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Stabilizing Uncertain Systems with Dynamic Quantization

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***Abstract*—We consider state feedback stabilization of uncertain linear systems with quantization. The plant uncertainty is dealt with by the supervisory control framework, which employs switching among a finite family of candidate controllers. For a static quantizer, we quantify a relationship between the quantization range and the quantization error bound to guarantees closed loop stability. Using a dynamic quantizer which can vary the quantization parameters in real time, we show that the closed loop can be asymptotically stabilized, provided that additional conditions on the quantization range and the quantization error bound are satisfied. Our results extend previous results on stabilization of known systems with quantization to the case of uncertain systems.**

I. INTRODUCTION

Most of the work in control over network and control with limited information (see, e.g., the survey papers [6], [16]) deal with known plants, and only recently, attempts have been made to study control of *uncertain systems* with limited information. While there are several aspects in control with limited information (such as signal quantization, sampling, transmission delays and data loss), dealing with both plant uncertainty and limited information at the same time is rather challenging. As a first step, we treat limited information as quantization only. Quantized control systems with known plants have been considered, for example, in [3], [4], [15], [17], [21]. In this paper, we consider the problem of *stabilizing uncertain systems* with *quantization*. This problem was also studied by Hayakawa et al in [5], where the authors provided a solution using a (static) logarithmic quantizer and a Lyapunov-based adaptive algorithm.

We are interested in the case of uncertain systems with large uncertainty so that robust control is not sufficient, and adaptive control is required. In this paper, we use the supervisory control framework [8] to deal with plant uncertainty. A supervisory control scheme employs switching among a finite family of candidate controllers, and the switching is orchestrated by a switching logic based on comparison of the estimation errors coming out of a multi-estimator; see [8] for discussions on advantages and applications of supervisory control.

For a static quantizer, we wish to find a relationship between the quantization range and the quantization error bound to guarantee closed loop stability. While it has been shown [7, Proposition 6] that supervisory control is robust to measurement noise, extending this result to quantization is not trivial because one needs to ensure that the information to be quantized does not exceed the quantization range. We will give a condition on the quantizer parameters to guarantee closed loop stability.

To achieve asymptotic stability, we utilize the *dynamic quantizers* [2], [12], which have the capability of varying the quantization parameters in real time (in particular, the quantizer can zoom in and zoom out while keeping the number of alphabets fixed). In the works [2], [12], the authors have applied dynamic quantization to asymptotically stabilize known linear plants (see also [11] for performance analysis of dynamic quantization). For known linear plants, asymptotic stability can also be achieved with a logarithmic quantizer [3] (which was the quantizer employed in [5] for the case of uncertain systems). Compared to the logarithmic quantizers, which have infinite alphabets, a dynamic quantizer has a finite alphabet. We show that for uncertain systems with quantization, asymptotic stability is achievable with supervisory control and dynamic quantization, provided that the quantizer satisfies a certain condition. While the tools for analyzing supervisory control and dynamic quantization have been reported separately [7], [12], the analysis of the combination of both is far from a trivial extension of [7] and [12].

The notations in this paper are fairly standard: $\mathbb{R}$ is the set of real numbers, $|\cdot|$ is the Euclidean norm, and $\|\cdot\|_{\mathcal{I}}$ is the supnorm of a signal over the interval $\mathcal{I} \subseteq [0, \infty)$.

II. QUANTIZED CONTROL SYSTEM

To convey our idea, we start with a simple setting in which 1) the uncertain plant is linear and belongs to a known finite set of plants and 2) the full state is available for measurement. We include discussions on how the result in this paper can be extended to more

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general settings such as continuum uncertainty sets and output feedback in later sections.

Consider an uncertain linear plant Γ_p parameterized by a parameter p, and denote by p^* the true but unknown parameter:

$$\Gamma_{p^*}: \begin{cases} \dot{x} = A_{p^*}x + B_{p^*}u \\ y = x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, and $y \in \mathbb{R}^{n_y}$ is the output (in the state feedback case, $n_y = n_x$). The parameter $p^* \in \mathbb{R}^{n_p}$ belongs to a known finite set $\mathcal{P} := \{p_1, \ldots, p_m\}$, where m is the cardinality of $\mathcal{P}$.

