned} \quad (15)
$$

(The dynamics of this observer are actually linear.)

IV. Classes of Systems that Admit Quasi-ISS Reduced-order Observers

In Section III we have constructed a quasi-ISS reduced-order observer under Assumption 1. However, Assumption 1 itself is not a constructive one because finding such functions V and l would be rather difficult in general. In addition, the class of plants (5) that admit Assumption 1 is not very clear.[3] In this section, we make use of the existing nonlinear observer design methods in the literature in order to characterize the classes of systems satisfying Assumption 1 and to obtain the required V and l. This job is done by way of the following lemma.

Lemma 1 *Suppose that the plant (5) has C^1 vector fields, and that a C^1 full-order observer in the form*

$$
\dot{\hat{z}} = g(y, \hat{z}, u), \qquad \hat{z} \in \mathbb{R}^n, \quad (16)
$$

is designed. If the error convergence is verified with a quadratic Lyapunov function

$$
W(\tilde{z}) = \frac{1}{2}\tilde{z}^T P \tilde{z}, \qquad \tilde{z} := \hat{z} - z, \quad P = P^T > 0 \quad (17)
$$

and the derivative of W is upper bounded by a quadratic term, i.e.,

$$
\dot{W} = \tilde{z}^T P[g(z_1, \tilde{z} + z, u) - f(z, u)] \le -k_1|\tilde{z}|^2, \quad \forall\, \tilde{z}, z, u \quad (18)
$$

with $k_1 > 0$, then Assumption 1 follows.

Proof: We note that, by the fact that $\hat{z}(t) = z(t)$ if $\hat{z}(0) = z(0)$, we should have that $g(y, \hat{z}, u) = f(\hat{z}, u) = f(z_1, \hat{z}_2, u)$ if $\tilde{z}_1 = \hat{z}_1 - y = 0$ (see [13], [16]).

[3] The work by Praly [13] has studied a necessary condition and a sufficient condition for the existence of such a state-independent error Lyapunov function $V(e)$.

Now, let us write

$$W(\tilde{z}) = \frac{1}{2}[\tilde{z}_1^T, \tilde{z}_2^T] \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}.$$

Then, from the assumption (18),

$$\dot{W} = [\tilde{z}_1^T, \tilde{z}_2^T] \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} g_1(y, \hat{z}, u) - f_1(z, u) \\ g_2(y, \hat{z}, u) - f_2(z, u) \end{bmatrix} \leq -k_1|\tilde{z}|^2.$$

We rewrite the above when $\tilde{z}_1 = 0$:

$$\begin{aligned} \dot{W}_{\tilde{z}_1=0} &= \tilde{z}_2^T P_2(f_1(z_1, \hat{z}_2, u) - f_1(z_1, z_2, u)) \\ &\quad + \tilde{z}_2^T P_3(f_2(z_1, \hat{z}_2, u) - f_2(z_1, z_2, u)) \\ &= \tilde{z}_2^T P_3[(f_2(z_1, \hat{z}_2, u) - f_2(z_1, z_2, u)) \\ &\quad + P_3^{-1} P_2(f_1(z_1, \hat{z}_2, u) - f_1(z_1, z_2, u))] \\ &\leq -k_1|\tilde{z}_2|^2. \end{aligned} \tag{19}$$

This inequality implies Assumption 1 with $e = \tilde{z}_2$, $V(e) = \frac{1}{2}e^T P_3 e$, and $l(z_1) = P_3^{-1} P_2 z_1$, which completes the proof. ■

Now we catalog some classes of systems for which the functions V and l of Assumption 1 are easily obtained.

1) Systems (1) that can be transformed into the form of

$$\dot{x} = Ax + f(y, u), \quad y = Cx$$

with (A, C) detectable. For this class, the observer design by Linearized Error Dynamics [10] can be applied to obtain the full-order observer $\dot{\hat{x}} = A\hat{x} + f(y, u) + L(C\hat{x} - y)$ with $(A + LC)$ Hurwitz. This design also yields a quadratic error Lyapunov function $W(e) = \frac{1}{2}e^T Pe$ with $\dot{W} \leq -k_1|e|^2$, $k_1 > 0$. Therefore, Lemma 1 guarantees Assumption 1.

