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Unified stability criteria for slowly time-varying and switched linear systems ☆

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ABSTRACT

This paper presents a unified approach to formulating stability conditions for slowly time-varying linear systems and switched linear systems. The concept of total variation is generalized to the case of matrix-valued functions. Using this generalized concept, a result extending existing stability conditions for slowly time-varying linear systems is derived. As special cases of this result, two sets of stability conditions are derived for switched linear systems, which match known results in the literature. A numerical example is included to further illustrate the application of the main result.

1. Introduction

Stability of slowly time-varying linear systems and switched linear systems has been extensively studied during the past decades. Earlier results on stability of slowly time-varying linear systems were derived via the frozen-time approach (Amato, Celentano, & Garofalo, 1993; Coppel, 1978; Desoer, 1969; Ilchmann, Owens, & Prätzel-Wolters, 1987; Ioannou & Sun, 1996). Specifically, if the system is stable for any frozen time and varies slowly enough, then the system is globally exponentially stable. There are two main ways in the literature to characterize the rate of system variation. First, in the work of Amato et al. (1993), Coppel (1978) and Desoer (1969), it is shown that the system is globally exponentially stable if the time derivative of the system matrix is sufficiently small. Second, in the work of Ilchmann et al. (1987) and Ioannou and Sun (1996), global exponential stability is established under either of the following two conditions: (1) the system matrix is globally Lipschitz in time and the Lipschitz constant is sufficiently small; (2) the time integral of the norm of the time derivative of the system matrix is bounded by some affine function of the length of the time interval, and the slope of the affine function is sufficiently small.

The above earlier results all impose conditions on the stability of the system matrix at each instant of time as well as on the continuity of the system matrix, which are somewhat conservative. More recent works on stability of linear time-varying systems have relaxed these conditions (Jetto & Orsini, 2009; Solo, 1994; Zhang, 1993). In these works, stability conditions were derived via a different approach, which was based on the "perturbed frozen-time form" of linear time-varying systems (Jetto & Orsini, 2009). In particular, it was assumed that the system matrix is stable at an infinite sequence of times. Then, the linear time-varying system can be viewed as a combination of a switched linear system with stable subsystems and a perturbation term. It was shown that the linear time-varying system is globally exponentially stable if, for each time interval between two consecutive stable times, the length of the time interval is long enough and the perturbation of the system matrix over the time interval is small enough.

For a switched linear system, the results by Morse (1996), as well as by Hespanha and Morse (1999), stated that if each subsystem is stable and if the system switches sufficiently slowly, then the switched system is also stable. The rate of switching is characterized by the *dwell-time*, or the *average dwell-time*, which describes the time, or average time, respectively, between two successive switches. One extension by Zhai et al. Zhai, Hu, Yasuda, and Michel (2001) relaxed the assumption on the stability of all subsystems, by allowing switched linear systems with unstable subsystems. It was shown that the system is exponentially stable if the average dwell-time is sufficiently large and the ratio between the activation time of unstable subsystems and the activation time

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of stable subsystems is sufficiently small. Following this line of extension, a stabilization result was derived in Zhao, Yin, Li, and Niu (2015) for switched linear systems with only unstable subsystems. Another branch of extensions is to allow different dwell-times or average dwell-times for each subsystem or each pair of subsystems. The former refers to the time (see Blanchini, Casagrande, & Miani, 2010), or the average time (see Zhao, Zhang, Shi, & Liu, 2012), that each subsystem is activated before the system switches to other subsystems. It was shown that the switched system is stable if the dwell-time or average dwell-time for each subsystem is large enough. The latter refers to the elapsed time (see Blanchini et al., 2010; Langerak & Polderman, 2005), or the average elapsed time (see Kundu & Chatterjee, 2015), before the system switches from one subsystem to another in each transition pair. It was shown that the switched system is stable if dwell-time or average dwell-time for each transition pair is large enough. (Other stability results in terms of constrained transition pairs governed by a graph can be found in Lee & Dullerud, 2007; Philippe, Essick, Dullerud, & Jungers, 2016 and the references therein.)

It is natural to view switched linear systems as a special class of linear time-varying systems. Although there are some similarities, to the best of our knowledge, there is no explicit relationship bridging the two sets of stability results. To be more specific, the stability conditions available in one set cannot be applied directly to the other.[1] With this in mind, we study in this paper the gap between the two sets of results. Our aim is to build connections between stability results for slowly time-varying linear systems derived by the frozen-time approach and stability results for switched linear systems with stable subsystems. Inspired by the total variation of vector-valued functions, we first extend the concept of total variation to matrix-valued functions (Section 2). Using this extended concept, we derive a generalized stability result for slowly time-varying linear systems (Section 2), where the system matrix could be piecewise differentiable, with discontinuities at the non-differentiable points. Then, we apply the derived result to switched linear systems, and obtain two sets of stability conditions (Section 3), which we show to match the existing results in Hespanha and Morse (1999) and Kundu and Chatterjee (2015). Next, we present a numerical example to further illustrate how our main results can be applied to establish stability of slowly time-varying linear systems (Section 4). Finally, we draw conclusions and discuss several future directions (Section 5).