Assumption 1 *(A_p, B_p) is stabilizable for every $p \in \mathcal{P}$.*

A (static) quantizer is a map $\mathrm{Q} : \mathbb{R}^{n_y} \to \{q_1, \ldots, q_N\}$, where $q_1, \ldots, q_N \in \mathbb{R}^{n_y}$ are quantization points, and Q has the following properties: 1) $|y| \leqslant M \Rightarrow |\mathrm{Q}(y) - y| \leqslant \Delta$, and 2) $|y| > M \Rightarrow |\mathrm{Q}(y)| > M - \Delta$. The numbers M and Δ are known as *the range* and the *error bound* of the quantizer Q. A *dynamic quantizer* Q_ν, which has an additional parameter ν that can be changed over time, is defined as

$$\mathrm{Q}_{\nu(t)}(z) := \nu(t)\mathrm{Q}\big(z/\nu(t)\big) \quad \forall t,$$

where Q is a static quantizer with the range M and the error bound Δ. From the property 1) of the quantizer, we have

$$|y| \leqslant \nu M \Rightarrow |\mathrm{Q}(y) - y| \leqslant \nu\Delta. \tag{2}$$

The parameter ν is known as a *zooming variable*: increasing ν corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing ν corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error.

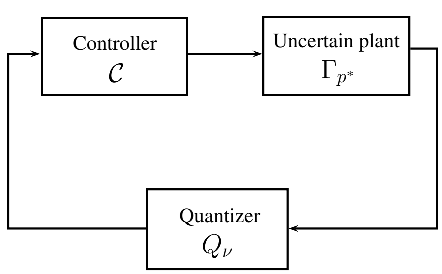

Fig. 1. Quantized control closed-loop system

Assuming that the plant is unstable, the objective is to asymptotically stabilize the plant while the information available to the controller is $\mathrm{Q}_\nu(y)$ instead of y. The quantized control system is depicted in Fig. 1, where $\mathcal{C}$ denotes the controller.

III. Supervisory control

A. Without quantization

We describe a supervisory (adaptive) control scheme (see Fig. 2) in which the uncertain plant belongs to a finite set of nominal models and there is no quantization; for further background on supervisory control, see, e.g., [14, Chapter 6] or [8] and the references therein.

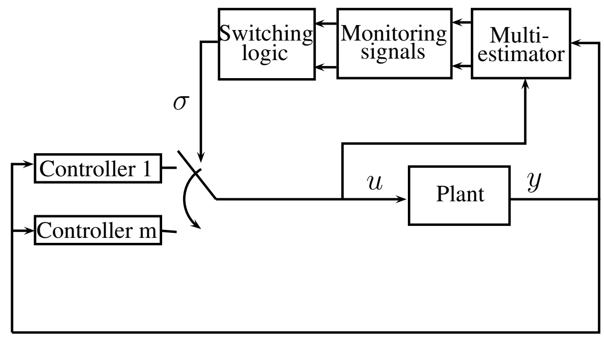

Fig. 2. The supervisory control framework

The supervisor comprises a multi-estimator, monitoring signals, and a switching logic. Below is one particular supervisory control design for the system (1).

- **Multi-estimator**: A multi-estimator is a collection of dynamics, one for each fixed parameter $p \in \mathcal{P}$. The multi-estimator takes in the input u and produces a bank of outputs $y_p, p \in \mathcal{P}$. The multi-estimator should have the following matching property: there is $\hat{p} \in \mathcal{P}$ such that

$$|y_{\hat{p}}(t) - y(t)| \leqslant c_e e^{-\lambda_e(t-t_0)}|y_{\hat{p}}(t_0) - y(t_0)| \tag{3}$$

for all $t \geqslant t_0$, for all u, and for some $c_e \geqslant 0$ and $\lambda_e > 0$. One such multi-estimator for (1) is the following dynamics

