Integral-Input-to-State Stability of Switched Nonlinear Systems under Slow Switching

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***Abstract*—In this paper we study integral-input-to-state stability (iISS) of nonlinear switched systems with jumps. We demonstrate by examples that, unlike ISS, iISS is not always preserved under slow enough dwell-time switching. We then present sufficient conditions for iISS to be preserved under slow switching. These conditions involve, besides a sufficiently large dwell time, some additional properties of comparison functions characterizing iISS of the individual modes. When these properties are only partially satisfied, we are able to conclude weaker variants of iISS, also introduced in this work. As an illustration, we show that switched systems with bilinear zero-input-stable modes are always iISS under sufficiently large dwell time.**

I. Introduction

A *switched system* is a dynamical system that consists of several subsystems and a logical rule, called a *switching signal*, which governs the switching between these subsystems. Due to their significance both in theory development and in practical applications (see, e.g., the book [1] or the survey [2] and the references therein), switched systems have received a great deal of attention in the last couple of decades. Nevertheless, the study of switched systems is still a challenging topic in the modern engineering literature because a switched system does not necessarily inherit the properties of its subsystems in general. For example, even if all the subsystems are stable, the switched system may still be unstable. Additional assumptions are needed to guarantee stability, either on the dynamics of all the subsystems — such as assuming the existence of a common Lyapunov function, which gives stability under arbitrary switching [1, Theorem 2.1] (the converse is also true under some mild assumptions, see [3]) — or on the switching signals, such as not allowing the switches to happen too often [4], [5], or a mix of the two.

When the subsystems are nonlinear and there are external inputs, input-to-state stability (ISS) [6] and integral-input-to-state stability (iISS) [7] can be used for the stability analysis. When all the subsystems are ISS with some mild assumptions, it has been shown using a Lyapunov function approach that the switched system is ISS as well under *dwell time* switching or *average dwell time* switching [5], [8], [9]; that is, the switching cannot happen too frequently. Intuitively, when the input is small compared with the magnitude of the state, the possible increase of the Lyapunov function due to the switches can be compensated by its decrease over the time when the ISS subsystems are active for sufficiently long between switches; hence asymptotically the Lyapunov function converges to some small value which is determined by the magnitude of the input.

While clean and satisfying results are available for ISS switched systems under slow switching, for iISS — which is known to have importance comparable to that of ISS — up to now no such results were known. One may initially expect the situation for iISS under slow switching to be similar to that for ISS. However, because the comparison function used to characterize the decay of an iISS-Lyapunov function is only positive definite, strictly weaker than that used in the case of ISS, the same proof will not go through. In fact, it has been shown in our recent work [10] that some switched systems with iISS subsystems are never iISS no matter how long the dwell time is. We showed that for such switched systems, bounds on the norm of initial state and input energy have to be known prior to determining how slow the switching should be so that the switched system is iISS. We will also provide another example in this paper to emphasize this problem. Unlike the example in [10], the best we can do in this example is derive an iISS-like estimate on the solution with some offsets; in other words, under any dwell time switching and when there are no inputs, the solution of the switched system may not converge to the equilibrium but only to a neighborhood of it. Those observations suggest that we may only conclude some weaker versions of iISS when analysing a switched system under slow switching, even if all its subsystems are iISS. On the other hand, there are still "good" switched nonlinear systems which inherit iISS property from their subsystems; for example, the switched bilinear systems which will also be discussed in this paper. With that being said, this paper aims to fill in the gap in the study of iISS of switched nonlinear systems with two main objectives:

1) Define some weaker variants of iISS for switched nonlinear systems under slow switching, and
2) Derive sufficient conditions so that the switched systems admit iISS or those weaker variants of iISS properties.

We briefly mention some relevant research here. ISS for switched systems is also studied in [11], [12] and the very recent work [13], just to name a few. ISS for hybrid systems

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is first studied in [14] and ISS for impulsive systems is studied in [15]; those studies can be generalized to switched systems. In terms of iISS results for switched systems, a converse theorem for iISS of switched systems is proposed in [16] while iISS is studied in the hybrid system framework in [17]. Both of them conclude iISS properties under arbitrary switching. State-dependent switching is studied in [18], [19] to guarantee iISS of switched systems. Some connections between ISS and iISS for switched systems are discussed in [20], and further characterizations of iISS for switched systems are presented by the same authors in [21]. A weaker variant of iISS, quasi-iISS which depends on the switching signal, is introduced in our recent work [10] and we have shown that under some mild assumptions, a switched system with all iISS subsystems is guaranteed to be quasi-iISS under slow switching. In this work we prove that those assumptions in fact guarantee uniform quasi-iISS with respect to all switching signals with the same dwell time condition. In addition, we identify other conditions which, when combined with the sufficient conditions for quasi-iISS, imply that the switched system is iISS.

This paper is organized as follows: Basic notations are introduced and the switched systems as well as variants of iISS are defined in Section II. Section III provides two examples of switched systems where iISS cannot be achieved by any dwell-time switching, which shows why we need some weaker variants of iISS. Our main results are then stated in Section IV and they are proven in Section V after some supporting lemmas are provided. With these results, the earlier examples are revisited in Section VI and a result on switched bilinear systems is derived. Section VII contains some remaining discussions and future work, followed by the conclusion in Section VIII.

II. Preliminaries

A. Notations

We use the convention that $\mathbb{N} = \{0, 1, \cdots\}$ is the set of all natural numbers including 0 and $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$. We denote the maximum between a and b by $a \vee b$, and the minimum between a and b by $a \wedge b$. For convenience, we also use $\bigvee_{i=1}^n a_i$ (resp. $\bigwedge_{i=1}^n a_i$) to denote the maximum (resp. minimum) in the set $\{a_1, \ldots, a_n\}$.

We say $\alpha \in \mathcal{PD}$ (positive definite) if $\alpha : [0, \infty) \to [0, \infty)$ is continuous and $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$. Some notations of comparison functions from [22] are adopted here. We say $\chi \in \mathcal{K}$ if $\chi \in \mathcal{PD}$ and it is strictly increasing. We say $\gamma \in \mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\lim_{s\to\infty} \gamma(s) = \infty$. We say $\gamma \in \mathcal{L}$ if γ is continuous, decreasing[1] and $\lim_{s\to\infty} \gamma(s) = 0$. We say $\beta(\cdot, \cdot) \in \mathcal{KL}$ if $\beta(\cdot, \cdot) : [0, \infty) \times [0, \infty) \to [0, \infty)$ is such that for any fixed t, $\beta(\cdot, t) \in \mathcal{K}$ and for any fixed s, $\beta(s, \cdot) \in \mathcal{L}$.

B. Switched systems

Let $\mathbb{P} \subset \mathbb{N}_+$ be a set of either finite or infinite cardinality. For each $p \in \mathbb{P}$, there is a locally Lipschitz vector field $f_p(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with the common equilibrium property that $f_p(0, 0) = 0$. The differential equations

$$\dot{x} = f_p(x, u), \quad p \in \mathbb{P} \tag{1}$$

are the dynamics of the *subsystems* or *modes* of the switched system. For each pair $(p, q) \in \mathbb{P}^2$, there is also a *jump map* $g_{p,q}(x) : \mathbb{R}^n \to \mathbb{R}^n$ with the common equilibrium property that $g_{p,q}(0) = 0$. Let $\overline{\Sigma}$ be the set of all left-continuous mappings from $[0, \infty)$ to $\mathbb{P}$, called *switching signals*. For each switching signal $\sigma \in \overline{\Sigma}$, define the ordered set $\{t_1, t_2, \cdots\} = \mathcal{T}(\sigma) := \{t \in \mathbb{R}_{>0} : \sigma(t^+) \neq \sigma(t)\}$; in other words, $\mathcal{T}(\sigma)$ is the collection of times when switches occur. Note that by this definition we have assumed that there is no switch at the initial time $t_0 = 0$. Define the class of switching signals with some *dwell time* $\tau_D > 0$:

$$\Sigma(\tau_D) := \{\sigma \in \overline{\Sigma} : |t_{i+1} - t_i| \geq \tau_D \ \forall t_i, t_{i+1} \in \mathcal{T}(\sigma)\} \tag{2}$$

Note that $\Sigma(\tau_D)$ is a more regular set of switching signals compared to $\overline{\Sigma}$ as the positive dwell time τ_D prevents accumulated switches and chattering. For a given $\sigma \in \Sigma(\tau_D)$, a *switched system* is defined by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)) \text{ if } t \notin \mathcal{T}(\sigma), \tag{3a}$$

$$x(t^+) = g_{\sigma(t),\sigma(t^+)}(x(t)) \text{ if } t \in \mathcal{T}(\sigma) \tag{3b}$$

where $x \in \mathbb{R}^n$ is the state variable of the system and $u \in U \subseteq \mathbb{R}^m$ is the input variable. We take the inputs to be locally essentially bounded, and denote the set of such input functions taking values in U by $\mathcal{M}_U$. We denote the solution of (3) with initial state x_0, input u and switching signal σ by $x(\cdot; x_0, u, \sigma)$. When x_0, u, σ are clear from the context, the solution is also abbreviated by $x(\cdot)$. Note that by definition of (3), $x(\cdot; x_0, u, \sigma)$ obeys some differential equation when there is no switch, and it jumps when there is a switch. When there are no state jumps, i.e., all $g_{p,q}$ functions are identity functions, (3a) is sufficient to describe the dynamics of the switched system. In this case we say that the switched system is *jump-free*. In general the jump maps $g_{p,q}$ could be functions of states and inputs so when there are jumps, the magnitude of the jumps could also depend on the input, which is discussed in the framework of hybrid systems in [14] or impulsive systems in [15]. The presence of inputs in the jump map would require additional care in treating $u(t_i)$ for $t_i \in \mathcal{T}(\sigma)$ when defining iISS. Our work can be easily extended to that more general framework; nevertheless, for the clarity of presentation we focus on the framework of (3) and the simpler stability definitions which are introduced in the next subsection.

C. Stability definitions

We start by defining iISS for switched systems:

Definition 1. The switched system (3) is *iISS under slow switching* if there exist a dwell time $\tau_D > 0$ and functions $\beta^*(\cdot, \cdot) \in \mathcal{KL}, \gamma^*(\cdot) \in \mathcal{K}_\infty$ and $\chi^*(\cdot) \in \mathcal{K}$ such that

$$|x(t; x_0, u, \sigma)| \leq \beta^*(|x_0|, t) + \gamma^* \left(\int_0^t \chi^*(|u(\tau)|) d\tau \right) \tag{4}$$

for all $t \geq 0, x_0 \in \mathbb{R}^n, u(\cdot) \in \mathcal{M}_U$ and all $\sigma \in \Sigma(\tau_D)$.

[1] It actually means "non-increasing" here.

The phrase "under slow switching" indicates that we only ask the iISS property to hold for sufficiently large τ_D. Note that our definition of iISS is uniform with respect to switching signal in the sense that β^*, γ^* are independent of σ. When there are no switches, the above definition reduces to the classical iISS notion for a single-mode system [7], which can also be characterized by an iISS-Lyapunov function:

Lemma II.1. *The system*

$$\dot{x} = f(x, u)$$

is iISS if and only if there exist functions $\alpha \in \mathcal{PD}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ *and there exists a* C^1 *function* $V : \mathbb{R}^n \to [0, \infty)$ *satisfying*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n \tag{5}$$

and

$$\nabla V(x) \cdot f(x, u) \le -\alpha(V(x)) + \chi(|u|) \quad \forall x \in \mathbb{R}^n, u \in U. \tag{6}$$

This result is proven in [7, Theorem 1]. Note that (6) is different from the more standard iISS-Lyapunov function condition

$$\nabla V(x) \cdot f(x, u) \le -\alpha(|x|) + \chi(|u|)$$

where α is evaluated on $|x|$, not $V(x)$. Nevertheless, in the proof of [7, Theorem 1] it is shown that the two conditions are equivalent, and hence we will always use (6) when defining iISS-Lyapunov functions. Moreover, we remark here that iISS is a strictly weaker property than ISS because $\alpha \in \mathcal{PD}$ rather than $\mathcal{K}_\infty$.

Next, we introduce two weaker variants of iISS:

Definition 2. The switched system (3) is *quasi-integral-input-to-state stable (qiISS) under slow switching* if for all $\delta_1, \delta_2 > 0$ there exist a dwell time $\tau_D > 0$ and functions $\beta^*(\cdot, \cdot) \in \mathcal{KL}, \gamma^*(\cdot) \in \mathcal{K}_\infty$ and $\chi^*(\cdot) \in \mathcal{K}$ such that the estimate (4) holds for all $t \ge 0$, $x_0 \in \mathbb{R}^n$ with $|x_0| \le \delta_1$, $u(\cdot) \in \mathcal{M}_U$ with $\int_0^\infty \chi(|u(\tau)|) d\tau \le \delta_2$ and all $\sigma \in \Sigma(\tau_D)$.

Definition 3. The switched system (3) is *integral-input-to-state practically stable (iISpS) under slow switching* if for each $\delta_3 > 0$ there exist a dwell time $\tau_D > 0$ and functions $\beta^*(\cdot, \cdot) \in \mathcal{KL}, \gamma^*(\cdot) \in \mathcal{K}_\infty$ and $\chi^*(\cdot) \in \mathcal{K}$ such that

$$|x(t; x_0, u, \sigma)| \le \beta^*(|x_0|, t) + \gamma^* \left(\int_0^t \chi^*(|u(\tau)|) d\tau \right) + \delta_3 \tag{7}$$

for all $t \ge 0$, $x_0 \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{M}_U$ and all $\sigma \in \Sigma(\tau_D)$.

The qiISS property is inspired by "quasi-disturbance-to-error stability" (qDES) in the work [23], and the prefix "quasi" means the nonlinear estimates are not global but depend on initial conditions and magnitude of inputs. The iISpS property is inspired by "input-to-state practical stability" (ISpS) in the work [24], and the qualifier "practical" refers to the presence of an offset term. Note that we also use the phrase "under slow switching" in the definitions of qiISS and iISpS, which indicates that we only ask the estimates to hold for sufficiently large dwell time τ_D.