2) Systems (1) that can be transformed into the form

$$\begin{aligned} y &= x_1 \\ \dot{x}_1 &= x_2 + f_1(x_1, u) \\ \dot{x}_2 &= x_3 + f_2(x_1, x_2, u) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_1, \ldots, x_{n-1}, u) \\ \dot{x}_n &= f_n(x, u) \end{aligned} \tag{20}$$

where f_i's are globally Lipschitz uniformly with respect to x. These systems admit the High-Gain Observer design, studied in [5] and [8]. It has been shown in [5] that the high-gain (full-order) observer yields a quadratic error Lyapunov function for the assumption of Lemma 1, hence Assumption 1 holds.

3) Systems (1) that can be transformed into the form

$$\dot{x} = Ax + G\gamma(Hx) + f(y, u), \qquad y = Cx$$

where (A, C) is detectable, $\gamma(\cdot)$ is decentralized (in the sense defined in [1]), and $\gamma_i(\cdot)$ is nondecreasing. For this class, the Circle Criterion Observer design has been proposed in [1] under the condition that there exist Γ, L and K such that the system $\dot{e} = (A + LC)e - Gv$, $z = \Gamma(H + KC)e$ has a strictly positive real transfer function. Then, the full-order observer is given by

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + G\gamma(H\hat{x} + K(C\hat{x} - y)) + f(y, u).$$

The proof in [1] uses the Circle Criterion, and since the Circle Criterion yields a quadratic Lyapunov function with a quadratic upper bound on its derivative, the assumption of Lemma 1 also holds for this case, from which Assumption 1 follows.

4) A motivation for the format of Assumption 1 dates back to [19], in which the following condition is given for nonlinear observer design: For the plant (5), there exists a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\tilde{x}^T P \frac{\partial f}{\partial x}(x, u)\tilde{x} \leq -k_1|\tilde{x}|^2,$$

$$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \forall \tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : \tilde{x}_1 = 0\}$$

with $k_1 > 0$, where $\tilde{x}^T = [\tilde{x}_1^T, \tilde{x}_2^T]$ with $\tilde{x}_1 \in \mathbb{R}^p$. Using the mean-value theorem, it can be shown that the above condition implies Assumption 1 similarly to the proof of Lemma 1.

V. QUANTIZED OUTPUT FEEDBACK

Let the plant be represented again in the form (5) after a suitable coordinate transformation. By an *output quantizer* we mean a piecewise constant function $q : \mathbb{R}^p \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^p$. We introduce the quantization error

$$d := q(y) - y. \tag{21}$$

As in [11], [12], we assume that there exist positive numbers M and Δ (called the quantizer's *range* and *error bound*) such that the following condition holds:

$$|y| \leq M \quad \Longrightarrow \quad |d| \leq \Delta. \tag{22}$$

Suppose that Assumption 1 holds and a quasi-ISS observer (8) has been designed as in Theorem 1. With d given by (21), this observer acts on the quantized output measurements

$$\bar{y} = q(y).$$

Since the quantizer saturates outside a bounded region in the output space (the ball of radius M around the origin), we must work on this bounded region and the quasi-ISS formulation will turn out to be adequate.

Next, suppose that a "nominal" controller (i.e., the controller that we would apply if the state z were exactly known) is given in the form of a static feedback $u = k(z)$. We can now define a (dynamic) quantized output feedback law by

$$u = k(\hat{z}) = k(z + \tilde{z})$$

where $\hat{z}$ is the state estimate generated by the observer and $\tilde{z} = \hat{z} - z$ is the state estimation error. We impose the following assumption on the feedback law k.