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u - (A_p + I)(y_p - y), \\ y_p &= x_p, \end{aligned} \tag{4}$$

for all $p \in \mathcal{P}$, and the property (3) is satisfied with $\hat{p} = p^*$, $c_e = 1$, and $\lambda_e = 1$ (since (4) with $p = p^*$ and (1) imply that $(d/dt)(x_{p^*} - x) = -(x_{p^*} - x)$ and $y = x$). The identity matrix I in (4) can be replaced by any Hurwitz matrix; we use the identity matrix there for convenience.
- **Multi-controller**: A family of *candidate feedback gains* $\{K_p\}$ is designed such that $A_p + B_pK_p$ are Hurwitz for every $p \in \mathcal{P}$. Then the *family of controllers* is

$$u_p = K_p x_p \quad p \in \mathcal{P}. \tag{5}$$

- **Monitoring signals**: *Monitoring signals* $\mu_p, p \in \mathcal{P}$ are norms of the output estimation errors, $y_p - y$. Here, the monitoring signals are generated as

$$\mu_p := \varepsilon + \int_0^t e^{-\lambda(t-s)}\gamma|y_p(s) - y(s)|^2 ds \quad (6)$$

for some $\gamma, \varepsilon, \lambda > 0$. The numbers γ, ε, and λ are design parameters and need to satisfy

$$0 < \lambda < \lambda_0 \quad (7)$$

for some constant λ_0 related to the eigenvalues of $A_p + B_p K_p, p \in \mathcal{P}$ (for detail on λ_0, see the appendix).

- **Switching logic**: A switching logic produces a switching signal that indicates at every time the active controller. In this paper, we use the *scale-independent hysteresis switching logic* [10]:

$$\sigma(t) := \begin{cases} \operatorname*{argmin}_{q\in\mathcal{P}} \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else,} \end{cases} \quad (8)$$

where $h > 0$ is a *hysteresis constant*; h is a design parameter and satisfies the following condition:

$$\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu_V}{\lambda_0 - \lambda} \quad (9)$$

for some constant μ_V (see the appendix for the definition of μ_V). The control signal applied to the plant is $u(t) = u_\sigma(t) := K_{\sigma(t)} x_{\sigma(t)}(t)$.

B. With quantization

With quantization, the multi-estimator (4) becomes

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u - (A_p + I)(y_p - \mathrm{Q}_\nu(y)), \\ y_p &= x_p, \end{aligned} \qquad p \in \mathcal{P}. \quad (10)$$

The above equation can be rewritten as $\dot{x}_p = A_p x_p + B_p u - (A_p + I)(y_p - y + y - \mathrm{Q}_\nu(y)), \quad p \in \mathcal{P}$. Due to the presence of $y - \mathrm{Q}_\nu(y)$ in the foregoing equation, the matching condition (3) becomes

$$\begin{aligned} |y_{\hat{p}}(t) - y(t)| &\leqslant c_e e^{-\lambda_e(t-t_0)}|y_{\hat{p}}(t_0) - y(t_0)| \\ &\quad + \gamma_e \|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)} \quad \forall t \geqslant t_0, \forall u \end{aligned} \quad (11)$$

for some $c_e, \gamma_e \geqslant 0$, $\lambda_e > 0$. Similarly as before, the condition (11) is satisfied with $\hat{p} = p^*$, $c_e = 1$, $\lambda_e = 1$, and $\gamma_e = \|A_{p^*} + I\|$. The monitoring signal generator (6) becomes

$$\mu_p = \varepsilon + \int_0^t e^{-\lambda(t-s)}\gamma|y_p(s) - \mathrm{Q}_\nu(y(s))|^2 ds. \quad (12)$$

IV. Stability of supervisory control with quantization

Denote by $\mathcal{K}(c)$ the class of continuous functions from $\mathbb{R}^\ell$ to $\mathbb{R}$ for some ℓ such that, if $f \in \mathcal{K}(c)$, then $f(z) \to c$ as $|z| \to 0$. We have the following result concerning a static quantizer (i.e. with a fixed zooming variable ν). Let $x_{\mathbb{E}} := (x_{p_1}, \ldots, x_{p_m})^T$ for some ordering $p_1, \ldots, p_m$ of $\mathcal{P}$.

Theorem 1 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section III with the design parameters satisfying* (7) *and* (9)*. Let t_0 be an arbitrary time, and suppose that $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$ and $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ for some constants $\bar{x}_0, \bar{\mu}_0 > 0$. Let $X_0 > 0$ and $\bar{y}_0 := X_0 + \bar{x}_0$. Suppose that the zooming variable ν is fixed. There exist a function $\chi_{\Delta,\nu} \in \mathcal{K}(a_1\sqrt{\varepsilon} + a_2\nu\Delta)$ for some positive constants a_1 and a_2, and a function $\psi_{\Delta,\nu} \in \mathcal{K}(a_3\sqrt{\varepsilon} + a_4\nu\Delta)$ for some positive constants a_3 and a_4 such that if*

$$\chi_{\Delta,\nu}(\bar{x}_0, \bar{\mu}_0, \bar{y}_0) < \nu M, \quad (13)$$

then $\forall |x(t_0)| \leqslant X_0$, all the closed-loop signals are bounded, and for every $\epsilon_x > 0$, $\exists T < \infty$ such that

$$|x(t)| \leqslant \psi_{\Delta,\nu}(\bar{\mu}_0, \bar{y}_0) + \epsilon_x \quad \forall t \geqslant t_0 + T. \quad (14)$$