It is observed that while iISS clearly implies both qiISS and iISpS, the other direction is unclear[2]. In the next section we point out the need for considering the two weaker variants of iISS through two examples of switched systems, in which, while all the subsystems are iISS, the switched systems are not iISS no matter how large τ_D is.

III. Motivating Examples

Example 1. Consider the two-dimensional jump-free switched system with two modes

$$\dot{x} = \frac{1}{1 + |x|^2} A_p x + u, \quad p = 1, 2 \tag{8}$$

where

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}. \tag{9}$$

This example is already mentioned in our earlier work [10] where it was shown that both subsystems are iISS (but not ISS). Meanwhile, it is already discussed in [1, Section 3.2] that the switched linear system with the modes

$$\dot{x} = A_p x, \quad p = 1, 2 \tag{10}$$

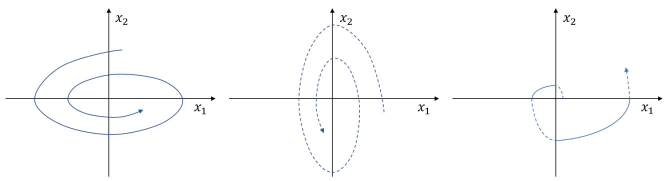
has no common Lyapunov function and for some particular switching signals the solution will diverge, as shown in Fig. 1c. When $u \equiv 0$, it is observed that the vector fields of the subsystems of (8) are the same as the vector fields of (10) except that the magnitude is scaled by $\frac{1}{1+|x|^2}$; hence the solution trajectories of (8) when $u \equiv 0$ will be the same as the trajectories of (10) up to a time re-parameterization. We now claim that the switched system generated by (8) is not iISS under slow switching. By a coordinate change of (8) into polar form we have

$$\dot{\theta} = \frac{d}{dt} \arctan \frac{x_2}{x_1} = \frac{\dot{x}_1 x_2 - x_1 \dot{x}_2}{x_1^2 + x_2^2} = \begin{cases} -\frac{x_2^2 + 2x_1^2}{(x_1^2+x_2^2)(1+|x|^2)} & \text{if } \sigma = 1 \\ -\frac{x_1^2 + 2x_2^2}{(x_1^2+x_2^2)(1+|x|^2)} & \text{if } \sigma = 2 \end{cases},$$

in either case we always have $|\dot{\theta}| \le \frac{2}{1+|x|^2}$. Note that in order to achieve a divergent solution trajectory as the one in Fig. 1c, we need the switches to occur when the state is on either axis; in other words, a switch occurs every time when θ increases by $\frac{\pi}{2}$. We also observe that in this divergent case, there exists $\rho \in \mathcal{K}_\infty$ such that $|x(t)| \ge \rho(|x(0)|)$ for all $t \ge 0$. Now for any $\tau_D > 0$, we pick the initial condition such that $|x(0)| \ge \rho^{-1} \left(\sqrt{\left(\frac{4\tau_D}{\pi} - 1 \right) \vee 0} \right)$ and it is on an axis. Suppose it requires Δt for θ to increase by $\frac{\pi}{2}$; in other words

$$\frac{\pi}{2} = \int_0^{\Delta t} |\dot{\theta}(\tau)| d\tau \le \int_0^{\Delta t} \frac{2}{1 + |x(\tau)|^2} d\tau \le \frac{2\Delta t}{1 + \rho(|x(0)|)^2} \le \frac{\pi \Delta t}{2\tau_D}$$

[2]It has drawn the authors' attention that the proof of equivalence between iISS and 0-GUAS (global uniform asymptotic stability under zero input) plus UBEBS (uniformly bounded energy of input implying bounded states) in [21] may give the implication from qiISS plus iISpS to iISS; nevertheless, since this is not the main focus of this paper, we postpone this analysis to the future work.



(a) Mode 1 only (b) Mode 2 only (c) An unstable switched case

Fig. 1: Solution trajectories of (10). Solid curve for $\sigma = 1$ and dashed curve for $\sigma = 2$.

Thus we have $\Delta t \geq \tau_D$, which means we can always find a switching signal $\sigma \in \Sigma(\tau_D)$ yet the solution is exactly as the one in Fig. 1c and it is divergent. Hence the switched system generated by (8) is not iISS under slow switching. Note that this fact can also be argued by picking an input with large but finite integral and only a very small support near $t = 0$ and showing that this input leads to an "initial" state with arbitrarily large magnitude; hence by a similar argument we can again find a divergent solution.

Example 2. Consider the two-dimensional jump-free switched system with two modes

$$\dot{x} = |x|A_p x + u, \quad p = 1, 2 \tag{11}$$

where A_p are the same as given in (9). Both its subsystems are iISS (in fact also ISS), which will be shown later in Section VI and hence not repeated here. Nevertheless, we still claim that the switched system generated by (11) is also not iISS under slow switching. To show this, we first observe that by the same argument as in the previous example, the solution trajectory of (11) when $u \equiv 0$ is the same as the trajectory of (10) up to a time re-parameterization. Again by coordinate change into polar form we can show that $|\dot{\theta}| \leq 2|x|$. For any $\tau_D > 0$, consider the threshold $c = \frac{\pi}{4\tau_D}$ and let $x(0)$ be on an axis and such that $|x(0)| < c$. Apply the switching signal which results in a locally divergent solution as in Fig. 1c until a time s at which $|x(s)| = c$. Keep the mode active at time s and let $\tau \geq \tau_D$ be the first time such that $x(s+\tau)$ is on an axis and $|x(s+\tau)| < c$ (such a τ exists because of how the modes behave for $u \equiv 0$). We can then treat $s + \tau$ as the initial time and repeat the above process. Note that for $|x| \leq c$, the time Δt needed for the state to travel from one axis to the other satisfies

$$\frac{\pi}{2} = \int_0^{\Delta t} |\dot{\theta}(\tau)| d\tau \leq \int_0^{\Delta t} 2|x(\tau)| d\tau \leq 2c\Delta t = \frac{\pi \Delta t}{2\tau_D}$$

Thus again $\Delta t \geq \tau_D$, and by construction our switching signal is in $\Sigma(\tau_D)$. On the other hand, the resulting solution satisfies $|x(t)| = c$ at infinitely many times t, hence it cannot converge to the origin. Therefore, the switched system generated by (11) is also not iISS under slow switching.

In both examples we see that there is no uniform stabilizing dwell time with respect to all initial states or inputs, and this is the major reason why these switched systems are not iISS under slow switching. In the first example, the convergence of subsystem solutions is slow when the magnitude of initial state or the integral of input is too large; in the second example, the convergence of subsystem solutions is slow when the states are too close to the origin. Recall that our definitions of qiISS and iISpS exactly deal with the cases of either small initial states plus small integral of inputs, or states sufficiently far away from the origin. We will revisit these two examples in Section VI and it is not to the readers' surprise that by applying the criteria derived in this work, the switched system generated by (8) can be shown to be indeed qiISS under slow switching and the switched system generated by (11) can be shown to be indeed iISpS under slow switching.

IV. Main results

Before we state the main result, we claim that a $\mathcal{PD}$ function can always be lower bounded by the product of a non-decreasing function and a non-increasing function. This claim will be used in our main theorem.

Lemma IV.1. *Let $\alpha \in \mathcal{PD}$ and locally Lipschitz. There exist a locally Lipschitz non-decreasing function $\rho_1 \in \mathcal{PD}$ and a locally Lipschitz non-increasing function $\rho_2 : [0, \infty) \to \mathbb{R}_{>0}$ such that $\alpha(v) \geq \rho_1(v)\rho_2(v)$ for all $v \geq 0$. In particular, if $\liminf_{v\to\infty} \alpha(v) > 0$, we can let*

$$\rho_1(v) := \inf_{w \geq v} \alpha(w), \quad \rho_2(v) := 1.$$

Otherwise if $\liminf_{v\to\infty} \alpha(v) = 0$, the maximum of α over $[0, \infty)$ exists and we can let

$$\rho_1(v) := \begin{cases} \min_{w\in[v,\overline{v}]} \alpha(w) & \text{if } v \in [0, \overline{v}], \\ \alpha(\overline{v}) & \text{if } v > \overline{v}, \end{cases}$$

$$\rho_2(v) := \begin{cases} 1 & \text{if } v \in [0, \overline{v}], \\ \min_{w\in[\overline{v},v]} \frac{\alpha(w)}{\alpha(\overline{v})} & \text{if } v > \overline{v} \end{cases}$$

where $\overline{v} \in \arg\max_{v\geq 0} \alpha(v)$.

Lemma IV.1 is adopted from Lemma IV.1 in [7] and its proof is straightforward and hence omitted.

We need the following two assumptions in order to state our main results:

Assumption 1. There exist a locally Lipschitz function $\alpha \in \mathcal{PD}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ such that for each mode $p \in \mathbb{P}$, there exists a C^1 function $V_p : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n. \tag{12}$$

and

$$\nabla V_p(x) \cdot f_p(x, u) \leq -\alpha(V_p(x)) + \chi(|u|) \quad \forall x \in \mathbb{R}^n, u \in U, p \in \mathbb{P}. \tag{13}$$

Assumption 2. The functions V_p in Assumption 1 satisfy

$$V_q(g_{p,q}(x)) \leq \mu(V_p(x))V_p(x) \quad \forall x \in \mathbb{R}^n, p, q \in \mathbb{P} \tag{14}$$

where $\mu(\cdot) : [0, \infty) \to [1, \infty)$ is a continuous and non-increasing function with the property that there exists $\delta > 0$ such that for any $s \geq t \geq 0$, $\mu(s)s - \mu(t)t \geq \delta(s - t)$.

We make some remarks regarding the two assumptions. Compared with Lemma II.1, Assumption 1 implies that the subsystems of the switched system are iISS. In general, the iISS estimation functions may vary from mode to mode; that is, instead of unique α and χ, we may have α_p, χ_p depending on $p \in \mathbb{P}$. However, if the set $\mathbb{P}$ is finite, we can always pick $\alpha(s) := \bigwedge_{p\in\mathbb{P}} \alpha_p(s)$ and $\chi(s) := \bigvee_{p\in\mathbb{P}} \chi_p(s)$ and the estimation (13) will hold uniformly. When $\mathbb{P}$ has infinite cardinality, our assumption is still valid as long as there exist a uniform lower bound on α_p and a uniform upper bound on χ_p and they belong to class $\mathcal{PD}$ and class $\mathcal{K}$, respectively.

On the other hand, assuming a constant gain μ in the value of Lyapunov functions when a switch occurs is a common practice in the literature (see, e.g., [8], [9]). The idea of using non-constant gain $\mu(\cdot)$ is borrowed from the work [13] and our Assumption 2 clearly includes the case of constant gain. It is observed that when the Lyapunov functions used for subsystems are all quadratic or polynomial functions of same degree, there is no advantage in assuming non-constant gain. However, in the case when $\lim_{|x|\to\infty} V_p(x)/V_q(x) = 1$, non-constant gain becomes critical for deriving the desired results.

We are now ready to state the first main result of the paper:

Theorem 1. *Consider a switched system defined via* (3) *with a set of modes $\mathbb{P}$ and assume both Assumption 1 and Assumption 2 hold. Let $\beta(s,t) \in \mathcal{KL}$ be the solution to the initial-value ODE problem $\dot{v} = -\rho_1(v)\rho_2(2v), v(0) = s$, where ρ_1, ρ_2 are derived from α as in Lemma IV.1. Define a function $h(s,t,\varepsilon) : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ by*

$$h(s,t,\varepsilon) := \beta\Big(\big((1+\varepsilon)\mu(s) - \varepsilon\big)s, t\Big). \tag{15}$$

We have the following implications:

1) *If*
$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \limsup_{s\to 0^+} \frac{h(s,t,\varepsilon)}{s} < 1, \tag{16}$$
then (3) *is qiISS under slow switching;*
2) *If*
$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \limsup_{s\to\infty} \big(h(s,t,\varepsilon) - s\big) < 0, \tag{17}$$
then (3) *is iISpS under slow switching;*
3) *If both* (16) *and* (17) *hold, then* (3) *is iISS under slow switching.*

Remark 1. The condition (17) can also be equivalently restated as
$$\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \limsup_{s\to\infty} \frac{e^{h(\ln s,t,\varepsilon)}}{s} < 1,$$
which has a similar form as (16).

We point out some drawbacks of Theorem 1 here. Firstly, it relies on Lemma IV.1 for computing the β function. In addition, it requires the knowledge of the non-constant gain μ between the Lyapunov functions, which needs to be computed and has to satisfy the conditions in Assumption 2. The next theorem provides a more direct qualitative statement on iISS-related properties for jump-free switched systems (recall that jump-free switched systems mean $g_{p,q}(x) = x \ \forall p,q \in \mathbb{P}$), which is only based on $\alpha, \alpha_1, \alpha_2$:

Theorem 2. *Consider a jump-free switched system defined via* (3a) *with a set of modes $\mathbb{P}$ and assume that Assumption 1 holds. Further assume that α_1, α_2 are C^1 and there exist $\overline{M} \geq \underline{M} > 0$ such that*

$$\underline{M} \leq \frac{\alpha_2'(v)}{\alpha_1'(v)} \leq \frac{\alpha_2(v)}{\alpha_1(v)} \leq \overline{M} \quad \forall v > 0. \tag{18}$$

We have the following implications:

1) *If α is differentiable at 0 with $\alpha'(0) > 0$, then* (3a) *is qiISS under slow switching;*
2) *If either*
   i. $\liminf_{v\to\infty} \frac{\alpha(v)}{v} > 0$, *or*
   ii. $\alpha \in \mathcal{K}$ *and* $\limsup_{v\to\infty} \big(\alpha_2(v) - \alpha_1(v)\big) < \infty$,

   then (3a) *is iISpS under slow switching;*
3) *If the assumptions in 1 and 2.i, or assumptions in 1 and 2.ii are satisfied, then* (3a) *is iISS under slow switching.*

The assumption $\alpha'(0) > 0$ is sometimes called super-linearity at the origin, which means the existence of $k, l > 0$ such that $\alpha(v) \geq kv$ for all $v \in [0, l]$. On the other hand, the assumption 2.i in Theorem 2 is super-linearity at infinity because it is equivalent to the existence of $k, l > 0$ such that $\alpha(v) \geq kv$ for all $v > l$. When both of them are true, we simply have

$$\nabla V_p(x) \cdot f_p(x,u) \leq -kV_p(x) + \chi(u), \quad p \in \mathbb{P} \tag{19}$$

which means the subsystems are all ISS with exponential convergence rate; hence the study in [8] in fact directly tells us that the switched system is ISS under slow switching. More discussion for this special case and how dwell time can be estimated will be given in Section VII.