Notation: We denote by $\|\cdot\|$ the Euclidean norm of a vector and the induced norm for a matrix. We write A^T for the transpose of a matrix A. We use I to denote the $n \times n$ identity matrix. For any complex number a, we use $\mathrm{Re}\{a\}$ to denote the real part of a. For an $n \times n$ matrix A, we use $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ to denote the eigenvalues of A. The matrix A is called Hurwitz if $\mathrm{Re}\{\lambda_i(A)\} < 0$, for all $i \in \{1, 2, \ldots, n\}$. A continuous function $\alpha(\cdot) : [0, \infty) \to [0, \infty)$ is said to be a class $\mathcal{K}_\infty$ function if $\alpha(0) = 0$, $\alpha(\cdot)$ is strictly monotone increasing, and $\alpha(\cdot)$ is unbounded.

2. Stability of slowly time-varying linear systems

Consider a real n-dimensional linear time-varying system described by

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

where $t \in [0, \infty)$ is the time, $x(\cdot) \in \mathbb{R}^n$ is the state vector, and $A(\cdot) \in \mathbb{R}^{n\times n}$ is the system matrix. The system described by (1) is said to be *globally exponentially stable* if there exist finite positive constants C and ρ such that for any initial condition $x(0) \in \mathbb{R}^n$, the state vector $x(\cdot)$ satisfies

$$\|x(t)\| \le C\|x(0)\|e^{-\rho t} \quad \forall\, t \ge 0.$$

In the special case when $A(t) = A\ \forall\ t \ge 0$, it is well known that the system (1) is globally exponentially stable if and only if A is Hurwitz. The conditions for global exponential stability of (1) have been widely studied (Amato et al., 1993; Coppel, 1978; Desoer, 1969; Ilchmann et al., 1987; Ioannou & Sun, 1996). Here, we present two existing results for slowly time-varying systems. We first introduce an assumption on the boundedness and Hurwitzness of the system matrix.

Assumption 1. The system matrix $A(\cdot)$ is such that

(i) There exists $L > 0$ such that $\|A(t)\| \le L\ \forall\, t \ge 0$.
(ii) There exists $\kappa > 0$ such that $\mathrm{Re}\big\{\lambda_i\big(A(t)\big)\big\} \le -\kappa \quad \forall\, t \ge 0,\ i \in \{1, 2, \ldots, n\}$.

By the above two conditions, for any $\lambda \in (0, \kappa)$, there exists $c > 0$ (which depends on L, κ, and λ, see Khalil (2002, Section 9.6, proof of Lemma 9.9) such that

$$\|e^{A(t)s}\| \le ce^{-\lambda s} \quad \forall\, t \ge 0,\ s \ge 0. \tag{2}$$

In Section 4, we will provide an example demonstrating how c and λ can be computed in a specific case.

Theorem 1 (Desoer, 1969)**.** *The system (1) is globally exponentially stable if the following conditions are satisfied:*

(i) *$A(\cdot)$ satisfies Assumption 1.*
(ii) *$A(\cdot)$ is differentiable and $\|\dot{A}(t)\| \le \frac{4\lambda^2}{3c^4}\ \ \forall\, t \ge 0$, where c and λ are from (2).*

Here $\|\dot{A}(\cdot)\|$ can be regarded as the rate at which the system changes over time. Hence, the result of Theorem 1 implies that a linear time-varying system (1) is globally exponentially stable if the system matrix is Hurwitz for each fixed time, uniformly bounded, and changes at a sufficiently small rate.

A more general sufficient condition is obtained by replacing $\|\dot{A}(\cdot)\|$ with the integral of $\|\dot{A}(\cdot)\|$ over a time interval, as follows.

Theorem 2 (*Theorem 3.4.11 in Ioannou & Sun, 1996*)**.** *The system (1) is globally exponentially stable if the following conditions are satisfied:*

(i) *$A(\cdot)$ satisfies Assumption 1.*
(ii) *$A(\cdot)$ is differentiable and there exist scalars $\alpha > 0$ and $0 < \mu < \frac{\beta_1}{2\beta_2^3}$ such that*

$$\int_t^{t+T} \big\|\dot{A}(s)\big\| ds \le \mu T + \alpha \quad \forall\, t \ge 0,\ T \ge 0,$$

where $\beta_1 = \frac{1}{2L}$, $\beta_2 = \frac{c^2}{2\lambda}$, and L, c, λ are from Assumption 1 and (2).

The third condition is in terms of the integral of $\|\dot{A}(\cdot)\|$ on each interval $[t, t+T]$, which is required to be bounded by some affine function of the length of the time interval and the slope of the affine function is sufficiently small. The third condition is also called "nondestabilizing condition" in Morse (1990).

All the sufficient conditions above assume that $A(\cdot)$ is differentiable over $[0, \infty)$. In the sequel, we will relax this assumption and consider a more general case in which $A(\cdot)$ is only piecewise differentiable. Our approach will entail appealing to total variation of piecewise differentiable functions.

[1] In Jetto and Orsini (2009), it was mentioned that the derived results can be applied to switched linear systems. It can be shown by applying the results in Jetto and Orsini (2009) that a switched linear system is globally exponentially stable if each subsystem is stable and the dwell time is large enough. However, it seems difficult to use this method to derive more advanced results on switched linear systems discussed in Section 3.2.

2.1. Total variation

To define total variation, we first introduce the standard concept of partition: A *partition* P of an interval $[a, b]$ is a finite set of points t_i, $i \in \{0, 1, 2, \ldots, k\}$ such that $a = t_0 < t_1 < \cdots < t_k = b$. We denote by $\mathbb{P}$ the set of all partitions of $[a, b]$. We next recall the notion of total variation for vector-valued functions.