Remark 1 *To better convey the idea and not get bogged down in complicated details, we do not give the explicit formulae for $\chi_{\Delta,\nu}$ and $\psi_{\Delta,\nu}$ in the theorem; see the appendix for details (equations* (25) *and* (26)*). Note from* (26) *that $\psi_{\Delta,\nu}$ implicitly depends on $\bar{x}_0$ via $\bar{y}_0$ in c_2. There are two interpretations of the condition* (13)*: 1) for a given M, Δ, and ν such that $\nu M > a_1\sqrt{\varepsilon} + a_2\nu\Delta$, there exists small enough $\bar{x}_0, \bar{y}_0$, and $\bar{\mu}_0$ such that* (13) *holds (this follows from the property that $\chi_{\Delta,\nu} \in \mathcal{K}(a_1\sqrt{\varepsilon} + a_2\nu\Delta)$), and 2) for a given $\bar{x}_0, \bar{y}_0$, and $\bar{\mu}_0$, the condition* (13) *holds if M is large enough ($\chi_{\Delta,\nu}$ does not depend on M).*

The proof of Theorem 1 comprises four main stages:

- We establish a bound on the signal $\mu_{\hat{p}}$ in terms of Δ using (2) and (11)
- We then establish a bound on the state $x_{\mathbb{E}}$ (which is known as the state of the *injected system*; see the appendix) in terms of the error bound Δ
- We show that the condition (13) on M and Δ ensures that the state x cannot get out of the ball of radius νM (and hence, the quantizer guarantees the error bounded for all time)
- From boundedness of $x_{\mathbb{E}}$, we finally conclude ultimate boundedness of the plant state x.

Technical details of the proof are interesting as it combines the techniques in supervisory control and dynamic quantization. Due to space limitation, the interested reader is referred to the full version of this paper [20] for the proof.

The importance of Theorem 1 is that it provides a condition on a static quantizer with fixed ν (this condition depends on the bounds on initial states) that guarantees closed loop stability. More precisely, we achieve not just boundedness but ultimate boundedness, characterized by (14). Note that the ultimate bound in (14) can be larger than X_0, which is due to quantization error. For state contraction, we need additional constraints on M, Δ, and the initial state bounds. If we do have state contraction for a fixed ν, then one can achieve asymptotic stability by using a dynamic quantizer, varying the zooming variable ν as well as ε in the supervisory control scheme as x gets closer to zero. Unlike the case for known systems [12] where one only needs to worry about the contraction of the plant state x, here one needs to take into account asymptotic behaviors of variables from the supervisory control scheme such as μ_p and $|y_p - y|$.

A *logarithmic* scalar variable ξ with a factor ρ and a period T is defined as follows (c.f. [3]):

$$\xi(t) := \begin{cases} \xi(kT) & \text{if } t \in [kT, (k+1)T) \\ \rho\xi(kT) & \text{if } t = (k+1)T, \end{cases} \quad k = 0, 1, \dots. \tag{15}$$

The following result says that using a dynamic quantizer with a logarithmic zooming variable, we can achieve asymptotic stability of the closed loop.