Note that if $\alpha \in \mathcal{PD}$ but $\lim_{v\to\infty} \alpha(v) = 0$, Theorem 2 is inconclusive. In fact, it can be shown that in this case a sufficient condition for the switched system to also be iISpS under slow switching is that $\alpha_2(v) - \alpha_1(v)$ converges to 0 as $v \to \infty$ at a rate comparable to the rate at which α converges to 0. We do not elaborate on this result because these properties appear to be too restrictive; examples satisfying such properties will be extremely artificial.

V. Proofs

A. Supporting lemmas on comparison functions

In this subsection we list several lemmas which will be used in the proof of Theorem 1. Lemma V.1 is a direct consequence of Assumption 2. Lemma V.2 contains some straightforward analysis observations. Corollary V.3 is a direct combination of the two statements of Lemma V.2. Lemma V.4 is cited from [7, Lemma IV.2]. Lemma V.5 is a result on combining two class $\mathcal{KL}$ functions and Lemma V.6 is a simple fact regarding splitting class $\mathcal{K}$ functions. The proofs of Lemma V.1, V.2, V.5 and V.6 are provided in Appendix A.

Lemma V.1. *Let h be defined as in* (15) *for some $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\mu(\cdot) : [0,\infty) \to [1,\infty)$ satisfying the properties stated in Assumption 2. There exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in [0,\bar{\varepsilon})$, $h(\cdot,\cdot,\varepsilon) \in \mathcal{KL}$.*

Lemma V.2. *Let h be defined as in* (15) *for some $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\mu(\cdot) : [0,\infty) \to [1,\infty)$ satisfying the properties stated in Assumption 2. Let $\bar{\varepsilon} > 0$ be given as in Lemma V.1.*

1) *The inequality* (16) *holds if and only if there exist $\lambda \in (0,1), \varepsilon_0 \in (0,\bar{\varepsilon})$ such that for any $b > 0$, there exists $\tau_D > 0$ such that*
$$h(s,t,\varepsilon) \leq \lambda s \tag{20}$$
for all $(s,t,\varepsilon) \in [0,b] \times [\tau_D,\infty) \times [0,\varepsilon_0]$.

2) *The inequality* (17) *holds if and only if there exist $\Delta > 0, \varepsilon_0 \in (0,\bar{\varepsilon})$ such that for any $b \geq 2\Delta$, there exists $\tau_D > 0$ such that*
$$h(s,t,\varepsilon) \leq s - \Delta \tag{21}$$
for all $(s,t,\varepsilon) \in [b,\infty) \times [\tau_D,\infty) \times [0,\varepsilon_0]$.

Remark 2. The second statement in Lemma V.2 can also be strengthened by stating that (17) holds if and only if there exist $\Delta_0 > 0, \varepsilon_0 \in (0,\bar{\varepsilon})$ such that for all $\Delta \in (0,\Delta_0], b \geq 2\Delta$, there exists $\tau_D > 0$ such that (21) holds for all $(s,t,\varepsilon) \in [b,\infty) \times [\tau_D,\infty) \times [0,\varepsilon_0]$. See the proof in the appendix for the details.

Corollary V.3. *Let h be defined as in* (15) *for some $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\mu(\cdot) : [0,\infty) \to [1,\infty)$ satisfying the properties stated in Assumption 2. Let $\bar{\varepsilon} > 0$ be given as in Lemma V.1. When both* (16) *and* (17) *hold, for any $\varepsilon_0 > 0$ there exist $\lambda \in (0,1), \Delta > 0, \varepsilon_0 \in (0,\bar{\varepsilon})$ and $\tau_D > 0$ such that*
$$h(s,t,\varepsilon) \leq \lambda s \vee (s - \Delta) \tag{22}$$
for all $(s,t,\varepsilon) \in [0,\infty) \times [\tau_D,\infty) \times [0,\varepsilon_0]$.

An inequality similar to (20) also appears in [11], where it is used as the essential assumption to show ISS of the switched nonlinear system. However, our assumption is weaker than theirs as we do not require (20) to hold for all $s \geq 0$. On the other hand, the bound on $h(s,t,\varepsilon)$ in (22) is strictly larger than the bound in (20). Later the reader will see that we are able to derive iISS as long as the weaker bound (22) holds for all $(s,t) \in [0,\infty) \times [\tau_D,\infty)$. It is also conjectured that an even weaker assumption that $h(s,t,\varepsilon) < s$ for some $\varepsilon > 0$ and all $(s,t) \in [0,\infty) \times [\tau_D,\infty)$ is sufficient to show the rest of the results in this paper. We will not discuss that assumption here.

Lemma V.4. *Let $\alpha \in \mathcal{PD}$ and consider two functions $y(t), z(t)$, $y : [0,\infty) \to \mathbb{R}$ being C^1 and $z : [0,\infty) \to [0,\infty)$ being continuous and non-decreasing such that*
$$\dot{y} \leq -\alpha((y+z) \vee 0) \tag{23}$$
for all $t \geq 0$. Then
$$y(t) \leq \beta(y(0) \vee 0, t) \vee z(t), \tag{24}$$
where $\beta(s,t) \in \mathcal{KL}$ is the solution to the initial value ODE problem $\dot{v} = -\rho_1(v)\rho_2(2v)$, $v(0) = s$ with ρ_1, ρ_2 derived from α as in Lemma IV.1.

Lemma V.5. *For any $\tau_D \geq 0, \beta_1, \beta_2 \in \mathcal{KL}$, the function*
$$\beta_3(s,t) := \sup_{\tau \geq \tau_D} \beta_2\big(\beta_1(s,\tau), (t-\tau) \vee 0\big) \tag{25}$$
is also a class $\mathcal{KL}$ function.

Lemma V.6. *Let $\alpha \in \mathcal{K}$ and $k_1, k_2, s_1, s_2 \geq 0$. Then for any $\varepsilon > 0$,*
$$\alpha(k_1 s_1 + k_2 s_2) \leq \alpha\big((k_1 + \varepsilon k_2)s_1\big) \vee \alpha\big((\varepsilon^{-1}k_1 + k_2)s_2\big) \tag{26}$$

The inequality (26) is inspired by [25, Lemma 10]. A coarser version of Lemma V.6 with fixed $\varepsilon = 1$ is used in our conference paper version of this work [10]. However, in this paper we can get tighter estimates on the dwell time by allowing arbitrarily small ε. See Section VII for more discussion on this.

B. Proof of Theorem 1

In our previous work [10] a similar but non-uniform qiISS is defined and proven for switched systems satisfying similar assumptions. For any chosen switching signal σ, qiISS is shown by providing an estimation of solutions in that work. In order to make the iISS-related properties uniform with respect to the switching signal $\sigma \in \Sigma(\tau_D)$, the estimation function needs be recursively defined as the supremum of some nested class $\mathcal{KL}$ functions and this new proof is presented in this section.

Throughout this section we will use the notation
$$\phi_\varepsilon(s) := (1+\varepsilon)\mu(s) - \varepsilon. \tag{27}$$
This is a non-increasing function on $[0,\infty)$ as μ is non-increasing, and $h(s,t,\varepsilon) = \beta(\phi_\varepsilon(s)s, t)$.

For any $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U$ and $\sigma \in \Sigma(\tau_D)$, define $v : [0,\infty) \to [0,\infty)$ by
$$v(t) = V_{\sigma(t)}(x(t)). \tag{28}$$
Then from (13), (14) and the dynamics (3) we have
$$\dot{v}(t) \leq -\alpha(v(t)) + \chi(|u(t)|) \quad \forall t \notin \mathcal{T}(\sigma), \tag{29a}$$
$$v(t^+) \leq \mu(v(t))v(t) \quad \forall t \in \mathcal{T}(\sigma). \tag{29b}$$

Define $z : [0,\infty) \to [0,\infty)$ by $z(0) = 0$, $\dot{z}(t) = \chi(|u(t)|)$ for all $t \geq 0$. We have $z(t) = \int_0^t \chi(|u(\tau)|)d\tau$ and it is continuous and increasing. Further define $\tilde{y}(t) := v(t) - z(t)$ and $y(t) := \tilde{y}(t) \vee 0$. We have $y(0) = v(0)$ and $y(t) \leq v(t)$ for all $t \geq 0$. Thus (29a) implies $\dot{\tilde{y}}(t) \leq -\alpha((\tilde{y}(t)+z(t)) \vee 0)$. Hence by Lemma V.4 we have $\tilde{y}(t) \leq \beta(\tilde{y}(t_i^+) \vee 0, t - t_i) \vee z(t)$ for any $t_i \in \mathcal{T}(\sigma)$ and any $t \in (t_i, t_{i+1}]$, where $\beta(\cdot,\cdot) \in \mathcal{KL}$ is constructed from α as in Lemma V.4. Because either $y(t) = \tilde{y}(t)$ or $y(t) = 0$, we further conclude that
$$y(t) \leq \beta(y(t_i^+), t - t_i) \vee z(t). \tag{30}$$
In addition, (29b) implies that $\tilde{y}(t_i^+) \leq \mu(v(t_i))\tilde{y}(t_i) + \big(\mu(v(t_i)) - 1\big)z(t_i)$ and because either $y(t_i^+) = \tilde{y}(t_i^+)$ or $y(t_i^+) = 0$,
$$y(t_i^+) \leq \mu(v(t_i))y(t_i) + \big(\mu(v(t_i)) - 1\big)z(t_i). \tag{31}$$

Combining (30) and (31), picking $\varepsilon \in (0, \varepsilon_0)$ where ε_0 comes from Lemma V.2 and applying Lemma V.6, we have

$$\begin{aligned}
y(t) &\leq \beta\Big(\mu(v(t_i))y(t_i) + \big(\mu(v(t_i)) - 1\big)z(t_i), t - t_i\Big) \vee z(t) \\
&\leq \beta\big(\phi_\varepsilon(v(t_i))y(t_i), t - t_i\big) \\
&\quad \vee \beta\big(\varepsilon^{-1}\phi_\varepsilon(v(t_i))z(t_i), t - t_i\big) \vee z(t) \\
&\leq \beta\big(\phi_\varepsilon(y(t_i))y(t_i), t - t_i\big) \\
&\quad \vee \beta\big(\varepsilon^{-1}\phi_\varepsilon(0)z(t_i), t - t_i\big) \vee z(t) \\
&\leq \beta_y(y(t_i), t - t_i) \vee \beta_z(z(t_i), t - t_i) \vee z(t)
\end{aligned}$$

where $\beta_y(s,t) := \beta\big(\phi_\varepsilon(s)s, t\big) = h(s,t,\varepsilon)$ and $\beta_z(s,t) := \beta\big(\varepsilon^{-1}\phi_\varepsilon(0)s, t\big)$. We need to invoke the following Lemma at this point:

Lemma V.7. *Let $\beta_y, \beta_z \in \mathcal{KL}$, $\sigma \in \Sigma(\tau_D)$ for some $\tau_D > 0$ and let $\mathcal{T}(\sigma) = \{t_1, t_2, \dots\}$ be the set of switching times. Let $z : [0,\infty) \to [0,\infty)$ be an increasing function and $y : [0,\infty) \to [0,\infty)$ with the properties that for any $t_i \in \mathcal{T}(\sigma)$ and any $t \in [t_i, t_{i+1})$,*

$$y(t) \leq \beta_y(y(t_i), t - t_i) \vee \beta_z(z(t_i), t - t_i) \vee z(t). \tag{32}$$

For $i \in \mathbb{N}_+$, recursively define two families of functions $h_i : [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\begin{aligned}
&h_1(s,t) := \beta_y(s,t), \\
&h_{i+1}(s,t) := \sup\nolimits_{\tau \geq \tau_D} h_i\big(\beta_y(s,\tau), (t-\tau) \vee 0\big)
\end{aligned} \tag{33}$$

and $l_i : [0,\infty) \to [0,\infty)$:

$$\begin{aligned}
&l_1(s) := \beta_z(s,0) \vee s, \\
&l_{i+1}(s) := h_i\big(\beta_z(s,\tau_D) \vee s, 0\big) \vee l_i(s).
\end{aligned} \tag{34}$$

Then $h_i \in \mathcal{KL}$ and $l_i \in \mathcal{K}_\infty$ for all $i \in \mathbb{N}_+$. In addition, for any $i \in \mathbb{N}$, $t_j \in \mathcal{T}(\sigma)$ and $t \in (t_{j+i-1}, t_{j+i}]$,

$$y(t) \leq h_i(y(t_j), t - t_j) \vee l_i(z(t)). \tag{35}$$

The proof of Lemma V.7 is given in Appendix B. Applying Lemma V.7, with $j = 1$, we conclude that $y(t) \leq h_i(y(t_1), t - t_1) \vee l_i(z(t))$ for any $t \in (t_i, t_{i+1}]$, where h_i, l_i are defined in (33), (34). In addition (30) also implies that $y(t_1) \leq \beta(y(0), t_1) \vee z(t_1)$ and therefore

$$y(t) \leq \sup_{t_1 \geq 0} h_i(\beta(y(0), t_1), t - t_1) \vee h_i(z(t), 0) \vee l_i(z(t)) \tag{36}$$

for any $t \in (t_i, t_{i+1}]$.

