ON ALMOST LYAPUNOV FUNCTIONS

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MOTIVATING REMARKS

$$\dot{x} = f(x)$$

- To verify $x(t) \to 0$ we typically look for a Lyapunov function V s.t. $\dot{V}(x) := \frac{\partial V}{\partial x}(x) \cdot f(x) < 0 \quad \forall x \neq 0$

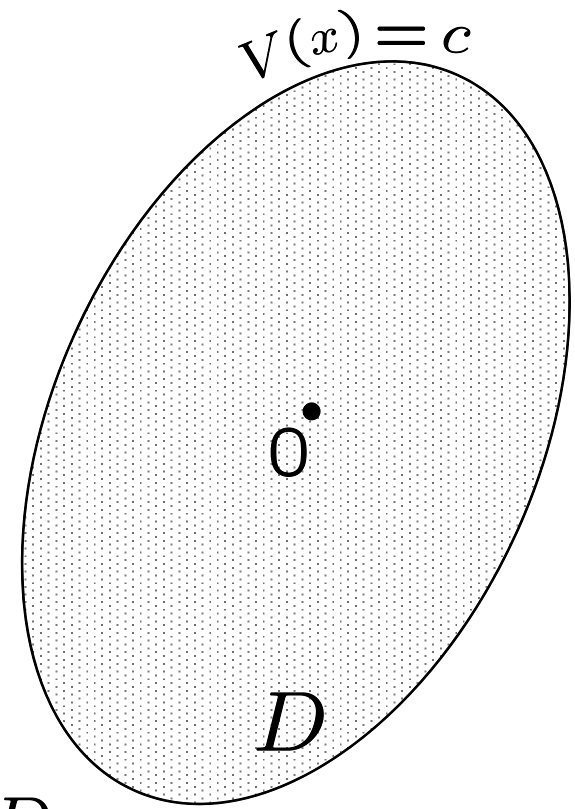
- Computing $\dot{V}(x)$ pointwise – easy, checking $\dot{V}(x) < 0 \; \forall x$ – hard

- For polynomial V and f, can use semidefinite programming (SOS)

- Randomized approach: check the inequality at sufficiently many randomly generated points. Then, with some confidence, we know that it holds outside a set of small volume (Chernoff bound; see, e.g., [Tempo et al.,'12]).

Taking this property as a starting point, what can we say about convergence of trajectories?

SET-UP



$$\dot{x} = f(x), \;\; x \in \mathbb{R}^n$$

$$V : \mathbb{R}^n \to \mathbb{R}, \;\; V(0) = 0, \; V(x) > 0 \;\; \forall x \neq 0$$