Theorem 2 *Consider the uncertain system* (1) *and the supervisory control scheme described in Section III with the design parameters satisfying* (7) *and* (9)*. Let t_0 be an arbitrary time, and suppose that $|x_{\mathbb{E}}(t_0)| \leqslant \bar{x}_0$ and $|\mu_{\hat{p}}(t_0)| \leqslant \bar{\mu}_0$ for some constants $\bar{x}_0, \bar{\mu}_0 > 0$. Let $X_0 > 0$ and $\bar{y}_0 := X_0 + \bar{x}_0$. There exist a function $\chi_{\Delta,\nu} \in \mathcal{K}(a_1\sqrt{\varepsilon} + a_2\nu\Delta)$ for some positive constants a_1 and a_2, a function $\psi_{\Delta,\nu} \in \mathcal{K}(a_3\sqrt{\varepsilon} + a_4\nu\Delta)$ for some positive constants a_3 and a_4, and positive constants a_5, a_6, and a_7 such that if* (13) *holds and*

$$\psi_{\Delta,\nu}(\bar{\mu}_0, \bar{y}_0) < \bar{x}_0, \tag{16a}$$

$$a_5\varepsilon + a_6\nu^2\Delta^2 < \bar{\mu}_0, \tag{16b}$$

$$a_7\nu\Delta < \bar{y}_0, \tag{16c}$$

then there exist $\rho \in (0, 1)$ and $0 < T < \infty$ such that under the logarithmic ε with factor ρ^2 and period T, and the logarithmic zooming variable μ with factor ρ and period T, for all $|x(0)| \leqslant X_0$, we have $|x(t)| \to 0$ as $t \to \infty$, and all the closed-loop signals are bounded.

Remark 2 *As before, for clarity of the presentation, we put the exact formulae for $\chi_{\Delta,\nu}$ and $\psi_{\Delta,\nu}$ in the appendix (they are the same as $\chi_{\Delta,\nu}$ and $\psi_{\Delta,\nu}$ in Theorem 1).*

Remark 3 *As discussed in Remark 1,* (13) *can always be satisfied for large enough M or small enough $\bar{x}_0, \bar{\mu}$, and $\bar{y}_0$. However, $\bar{x}_0, \bar{\mu}$, and $\bar{y}_0$ also need to be lower bounded as in* (16a), (16b), *and* (16c). *Nevertheless, $\psi_{\Delta,\nu} \to 0$ as $\{\Delta, \varepsilon\} \to 0$ so for a given $\bar{x}_0, \bar{\mu}$, and $\bar{y}_0$,* (16a), (16b), *and* (16c) *hold if Δ is small enough. Compared to Theorem 1, the extra conditions* (16a), (16b), *and* (16c) *place an upper bound on Δ for given $\bar{x}_0, \bar{\mu}$, and $\bar{y}_0$ to ensure that the signals in the supervisory control system are contracting after a certain time. Combining this contracting property with the zooming-in technique, we achieve asymptotic stability. In Theorem 1, this contraction is not needed when one is only concerned with stability, not asymptotic stability.*

Remark 4 *The conditions* (13) *and* (16) *on M and Δ imply a lower bound on the number of quantization bits. Suppose that each component of x has the same range and is equally quantized into 2^{n_Q} regions using n_Q quantization bits. Then $n_Q = \log_2\lceil M/\Delta\rceil$. Then the condition* (13) *and* (16) *can be rewritten into the form $n_Q > \log_2\lceil \chi_{\Delta,\nu}(\bar{x}_0, \bar{\mu}_0, \bar{y}_0)/(\nu\Delta)\rceil$.*

Remark 5 *If the bound X_0 on the initial state is not available, we can include a zooming-out stage at the beginning (see [12]) so that after a certain time t_0, we guarantee $|x(t_0)| < \nu M$. This means increasing ν faster than the system can blow up (for any value of $p \in \mathcal{P}$) until the quantizer no longer saturates.*

Remark 6 *The result in this section can also be extended to the output feedback case, where the uncertain plant with partial output measurement is*

$$\Gamma : \begin{cases} \dot{x} = A_{p^*}x + B_{p^*}u \\ y = C_{p^*}x. \end{cases} \tag{17}$$

Details are omitted due to space limitation. See the full version [20] of this paper for more details.