Assumption 2 (ISS controller) *The system*

$$\dot{z} = f(z, k(\hat{z})) = f(z, k(z + \tilde{z}))$$

is ISS with respect to $\tilde{z}$, i.e.,

$$|z(t)| \leq \hat{\beta}(|z(0)|, t) \vee \hat{\gamma}\left(\|\tilde{z}\|_{[0,t]}\right) \tag{23}$$

for some $\hat{\beta} \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}_\infty$.

In other words, our state feedback law should provide ISS with respect to a measurement disturbance, which in our case is the observer's state estimation error. We refer the reader to [12] for a detailed discussion of Assumption 2 and an overview of relevant literature.

The overall closed-loop system obtained by combining the plant, the observer, and the control law can be written as

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_1(z_1, z_2, k(\hat{z})) \\ f_2(z_1, z_2, k(\hat{z})) \end{bmatrix} \\ \dot{\hat{\xi}} &= f_2(q(z_1), \hat{\xi} - l(q(z_1)), k(\hat{z})) \\ &\quad + \frac{\partial l}{\partial z_1}(q(z_1)) f_1(q(z_1), \hat{\xi} - l(q(z_1)), k(\hat{z})) \\ \hat{z} &= \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \begin{bmatrix} q(z_1) \\ \hat{\xi} - l(q(z_1)) \end{bmatrix}. \end{aligned} \tag{24}$$

We know from the proof of Theorem 1 that for $e = \hat{\xi} - \xi$, where $\xi = z_2 + l(z_1)$, the bound (10) holds. Combining this with (23) and the first inequality in (11), we can show that

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \beta\left(\left|\begin{pmatrix} z(0) \\ \hat{\xi}(0) \end{pmatrix}\right|, t\right) \vee \gamma_K\left(\|d\|_{[0,t]}\right) \tag{25}$$

where $\beta \in \mathcal{KL}$ and γ_K a class-$\mathcal{K}_\infty$ function valid for $\|z\|_{[0,t]} \leq K$ and $\|u\|_{[0,t]} = \|k(\hat{z})\|_{[0,t]} \leq K$. To arrive at (25), it is enough to apply a standard ISS cascade argument (cf. [17]) working in the (z, e)-coordinates. Below we will use the obvious fact that $\beta(s, 0) \geq s$ for all s.

Take κ_l to be some class-$\mathcal{K}_\infty$ function with the property that

$$|l(z_1)| \leq \kappa_l(|z_1|) \qquad \forall\, z_1.$$

Similarly, take κ_u to be some class-$\mathcal{K}_\infty$ function with the property that

$$|k(z)| \leq \kappa_u(|z|) \qquad \forall\, z.$$

Let

$$K := M \vee \kappa_u(2M + \Delta + \kappa_l(M + \Delta)). \tag{26}$$

We are now ready to state the following result, which provides an ultimate bound on the solutions of the closed-loop system starting in a suitable region.

Proposition 1 *With M and Δ as in* (22) *and K as defined in* (26)*, assume that*

$$\gamma_K(\Delta) < M. \tag{27}$$

Suppose that the initial condition of the closed-loop system (24) *satisfies*

$$\left|\begin{pmatrix} z(0) \\ \hat{\xi}(0) \end{pmatrix}\right| < E_0 \tag{28}$$

where $E_0 > 0$ is such that

$$\beta(E_0, 0) = M. \tag{29}$$

Then the corresponding solution satisfies

$$\limsup_{t \to \infty} \left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \gamma_K(\Delta). \tag{30}$$

Proof: As long as the inequality

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq M$$

is true, we have

$$|y(t)| = |z_1(t)| \leq M$$

hence by (22)

$$|d(t)| = |q(z_1(t)) - z_1(t)| \leq \Delta\,, \tag{31}$$

and we also have

$$|z(t)| \leq M \leq K$$

and

$$\begin{aligned} |u(t)| = |k(\hat{z}(t))| &\leq \kappa_u(|\hat{z}(t)|) \\ &\leq \kappa_u(|q(z_1(t))| + |\hat{\xi}(t)| + |l(q(z_1(t)))|) \\ &\leq \kappa_u(M + \Delta + M + \kappa_l(M + \Delta)) \leq K. \end{aligned}$$