$$D := V^{-1}([0, c]) \subset \mathbb{R}^n, \;\; c > 0$$

$$|f(x) - f(y)| \leq L|x - y| \quad \forall\, x, y \in D$$

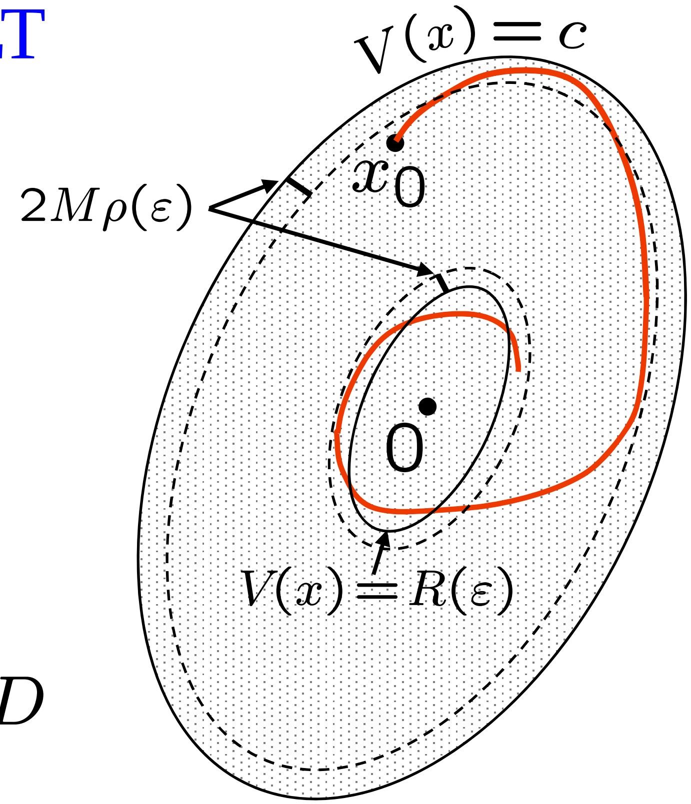
$$M := \max_{x \in D} |V_x(x)|$$

Assume: for some $a > 0$ and subset $\Omega \subset D$,

$$\dot{V}(x) \leq -aV(x) \quad \forall\, x \in D \setminus \Omega$$

For $\varepsilon > 0$, let $\rho(\varepsilon)$ be the radius of the ball in $\mathbb{R}^n$ with volume ε

MAIN RESULT



$M = \max_{x \in D} |V_x(x)|$

$\dot{V}(x) \leq -aV(x) \quad \forall x \in D \setminus \Omega$

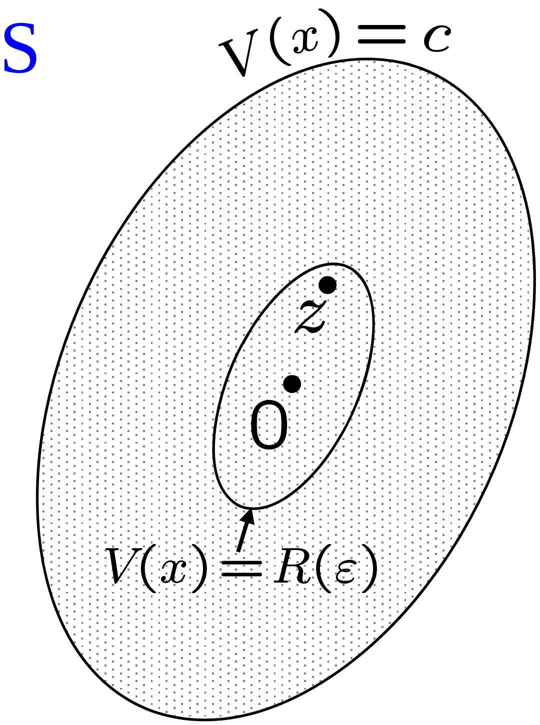
$\rho(\varepsilon) =$ radius of the ball with volume ε

Theorem: $\exists \bar{\varepsilon} > 0$ & a cont., incr. fcn $R : [0, \bar{\varepsilon}] \to [0, \infty)$ with $R(0) = 0$ s.t. if $\mathrm{vol}(\Omega) < \varepsilon \leq \bar{\varepsilon}$ then $\forall x_0 \in D$ with $V(x_0) < c - 2M\rho(\varepsilon)$ we have:

- $V(x(t)) \leq V(x_0) + 2M\rho(\varepsilon) \ (\Rightarrow x(t) \in D) \ \forall t \geq 0$
- $V(x(T)) \leq R(\varepsilon)$ for some $T \geq 0$
- $V(x(T)) \leq R(\varepsilon) + 2M\rho(\varepsilon) \ \forall t \geq T$

CLARIFYING REMARKS

$$\dot{V}(x) \leq -aV(x) \quad \forall x \in D \setminus \Omega$$



- vol$(\Omega) < \varepsilon$ – nothing else known about Ω

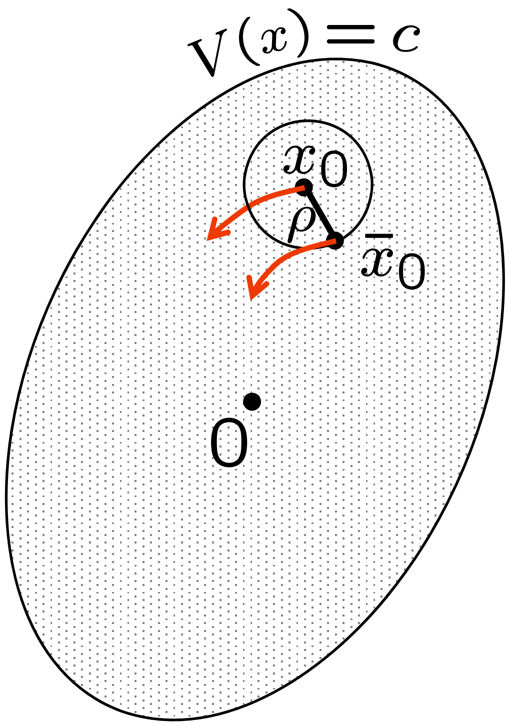
- Gives a meaningful result for ε small enough