Remark 7 *The results in this paper can also be extended to cover nonlinear plants, under certain assumptions. For nonlinear plants, the input-to-state stability (ISS) framework (see, e.g., [18]) is used instead of* *the affine relation between inputs and states. For example, the constant gain* γ_e *in* (11) *is replaced by a function* γ_e *of certain class of functions (technically, the estimator is ISS with respect to the quantization error). The work [19] has applied the ISS framework to treat supervisory control of nonlinear plants with disturbances. Likewise, using ISS, the work [12] has provided a general framework for handling quantized feedback control in nonlinear systems (state feedback case). Our framework in this paper allows us to put the developments of [12] and [19] together, but there is the complication that we need to keep the state in the range of the quantizer. This technical difficulty can be handled in the same way it has done in Theorem 1 and Theorem 2 for the linear case, although the details are a bit messy. Due to space limitation, such results for nonlinear systems are not included in this paper; see the full version [20] for further details.*

V. Continuum uncertainty set

So far, we have assumed that the set $\mathcal{P}$ is finite. For the case of continuum uncertainty sets, under a certain robustness assumption, we can still achieve asymptotic stability. To utilize notations in the previous sections, denote a continuum uncertainty set by $\Omega \subseteq \mathbb{R}^{n_p}$ and denote by $\mathcal{P}$ a finite index set such that $\bigcup_{i\in\mathcal{P}} \Omega_i = \Omega$ for some Ω_i, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. How to divide Ω into Ω_i and what the number of subsets is are interesting research questions of their own and are not pursued here (see [1]). For every subset Ω_i, pick a nominal value p_i. By this procedure, we obtain a finite family of nominal plants, $\{P(p_1), \dots, P(p_m)\}$. The difference here compared to the assumption of $p^* \in \mathcal{P}$ is that we may have no exact matching i.e., $p^* \notin \{p_1, \dots, p_m\}$.

Assumption 2 *There exists an index* $\hat{p} \in \mathcal{P}$ *such that for the plant* $P(p^*)$ *with the observer*

$$\begin{cases} \dot{\hat{x}}_{\hat{p}} = A_{\hat{p}}\hat{x}_{\hat{p}} + B_{\hat{p}}u + L_{\hat{p}}(y_{\hat{p}} - Q_\nu(y)) \\ y_{\hat{p}} = C_{\hat{p}}\hat{x}_{\hat{p}}, \end{cases} \tag{18}$$

we have

$$\begin{aligned} |x(t) - x_{\hat{p}}(t)| \leqslant \bar{c}_e e^{-\lambda_e(t-t_0)}|y(t_0) - y_{\hat{p}}(t_0)| \\ + \gamma_e\|y - Q_\nu(y)\|_{[t_0,t)} \end{aligned} \tag{19}$$

for some $\bar{c}_e, \lambda_e, \gamma_e > 0$ *for all inputs* u.

Basically, the assumption says that there is a robust state estimator in the set $\mathcal{P}$ for the original plant, even if $p^* \notin \mathcal{P}$ (note that for the case $p^* \in \mathcal{P}$, the assumption is exactly the same as (11)). For example, if p^* enters the system matrices continuously, then the assumption above is true if $|p^* - \hat{p}|$ is small enough, and the set $\mathcal{P}$ is finite if, for example, Ω is compact. If Assumption 2 holds, then all the reasonings and results for linear systems in Section IV hold without any modification for a continuum uncertainty set Ω.

For more general continuum uncertainty sets Ω, Assumption 2 above can be relaxed further to include unmodeled dynamics resulting from the parameter mismatch. For example, we may want to require the estimator (18) to be robust with respect to small unmodeled dynamics such that $|x(t) - x_{\hat{p}}(t)| \leqslant \bar{c}_e e^{-\lambda_e(t-t_0)}|y(t_0) - y_{\hat{p}}(t_0)| + \gamma_e\|y - \mathrm{Q}_\nu(y)\|_{[t_0,t)} + \Delta_u\|u\|_{[t_0,t)} + \Delta_x\|\hat{x}_p\|_{[t_0,t)}$ where Δ_u, Δ_x are such that $\{\Delta_u, \Delta_x\} \to 0$ as $\hat{p} \to p^*$; see [9]. Another approach is to use the so-called hierarchical hysteresis switching logic as in [7].

VI. Conclusions

We treated the problem of stabilizing uncertain systems with quantization. We used the supervisory control framework to deal with plant uncertainty. For a static quantizer, we provided a condition between the quantization range and the quantization error bound to guarantee closed loop stability. With a dynamic quantizer, we provided a zooming strategy on the quantization zooming variable ν and on the parameter ε of the supervisory control scheme to achieve closed-loop asymptotic stability.

This work is the first to explore the use of the supervisory control framework and dynamic quantization to tackle the problem of controlling uncertain systems with limited information. Future research can extend this work in several directions. One direction is to consider other types of limited information, such as sampling, delay, or package loss, or a combination of those with quantization. In this direction, it may be fruitful to combine the approach in this paper with the result in [13]. Yet another direction could be relaxing the matching condition and treating the case of supervisory control of uncertain plants with unmodeled dynamics using dynamic quantization.