In Hu and Yang (2013, Definition 3.1.1), given a vector-valued function $v(\cdot) : \mathbb{R} \to \mathbb{R}^n$, its *total variation* over $[a, b]$ is defined as follows:

$$\sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|v(t_i) - v(t_{i-1})\| .$$

The norm here is the Euclidean norm.

Following this definition, we generalize the concept of total variation to the case of matrix-valued functions. Specifically, given a matrix-valued function $A(\cdot) : \mathbb{R} \to \mathbb{R}^{m\times n}$, its *total variation* over $[a, b]$ is defined as follows:

$$\int_a^b \|dA\| := \sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| .$$

The norm in the above expression is the matrix norm induced by the Euclidean norm. We use the notation $\int_a^b \|dA\|$ for the total variation to make our main results comparable with the results in Theorem 2.

In a special case where $A(\cdot)$ satisfies the following *regularity conditions* (Assumption 2), the total variation of $A(\cdot)$ on $[a, b]$ has an elegant expression (described in Lemma 1).

Assumption 2. Given an interval $[a, b]$, the matrix-valued function $A(\cdot)$ satisfies

(i) $A(\cdot)$ is continuous from the right everywhere on $[a, b)$ and has left limits everywhere on $(a, b]$.
(ii) $A(\cdot)$ has a finite number of discontinuities $d_1, d_2, \ldots, d_m$, $m \in \mathbb{N}$ over (a, b), and $a =: d_0 < d_1 < d_2 < \cdots < d_m < d_{m+1} := b$.
(iii) $A(\cdot)$ is continuously differentiable on (d_i, d_{i+1}), and $\dot{A}(\cdot)$ is Riemann integrable on $[d_i, d_{i+1}]$, for all $i \in \{0, 1, \ldots, m\}$.
(iv) $\|\dot{A}(\cdot)\|$ is Riemann integrable on $[d_i, d_{i+1}]$, for all $i \in \{0, 1, \ldots, m\}$.

We have two remarks here. (i) By convention, the continuity and differentiability of $A(\cdot)$, together with the integrability of $\dot{A}(\cdot)$, are defined elementwise. (ii) By Ross (1980, Section 34), a real-valued function defined on (a, b) is called integrable on $[a, b]$ if any extension of the function to $[a, b]$ is integrable. Furthermore, it can be shown that the integral does not depend on the extension of the function at a or b.

We use $A(t^-)$ to denote the left limit of $A(\cdot)$ at a time $t > 0$. Then, we have the following lemma.

Lemma 1. *Consider a matrix-valued function $A(\cdot)$ satisfying Assumption 2. The total variation of $A(\cdot)$ on $[a, b]$ is given by the following expression:*

$$\int_a^b \|dA\| = \sum_{i=0}^{m} \int_{d_i}^{d_{i+1}} \|\dot{A}(t)\| dt + \sum_{i=1}^{m+1} \left\| A(d_i) - A(d_i^-) \right\| .$$

The proof of Lemma 1 is given in Appendix. It borrows some elements from known proofs for the scalar case (see, e.g., Appell, Banaś, and Merentes (2014, Theorem 3.19)). The novelty here is that (i) we consider matrix-valued functions with *discontinuities*; and (ii) we work with the *matrix norm*, which does not have a closed-form expression in terms of the elements of the matrix. This creates technical challenges and we need to apply some matrix properties in the proof.

Intuitively, the total variation of $A(\cdot)$ equals the sum of variations over smooth portions and variations at jumps.

2.2. Main result

With the extended concept of total variation introduced in the previous subsection, we are able now to generalize the existing results in the literature (i.e., Theorems 1 and 2), as stated in the following theorem.

Theorem 3. *The system (1) is globally exponentially stable if the following conditions are satisfied:*

(i) *$A(\cdot)$ satisfies Assumption 1.*
(ii) *For any interval $[a, b]$, $A(\cdot)$ satisfies Assumption 2.*
(iii) *There exist $\alpha > 0$ and $0 < \mu < \frac{\beta_1}{2\beta_2^3}$ such that*

$$\int_t^{t+T} \|dA\| \le \mu T + \alpha \quad \forall\, t \ge 0,\ T \ge 0,$$

where

$$\beta_1 = \frac{1}{2L}, \quad \beta_2 = \frac{c^2}{2\lambda}, \tag{3}$$

and L, c, λ are from Assumption 1 and (2).

In the case when $A(\cdot)$ is differentiable over $[0, \infty)$, Theorem 3 collapses to Theorem 2.

Remark 1. A condition equivalent to (iii) in Theorem 3 has been proposed in Pait and Kassab (2001). In Pait and Kassab (2001), this condition was used to analyze stability of a switched adaptive control system, and thus the problem formulation was different. Furthermore, the condition proposed in Pait and Kassab (2001) directly used the expression of total variation without stating the concept. Although the concept of total variation does not affect the strength of Theorem 3, it provides a clear intuition on the nature of the result. To be more specific, no matter whether a system is a linear time-varying system or a switched linear system, the variation of the system should be small enough to guarantee stability, and the variation is characterized by total variation of the system matrix.

To prove the theorem, we need the two lemmas and the proposition given below.