- If $\exists$ another equilibrium z then $V(z) \leq R(\varepsilon)$
  $\dot{V}(z) = 0$, $V(z) > 0$ $\Rightarrow$ Ω contains a nbhd of z and ε cannot be arb. small

- vol$(\Omega) < \varepsilon \Rightarrow \Omega$ cannot contain a ball of radius $\rho(\varepsilon)$
  In fact, this weaker condition is enough for the theorem to hold, and it can be checked by sampling enough points (on a lattice).

- If $\dot{V}(x) \leq -aV(x)\ \forall x \in D$ then we can take $\varepsilon \to 0$ and recover the classical asymptotic stability theorem

IDEA of PROOF



- For any $x_0 \in D$ consider $\rho(\varepsilon)$-ball around it

- In this ball $\exists\, \bar{x}_0$ s.t. $\dot{V}(\bar{x}_0) \leq -aV(\bar{x}_0)$

- Corresp. solution $\bar{x}(t)$ satisfies, for some time,

$$\dot{V}(\bar{x}(t)) \leq -\frac{a}{2}V(\bar{x}(t)) \Rightarrow V(\bar{x}(t)) \leq e^{-\frac{a}{2}t}V(\bar{x}_0)$$

- Distance between the two trajectories grows as

$$|x(t) - \bar{x}(t)| \leq |x_0 - \bar{x}_0|e^{Lt} \leq \rho\, e^{Lt} \qquad (L = \text{Lip const of } f)$$

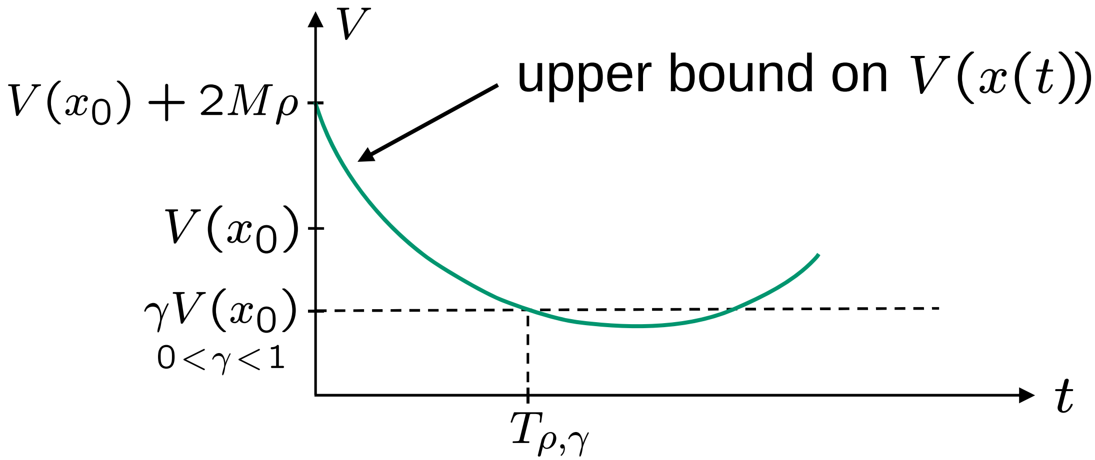
- If $V(\bar{x}_0)$ is large compared to ρ then decay of $V(\bar{x}(t))$ initially dominates and $\bar{x}(t)$ “pulls” $x(t)$ towards 0