Notice that the estimation of $y(t)$ given in (36) is only true when there are i switches up to time t. In order to obtain an estimation of $y(t)$ for arbitrary number of switches up to any time $t \geq 0$, define

$$H_\infty(s,t) := \lim_{N \to \infty} \bigvee_{i=1}^{N} h_i(s,t). \tag{37}$$

We claim that whenever $H_\infty(s,t)$ is finite,

$$\begin{aligned}
y(t) \leq &\sup_{t_1 \geq 0} H_\infty(\beta(y(0), t_1), (t - t_1) \vee 0) \\
&\vee H_\infty\big(\varepsilon^{-1}\phi_\varepsilon(0)z(t) \vee z(t), 0\big) \vee l_1(z(t))
\end{aligned} \tag{38}$$

for all $t \geq 0$. Clearly by comparing (38) with (36), in order to show the claim it suffices to show that $l_i(s) \leq H_\infty\big(\varepsilon^{-1}\phi_\varepsilon(0)s \vee s, 0\big)$ for all $i \geq 2$ and all $s \geq 0$. This is indeed guaranteed by the recursive definition of l_i in (34), and the fact that while either (16) or (17) hold, by Lemma V.2 we have

$$\begin{aligned}
\beta_z(s,\tau_D) = \beta(\varepsilon^{-1}\phi_\varepsilon(0)s, \tau_D) &= h(\varepsilon^{-1}\phi_\varepsilon(0)s, \tau_D, 0) \\
&\leq h(\varepsilon^{-1}\phi_\varepsilon(0)s, \tau_D, \varepsilon) \leq \varepsilon^{-1}\phi_\varepsilon(0)s.
\end{aligned}$$

At this point we make some discussions about H_∞ before continuing the proof of Theorem 1. Since each h_i function is a class $\mathcal{KL}$ function, we would like to see if $H_\infty \in \mathcal{KL}$ as well. Notice that if define

$$H_N(s,t) := \bigvee_{i=1}^{N} h_i(s,t), \tag{39}$$

we have $H_N \in \mathcal{KL}$ for all $N \in \mathbb{N}_+$ and H_∞ is the point-wise limit of the sequence of functions H_N. In addition, for each pair (s,t), $H_N(s,t)$ is increasing with respect to N, and it may increase to infinity as N approaches to infinity. However, if $H_N(s,t)$ can be shown to be uniformly bounded with respect to N on a domain $D \subseteq [0,\infty) \times [0,\infty)$, then $H_\infty(s,t)$ is finite on the domain D and its monotonicity with respect to each argument will be inherited from the functions H_N. Therefore, $H_\infty(\cdot, t)$ can be shown to be upper bounded by a class $\mathcal{K}$ function on the domain D without checking its continuity. Nevertheless, $H_\infty(s, \cdot)$ cannot be guaranteed to be upper bounded by a class $\mathcal{L}$ function as the property of convergence to 0 may be lost when passing to the limit. Therefore even though each $H_N \in \mathcal{KL}$, the point-wise limit H_∞ may not be a class $\mathcal{KL}$ function. Nevertheless, the following Lemma tells that we do have some class $\mathcal{KL}$-like properties for the function H_∞ under some assumptions:

Lemma V.8. *Let h be defined as in (15) for some $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\mu(\cdot) : [0,\infty) \to [1,\infty)$ satisfying the properties stated in Assumption 2. Let $\varepsilon \in (0, \varepsilon_0)$ where ε_0 comes from Lemma V.2, and recursively define a family of functions $h_i : [0,\infty) \times [0,\infty) \to [0,\infty)$, $i \in \mathbb{N}_+$ as in (33) with $\beta_y(s,t) := h(s,t,\varepsilon)$. Further define $H_\infty(s,t)$ as in (37). Then there exists $\overline{\beta}(\cdot,\cdot) \in \mathcal{KL}$ such that we have the following implications on H_∞:*

1) *if there exist $\lambda < 1, b > 0$ such that h satisfies the inequality (20) for all $(s,t) \in [0,b] \times [\tau_D, \infty)$, then $H_\infty(s,t) \leq \overline{\beta}(s,t)$ for all $(s,t) \in [0,b] \times [0,\infty)$.*
2) *if there exist $\Delta > 0, b \geq 2\Delta$ such that h satisfies the inequality (21) for all $(s,t) \in [b,\infty) \times [\tau_D, \infty)$, then $H_\infty(s,t) \leq \overline{\beta}(s,t) + h(b - \Delta, 0, \varepsilon)$ for all $(s,t) \in [0,\infty) \times [0,\infty)$.*
3) *if there exist $\lambda < 1, \Delta > 0$ such that h satisfies the inequality (22) for all $(s,t) \in [0,\infty) \times [\tau_D, \infty)$, then $H_\infty(s,t) \leq \overline{\beta}(s,t)$ for all $(s,t) \in [0,\infty) \times [0,\infty)$.*

The proof of Lemma V.8 is also given in Appendix B. With this lemma, we can finally finish the proof of Theorem 1.

To show qiISS when (16) holds, let $\delta_1, \delta_2 > 0$ be arbitrary. By the first statement in Lemma V.2, (16) implies the existence of $\lambda \in (0,1)$ such that for

$$b := \alpha_2(\delta_1) \vee \varepsilon^{-1}\phi_\varepsilon(0)\delta_2 \vee \delta_2, \tag{40}$$

there exists $\tau_D > 0$ such that (20) holds for all $(s,t) \in [0,b] \times [\tau_D, \infty)$. Thus by the first conclusion in Lemma V.8 there exists $\overline{\beta}(\cdot,\cdot) \in \mathcal{KL}$ that $H_\infty(s,t) \le \overline{\beta}(s,t)$ for all $(s,t) \in [0,b] \times [0,\infty)$. Denote

$$\beta^\dagger(s,t) := \sup_{t_1 \ge 0} \overline{\beta}(\beta(s,t_1),(t-t_1) \vee 0) \tag{41}$$

which is class $\mathcal{KL}$ by Lemma V.5. Also denote

$$\gamma^\dagger(s) := \overline{\beta}(\varepsilon^{-1}\phi_\varepsilon(0)s \vee s, 0) \vee l_1(s) \tag{42}$$

which is class $\mathcal{K}_\infty$. Following from (38), we conclude that

$$v(t) \le y(t) + z(t) \le \beta^\dagger(v(0),t) + \gamma^\dagger(z(t)) + z(t)$$

for all $v(0), z(t)$ such that the first arguments in the two H_∞ functions in (38) are in the domain $[0,b]$. In addition, from (12) and (28) we have

$$\begin{aligned}|x(t)| &\le \alpha_1^{-1}(V_{\sigma(t)}(x(t))) = \alpha_1^{-1}(v(t))\\ &\le \alpha_1^{-1}\big(2\beta^\dagger(v(0),t)\big) \vee \alpha_1^{-1}\big(2\gamma^\dagger(z(t)) + 2z(t)\big)\\ &\le \alpha_1^{-1}\big(2\beta^\dagger(\alpha_2^{-1}(|x_0|),t)\big) \vee \alpha_1^{-1}\big(2\gamma^\dagger(z(t)) + 2z(t)\big)\end{aligned}$$

where we have also used the fact that $\alpha\big(\sum_{i=1}^n s_i\big) \le \bigvee_{i=1}^n \alpha(ns_i)$ for any $\alpha \in \mathcal{K}$. Notice that the previously mentioned constraints on the first arguments in the two H_∞ functions in (38) are also satisfied since we have assumed that $|x_0| \le \delta_1$ and $\int_0^\infty \chi(|u(\tau)|) \le \delta_2$. These assumptions imply that $\beta(y(0),t_1) \le y(0) = v(0) \le \alpha_2(|x_0|) \le \alpha_2(\delta_1) \le b$ and $\varepsilon^{-1}\phi_\varepsilon(0)z(t) \vee z(t) \le \varepsilon^{-1}\phi_\varepsilon(0)\delta_2 \vee \delta_2 \le b$ for all $t \ge 0$. Hence the system (3) is qiISS under slow switching with $\beta^*(s,t) := \alpha_1^{-1}\big(2\beta^\dagger(\alpha_2^{-1}(s),t)\big), \gamma^*(s) := \alpha_1^{-1}\big(2\gamma^\dagger(s) + 2s\big)$ and $\chi^*(s) := \chi(s)$.

To show iISpS when (17) holds, let $\delta_3 > 0$ be arbitrary. By the second statement in Lemma V.2 and Remark 2, (17) implies the existence of $\Delta_0 > 0$ such that for some $b > 0$ satisfying

$$3h(b,0,\varepsilon) = \alpha_1(\delta_3) \tag{43}$$

and some $\Delta \le \Delta_0 \wedge \frac{b}{2}$, there exists $\tau_D > 0$ such that (21) holds for all $(s,t) \in [b,\infty) \times [\tau_D,\infty)$. Thus by the second conclusion in Lemma V.8, there exists $\overline{\beta}(\cdot,\cdot) \in \mathcal{KL}$ such that $H_\infty(s,t) \le \overline{\beta}(s,t) + h(b,0,\varepsilon)$ for all $(s,t) \in [0,\infty) \times [0,\infty)$. Again denoting $\beta^\dagger, \gamma^\dagger$ as in (41), (42) and following from (38), we conclude that

$$v(t) \le \beta^\dagger(v(0),t) + \gamma^\dagger(z(t)) + z(t) + h(b,0,\varepsilon).$$

Following similar derivation as in the previous case we conclude that

$$\begin{aligned}|x(t)| \le \alpha_1^{-1}&\big(3\beta^\dagger(\alpha_2^{-1}(|x_0|),t)\big)\\ &\vee \alpha_1^{-1}\big(3\gamma^\dagger(z(t)) + 3z(t)\big) \vee \delta_3\end{aligned}$$

Because there are no constraints on x_0 and $z(t)$, we have proven that the system (3) is iISpS under slow switching in this case, with $\beta^*(s,t) := \alpha_1^{-1}\big(3\beta^\dagger(\alpha_2^{-1}(s),t)\big), \gamma^*(s) := \alpha_1^{-1}\big(3\gamma^\dagger(s) + 3s\big)$ and $\chi^*(s) := \chi(s)$.

Lastly we need to show iISS when both (16) and (17) hold. By Corollary V.3, (16) plus (17) implies the existence of $\lambda \in (0,1), \Delta > 0$ and $\tau_D > 0$ such that (22) holds for all $(s,t) \in [0,\infty) \times [\tau_D,\infty)$. Thus by the third conclusion in Lemma V.8, there exists $\overline{\beta}(\cdot,\cdot) \in \mathcal{KL}$ such that $H_\infty(s,t) \le \overline{\beta}(s,t)$ for all $(s,t) \in [0,\infty) \times [0,\infty)$. Again denoting $\beta^\dagger, \gamma^\dagger$ as in (41), (42) and following from (38), we conclude that

$$v(t) \le \beta^\dagger(v(0),t) + \gamma^\dagger(z(t)) + z(t)$$

and consequently

$$|x(t)| \le \alpha_1^{-1}\big(2\beta^\dagger(\alpha_2^{-1}(|x_0|),t)\big) \vee \alpha_1^{-1}\big(2\gamma^\dagger(z(t)) + 2z(t)\big).$$

Again because there are no constraints on x_0 and $z(t)$, we have proven that the system (3) is iISS under slow switching in this case, with $\beta^*(s,t) := \alpha_1^{-1}\big(2\beta^\dagger(\alpha_2^{-1}(s),t)\big), \gamma^*(s) := \alpha_1^{-1}\big(2\gamma^\dagger(s) + 2s\big)$ and $\chi^*(s) := \chi(s)$.

C. Proof of Theorem 2

Let

$$\mu(v) = \begin{cases} \frac{\alpha_2 \circ \alpha_1^{-1}(v)}{v} & \text{if } v > 0,\\ \limsup_{v \to 0^+} \frac{\alpha_2 \circ \alpha_1^{-1}(v)}{v} & \text{if } v = 0.\end{cases} \tag{44}$$

We first show that (14) holds for all $x \in \mathbb{R}^n, p, q \in \mathbb{P}$. For nontrivial case when $x \ne 0$, recall $g_{p,q}(x) = x$ for a jump-free switched system, so (12) implies that for any $p, q \in \mathbb{P}$,

$$\begin{aligned}V_q(x) &\le \alpha_2(|x|) \le \alpha_2 \circ \alpha_1^{-1}(V_p(x))\\ &= \frac{\alpha_2 \circ \alpha_1^{-1}(V_p(x))}{V_p(x)} V_p(x) = \mu(V_p(x))V_p(x).\end{aligned}$$

We then show that μ defined via (44) satisfies the conditions in Assumption 2. Denoting $w = \alpha_1^{-1}(v)$, we have

$$\begin{aligned}&\mu'(v) = \frac{1}{v^2}\Big(\frac{\alpha_2'(w)}{\alpha_1'(w)}v - \alpha_2(w)\Big)\\ &\le \frac{1}{v^2}\Big(\frac{\alpha_2(w)}{\alpha_1(w)}v - \alpha_2(w)\Big) = \frac{1}{v^2}\Big(\frac{\alpha_2(w)}{v}v - \alpha_2(w)\Big) = 0,\end{aligned}$$

where we have used the inequality $\frac{\alpha_2'(w)}{\alpha_1'(w)} \le \frac{\alpha_2(w)}{\alpha_1(w)}$ from (18). Hence $\mu(v)$ is non-increasing on $(0,\infty)$. Thus the $\limsup$ in (44) is in fact $\lim$ and

$$\lim_{v \to 0^+} \frac{\alpha_2 \circ \alpha_1^{-1}(v)}{v} = \lim_{w \to 0^+} \frac{\alpha_2(w)}{\alpha_1(w)} \le \overline{M}.$$

Therefore μ is continuous and non-increasing on $[0,\infty)$. In addition, (18) also implies that

$$\begin{aligned}&\mu(s)s - \mu(t)t = \alpha_2 \circ \alpha_1^{-1}(s) - \alpha_2 \circ \alpha_1^{-1}(t)\\ &= \int_t^s \frac{d}{d\tau}\alpha_2 \circ \alpha_1^{-1}(\tau)d\tau = \int_t^s \frac{\alpha_2' \circ \alpha_1^{-1}(\tau)}{\alpha_1' \circ \alpha_1^{-1}(\tau)}d\tau \ge \underline{M}(s-t)\end{aligned}$$

Hence μ satisfies the conditions in Assumption 2.