Lemma 2 (*Lemma 9.9 in Khalil, 2002*)**.** *Suppose that $A(\cdot)$ satisfies Assumption 1. For each fixed $t \ge 0$, let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$ and consider the candidate Lyapunov function $V(t, x) = x^T P(t)x$. Then,*

$$\beta_1 \le \|P(t)\| \le \beta_2 \quad \forall\, t \ge 0,$$
$$\beta_1\|x\|^2 \le V(t, x) \le \beta_2\|x\|^2 \quad \forall\, x \in \mathbb{R}^n,\ t \ge 0,$$

where β_1, β_2 are defined in (3).

Proposition 1. *Assume that two $n \times n$ matrices A_1 and A_2 satisfy Assumption 1, namely, for $k = 1, 2$,*

$$\|A_k\| \le L,$$
$$\mathrm{Re}\left\{\lambda_i\big(A_k\big)\right\} \le -\kappa \quad \forall\, i \in \{1, 2, \ldots, n\},$$
$$\left\|e^{A_k s}\right\| \le ce^{-\lambda s} \quad \forall\, s \ge 0.$$

Let P_1 and P_2 be respectively the solutions of the Lyapunov equations $P_1A_1 + A_1^TP_1 = -I$ and $P_2A_2 + A_2^TP_2 = -I$, and consider the *candidate Lyapunov functions $V_1(x) := x^T P_1 x$ and $V_2(x) := x^T P_2 x$. Then,*

$$\begin{aligned}&\|P_1 - P_2\| \le 2\beta_2^2 \|A_1 - A_2\|,\\ &V_1(x) \le e^{2\beta_2^2\beta_1^{-1}\|A_1-A_2\|} V_2(x) \quad \forall\, x \in \mathbb{R}^n,\end{aligned} \tag{4}$$

with β_1, β_2 as defined in (3).

Proof. [2] Since P_1 and P_2 satisfy the Lyapunov equations, we have

$$\begin{aligned}&(P_1 - P_2)A_2 + A_2^T(P_1 - P_2)\\ =\ & P_1A_2 + A_2^TP_1 - P_2A_2 - A_2^TP_2\\ =\ & P_1A_2 + A_2^TP_1 - P_1A_1 - A_1^TP_1\\ =\ & P_1(A_2 - A_1) + (A_2^T - A_1^T)P_1\\ =:\ & -Q,\end{aligned} \tag{5}$$

where Q is a symmetric matrix. Since A_2 is Hurwitz, by Chen (1995, Section 3.7), the Lyapunov equation (5) in terms of $P_1 - P_2$ has a unique solution, which is

$$P_1 - P_2 = \int_0^\infty e^{A_2^T s} Q e^{A_2 s} ds.$$

Hence,

$$\begin{aligned}\|P_1 - P_2\| &\le \|Q\| \int_0^\infty \|e^{A_2^T s}\| \cdot \|e^{A_2 s}\| ds\\ &\le \|Q\| \int_0^\infty c^2 e^{-2\lambda_0 s} ds\\ &= \frac{c^2}{2\lambda}\|Q\| = \beta_2\|Q\|.\end{aligned} \tag{6}$$

By Lemma 2 and the definition of Q in (5),

$$\begin{aligned}\|Q\| &\le \|P_1(A_1 - A_2)\| + \|(A_1^T - A_2^T)P_1\|\\ &\le 2\|P_1\|\ \|A_1 - A_2\|\\ &\le 2\beta_2\|A_1 - A_2\|.\end{aligned} \tag{7}$$

Combining (6) and (7), we have

$$\|P_1 - P_2\| \le 2\beta_2^2\|A_1 - A_2\|. \tag{8}$$

This completes the proof of the first inequality in (4).

To prove the second inequality, consider the function

$$g(y) = e^{y-1} - y,$$

where $y \in \mathbb{R}$. It can be checked that $g'(y) > 0$ when $y > 1$, $g'(y) < 0$ when $y < 1$, and $g'(y) = 0$ when $y = 1$. Hence, $g(\cdot)$ attains its global minimum at $y = 1$, and $g(1) = 0$. Then, we have

$$y \le e^{y-1} \quad \forall\, y \in \mathbb{R},$$

which implies that

$$\frac{V_1(x)}{V_2(x)} \le e^{\frac{V_1(x)}{V_2(x)} - 1} = e^{\frac{V_1(x)-V_2(x)}{V_2(x)}} \quad \forall\, x \in \mathbb{R}^n, x \ne 0. \tag{9}$$

By Lemma 2, we have

$$0 < \beta_1\|x\|^2 \le V_2(x) \quad \forall\, x \in \mathbb{R}^n, x \ne 0.$$

Hence,

$$0 < \frac{1}{V_2(x)} \le \frac{1}{\beta_1\|x\|^2}. \tag{10}$$

[2] The contents of Lemmas 3 and 4 in the conference version (Gao, Liberzon, Liu, & Başar, 2015) have been rearranged here as Proposition 1 and Lemma 3, and accordingly the proofs have been rearranged as well.