Define the time

$$T := \sup\left\{t \geq 0 : \left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| < M\right\} \leq \infty$$

which is well defined because of (28) and since $E_0 \leq \beta(E_0, 0) = M$. For $t \in [0, T]$, by the above calculations we know that the bounds (25) and (31) are valid. In view of (27), (28), and (29) we have

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| < M \qquad \forall\, t \in [0, T].$$

If T were finite, this would be a contradiction, hence $T = \infty$ and the above analysis is valid for all time.

Since $\beta \in \mathcal{KL}$, for every $\varepsilon > 0$ there exists a time $T(\varepsilon)$ such that

$$\beta\left(\left|\begin{pmatrix} z(0) \\ \hat{\xi}(0) \end{pmatrix}\right|, t\right) \leq \varepsilon \qquad \forall\, t \geq T(\varepsilon) \tag{32}$$

hence

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \varepsilon \vee \gamma_K(\Delta) \qquad \forall\, t \geq T(\varepsilon).$$

This proves (30). ■

For the ultimate bound (30) to guarantee contraction, we need to know that

$$\gamma_K(\Delta) < E_0.$$

In view of (29) this requires

$$\beta(\gamma_K(\Delta), 0) < M \tag{33}$$

which is a strengthening of (27). Note that γ_K depends on K which in turn depends on M, i.e., M affects both sides of the inequality (33). However, we can always satisfy (33) by selecting Δ to be small enough (compared to M). Or, if the left-hand side of (33) grows slower than linearly in M, it is enough to choose M large enough. Basically, (33) then means that we must have sufficiently many quantization regions. The same comments apply to the condition (27).

The above result is especially useful in situations where the quantization can be *dynamic*, in the sense that the parameters of the quantizer can be changed on-line by the control designer [11]. We can then improve on the ultimate bound (30) by using a "zooming" strategy. Consider a dynamic quantizer

$$q_\mu(y) := \mu q\Big(\frac{y}{\mu}\Big)$$

where μ is a "zoom" variable. This new quantizer has range $M\mu$ and error bound $\Delta\mu$. Because of (33) we can find a value $\mu < 1$ for which

$$\beta(\gamma_K(\Delta), 0) < M\mu.$$

We know there is a time $\bar{t}$ at which we will have

$$\left|\begin{pmatrix} z(\bar{t}) \\ \hat{\xi}(\bar{t}) \end{pmatrix}\right| < E_\mu$$

where E_μ is such that

$$\beta(E_\mu, 0) = M\mu.$$

Calculating this time is a matter of having some knowledge of β and using its property (32) with ε small enough. (An alternative approach using ISS-Lyapunov functions is described in [12].)

Redefine K using $M\mu$ and $\Delta\mu$ instead of M and Δ, and call it K_μ. Applying the same analysis as before with $M\mu$, K_μ, $\Delta\mu$ instead of M, K, Δ, respectively, for $t \geq \bar{t}$, we obtain the smaller ultimate bound $\gamma_{K_\mu}(\Delta\mu)$. We can then pick a smaller value of μ and repeat the procedure. In principle, we can decrease μ to 0 in this way and obtain asymptotic convergence. We can also use "zooming out" to increase μ initially if (28), (29) do not hold at $t = 0$. Being able to do this requires strengthening (33) to

$$\beta(\gamma_{K_\mu}(\Delta\mu), 0) < M\mu \qquad \forall\, \mu > 0.$$

Further details on this zooming procedure (in a slightly different setting) can be found in [12]. In practice, there are limitations on how small or how large the zoom variable can be, and these would determine the size of the ultimate bound and the region of attraction.

Finally, we revisit Example 2 from Section III. It is shown in [2] that for the system (14), the feedback law

$$u = k(z) = -z_1 - z_2 - z_2^3$$

satisfies our Assumption 2. Therefore, by using this nominal controller together with the observer (15), the system can be stabilized by quantized output feedback as shown in this section.