To show the first implication in Theorem 2, it suffices to show that (16) holds if we construct a $\beta(\cdot,\cdot) \in \mathcal{KL}$ from α. From the statement of Lemma IV.1 we see that there exists $\overline{v} > 0$ such that $\rho_1(v) = \min_{v \le w \le \overline{v}} \alpha(w), \rho_2(v) = 1$ for all $v \in [0,\overline{v}]$. Because of the assumption $\alpha'(0) > 0$, define $\zeta(v) : [0, \frac{\overline{v}}{2}] \to [0,\infty)$ by

$$\zeta(v) := \begin{cases} \frac{\rho_1(v)}{v} & \text{if } v \in (0, \frac{\overline{v}}{2}],\\ \alpha'(0) & \text{if } v = 0\end{cases}$$

It is easy to check that $\zeta(v)$ is positive and continuous over $[0, \frac{\overline{v}}{2}]$. Hence we can define $k := \min_{s\in[0,\frac{\overline{v}}{2}]} \zeta(s) > 0$. Recall that in Theorem 1, $\beta(s,t)$ is the solution to the initial value ODE problem $\dot{v} = -\rho_1(v)\rho_2(2v)$, $v(0) = s$. When $s \le \frac{\overline{v}}{2}$, $\dot{v} = -\zeta(v)v \le -kv$ and thus by comparison principle we have $\beta(s,t) \le e^{-kt}s$. Hence for s, ε sufficiently small,

$$h(s,t,\varepsilon) = \beta\big(\phi_\varepsilon(s)s, t\big) \le e^{-kt}\phi_\varepsilon(s)s \le e^{-kt}\phi_\varepsilon(0)s$$

where recall ϕ_ε is defined in (27). The inequality (16) can be shown subsequently and the first implication in Theorem 2 is proven by applying the first result in Theorem 1.

To show that 2.i implies iISpS, recall that $\lim_{v\to\infty} \alpha(v)/v > 0$ means the existence of $k, l > 0$ such that $\alpha(v) \ge kv$ for all $v \in [0, l]$. Thus we can construct ρ_1, ρ_2 such that $\rho_1(v) = kv, \rho_2(v) = 1$ for all $v \ge l$ and $\alpha(v) \ge \rho_1(v)\rho_2(v)$ for all $v \ge 0$. Because $\beta(s,t)$ is the solution to the initial value ODE problem $\dot{v} = -\rho_1(v)\rho_2(2v)$, $v(0) = s$, when the initial condition $s \ge l$, $\beta(s,t) \le se^{-kt} \vee l$ for all $t \ge 0$. Pick $\tau_D = \frac{1}{k}\big(\ln \phi_\varepsilon(0) + \ln(l+1) - \ln l\big)$. By this construction we have $e^{-k\tau_D} = \frac{l}{\phi_\varepsilon(0)(l+1)}$. Thus for any $s \ge l+1$ and $t \ge \tau_D$, $\phi_\varepsilon(0)s \ge s \ge l$ and

$$h(s,t,\varepsilon) = \beta\big(\phi_\varepsilon(s)s, t\big) \le \phi_\varepsilon(s)se^{-kt} \vee l \le \frac{sl}{l+1} \vee l$$

Because $s \ge l+1$, $\frac{sl}{l+1} \le s-1$ and therefore $h(s,t,\varepsilon) \le s-1$. Subsequently (17) can be shown and hence by the second result in Theorem 1 the system is iISpS under slow switching.

Next we show that 2.ii implies iISpS. Because $\alpha \in \mathcal{K}_\infty$ can always be replaced by $\alpha \in \mathcal{K}$ while preserving (13), we can always assume $\lim_{s\to\infty} \alpha(s) =: M_1 < \infty$. Because $\alpha \in \mathcal{K}$, $M_1 > 0$. Now by definition (44) and the assumption on $\limsup$ in this case,

$$\begin{aligned} M_2 :&= \limsup_{v\to\infty} (\mu(v) - 1)v \\ &= \limsup_{v\to\infty} \alpha_2 \circ \alpha_1^{-1}(v) - v \\ &= \limsup_{w\to\infty} \alpha_2(w) - \alpha_1(w) < \infty \end{aligned}$$

As a result, for any $\delta > 0$, there exists $\underline{v}$ such that as long as $v \ge \underline{v}$, $0 \le M_1 - \alpha(v) \le \delta$ and $\mu(v)v - v \le M_2 + \delta$. We pick large enough $\underline{v}$ such that $\underline{v} \ge \frac{(2+\varepsilon)(M_2+\delta)M_1}{M_1 - \varepsilon}$. Let $T = \frac{\underline{v}}{M_1}$. For any $v(0) \ge 2\underline{v}$, $\dot{v} = -\alpha(v) \ge -M_1$ so by comparison principle we have $v(t) \ge v(0) - M_1 t \ge 2\underline{v} - \frac{\underline{v}t}{T} \ge \underline{v}$ for all $t \in [0, T]$ so indeed the comparison principle can be applied over this time interval. In addition, $\dot{v} = -\alpha(v) \le -M_1 + \delta$ so $v(T) \le v(0) - T(M_1 - \delta) = v(0) - \frac{\underline{v}(M_1-\varepsilon)}{M_1} \le v(0) - (2 + \varepsilon)(M_2+\delta)$. As a result for all $s \ge 2\underline{v}$, $\big((1+\varepsilon)\mu(s) - 1\big)s \ge s \ge 2\underline{v}$ so the initial condition is large enough and consequently for all $t \ge T$ we have

$$\begin{aligned} h(s,t,\varepsilon) - s &= \beta\big(\phi_\varepsilon(s)s, t\big) - s \\ &\le \beta\Big(\big(\phi_\varepsilon(s)s, T\Big) - s \\ &\le \phi_\varepsilon(s)s - (2+\varepsilon)(M_2+\delta) - s \\ &= (1+\varepsilon)(\mu(s)s - s) - (2+\varepsilon)(M_2+\delta) \\ &\le (1+\varepsilon)(M_2+\delta) - (2+\varepsilon)(M_2+\delta) \\ &= -(M_2+\delta) \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0^+} \lim_{t\to\infty} \lim_{s\to\infty} h(s,t,\varepsilon) - s \le -(M_2+\delta) < 0$$

So the property (17) holds and the system is iISpS by the second result in Theorem 1.

The last implication in Theorem 2 is a direct result of the last result in Theorem 1.

VI. Examples revisited

In this section we apply our main results to the two examples aforementioned in Section III to draw some conclusions about their iISS properties.

We use the Lyapunov function $V_p = \sqrt{x^\top M_p x}$ to show that the subsystems (8) are iISS, where M_p is the solution to the Lyapunov equation

$$M_p A_p + A_p^\top M_p + 2I = 0. \tag{45}$$

Denote

$$\overline{a} := \max_{p\in\mathbb{P}} \overline{\sigma}(M_p), \quad \underline{a} := \min_{p\in\mathbb{P}} \underline{\sigma}(M_p) \tag{46}$$

where $\overline{\sigma}(M)$ is the largest singular value of M and $\underline{\sigma}(M)$ is the smallest singular value of M. This choice of Lyapunov functions gives us the condition (12) where $\alpha_1(v) = \sqrt{\underline{a}}v, \alpha_2(v) = \sqrt{\overline{a}}v$. Thus $\frac{\alpha_2'(v)}{\alpha_1'(v)} = \frac{\alpha_2(v)}{\alpha_1(v)} = \sqrt{\frac{\overline{a}}{\underline{a}}}$ and the assumption (18) is satisfied. In addition,

$$\begin{aligned} \nabla V_p(x) \cdot f_p(x,u) &= \frac{x^\top M_p}{\sqrt{x^\top M_p x}} \left(\frac{1}{1+|x|^2} A_p x + u \right) \\ &= -\frac{|x|^2}{\sqrt{x^\top M_p x}(1+|x|^2)} + \frac{x^\top M_p}{\sqrt{x^\top M_p x}} u \\ &= -\frac{1}{V_p(x)\left(\frac{1}{|x|^2} + 1\right)} + \frac{x^\top M_p}{\sqrt{x^\top M_p x}} u \\ &\le -\frac{1}{V_p(x)\left(\frac{\overline{a}}{V_p(x)^2} + 1\right)} + \frac{\overline{a}}{\sqrt{\underline{a}}}|u| \\ &= -\frac{V_p(x)}{\overline{a} + V_p(x)^2} + \frac{\overline{a}}{\sqrt{\underline{a}}}|u| \end{aligned}$$

Hence we have $\alpha(v) = \frac{v}{\overline{a}+v^2}, \chi(v) = \frac{\overline{a}}{\sqrt{\underline{a}}}v$. It is easy to compute that $\alpha'(0) = \frac{1}{\overline{a}} > 0$ so by the first implication in Theorem 2, the jump-free switched system generated by (8) is qiISS under slow switching. On the other hand, neither assumption in the second implication in Theorem 2 applies here because α is not super-linear at infinity (in fact it converges to 0 at infinity) and $\alpha_2(s) - \alpha_1(s)$ diverges at infinity. Indeed, we have already shown in Section III that this switched system is not iISS under slow switching.

For completeness, we provide a numerical estimation of the dwell time for this qiISS system with A_1, A_2 given by (9). Solving the Lyapunov equation (45) for M_p and using the definitions (46), it is not hard to compute that $\overline{a} \approx 14.98, \underline{a} \approx 7.50$. Hence $\alpha(v) = \frac{v}{14.98+v^2}, \chi(v) = 5.47v, \mu(v) = 1.41$. Pick $x_0 = (1,0)^\top$ and $u(t) = 0.001(e^{-t}, e^{-t})^\top$; set $\delta_1 := 1 = |x_0|, \delta_2 := 0.0080 > 0.0077 = \int_0^\infty \chi(|u(\tau)|)d\tau$ and $\lambda := 0.9$. It follows from (40) that for $\varepsilon = 0.01$, $b = 3.87$. In addition, with such a choice of parameters, it is numerically computed from the condition (20) that $\tau_D = 42.72$.

For the other jump-free switched system with the modes (11), we pick the Lyapunov function $V_p = |x|x^\top M_p x$ where M_p again solves (45). Again the assumption (12) holds with $\alpha_1(v) = \underline{a}v^3, \alpha_2(v) = \overline{a}v^3$ where $\overline{a}, \underline{a}$ are defined in (46). Thus $\frac{\alpha_2'(v)}{\alpha_1'(v)} = \frac{\alpha_2(v)}{\alpha_1(v)} = \frac{\overline{a}}{\underline{a}}$ and the assumption (18) in Theorem 2 is satisfied. In addition,

$$\begin{aligned}
&\nabla V_p(x) \cdot f_p(x,u) \\
&\quad = \left((x^\top M_p x)\frac{x^\top}{|x|} + 2|x|x^\top M_p \right) (|x|A_p x + u) \\
&\quad = x^\top M_p \left(\frac{xx^\top}{|x|} + 2|x|I \right) (|x|A_p x + u) \\
&\quad = x^\top \left(\frac{xx^\top}{|x|} + 2|x|I \right) M_p(|x|A_p x + u) \\
&\quad = 3|x|x^\top M_p(|x|A_p x + u) \\
&\quad \le -3|x|^4 + 3\overline{a}|x|^2|u| \\
&\quad \le -\frac{3}{2}|x|^4 + \frac{3\overline{a}^2}{2}|u|^2 \le -\frac{3}{2}\left(\frac{V_p(x)}{\overline{a}}\right)^{\frac{4}{3}} + \frac{3\overline{a}^2}{2}|u|^2
\end{aligned}$$

Hence we conclude $\alpha(v) = \frac{3}{2}\left(\frac{v}{\overline{a}}\right)^{\frac{4}{3}}, \chi(v) = \frac{3\overline{a}^2}{2}v^2$ and each mode (11) is iISS (in fact also ISS). Clearly this α satisfies the assumption in 2.i of Theorem 2; hence the switched system (11) is iISpS under slow switching. On the other hand, the assumption in the first implication in Theorem 2 does not apply because $\alpha'(0) = 0$. Indeed, we have already shown in Section III that this switched system is not iISS under slow switching. Similarly to what has been done for the qiISS example, let us provide a numerical estimation of the dwell time for this iISpS system. For this example, we have $\alpha(v) = \frac{3}{2}\left(\frac{v}{14.98}\right)^{\frac{4}{3}}, \mu(v) = \frac{\overline{a}}{\underline{a}} = 1.99$; we also pick $\Delta := 0.001$ and $\delta_3 := 2$ which, It follows from (43) that for $\varepsilon = 0.001$, $b = 10.02$. With such a choice of parameters, it is numerically computed from the condition (21) that $\tau_D = 7.12$.

At last, we draw some conclusions on the switched bilinear systems with inputs, generated from finitely many modes:

$$\dot{x} = f_p(x,u) = A_p x + \sum_{j=1}^{m} B_{p,j} x u_j + C_p u \tag{47}$$

where $p \in \mathbb{P} = \{1, 2, \dots, P\}$ are the modes, $x \in \mathbb{R}^n$, $u = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^\top \in \mathbb{R}^m$ and $A_p \in \mathbb{R}^{n\times n}, B_{p,j} \in \mathbb{R}^{n\times n}$ and $C_p \in \mathbb{R}^{n\times m}$.