Moreover, by (8), we have

$$\begin{aligned}V_1(x) - V_2(x) &= x^T\left(P_1 - P_2\right)x\\ &\le \|P_1 - P_2\|\ \|x\|^2\\ &\le 2\beta_2^2\|A_1 - A_2\|\ \|x\|^2\end{aligned} \tag{11}$$

Combining inequalities (9), (10), and (11), we have

$$\frac{V_1(x)}{V_2(x)} \le e^{\frac{V_1(x)-V_2(x)}{V_2(x)}} \le e^{\frac{2\beta_2^2\|A_1-A_2\|\|x\|^2}{\beta_1\|x\|^2}} \quad \forall\, x \in \mathbb{R}^n, x \ne 0,$$

which implies that

$$V_1(x) \le e^{2\beta_2^2\beta_1^{-1}\|A_1-A_2\|} V_2(x) \quad \forall\, x \in \mathbb{R}^n, x \ne 0.$$

It is easy to see that the inequality also holds when $x = 0$, which completes the proof of the second inequality in (4). □

Remark 2. Our approach to proving the first inequality in (4) is a discrete-time version of the approach to proving the intermediate results of Theorem 3.4.11 in Ioannou and Sun (1996), where an upper bound of $\|\dot{P}(t)\|$ is derived in terms of $\|\dot{A}(t)\|$.

Lemma 3. *Suppose that $A(\cdot)$ satisfies Assumption 1 and 2. For each fixed $t \ge 0$, let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$. Consider the candidate Lyapunov function $V(t, x) = x^T P(t)x$, and let $V(t) = V(t, x(t))$ be the candidate Lyapunov function evaluated along the state trajectory $x(t)$. For any $t_b > 0$, introduce*

$$V(t_b^+) := \lim_{t \to t_b^+} V(t), \qquad V(t_b^-) := \lim_{t \to t_b^-} V(t).$$

Then, $V(t_b^+)$ and $V(t_b^-)$ exist and satisfy

$$V(t_b^+) = V(t_b) \le e^{2\beta_2^2\beta_1^{-1}\|A(t_b)-A(t_b^-)\|} V(t_b^-),$$

where β_1 and β_2 are defined in (3).

Proof. We still use $A(t_b^-)$ to denote the left limit of $A(\cdot)$ at t_b. In addition, we use $A(t_b^+)$ to denote the right limit of $A(\cdot)$ at t_b. By Assumption 2, $A(t_b^+)$ and $A(t_b^-)$ exist, and $A(t_b^+) = A(t_b)$. Moreover, $\|A\|$, $\mathrm{Re}\{\lambda_i(A)\}$ and $\|e^{As}\|$ are continuous functions of A, and thus $A(t_b^+)$ and $A(t_b^-)$ satisfy Assumption 1, namely

$$\begin{aligned}\left\|A(t_b^+)\right\|, \left\|A(t_b^-)\right\| &\le L,\\ \mathrm{Re}\left\{\lambda_i\left(A(t_b^+)\right)\right\}, \mathrm{Re}\left\{\lambda_i\left(A(t_b^-)\right)\right\} &\le -\kappa,\\ \|e^{A(t_b^+)s}\|, \|e^{A(t_b^-)s}\| &\le ce^{-\lambda s},\end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$ and $s \ge 0$.

Since $A(t_b^+)$ and $A(t_b^-)$ are Hurwitz, the Lyapunov equations

$$\begin{aligned}&P(t_b^+)A(t_b^+) + A^T(t_b^+)P(t_b^+) = -I,\\ &P(t_b^-)A(t_b^-) + A^T(t_b^-)P(t_b^-) = -I\end{aligned}$$

have unique solutions $P(t_b^+)$ and $P(t_b^-)$. By the definitions of $A(t_b^+)$ and $A(t_b^-)$, given any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}&\|A(t_b^+) - A(t)\| \le \frac{\epsilon}{2\beta_2^2} \quad \forall\, t \in (t_b, t_b + \delta_1),\\ &\|A(t_b^-) - A(t)\| \le \frac{\epsilon}{2\beta_2^2} \quad \forall\, t \in (t_b - \delta_2, t_b).\end{aligned}$$

Since $A(t_b^+)$, $A(t_b^-)$, and $A(t)$ satisfy Assumption 1, by (4) in Proposition 1, we have

$$\begin{aligned}&\|P(t_b^+) - P(t)\| \le \epsilon \quad \forall\, t \in (t_b, t_b + \delta_1),\\ &\|P(t_b^-) - P(t)\| \le \epsilon \quad \forall\, t \in (t_b - \delta_2, t_b),\end{aligned}$$

which imply that

$$\lim_{t \to t_b^+} P(t) = P(t_b^+), \quad \lim_{t \to t_b^-} P(t) = P(t_b^-).$$

Since $A(t_b^+)$, $A(t_b)$ are Hurwitz and $A(t_b^+) = A(t_b)$, the Lyapunov equations

$$P(t_b^+)A(t_b^+) + A^T(t_b^+)P(t_b^+) = -I,$$
$$P(t_b)A(t_b) + A^T(t_b)P(t_b) = -I$$

have unique solutions $P(t_b^+)$, $P(t_b)$, and $P(t_b^+) = P(t_b)$.