VI. Conclusions

We formulated and studied the notion of a quasi-ISS observer for nonlinear systems with additive output disturbances. We proposed a quasi-ISS reduced-order observer design based on a coordinate transformation and the existence of a state-independent error Lyapunov function. We showed that this construction applies to several classes of nonlinear systems previously studied in the observer design literature. As an application, we proved that quantized output feedback stabilization is achievable when the system possesses a quasi-ISS reduced-order observer and a state feedback controller providing ISS with respect to measurement errors. We included an example for which both of these assumptions are satisfied and the design could be carried through.

References

[1] M. Arcak and P.V. Kokotovic, "Nonlinear observers: a circle criterion design and robustness analysis," *Automatica*, vol. 37, pp. 1923–1930, 2001.

[2] C. Ebenbauer, T. Raff, and F. Allgower, "Certainty-equivalence feedback design with polynomial-type feedbacks which guarantee ISS," *IEEE Trans. Automat. Control*, vol. 52, pp. 716–720, 2007.

[3] R.A. Freeman and P.V. Kokotovic, "Global robustness of nonlinear systems to state measurement disturbances," *Proc. 32nd IEEE Conf. on Decision and Control,* pp. 1507–1512, 1993.

[4] R.A. Freeman and P.V. Kokotovic, *Robust Nonlinear Control Design: State-space and Lyapunov Techniques*, Birkhauser, Boston, 1996.

[5] J.P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems: Applications to bioreactors," *IEEE Trans. Automat. Contr.*, vol. 37, no. 6, pp. 875-880, 1992.

[6] Z.-P. Jiang, I.M.Y. Mareels, and D. Hill, "Robust control of uncertain nonlinear systems via measurement feedback," *IEEE Trans. Automat. Control*, vol. 44, pp. 807–812, 1999.

[7] D. Karagiannis and A. Astolfi, "Nonlinear observer design using invariant manifolds and applications," *Proc. 44th IEEE Conf. on Decision and Control and European Control Conference*, Spain, pp. 7775-7780, 2005.

[8] H. Khalil, "High-gain observers in nonlinear feedback control," in *New Directions in Nonlinear Observer Design* (H. Nijmeijer and T. I. Fossen, Eds.), Springer, Berlin, 1999, pp. 249–268.

[9] H.K. Khalil, *Nonlinear Systems*, Prentice Hall, New Jersey, 3rd edition, 2002.

[10] A.J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Systems & Control Letters,* vol. 3, pp. 47–52, 1983.

[11] D. Liberzon, Hybrid feedback stabilization of systems with quantized signals, *Automatica*, 39:1543–1554, 2003.

[12] D. Liberzon, "Observer-based quantized output feedback control of nonlinear systems," *Proc. 17th IFAC World Congress*, 2008.

[13] L. Praly, "On observers with state independent error Lyapunov function," *IFAC Symp. on Nonlinear Control Systems Design (NOLCOS)*, pp. 1425–1430, 2001.

[14] R.G. Sanfelice and A.R. Teel, "On hybrid controllers that induce input-to-state stability with respect to measurement noise," *Proc. 44th IEEE Conf. on Decision and Control*, pp. 4891–4896, 2005.

[15] H. Shim and L. Praly, "Remarks on equivalence between full order and reduced order nonlinear observers," *Proc. 42nd IEEE Conf. on Decision and Control*, pp. 5837–5840, 2003.

[16] H. Shim, J.H. Seo, and A.R. Teel, "Nonlinear observer design via passivation of error dynamics," *Automatica*, vol. 39, pp. 885-892, 2003.

[17] E.D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, pp. 435–443, 1989.

[18] E.D. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization," *Systems & Control Letters,* vol. 13, pp. 117–123, 1989.

[19] J. Tsinias, "Observer design for nonlinear systems," *Systems & Control Letters,* vol. 13, pp. 135–142, 1989.