Proposition VI.1. *The jump-free switched bilinear system generated from finitely many modes* (47)*, where A_p are all Hurwitz, is iISS under slow switching.*

Proof. It is known from [26] that bilinear systems with Hurwitz matrices are iISS. For each $p \in \mathbb{P}$, define the iISS-Lyapunov function $V_p(x)$ by (see, e.g., [27])

$$V_p(x) := \ln(1 + x^\top M_p x) \tag{48}$$

where M_p solves the Lyapunov equation (45). Again using the notation (46), the assumption (12) is satisfied with $\alpha_1(v) = \ln(1 + \underline{a}v^2), \alpha_2(v) = \ln(1 + \overline{a}v^2)$. In addition,

$$\begin{aligned}
&\nabla V_p(x) \cdot f_p(x,u) \\
&\quad = \frac{1}{1 + x^\top M_p x}\Big(x^\top \big(A_p^\top M_p + M_p A_p\big)x \\
&\qquad + 2\Big(\sum_{j=1}^{m} x^\top B_{p,j}^\top u_j\Big) M_p x + 2x^\top M_p C_p u \Big) \\
&\quad = \frac{2}{1 + x^\top M_p x}\Big(-|x|^2 \\
&\qquad + \sum_{j=1}^{m} x^\top B_{p,j}^\top M_p x u_j + x^\top M_p C_p u \Big) \\
&\quad \le \frac{2}{1 + x^\top M_p x}\Big(-|x|^2 \\
&\qquad + \sum_{j=1}^{m} \overline{\sigma}\big(B_{p,j}^\top M_p\big)|x|^2 u_j + x^\top M_p C_p u \Big) \\
&\quad \le \frac{2}{1 + x^\top M_p x}(-|x|^2 + c_1|x|^2|u| + c_2|x||u|) \\
&\quad = -\frac{2|x|^2}{1 + x^\top M_p x} + \frac{2(c_1|x|^2 + c_2|x|)}{1 + x^\top M_p x}|u|
\end{aligned}$$

where $c_1 := \max_{p\in P}\left(\sum_{j=1}^{m} \overline{\sigma}(B_{p,j}^\top M_p)^2\right)^{\frac{1}{2}}, c_2 := \max_{p\in P} \overline{\sigma}(M_p C_p)$. Now notice that $\underline{a}|x|^2 \le x^\top M_p x \le \overline{a}|x|^2$ so

$$-\frac{|x|^2}{1 + x^\top M_p x} \le -\frac{x^\top M_p x}{\overline{a}(1 + x^\top M_p x)} = -\frac{e^{V_p(x)} - 1}{\overline{a}e^{V_p(x)}},$$

$$\begin{aligned}
&\frac{c_1|x|^2 + c_2|x|}{1 + x^\top M_p x} \le \frac{c_1|x|^2 + c_2|x|}{1 + \underline{a}|x|^2} \\
&\quad = \frac{c_1}{\underline{a}}\left(\frac{\underline{a}|x|^2}{1 + \underline{a}|x|^2}\right) + \frac{c_2}{\frac{1}{|x|} + \underline{a}|x|} \le \frac{c_1}{\underline{a}} + \frac{c_2}{2\sqrt{\underline{a}}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\nabla V_p \cdot f_p \le -\frac{2|x|^2}{1 + x^\top M_p x} + \frac{2(c_1|x|^2 + c_2|x|)}{1 + x^\top M_p x}|u|^2 \\
&\le -\frac{2(e^{V_p(x)} - 1)}{\overline{a}e^{V_p(x)}} + 2\left(\frac{c_1}{\underline{a}} + \frac{c_2}{2\sqrt{\underline{a}}}\right)u^2 = -\alpha(V_p) + \chi(u)
\end{aligned}$$

for all $p \in \mathbb{P}$, where $\alpha(v) = \frac{2(e^v - 1)}{\overline{a}e^v}$, $\chi(|u|) = 2\left(\frac{c_1}{\underline{a}} + \frac{c_2}{2\sqrt{\underline{a}}}\right)|u|$. In addition, by definition we see that $\alpha \in \mathcal{PD}$, $\chi \in \mathcal{K}$ so we conclude that each subsystem of (47) is iISS.

Now we want to show that the switched bilinear system is iISS. We start by verifying (18). Define $g : [0,\infty) \to [0,\infty)$ by $g(a) := (1 + av^2)\ln(1 + av^2)$ for some $v > 0$. It can be computed that $g''(a) = v^4/(1 + av^2) > 0$ so g is convex and for $\overline{a} \ge \underline{a} > 0$, we have $g(\underline{a}) \le (\underline{a}/\overline{a})g(\overline{a}) + (1 - \underline{a}/\overline{a})g(0)$, which gives the inequality $(1 + \underline{a}v^2)\ln(1 + \underline{a}v^2) \le (\underline{a}/\overline{a})(1 + \overline{a}v^2)\ln(1 + \overline{a}v^2)$. Thus we conclude that

$$\frac{\alpha_2'(v)}{\alpha_1'(v)} = \frac{\overline{a}(1 + \underline{a}v^2)}{\underline{a}(1 + \overline{a}v^2)} \le \frac{\ln(1 + \overline{a}v^2)}{\ln(1 + \underline{a}v^2)} = \frac{\alpha_2(v)}{\alpha_1(v)}.$$

Further we have $\frac{d}{dv}\left(\frac{\alpha_2(v)}{\alpha_1(v)}\right) = \frac{\alpha_2'(v)\alpha_1(v)-\alpha_1'(v)\alpha_2(v)}{(\alpha_1(v))^2} \leq 0$ so the function $\frac{\alpha_2(v)}{\alpha_1(v)}$ is decreasing and

$$\frac{\alpha_2(v)}{\alpha_1(v)} \leq \lim_{s\to 0}\frac{\alpha_2(s)}{\alpha_1(s)} = \frac{\alpha_2'(0)}{\alpha_1'(0)} = \frac{\overline{a}}{\underline{a}}$$

for all $v > 0$. As a result, the assumption of (18) holds with $\underline{M} = 1, \overline{M} = \frac{\overline{a}}{\underline{a}}$. To check the remaining conditions, notice that $\alpha'(0) = \frac{2}{\overline{a}} > 0$ so the assumption in the first implication of Theorem 2 is satisfied. In addition, clearly $\alpha \in \mathcal{K}$ and

$$\lim_{v\to\infty}\big(\alpha_2(v)-\alpha_1(v)\big) = \lim_{v\to\infty}\ln\left(\frac{1+\overline{a}v^2}{1+\underline{a}v^2}\right) = \ln\left(\frac{\overline{a}}{\underline{a}}\right) < \infty$$

so the assumption in 2.ii of Theorem 2 is also satisfied. Therefore by the third implication in Theorem 2 the switched bilinear system is iISS under slow switching.

□

VII. Discussion and future work

First of all, since the major focus of this paper is to qualitatively determine whether a switched nonlinear system is iISS when it switches sufficiently slowly, the quantitative value of dwell time is not emphasized and it can be investigated in the future research. Nevertheless, we point out here that the τ_D in Corollary V.3 is a dwell time for the switched systems to be iISS. In the special case when the assumptions in the cases 1 and 2.i in Theorem 2 both hold or, equivalently, when (19) holds for the subsystems, we have $\beta(s,t) = se^{-kt}$, given by Lemma V.4. Thus, taking μ to be a constant and ignoring Δ, the inequality (22) gives $\big((1+\varepsilon)\mu - \varepsilon\big)se^{-kt} \leq \lambda s$, which implies $t > \frac{\ln((1+\varepsilon)\mu-\varepsilon)-\ln\lambda}{k}$. Taking $\varepsilon \to 0$ and $\lambda \to 1$, we see that the infimum of dwell time is $\frac{\ln\mu}{k}$, same as the lower bound on dwell time found in [8] which is based on ISS analysis. This suggests that our approach may also be promising for quantitatively analysing general switched nonlinear systems in the sense that it can give tight results on the minimum dwell time.

Another possible future work direction is of course to extend dwell time to average dwell time. It is observed in the literature that linear decay rates in Lyapunov functions near the origin (such as the assumption in 2.i in Theorem 2) are essential when deriving the average dwell time. When this does not hold in general, nonlinear decay rates can be transformed into linear ones as studied in [28]. In addition, the recent work [13] also discusses how to formulate average dwell time conditions for ISS of hybrid systems regardless of the nonlinear estimation function α, based on that transformation. Using similar techniques, some promising results are presented in [29].

We can also study sufficient conditions for iISS of switched systems when there are unstable subsystems. Inspired by [9], the divergence of solutions due to the unstable subsystems can be either compensated by the convergence when the switched system is dwelling in a stable subsystem, or the stable jumps during switches. Reverse dwell time conditions may be concluded in this case to guarantee overall stability of the system.

One drawback of our criteria is that they rely on the Lyapunov functions of subsystems. This suggests that potentially the sufficient conditions based on $\alpha, \alpha_1, \alpha_2$ may not be invariant with respect to the choice of Lyapunov functions. Hence better criteria to test whether a switched system is iISS or not will be directly relying on the stability properties of the subsystems. Certainly as discussed in this work, we need properties stronger than iISS for all the subsystems to hold in order for the switched system to be iISS under slow switching. It is noticed that in the paper [30] iISS is shown to be equivalent to 0-GUAS (global uniform asymptotic stability under zero input) plus UBEBS (uniformly bounded energy of input implying bounded states). By a comparison to our "bad" systems (8) and (11) and the "good" bilinear system (47), an interesting question to ask is whether it is true that if the subsystems of a switched system are all 0-GES (globally exponentially stable under zero input) and UBEBS, then the switched system is iISS. More research can be done in this direction.

VIII. Conclusion

In this paper we have defined iISS, qiISS and iISpS under slow switching for switched nonlinear systems. We then provided two sets of sufficient conditions such that the switched system will have one of the aforementioned stability properties when either set of the proposed conditions is satisfied. In addition, if a switched system satisfies both sets of the proposed conditions, then it is iISS under slow switching. As a direct consequence from our result, we have shown that switched systems whose subsystems are 0-input stable bilinear ones are iISS under slow switching.

Appendix

A. Proofs of lemmas in Section V-A

Proof of Lemma V.1. It suffices to show $\phi_\varepsilon(s)s \in \mathcal{K}$ in order for $h(\cdot,\cdot,\varepsilon) \in \mathcal{KL}$, where ϕ_ε is defined in (27). Clearly $\phi_\varepsilon(s)s|_{s=0} = 0$. Without loss of generality, assume that $\delta < 1$ where δ is given by Assumption 2. Set $\bar{\varepsilon} = \frac{\delta}{1-\delta}$ and for all $\varepsilon \in [0,\bar{\varepsilon})$, we have $\varepsilon(1-\delta) < \delta$. Therefore for all $s > t \geq 0$,

$$\begin{aligned}\phi_\varepsilon(s)s - \phi_\varepsilon(t)t &= (1+\varepsilon)\big(\mu(s)s - \mu(t)t\big) - \varepsilon(s-t)\\ &\geq \big((1+\varepsilon)\delta - \varepsilon\big)(s-t) > 0.\end{aligned}$$

Thus $\phi_\varepsilon(s)s \in \mathcal{K}$ and Lemma V.1 is proven. □

Proof of Lemma V.2. We start with showing the first equivalence. Note $h(s,t,\varepsilon)$ is increasing in ε and by Lemma V.1 when $\varepsilon \in (0,\bar{\varepsilon})$, $h(\cdot,\cdot,\varepsilon) \in \mathcal{KL}$. To show necessity, pick an arbitrary $b > 0$. The condition (16) implies the existence of $\varepsilon_0 \in (0,\bar{\varepsilon}), \tau_1 \geq 0, b' > 0$ and $\delta > 0$ such that

$$\frac{h(s,t,\varepsilon)}{s} \leq 1-\delta$$

for all $(s,t,\varepsilon) \in (0,b') \times [\tau_1,\infty) \times [0,\varepsilon_0]$. We are done in this direction of the proof with $\lambda = 1-\delta$ and $\tau_D = \tau_1$ if $b < b'$. For $b \geq b'$ and each $s \in [b',b]$, pick $\tau(s)$ such that $\frac{h(s,\tau(s),\varepsilon_0)}{s} \leq 1-2\delta$. Because $\lim_{t\to\infty} h(s,t,\varepsilon_0) = 0$, such $\tau(s)$ always exists. By continuity of $\frac{h(s,t,\varepsilon_0)}{s}$ in s, we have $r(s) > 0$ such that $\frac{h(s',\tau(s),\varepsilon_0)}{s'} \leq 1-\delta = \lambda$ for all $s' \in (s-r(s), s+r(s)) \cap [b', b] =: B(s)$. Because $[b', b]$ is compact, there is a finite subcover $\{B(s)\}_{s\in I}$ with index set $I \subseteq [b', b]$ and $\cup_{s\in I} B(s) = [b', b]$. Let $\tau_2 := \bigvee_{s\in I} \tau(s)$, then for all $s' \in [b', b]$, $s' \in B(s)$ for some $s \in I$ and

$$h(s', t, \varepsilon) \leq h(s', \tau_2, \varepsilon_0) \leq h(s', \tau(s), \varepsilon_0) \leq \lambda s'$$

for all $t \geq \tau_2$. Hence (20) holds with $\tau_D = \tau_1 \vee \tau_2$. For sufficiency, let $\varepsilon_0 > 0, \lambda < 1$ be given. Fix some $b > 0$, there is $\tau_D \geq 0$ such that for all $(s,t,\varepsilon) \in [0,b] \times [\tau_D, \infty) \times [0, \varepsilon_0]$, $h(s,t,\varepsilon) \leq \lambda s$. Thus

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \limsup_{s\to 0^+} \frac{h(s,t,\varepsilon)}{s} \\
&\quad = \limsup_{\varepsilon\to 0^+, \varepsilon\leq\varepsilon_0} \limsup_{t\to\infty, t\geq\tau_D} \limsup_{s\to 0^+, s\leq b} \frac{h(s,t,\varepsilon)}{s} \\
&\quad\quad \leq \limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty, t\geq\tau_D} \limsup_{s\to 0^+, s\leq b} \frac{\lambda s}{s} = \lambda < 1.
\end{aligned}$$

We then show the second equivalence. To show necessity, again pick an arbitrary $b > 0$. The condition (17) implies that there exist $\varepsilon_0 \in (0, \bar{\varepsilon}), \tau_1 \geq 0, b' > 0$ and $\Delta > 0$ such that

$$h(s, \tau_1, \varepsilon) - s \leq -\Delta \tag{49}$$

for all $(s,t,\varepsilon) \in (b', \infty) \times [\tau_1, \infty) \times [0, \varepsilon_0]$. It is not hard to see that the choice of Δ is not unique; there exists $\Delta_0 > 0$ such that (17) implies (49) for any $\Delta \in (0, \Delta_0]$. We are done with this direction of the proof with $\tau_D = \tau_1$ if $b > b'$. For $b \leq b'$ and each $s \in [b, b']$, pick $\tau(s)$ such that $h(s, \tau(s), \varepsilon_0) - s \leq -\frac{3}{2}\Delta$. Because $\lim_{t\to\infty} h(s,t,\varepsilon_0) = 0$ and $s \geq b \geq 2\Delta$, such $\tau(s)$ always exists. By continuity of $h(s,t,\varepsilon_0) - s$ in s, we have $r(s) > 0$ such that $h(s', \tau(s), \varepsilon_0) - s' \leq -\Delta$ for all $s' \in (s - r(s), s + r(s)) \cap [b, b'] =: B(s)$. Because $[b, b']$ is compact, there is a finite subcover $\{B(s)\}_{s\in I}$ with index set $I \subseteq [b, b']$ and $\cup_{s\in I} B(s) = [b, b']$. Let $\tau_2 := \bigvee_{s\in I} \tau(s)$, then for all $s' \in [b', b]$, $s' \in B(s)$ for some $s \in I$ and

$$h(s', t, \varepsilon) \leq h(s', \tau_2, \varepsilon_0) \leq h(s', \tau(s), \varepsilon_0) \leq s' - \Delta$$

for all $t \geq \tau_2$. Hence (21) holds with $\tau_D = \tau_1 \vee \tau_2$. For sufficiency, let $\varepsilon_0 > 0, \Delta > 0$ be given. Fix some $b \geq 2\Delta$, there is $\tau_D \geq 0$ such that for all $(s,t,\varepsilon) \in [b, \infty) \times [\tau_D, \infty) \times [0, \varepsilon_0]$, $h(s,t,\varepsilon) \leq s - \Delta$. Thus