Furthermore, $A(\cdot)$ is piecewise continuous, and hence $x(\cdot)$ is always continuous regardless of the initial condition. Then,

$$x(t_b^+) := \lim_{t\to t_b^+} x(t) = x(t_b) = \lim_{t\to t_b^-} x(t) =: x(t_b^-).$$

Now consider $V(t_b^+)$ and $V(t_b^-)$, which can be expressed as

$$V(t_b^+) = \lim_{t\to t_b^+} V(t) = \lim_{t\to t_b^+} x^T(t)P(t)x(t)$$
$$V(t_b^-) = \lim_{t\to t_b^-} V(t) = \lim_{t\to t_b^-} x^T(t)P(t)x(t)$$

We have already shown that the left and right limits of $x(\cdot)$ and $P(\cdot)$ at t exist. Hence, $V(t_b^+)$ and $V(t_b^-)$ exist, and

$$V(t_b^+) = x^T(t_b^+)P(t_b^+)x(t_b^+) = x^T(t_b)P(t_b^+)x(t_b)$$
$$V(t_b^-) = x^T(t_b^-)P(t_b^-)x(t_b^-) = x^T(t_b)P(t_b^-)x(t_b)$$

Furthermore, we have

$$V(t_b^+) = x^T(t_b)P(t_b)x(t_b) = V(t_b).$$

Since $A(t_b)$ and $A(t_b^-)$ satisfy Assumption 1, by (4) in Proposition 1, we have

$$V(t_b) \le e^{2\beta_2^2\beta_1^{-1}\|A(t_b)-A(t_b^-)\|}V(t_b^-),$$

which completes the proof of this lemma. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3. [3] For each fixed $t \ge 0$, let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$. Consider the candidate Lyapunov function $V(t, x) = x^T P(t)x$, and let $V(t) = V(t, x(t))$ be the candidate Lyapunov function evaluated along the state trajectory $x(t)$. By Assumption 2, within any time interval (a, b), there are finitely many discontinuities of $A(\cdot)$, denoted by $\{d_1, d_2, \ldots, d_m\}$, where $a =: d_0 < d_1 < d_2 < \cdots < d_m < d_{m+1} := b$. Consider any sub-interval of $[a, b]$ among $[d_i, d_{i+1}]$, $i = 0, 1, \ldots, m$. Then, $A(\cdot)$ has no discontinuity on (d_i, d_{i+1}). By the proof of Theorem 3.4.11 in Ioannou and Sun (1996), we have

$$V(d_{i+1}^-) \le e^{-\int_{d_i}^{d_{i+1}} \beta_2^{-1}dt + 2\beta_2^2\beta_1^{-1}\left(\int_{d_i}^{d_{i+1}} \|\dot{A}(t)\|dt\right)}V(d_i^+),$$

where $V(d_{i+1}^-)$ and $V(d_i^+)$ are introduced in Lemma 3. Combining the inequality above with the inequality in Lemma 3, we have

$$\begin{aligned} V(d_{i+1}) \le\ & e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_i}^{d_{i+1}} \|\dot{A}(t)\|dt + \|A(d_{i+1})-A(d_{i+1}^-)\|\right)} \\ & \cdot e^{-\beta_2^{-1}(d_{i+1}-d_i)}\, V(d_i). \end{aligned} \tag{12}$$

[3] A version of this proof has appeared earlier in Gao et al. (2015).

Applying inequality (12) to sub-intervals $[d_0, d_1]$, $[d_1, d_2]$, …, and $[d_m, d_{m+1}]$, we have

$$\begin{aligned} V(d_1) \le\ & e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_0}^{d_1} \|\dot{A}(t)\|dt + \|A(d_1)-A(d_1^-)\|\right)} \\ & \cdot e^{-\beta_2^{-1}(d_1-d_0)}V(d_0), \\ V(d_2) \le\ & e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_1}^{d_2} \|\dot{A}(t)\|dt + \|A(d_2)-A(d_2^-)\|\right)} \\ & \cdot e^{-\beta_2^{-1}(d_2-d_1)}V(d_1), \\ & \vdots \\ V(d_{m+1}) \le\ & e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_m}^{d_{m+1}} \|\dot{A}(t)\|dt + \|A(d_{m+1})-A(d_{m+1}^-)\|\right)} \\ & \cdot e^{-\beta_2^{-1}(d_{m+1}-d_m)}V(d_m). \end{aligned}$$

Combining all inequalities above and the expression of total variation in Lemma 1, we have

$$V(b) \le e^{-\beta_2^{-1}(b-a) + 2\beta_2^2\beta_1^{-1}\left(\int_a^b \|dA\|\right)}V(a)$$

By condition (iii) in Theorem 3,

$$\begin{aligned} V(b) \le\ & e^{-\beta_2^{-1}(b-a) + 2\beta_2^2\beta_1^{-1}\left(\mu(b-a)+\alpha\right)}V(a) \\ =\ & e^{2\beta_2^2\beta_1^{-1}\alpha} \cdot e^{-(\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu)(b-a)}V(a), \end{aligned}$$

where $e^{2\beta_2^2\beta_1^{-1}\alpha} > 0$, and $\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu > 0$. Hence, the Lyapunov function decays exponentially along system solutions, which leads to the global exponential stability of the system. □

3. Stability of switched linear systems

In this section, we will apply the generalized stability conditions for slowly time-varying linear systems (Theorem 3) to derive two sets of stability conditions for switched linear systems. Then, we will compare the derived results with the existing stability conditions, thereby bridging the gap between the two groups of results.