LAST STEP in MORE DETAIL

$$V(x(t)) \le V(\bar{x}(t)) + M|x(t) - \bar{x}(t)|$$ (using MVT)

$$\le e^{-\frac{a}{2}t} V(\bar{x}_0) + M|x_0 - \bar{x}_0| e^{Lt}$$ (for $0 \le t \le T^+$)

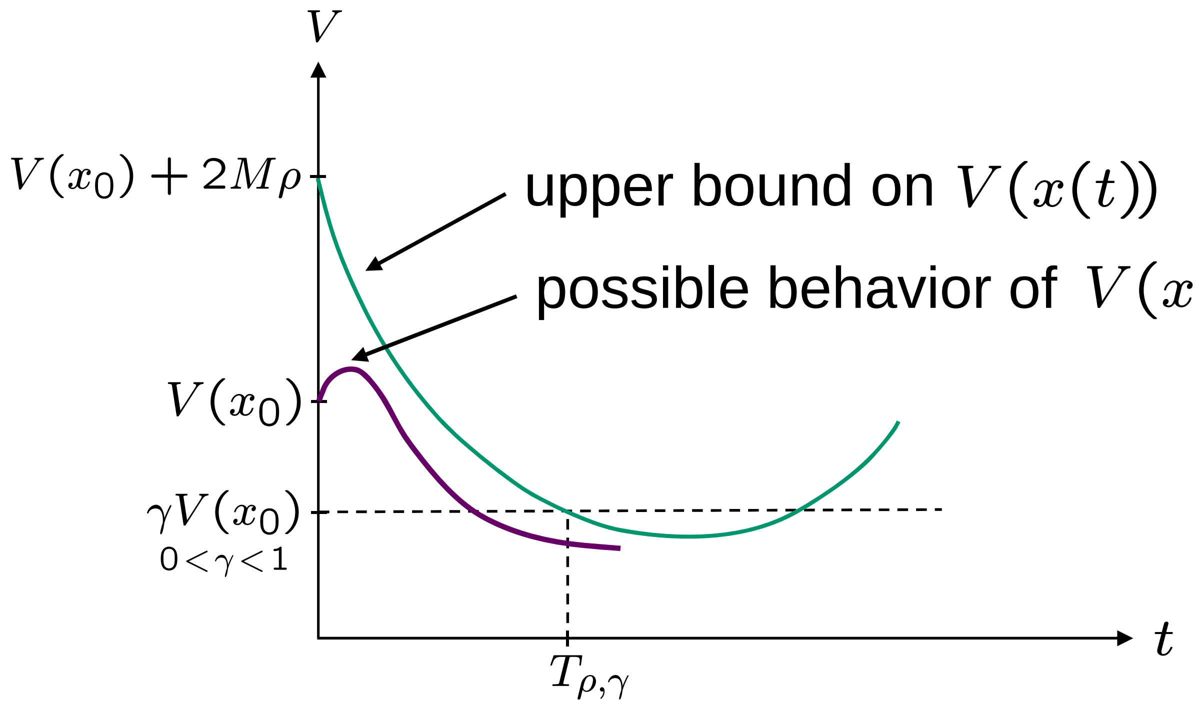
$$\le e^{-\frac{a}{2}t} (V(x_0) + M\rho) + M\rho\, e^{Lt}$$ (from $|x_0 - \bar{x}_0| \le \rho$ and MVT again)



If $V(x_0) \ge R$ where R is large enough compared to ρ then crossing time $T_{\rho,\gamma}$ exists and is smaller than T^+

We can then repeat the procedure with $x(T_{\rho,\gamma})$ in place of x_0 and iterate until $V(x(t)) = R$

OPEN QUESTION



Does our result actually allow this to happen?

$$\dot{V}(x) \leq -aV(x) \quad \forall\, x \in D \setminus \Omega$$

but $\dot{V}(x) \leq 0$ holds on a larger set

Can this set still be smaller than D?

CONCLUSIONS

- Developed a stability result that calls for $\dot{V}(x) < 0$ to hold only outside a set of small volume

- Proof compares convergence rate of nearby stable trajectories with expansion rate of distance between trajectories (entropy)

- Remains to test our ability to handle situations where $\dot{V}(x) > 0$ at some points in the region of interest

- Less conservative results will need to be tailored to specific system structure