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty} \limsup_{s\to\infty} \Big(h(s,t,\varepsilon) - s\Big) \\
&\quad = \limsup_{\varepsilon\to 0^+, \varepsilon\leq\varepsilon_0} \limsup_{t\to\infty, t\geq\tau_D} \limsup_{s\to\infty, s\geq b} \Big(h(s,t,\varepsilon) - s\Big) \\
&\quad \leq \limsup_{\varepsilon\to 0^+} \limsup_{t\to\infty, t\geq\tau_D} \limsup_{s\to\infty, s\geq b} s - \Delta - s = -\Delta < 0.
\end{aligned}$$

□

Proof of Lemma V.5. Define

$$g(s,t,\tau) := \beta_2\big(\beta_1(s,\tau), (t-\tau) \vee 0\big).$$

Denote $B_\delta(s,t) := \{(s',t') \in [0,\infty) \times [0,\infty) : |s-s'|^2 + |t-t'|^2 \leq \delta^2\}$. We show that $g(s,t,\tau)$ is continuous in (s,t), uniformly over $B_{\delta_0}(0,0) \times [\tau_D, \infty)$ for any $\delta_0 > 0$. Pick $\varepsilon > 0$ and let $T > \tau_D$ be such that $\beta_2(\beta(\delta_0, T), 0) \leq \varepsilon$. Then for all $(s,t), (s',t') \in B_{\delta_0}(s,t), \tau \geq T$ we have $|g(s',t',\tau) - g(s,t,\tau)| \leq g(s',t',\tau) \leq \beta_2(\beta_1(\delta_0, T), 0) \leq \varepsilon$. For each $\tau \in [\tau_D, T]$, there exists $\delta = \delta(\tau) > 0$ such that $|g(s',t',\tau) - g(s,t,\tau)| \leq \frac{\varepsilon}{3}$ for all $(s,t),(s',t') \in B_{\delta_0}(0,0)$. In addition there is $r = r(\tau)$ such that $|g(s,t,\tau') - g(s,t,\tau)| \leq \frac{\varepsilon}{3}$ for all $(s,t) \in B_{\delta_0}(0,0)$, $\tau' \in [\tau_D, T]$ with $|\tau' - \tau| \leq r(\tau)$. Therefore

$$\begin{aligned}
|g(s',t',\tau') - g(s,t,\tau')| &\leq |g(s',t',\tau') - g(s',t',\tau)| \\
&\quad + |g(s',t',\tau) - g(s,t,\tau)| \\
&\quad + |g(s,t,\tau) - g(s,t,\tau')| \\
&= \varepsilon
\end{aligned}$$

Notice that $\{[\tau - r(\tau), \tau + r(\tau)] \cap [0,T]\}_{\tau\in[\tau_D,T]}$ is a cover of the interval $[\tau_D, T]$ so there is a finite subcover with index set $I \subset [\tau_D, T]$. Let $\bar{\delta} = \delta_0 \wedge \big(\bigwedge_{\tau\in I} \delta(\tau)\big)$. For any $(s',t') \in B_{\bar{\delta}}(s,t)$ and any $\tau \geq \tau_D$, we will always have $|g(s',t',\tau) - g(s,t,\tau)| \leq \varepsilon$ so the continuity of $g(s,t,\tau)$ in (s,t) is uniform over $B_{\delta_0}(0,0) \times [\tau_D, \infty)$.

We also have other properties for $g(s,t,\tau)$. For each pair (s,t), $g(s,t,\tau)$ is bounded and continuous in τ and $\lim_{\tau\to\infty} g(s,t,\tau) = 0$ so the supremum in $\tau \geq \tau_D$ is attainable at a finite τ. For any $s, s', t, t' \geq 0$, let $\delta_0 > 0$ be such that both $(s,t),(s',t') \in B_{\delta_0}(0,0)$. Let $\varepsilon > 0$ be arbitrary. There is $\tau^* \geq \tau_D$ such that $\beta_3(s,t) = \sup_{\tau\geq\tau_D} g(s,t,\tau) = g(s,t,\tau^*)$. By earlier result on uniform continuity we know that there exists $\delta > 0$ which only depends on ε and δ_0 such that $|g(s',t',\tau^*) - g(s',t',\tau^*)| \leq \varepsilon$ if $(s',t') \in B_\delta(s,t)$. Hence

$$\beta_3(s,t) = g(s,t,\tau^*) \leq g(s',t',\tau^*) + \varepsilon \leq \beta_3(s',t') + \varepsilon.$$

Swapping (s,t) with (s',t') we can conclude $\beta_3(s',t') \leq \beta_3(s,t) + \varepsilon$. Therefore $|\beta_3(s,t) - \beta_3(s',t')| \leq \varepsilon$ and continuity of β_3 is shown.

Monotonicity of $\beta_3(s,t)$ in s and t comes from the fact that $g(s,t,\tau)$ is monotone in s and t for each fixed τ and it can be shown in a similar manner as continuity.

Lastly to show $\lim_{t\to\infty} \beta_3(s,t) = 0$, again we fix $s \geq 0$ and let $\varepsilon > 0$ be arbitrary. Pick $T > \tau_D$ such that $\beta_2(\beta_1(s,T), 0) \leq \varepsilon$. Further pick $\bar{t} \geq T$ such that $\beta_1(\beta_2(s, \tau_D), \bar{t} - T) \leq \varepsilon$. Then for all $\tau \geq \tau_D, t \geq \bar{t}$, either $\tau \geq T$ so $g(s,t,\tau) \leq \beta_2(\beta_1(s,T), 0) \leq \varepsilon$, or $\tau \in [\tau_D, T]$ and $g(s,t,\tau) \leq \beta_2(\beta_1(s,\tau_D), \bar{t} - T) \leq \varepsilon$. Hence $\beta_3(s,t) = \sup_{\tau\geq\tau_D} g(s,t,\tau) \leq \varepsilon$ for all $t \geq \bar{t}$ and since ε is arbitrary, $\lim_{t\to\infty} \beta_3(s,t) = 0$. □

Proof of Lemma V.6. When $s_2 \leq \varepsilon s_1$, $\alpha(k_1 s_1 + k_2 s_2) \leq \alpha\big((k_1 + \varepsilon k_2) s_1\big)$; otherwise $s_1 < \varepsilon^{-1} s_2$ and $\alpha(k_1 s_1 + k_2 s_2) \leq \alpha\big((\varepsilon^{-1} k_1 + k_2) s_2\big)$. In both cases we conclude the inequality (26). □

B. Proofs of lemmas in Section V-B

Proof of Lemma V.7. The claim that $h_i \in \mathcal{KL}$ is a direct result of Lemma V.5 and the claim that $l_i \in \mathcal{K}_\infty$ follows from the definition (34). We now use induction on j to show (35). The base cases when $i \in \mathbb{N}_+, j = 1$ are trivially given by (32). Suppose the estimate (35) is true for all $i \in \mathbb{N}$ and all $j \leq N$. We now want to find an upper bound on $y(t)$ for $t \in$

$(t_{i+N}, t_{i+N+1}]$. Notice that there are N switches from time t_{i+1} to time t so by induction hypothesis we have

$$\begin{aligned}y(t) &\le h_N(y(t_{i+1}), t-t_{i+1}) \vee l_N(z(t))\\ &\le h_N\big(\beta_y(y(t_i), t_{i+1}-t_i) \vee \beta_z(z(t_i), t_{i+1}-t_i)\\ &\quad \vee z(t_{i+1}), t-t_{i+1}\big) \vee l_N(z(t))\\ &\le h_N\big(\beta_y(y(t_i), t_{i+1}-t_i), t-t_{i+1}\big)\\ &\quad \vee h_N\big(\beta_z(z(t_i), t_{i+1}-t_i) \vee z(t_{i+1}), t-t_{i+1}\big)\\ &\quad \vee l_N(z(t))\end{aligned}$$

where the second inequality comes from bounding $y(t_{i+1})$ using (32) and it is split into two terms in the third inequality. In addition since $\sigma \in \Sigma(\tau_D)$, $t_{i+1}-t_i \ge \tau_D$ so

$$\begin{aligned}&h_N\big(\beta_y(y(t_i), t_{i+1}-t_i), t-t_{i+1}\big)\\ &\quad \le \sup_{\tau \ge \tau_D} h_N\big(\beta_y(y(t_i), \tau), t-\tau\big) = h_{N+1}(y(t_i), t).\end{aligned}$$

Thus by the monotonicity of z, β_z and h_N, we have

$$\begin{aligned}&h_N\big(\beta_z(z(t_i), t_{i+1}-t_i) \vee z(t_{i+1}), t-t_{i+1}\big) \vee l_N(z(t))\\ &\le h_N\big(\beta_z(z(t), \tau_D) \vee z(t), 0\big) \vee l_N(z(t)) = l_{N+1}(z(t)).\end{aligned}$$

Therefore $y(t) \le h_{N+1}(y(t_i), t-t_i) \vee l_{N+1}(z(t))$ and Lemma V.7 is proven by induction. □

Proof of Lemma V.8. Define $H_N(s,t)$ as in (39). As discussed in the proof of Theorem 1, we need to show uniform boundedness of $H_N(s,t)$ in the domain $[0,b)\times[0,\infty)$ for the first case or in the domain $[0,\infty)\times[0,\infty)$ for the second and third case. We also need to show the properties that $H_N(s,\cdot)$ eventually uniformly converges to 0 for the first and third case, or it is eventually uniformly bounded from above by $h(b-\Delta, 0, \varepsilon)$ for the second case. We discuss the three cases individually.

1) In this case, we need to show that $H_\infty(s,t)$ is finite for all $(s,t) \in [0,b]\times[0,\infty)$ and $\lim_{t\to\infty} H_\infty(s,t) = 0$ for all $s \in [0,b]$. To show finiteness we claim

$$h_i(s,t) \le h(\lambda^{i-1}s, 0, \varepsilon) \tag{50}$$

for all $i \in \mathbb{N}_+$ and all $(s,t) \in [0,b]\times[0,\infty)$. The base case is trivial as $h_1(s,t) = h(s,t,\varepsilon) \le h(s,0,\varepsilon)$. When the claim is true for the incidence i,

$$\begin{aligned}h_{i+1}(s,t) &= \sup_{\tau \ge \tau_D} h_i\Big(h(s,\tau,\varepsilon), (t-\tau)\vee 0\Big)\\ &\le h_i(h(s,\tau_D,\varepsilon),0) \le h_i(\lambda s, 0) \le h(\lambda^i s, 0, \varepsilon)\end{aligned}$$

where we have used the property (20) for the second inequality above. Thus we have proven the claim and we conclude that

$$\begin{aligned}H_N(s,t) &= \bigvee_{i=1}^N h_i(s,t)\\ &\le \bigvee_{i=1}^N h(\lambda^{i-1}s, 0, \varepsilon) \le h(s,0,\varepsilon)\end{aligned}$$

And hence the point-wise limit $H_\infty(s,t)$ is finite.
To show $\lim_{t\to\infty} H_\infty(s,t) = 0$, take any $\delta > 0$ and sufficiently large $M \in \mathbb{N}$ such that $h(\lambda^{M-1}b, 0, \varepsilon) < \delta$. Since $\lim_{t\to\infty} h_i(s,t) = 0$, for any $i \in \mathbb{N}_+, i \le M$, there exists τ_i such that $h_i(b,t) < \delta$ for all $t \ge \tau_i$. Denote $\bar\tau := \bigvee_{j=1}^M \tau_j$. For all $(s,t) \in [0,b]\times[\bar\tau,\infty)$, and all $i \in \mathbb{N}_+$, either $i \le M$ so

$$h_i(s,t) \le h_i(b,\bar\tau) < \delta,$$

or $i > M$ and by the earlier claim (50) we have

$$\begin{aligned}h_i(s,t) \le h_i(b,t) &\le h(\lambda^{i-1}b, 0, \varepsilon)\\ &\le h(\lambda^{M-1}b, 0, \varepsilon) < \delta.\end{aligned}$$

Hence all $h_i(s,\cdot)$ converge to 0 uniformly and consequently $\lim_{t\to\infty} H_\infty(s,t) = 0$. We conclude that there exists $\bar\beta \in \mathcal{KL}$ such that $H_\infty(s,t) \le \bar\beta(s,t)$ for all $(s,t) \in [0,b]\times[0,\infty)$.

2) In this case we need to show that $H_\infty(s,t)$ is finite for all $(s,t) \in [0,\infty)\times[0,\infty)$ and $\lim_{t\to\infty} H_\infty(s,t) \le h(b-\Delta, 0, \varepsilon)$ for all $s \ge 0$. To show finiteness we claim that

$$h_i(s,t) \le h\big((s-(i-1)\Delta) \vee (b-\Delta), 0, \varepsilon\big) \tag{51}$$

for all $i \in \mathbb{N}_+$ and all $(s,t) \in [0,\infty)\times[0,\infty)$. Again the base case is trivial. Recall the property (21) and notice that when $s \ge 0$ and $t \ge \tau_D$, either $s \le b$ and hence $h(s,t,\varepsilon) \le h(b,\tau_D,\varepsilon) \le b-\Delta$, or $s \ge b$ and $h(s,t,\varepsilon) \le s-\Delta$. Hence when the claim is true for the incidence i,

$$\begin{aligned}h_{i+1}(s,t) &= \sup_{\tau \ge \tau_D} h_i\Big(h(s,\tau,\varepsilon), (t-\tau)\vee 0\Big)\\ &\le h_i(h(s,\tau_D,\varepsilon),0) \le h_i\big((s-\Delta)\vee(b-\Delta), 0\big)\\ &\le h\Big(\big((s-\Delta)\vee(b-\Delta)-(i-1)\Delta\big)\vee(b-\Delta), 0, \varepsilon\Big)\\ &\le h\big((s-i\Delta)\vee(b-\Delta), 0, \varepsilon\big).\end{aligned}$$

Thus we have proven the claim and we conclude that

$$\begin{aligned}H_N(s,t) &= \bigvee_{i=1}^N h_i(s,t)\\ &\le \bigvee_{i=1}^N h\big((s-(i-1)\Delta)\vee(b-\Delta), 0, \varepsilon\big)\\ &\le h(s \vee (b-\Delta), 0, \varepsilon)\end{aligned}$$

and hence the point-wise limit $H_\infty(s,t)$ is finite.
To show $\lim_{t\to\infty} H_\infty(s,t) \le h(b-\Delta, 0, \varepsilon)$, fix any $s \ge 0$ and pick sufficiently large $M \in \mathbb{N}$ such that $s - M\Delta \le b$. Since $\lim_{t\to\infty} h_i(s,t) = 0$, for any $i \in \mathbb{N}_+, i \le M$, there exists τ_i such that $h_i(s,t) < h(b-\Delta, 0, \varepsilon)$ for all $t \ge \tau_i$. Denote $\bar\tau := \bigvee_{j=1}^M \tau_j$. For any $t \ge \bar\tau$ and $i \in \mathbb{N}$, either $i \le M$ so

$$h_i(s,t) \le h_i(s,\bar\tau) < h(b-\Delta, 0, \varepsilon),$$

or $i > M$ and by the earlier claim (51) we have

$$\begin{aligned}&h_i(s,t) \le h\big((s-(i-1)\Delta)\vee(b-\Delta), 0, \varepsilon\big)\\ &\le h\big((s-(M-1)\Delta)\vee(b-\Delta), 0, \varepsilon\big) \le h(b-\Delta, 0, \varepsilon).\end{aligned}$$

Hence for each $s \ge 0$, $h_i(s,t)$ are uniformly bounded from above by $h(b-\Delta, 0, \varepsilon)$ when $t \ge \bar\tau$ and thus

$\lim_{t\to\infty} H_\infty(s,t) \le h(b-\Delta,0,\varepsilon)$. We conclude that there exists $\bar\beta(s,t) \in \mathcal{KL}$ such that $H_\infty(s,t) \le \bar\beta(s,t) + h(b-\Delta,0,\varepsilon)$ for all $(s,t) \in [0,\infty)\times[0,\infty)$.

3) In the last case we need to show that $H_\infty(s,t)$ is finite for all $(s,t) \in [0,\infty)\times[0,\infty)$ and $\lim_{t\to\infty} H_\infty(s,t) = 0$ for all $s \ge 0$. To show finiteness we claim that

$$h_i(s,t) \le h\Big(\bigvee_{j=0}^{i-1} p(s,j,i-1-j),0,\varepsilon\Big) \qquad (52)$$

for all $i \in \mathbb{N}_+$ and all $(s,t) \in [0,\infty)\times[0,\infty)$, where $p(s,j,k) := \lambda^j\big(s-k\Delta\big) \vee 0$. We see that $p(s,j,k) \le s$ for all $j,k \in \mathbb{N}$ and it is decreasing in both j,k and it converges to 0 when either j or k converges to infinity. In addition,

$$\begin{aligned} p(\lambda s \vee (s-\Delta),j,k) &= \lambda^j(\lambda s - k\Delta) \\ &\quad \vee \lambda^j(s-(k+1)\Delta) \vee 0 \\ &\le \lambda^{j+1}(s-k\Delta) \quad \vee \lambda^j(s-(k+1)\Delta)\vee 0 \\ &= p(s,j+1,k) \vee p(s,j,k+1). \end{aligned}$$

The base case of (52) is trivial. When the claim is true for the incidence i,

$$\begin{aligned} h_{i+1}(s,t) &= \sup_{\tau \ge \tau_D} h_i\Big(h(s,\tau,\varepsilon),(t-\tau)\vee 0\Big) \\ &\le h_i(h(s,\tau_D,\varepsilon),0) \\ &\le h_i(\lambda s \vee (s-\Delta),0) \\ &\le h\Big(\bigvee_{j=0}^{i-1} p(\lambda s \vee (s-\Delta),j,i-1-j),0,\varepsilon\Big) \\ &\le h\Big(\bigvee_{j=0}^{i-1}\big(p(s,j+1,i-1-j)\vee p(s,j,i-j)\big),0,\varepsilon\Big) \\ &\le h\Big(\Big(\bigvee_{j=1}^{i} p(s,j,i-j)\Big) \vee \Big(\bigvee_{j=0}^{i-1} p(s,j,i-j)\Big),0,\varepsilon\Big) \\ &= h\Big(\bigvee_{j=0}^{i} p(s,j,i-j),0,\varepsilon\Big) \end{aligned}$$

where we have used the property (20) for the second inequality above. Thus we have proven the claim and we conclude that

$$\begin{aligned} H_N(s,t) &= \bigvee_{i=1}^{N} h_i(s,t) \\ &\le \bigvee_{i=1}^{N} h\Big(\bigvee_{j=0}^{i-1} p(s,j,i-1-j),0,\varepsilon\Big) \le h(s,0,\varepsilon) \end{aligned}$$

And hence the point-wise limit $H_\infty(s,t)$ is finite. The proof for showing $\lim_{t\to\infty} H_\infty(s,t) = 0$ is similar to the first case but more complicated. Denote $P(s,i) := \bigvee_{j=0}^{i} p(s,j,i-j) = \bigvee_{j=0}^{i} \lambda^j\big(s-(i-j)\Delta\big)$. We claim that $P(s,i)$ is decreasing in i and $\lim_{i\to\infty} P(s,i) = 0$ for all $s \ge 0$.

$$\begin{aligned} P(s,i+1) &= \bigvee_{j=0}^{i+1} p(s,j,i+1-j) \\ &= \Big(\bigvee_{j=0}^{i} p(s,j,i+1-j)\Big) \vee \Big(\bigvee_{j=1}^{i+1} p(s,j,i+1-j)\Big) \\ &\le \Big(\bigvee_{j=0}^{i} p(s,j,i-j)\Big) \vee \Big(\bigvee_{j=1}^{i+1} p(s,j-1,i+1-j)\Big) \\ &= \bigvee_{j=1}^{i} p(s,j,i-j) = P(s,i) \end{aligned}$$

and hence $P(s,i+1)$ is decreasing in i. To show $\lim_{i\to\infty} P(s,i) = 0$, pick any $\varepsilon > 0$ and $M = \lceil \frac{s}{\Delta} + \frac{\ln\varepsilon - \ln s}{\ln\lambda} \rceil$. Notice that $p(s,j,k) \le s$ for all $j,k \in \mathbb{N}$ so if $s \le \varepsilon$, $P(s,i) \le \varepsilon$ for all $i \in \mathbb{N}_+$. Otherwise when $s \ge \varepsilon$, either $j < M - \frac{s}{\Delta}$ so $p(s,j,M-j) \le 0$, or $j \ge M - \frac{s}{\Delta}$ and

$$\begin{aligned} p(s,j,M-j) = \lambda^j(s-(M-j)\Delta) &\le (\lambda^{M-\frac{s}{\Delta}})s \\ &\le (\lambda^{\lceil \frac{s}{\Delta} + \frac{\ln\varepsilon-\ln s}{\ln\lambda}\rceil - \frac{s}{\Delta}})s \le \varepsilon. \end{aligned}$$

Thus we conclude that for all $i \ge M$, $P(s,i) \le P(s,M) = \bigvee_{j=1}^{M} p(s,j,M-j) \le \varepsilon$ so $\lim_{i\to\infty} P(s,i) = 0$. Following from this claim, for any $s \ge 0$ and any $\delta > 0$, there exists a sufficiently large M such that $h\big(P(s,i-1),0,\varepsilon\big) < \delta$ for all $i > M$. On the other hand, because $h_i \in \mathcal{KL}$, there exists τ_i such that $h_i(s,t) < \delta$ for all $t \ge \tau_i$. Denote $\bar\tau := \bigvee_{j=1}^{M} \tau_j$. For all $t \ge \bar\tau$ and $i \in \mathbb{N}_+$, either $i \le M$ so

$$h_i(s,t) \le h_i(s,\bar\tau) < \delta,$$

or $i > M$ and by the earlier claim (52) we have

$$h_i(s,t) \le h\Big(P(s,i-1),0,\varepsilon\Big) < \delta$$

Hence all $h_i(s,\cdot)$ converge to 0 uniformly and we have proven $\lim_{t\to\infty} H_\infty(s,t) = 0$. We conclude that there exists $\bar\beta \in \mathcal{KL}$ such that $H_\infty(s,t) \le \bar\beta(s,t)$ for all $(s,t) \in [0,\infty)\times[0,\infty)$.

□

References

[1] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.

[2] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, Feb 2009.

[3] J. Mancilla-Aguilar and R. García, "A converse Lyapunov theorem for nonlinear switched systems," *Systems & Control Letters*, vol. 41, no. 1, pp. 67 – 71, 2000.

[4] A. S. Morse, "Dwell-time switching," in *Proceedings of the 2nd European Control Conference*, 1993, pp. 176–181.

[5] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control*, vol. 3, Dec 1999, pp. 2655–2660.

[6] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, Apr 1989.

[7] D. Angeli, E. D. Sontag, and Y. Wang, “A characterization of integral input-to-state stability,” *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1082–1097, June 2000.
[8] L. Vu, D. Chatterjee, and D. Liberzon, “Input-to-state stability of switched systems and switching adaptive control,” *Automatica*, vol. 43, no. 4, pp. 639 – 646, 2007.
[9] M. A. Müller and D. Liberzon, “Input/output-to-state stability and state-norm estimators for switched nonlinear systems,” *Automatica*, vol. 48, no. 9, pp. 2029 – 2039, 2012.
[10] A. Russo, S. Liu, D. Liberzon, and A. Cavallo, “Quasi-integral input-to-state stability for switched nonlinear systems,” accepted by the 21st IFAC World Congress, 2020, (IFAC 2020).
[11] W. Xie, C. Wen, and Z. Li, “Input-to-state stabilization of switched nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1111–1116, July 2001.
[12] G. Yang and D. Liberzon, “A Lyapunov-based small-gain theorem for interconnected switched systems,” *Systems & Control Letters*, vol. 78, pp. 47–54, 2015.
[13] G. Zhang and A. Tanwani, “ISS Lyapunov functions for cascade switched systems and sampled-data control,” *Automatica*, vol. 105, pp. 216 – 227, 2019.
[14] C. Cai and A. R. Teel, “Characterizations of input-to-state stability for hybrid systems,” *Systems & Control Letters*, vol. 58, no. 1, pp. 47 – 53, 2009.
[15] J. P. Hespanha, D. Liberzon, and A. R. Teel, “Lyapunov conditions for input-to-state stability of impulsive systems,” *Automatica*, vol. 44, no. 11, pp. 2735 – 2744, 2008.
[16] J. Mancilla-Aguilar and R. García, “On converse Lyapunov theorems for ISS and iISS switched nonlinear systems,” *Systems & Control Letters*, vol. 42, no. 1, pp. 47 – 53, 2001.
[17] N. Noroozi, A. Khayatian, and R. Geiselhart, “A characterization of integral input-to-state stability for hybrid systems,” *Mathematics of Control, Signals, and Systems*, vol. 29, no. 3, p. 13, Jun 2017.
[18] C. D. Persis, R. D. Santis, and A. Morse, “Switched nonlinear systems with state-dependent dwell-time,” *Systems & Control Letters*, vol. 50, no. 4, pp. 291 – 302, 2003.
[19] L. Long, “Integral ISS for switched nonlinear time-varying systems using indefinite multiple Lyapunov functions,” *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 404–411, Jan 2019.
[20] H. Haimovich and J. L. Mancilla-Aguilar, “ISS implies iISS even for switched and time-varying systems (if you are careful enough),” *Automatica*, vol. 104, pp. 154 – 164, 2019.
[21] H. Haimovich and J. L. Mancilla-Aguilar, “A characterization of integral ISS for switched and time-varying systems,” *IEEE Transactions on Automatic Control*, vol. 63, no. 2, pp. 578–585, Feb 2018.
[22] H. Khalil, *Nonlinear Systems, 3rd ed.* Prentice Hall, 2002.
[23] H. Shim and D. Liberzon, “Nonlinear observers robust to measurement disturbances in an ISS sense,” *IEEE Transactions on Automatic Control*, vol. 61, no. 1, pp. 48–61, Jan 2016.
[24] Z. P. Jiang, A. R. Teel, and L. Praly, “Small-gain theorem for ISS systems and applications,” *Mathematics of Control, Signals and Systems*, vol. 7, no. 2, pp. 95–120, Jun 1994.
[25] C. M. Kellett, “A compendium of comparison function results,” *Mathematics of Control, Signals, and Systems*, vol. 26, no. 3, pp. 339–374, Sep 2014.
[26] E. D. Sontag, “Comments on integral variants of ISS,” *Systems & Control Letters*, vol. 34, no. 1, pp. 93 – 100, 1998.
[27] A. Chaillet, D. Angeli, and H. Ito, “Combining iISS and ISS with respect to small inputs: The strong iISS property,” *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2518–2524, Sep. 2014.
[28] L. Praly and Y. Wang, “Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability,” *Mathematics of Control, Signals and Systems*, vol. 9, no. 1, pp. 1–33, Mar 1996.
[29] S. Liu, A. Tanwani, and D. Liberzon, “Average dwell-time bounds for ISS and integral ISS of switched systems using Lyapunov functions,” submitted to CDC 2020.
[30] D. Angeli, E. D. Sontag, and Y. Wang, “Further equivalences and semiglobal versions of integral input to state stability,” *Dynamics and Control*, vol. 10, no. 2, pp. 127–149, Apr 2000.