3.1. Applications of generalized stability conditions for slowly time-varying linear systems

Suppose that we are given a family of linear systems

$$\dot{x} = A_p x, \quad p \in \mathcal{P},$$

where $x \in \mathbb{R}^n$ is the system state, p is the index of the linear system in the family, and $\mathcal{P}$ is the index set. Now consider a switched linear system in the form of

$$\dot{x}(t) = A_{\sigma(t)}x(t). \tag{13}$$

For each fixed $t \ge 0$, $A_{\sigma(t)}$ is an $n \times n$ real matrix. Furthermore, $A_{\sigma(t)}$ belongs to $\{A_p, p \in \mathcal{P}\}$, which is the set of system matrices. The function $\sigma : [0, \infty) \to \mathcal{P}$ is called the switching signal. We assume that $\sigma(\cdot)$ is a piecewise constant function, which has a finite number of discontinuities on every bounded time interval. It is also assumed that $\sigma(\cdot)$ is continuous from the right everywhere. Denote by $N_\sigma(t, t+T)$ the number of switches (number of discontinuities of $\sigma(\cdot)$) over the time interval $(t, t+T]$. If there exist two constants $\tau_a > 0$ and $N_0 > 0$ such that

$$N_\sigma(t, t+T) \le N_0 + \frac{T}{\tau_a} \quad \forall\, t \ge 0,\ T \ge 0, \tag{14}$$

then the switching signal $\sigma(\cdot)$ is said to have the *average dwell time* τ_a (Hespanha & Morse, 1999). The switched linear system can be viewed as a special case of linear time-varying systems, and by Theorem 3 we have the following set of stability conditions for switched linear systems. We use $A(t)$ to represent the system matrix instead of $A_{\sigma(t)}$ to be consistent with the statements in Theorem 3.

Corollary 1. *The system (13) is globally exponentially stable if:*

(i) *Assumption 1 holds for* $A(\cdot) := A_{\sigma(\cdot)}$.
(ii) *The switching signal* $\sigma(\cdot)$ *has an average dwell time* τ_a *such that*

$$\tau_a > \frac{2\beta_2^3}{\beta_1^2}, \tag{15}$$

where $\beta_1 = \frac{1}{2L}$ *and* $\beta_2 = \frac{c^2}{2\lambda}$.

Proof. Due to the piecewise constant property of $\sigma(\cdot)$, $A(\cdot)$ satisfies Assumption 2. Then, to establish the exponential stability of system (13), we only need to show that $A(\cdot)$ satisfies the last condition in Theorem 3.

Consider any time interval $[t, t+T]$, and denote by $\{d_1, d_2, \ldots, d_m\}$ the discontinuities of $A(t)$ on $(t, t+T]$, where $t < d_1 < d_2 < \cdots < d_m \le t+T$ and $m = N_\sigma(t, t+T)$. Since $A(\cdot)$ is piecewise constant, $\|\dot{A}(\cdot)\|$ remains zero on (t, d_1), $(d_m, t+T)$, and (d_i, d_{i+1}) for all $i \in \{1, 2, \ldots, m-1\}$. Moreover, by the triangle inequality,

$$\|A_p - A_q\| \le \|A_p\| + \|A_q\| \le 2L \quad \forall\, p, q \in \mathcal{P}.$$

Hence, the total variation of $A(\cdot)$ on $[t, t+T]$ is

$$\begin{aligned}\int_t^{t+T} \|dA\| &= \sum_{i=1}^{m} \left\|A(d_i) - A(d_i^-)\right\| \\ &\le m \sup_{p,q\in\mathcal{P}} \|A_p - A_q\| \\ &\le m(2L) = m\beta_1^{-1}.\end{aligned}$$

By definition of average dwell time, there exists some positive number N_0 such that

$$m = N_\sigma(t, t+T) \le N_0 + \frac{T}{\tau_a}.$$

Hence,

$$\int_t^{t+T} \|dA\| \le m\beta_1^{-1} \le \left(N_0 + \frac{T}{\tau_a}\right)\beta_1^{-1} =: \mu T + \alpha,$$

where $\mu = \frac{1}{\tau_a}\beta_1^{-1} > 0$ and $\alpha = N_0\beta_1^{-1} > 0$. Moreover, since

$$\tau_a > \frac{2\beta_2^3}{\beta_1^2},$$

we have

$$\mu = \frac{1}{\tau_a}\beta_1^{-1} < \frac{\beta_1}{2\beta_2^3}.$$

Therefore, the last condition in Theorem 3 is satisfied, and the switched linear system is globally exponentially stable. □

In a more general case where the switching signal does not have an average dwell time, we can still apply Theorem 3 to derive stability conditions for switched linear systems. We first use $N_\sigma^{pq}(t, t+T)$ to denote the number of transitions from subsystem p to subsystem q during the time interval $(t, t+T]$. Then, we have the following corollary:

Corollary 2. *The system (13) is globally exponentially stable if:*

(i) *Assumption 1 holds for* $A(\cdot) := A_{\sigma(\cdot)}$.
(ii) *There exist scalars* $\alpha > 0$ *and* $0 < \mu < \frac{\beta_1}{2\beta_2^3}$ *such that*

$$\sum_{p,q\in\mathcal{P},p\ne q} N_\sigma^{pq}(t, t+T)\|A_p - A_q\| \le \mu T + \alpha$$

for all $t \ge 0$, $T \ge 0$, *where* $\beta_1 = \frac{1}{2L}$ *and* $\beta_2 = \frac{c^2}{2\lambda}$.

Proof. It is easy to see that

$$\sum_{p,q\in\mathcal{P},p\ne q} N_\sigma^{pq}(t, t+T)\|A_p - A_q\| = \int_t^{t+T} \|dA\|,$$

which establishes the desired result. □

3.2. Comparison with the existing stability conditions for switched linear systems

Given a family of systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P},$$

consider a general switched system

$$\dot{x} = f_{\sigma(t)}(x), \tag{16}$$

where $x \in \mathbb{R}^n$ is the system state. For each fixed $t \ge 0$, $f_{\sigma(t)}(\cdot)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $f_{\sigma(t)} \in \{f_p,\ p \in \mathcal{P}\}$, which is the set of subsystems. As in Section 3.1, $\mathcal{P}$ is the index set, and $\sigma : [0, \infty) \to \mathcal{P}$ is the switching signal. The definition of average dwell time is the same as above. It is well known that if each subsystem is globally asymptotically stable and the average dwell time of the switching signal is large enough, then the switched system is globally asymptotically stable. The earliest result on this line of research has been derived in Hespanha and Morse (1999). We recover this result in Theorem 4. Then, we compare Theorem 4 with Corollary 1 when specialized to switched linear systems.

Theorem 4 (*Theorem 4 in Hespanha & Morse, 1999*). *The switched system (16) is globally asymptotically stable if:*

(i) *There exist continuously differentiable functions* $V_p(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}$ *and two class* $\mathcal{K}_\infty$ *functions* α_1 *and* α_2 *such that*

$$\alpha_1(\|x\|) \le V_p(x) \le \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

(ii) *There exists a constant* $\gamma > 0$ *such that*

$$\frac{\partial V_p}{\partial x} f_p(x) \le -\gamma V_p(x) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

(iii) *There exists a constant* $\nu > 0$ *such that*

$$V_p(x) \le \nu V_q(x) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P},\ q \in \mathcal{P}.$$

(iv) *The switching signal* $\sigma(\cdot)$ *has an average dwell time* τ_a *such that*

$$\tau_a > \frac{\ln \nu}{\gamma}.$$

Note that a switched linear system being globally asymptotically stable implies that it is also globally exponentially stable. We now apply Theorem 4 to the switched linear system (13) satisfying Assumption 1. For each subsystem matrix A_p, define $V_p(x) := x^T P_p x$, where P_p is the solution to the Lyapunov equation

$$P_p A_p + A_p^T P_p = -I.$$

By Lemma 2, we have

$$\beta_1\|x\|^2 \le V_p(x) \le \beta_2\|x\|^2 \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

Therefore, we have

$$\alpha_1(\|x\|) = \beta_1\|x\|^2, \quad \alpha_2(\|x\|) = \beta_2\|x\|^2.$$

Furthermore,

$$\frac{\partial V_p}{\partial x} f_p(x) = -\|x\|^2 \le -\frac{1}{\beta_2} V_p(x) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

Then,

$$\gamma = \frac{1}{\beta_2}.$$

Moreover,

$$V_p(x) \le \beta_2\|x\|^2 \le \frac{\beta_2}{\beta_1}V_q(x) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P},\ q \in \mathcal{P}.$$

Hence,

$$\nu = \frac{\beta_2}{\beta_1}.$$

By Theorem 4, the switched linear system is stable if

$$\tau_a > \frac{\ln \nu}{\gamma} = \beta_2 \ln \frac{\beta_2}{\beta_1}. \tag{17}$$

Comparing (15) in Corollary 1 and (17),

$$\frac{2\beta_2^3}{\beta_1^2} = 2 \cdot \beta_2 \cdot \frac{\beta_2}{\beta_1} \cdot \frac{\beta_2}{\beta_1} > 1 \cdot \beta_2 \cdot \ln \frac{\beta_2}{\beta_1} \cdot 1.$$

Therefore, the set of stability conditions in terms of average dwell time for switched linear systems (Corollary 1), which is derived from generalized stability conditions for slowly time-varying linear systems, matches the existing result qualitatively but not quantitatively.

The comparison above does not imply the conservativeness of the generalized stability conditions for slowly time-varying linear systems. By Theorem 3, a switched linear system is stable if the total variation of system matrix over a long time interval is small enough. Small variation of system matrix can be achieved in two ways: (1) The variation caused by each switch is large while the system switches slowly enough; (2) The system switches fast while the variation caused by each switch is small enough. The comparison above is under the first scenario. However, under the second scenario, the switching signal might not even have an average dwell time. In that case, we cannot apply Theorem 4, but can apply Corollary 2 to establish the stability result, as illustrated in the following example.

Example 1. Let the constants L, κ, c, λ introduced in Assumption 1 be given. Then, $\beta_1 = \frac{1}{2L}$, $\beta_2 = \frac{c^2}{2\lambda}$, and $\mu = \frac{\beta_1}{2\beta_2^3}$ are well defined. Suppose that there exists a family of linear systems with system matrices $\{A_i | i \in \mathbb{N}\}$ satisfying Assumption 1 with constants L, κ, c, λ. Furthermore, assume that

$$\|A_i - A_0\| < \frac{\mu}{2i} \quad \forall\, i \in \mathbb{N}_+. \tag{18}$$

The above two assumptions can be satisfied by first choosing A_0 that satisfies Assumption 1 with strict inequalities, and then letting $A_i = A_0 + \epsilon_i I$ with ϵ_i small enough.

Consider the switching signal $\sigma(\cdot)$ such that during the time interval $[i-1, i]$, $i \in \mathbb{N}_+$, $A_{\sigma(\cdot)}$ switches $2i$ times between A_0 and A_i, uniformly over $[i-1, i]$. