ON ALMOST LYAPUNOV FUNCTIONS

Daniel Liberzon

Joint work with Charles Ying and Vadim Zharnitsky

University of Illinois, Urbana-Champaign, U.S.A.
• To verify $x(t) \to 0$ we typically look for a Lyapunov function $V$ s.t. $\dot{V}(x) := \frac{\partial V}{\partial x}(x) \cdot f(x) < 0 \; \forall x \neq 0$

• Computing $\dot{V}(x)$ pointwise – easy, checking $\dot{V}(x) < 0$ – hard

• For polynomial $V$ and $f$, can use semidefinite programming (SOS)

• Randomized approach: check the inequality at sufficiently many randomly generated points. Then, with some confidence, we know that it holds outside a set of small volume (Chernoff bound; see, e.g., [Tempo et al.,’12]).

Taking this property as a starting point, what can we say about convergence of trajectories?
SET-UP

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n \]

\[ V : \mathbb{R}^n \to \mathbb{R}, \quad V(0) = 0, \quad V(x) > 0 \quad \forall x \neq 0 \]

\[ D \coloneqq V^{-1}([0, c]) \subset \mathbb{R}^n, \quad c > 0 \]

\[ |f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D \]

\[ M \coloneqq \max_{x \in D} |V_x(x)| \]

Assume: for some \( a > 0 \) and subset \( \Omega \subset D \),

\[ \dot{V}(x) \leq -aV(x) \quad \forall x \in D \setminus \Omega \]

For \( \varepsilon > 0 \), let \( \rho(\varepsilon) \) be the radius of the ball in \( \mathbb{R}^n \) with volume \( \varepsilon \).
MAIN RESULT

\[ M = \max_{x \in D} |V_x(x)| \]
\[ \dot{V}(x) \leq -aV(x) \quad \forall x \in D \setminus \Omega \]
\[ \rho(\varepsilon) = \text{radius of the ball with volume } \varepsilon \]

**Theorem:** \( \exists \bar{\varepsilon} > 0 \ & \text{a cont., incr. fcn} \)
\[ R : [0, \bar{\varepsilon}] \rightarrow [0, \infty) \text{ with } R(0) = 0 \]
s.t. if \( \text{vol}(\Omega) < \varepsilon \leq \bar{\varepsilon} \) then \( \forall x_0 \in D \)
with \( V(x_0) < c - 2M\rho(\varepsilon) \) we have:

- \( V(x(t)) \leq V(x_0) + 2M\rho(\varepsilon) \) (\( \Rightarrow x(t) \in D \)) \( \forall t \geq 0 \)
- \( V(x(T)) \leq R(\varepsilon) \) for some \( T \geq 0 \)
- \( V(x(T)) \leq R(\varepsilon) + 2M\rho(\varepsilon) \) \( \forall t \geq T \)
CLARIFYING REMARKS

\[ \dot{V}(x) \leq -a V(x) \quad \forall \ x \in D \setminus \Omega \]

• \( \text{vol}(\Omega) < \varepsilon \) – nothing else known about \( \Omega \)

• Gives a meaningful result for \( \varepsilon \) small enough

• If \( \exists \) another equilibrium \( z \) then \( V(z) \leq R(\varepsilon) \)

\[ \dot{V}(z) = 0 \Rightarrow \Omega \text{ contains a nbhd of } z \text{ and } V(z) > 0 \]

\( \varepsilon \) cannot be arb. small

• \( \text{vol}(\Omega) < \varepsilon \) \( \Rightarrow \) \( \Omega \) cannot contain a ball of radius \( \rho(\varepsilon) \)

In fact, this weaker condition is enough for the theorem to hold, and it can be checked by sampling enough points (on a lattice).

• If \( \dot{V}(x) \leq -a V(x) \quad \forall \ x \in D \) then we can take \( \varepsilon \to 0 \) and recover the classical asymptotic stability theorem
IDEA of PROOF

• For any $x_0 \in D$ consider $\rho(\varepsilon)$-ball around it

• In this ball $\exists \bar{x}_0$ s.t. $\dot{V}(\bar{x}_0) \leq -aV(\bar{x}_0)$

• Corresp. solution $\bar{x}(t)$ satisfies, for some time,

$$\dot{V}(\bar{x}(t)) \leq -\frac{a}{2}V(\bar{x}(t)) \Rightarrow V(\bar{x}(t)) \leq e^{-\frac{a}{2}t}V(\bar{x}_0)$$

• Distance between the two trajectories grows as

$$|x(t) - \bar{x}(t)| \leq |x_0 - \bar{x}_0|e^{Lt} \leq \rho e^{Lt} \quad (L = \text{Lip const of } f)$$

• If $V(\bar{x}_0)$ is large compared to $\rho$ then decay of $V(\bar{x}(t))$ initially dominates and $\bar{x}(t)$ “pulls” $x(t)$ towards 0
LAST STEP in MORE DETAIL

\[ V(x(t)) \leq V(\bar{x}(t)) + M|x(t) - \bar{x}(t)| \]
\[ \leq e^{-\frac{a}{2}t}V(\bar{x}_0) + M|x_0 - \bar{x}_0|e^{Lt} \]
\[ \leq e^{-\frac{a}{2}t}(V(x_0) + M\rho) + M\rho e^{Lt} \]

(using MVT)

(for \(0 \leq t \leq T^+\))

(from \(|x_0 - \bar{x}_0| \leq \rho\) and MVT again)

If \(V(x_0) \geq R\) where \(R\) is large enough compared to \(\rho\) then crossing time \(T_{\rho,\gamma}\) exists and is smaller than \(T^+\)

We can then repeat the procedure with \(x(T_{\rho,\gamma})\) in place of \(x_0\) and iterate until \(V(x(t)) = R\)
**OPEN QUESTION**

Does our result actually allow this to happen?

\[ \dot{V}(x) \leq -aV(x) \quad \forall x \in D \setminus \Omega \]

But \( \dot{V}(x) \leq 0 \) holds on a larger set

Can this set still be smaller than \( D \)?
CONCLUSIONS

• Developed a stability result that calls for $\dot{V}(x) < 0$ to hold only outside a set of small volume

• Proof compares convergence rate of nearby stable trajectories with expansion rate of distance between trajectories (entropy)

• Remains to test our ability to handle situations where $\dot{V}(x) > 0$ at some points in the region of interest

• Less conservative results will need to be tailored to specific system structure