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APPENDIX

The parameter λ_0 and μ

An *injected system* is obtained by combining the multi-estimator and a candidate controller. For a fixed controller u_q, $q \in \mathcal{P}$, from (5) and (10), the injected system is

$$\dot{x}_p = A_p x_p + B_p K_q x_q - (A_p + I)(y_p - \mathrm{Q}_\nu(y)), \quad \forall\, p \in \mathcal{P}.$$

Recalling that $x_p = y_p$, we can write the above injected system explicitly as

$$\dot{x}_q = (A_q + B_q K_q)x_q - (A_q + I)(x_q - \mathrm{Q}_\nu(x)), \tag{20}$$

$$\dot{x}_p = -x_p + L_{p,q} x_q - (A_p + I)(x_q - \mathrm{Q}_\nu(x)), \quad p \neq q \tag{21}$$

This takes the form

$$\dot{x}_{\mathbb{E}} = \mathbf{A}_q x_{\mathbb{E}} + \mathbf{B}_q (x_q - \mathrm{Q}_\nu(x)), \tag{22}$$

where the definitions of $\mathbf{A}_q$ and $\mathbf{B}_q$ are obvious. It is clear that if $x_q - \mathrm{Q}_\nu(x) = 0$ then $x_q \to 0$ by (20) and then all $x_p \to 0$ by (21), which means that $\mathbf{A}_q$ is Hurwitz (since the system is linear).

Since $\mathbf{A}_p$ are Hurwitz for all p, $\exists\ V_p(x_{\mathbb{E}}) = x_{\mathbb{E}}^T P_p x_{\mathbb{E}}, P_p^T = P_p > 0$ such that

$$\underline{a}|x_{\mathbb{E}}|^2 \leqslant V_p(x_{\mathbb{E}}) \leqslant \overline{a}|x_{\mathbb{E}}|^2 \tag{23a}$$

$$\frac{\partial V_p(x_{\mathbb{E}})}{\partial x}(\mathbf{A}_p x_{\mathbb{E}} + \mathbf{B}_p \tilde{y}_p) \leqslant -\lambda_0 V_p(x_{\mathbb{E}}) + \gamma|\tilde{x}_p|^2 \tag{23b}$$

for some constants $\underline{a}, \overline{a}, \lambda_0, \gamma_0 > 0$ (the existence of such common constants for the family of Lyapunov functions is guaranteed since $\mathcal{P}$ is finite). $\exists\ \mu_V \geqslant 1$ such that

$$V_q(x) \leqslant \mu_V V_p(x) \quad \forall x \in \mathbb{R}^n, \forall p, q \in \mathcal{P}. \tag{24}$$

We can always pick $\mu_V = \overline{a}/\underline{a}$ but there may be other smaller μ_V satisfying (24) (for example, $\mu_V = 1$ if V_p are the same for all p even though $\overline{a}/\underline{a} > 1$).

Formulae for χ_Δ and $\psi_{\Delta,\nu}$

$$\begin{aligned}
\overline{N}_0 &:= 1 + m + \frac{m}{\ln(1+h)} \times \\
&\quad \ln((\varepsilon + \bar{\mu}_0 + 2\gamma c_e^2 \bar{y}_0^2 + 2(\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2)/\varepsilon) \\
c_1 &:= \mu_V^{1+\overline{N}_0} \overline{a}/\underline{a} \\
c_2 &:= \mu_V^{1+\overline{N}_0} m(1+h)/\underline{a} \\
\bar{x}^2 &:= c_1 \bar{x}_0^2 + c_2\varepsilon + c_2\bar{\mu}_0 + 2c_2\gamma c_e^2 \bar{y}_0^2 \\
&\quad + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2
\end{aligned}$$

$$\chi_{\Delta,\nu} := (\bar{x} + c_e\bar{y}_0 + \gamma_e\nu\Delta). \tag{25}$$

$$\psi_{\Delta,\nu} := \gamma_e\nu\Delta + (c_2\varepsilon + (2c_2\gamma(1+\gamma_e)^2/\lambda)\nu^2\Delta^2)^{1/2}. \tag{26}$$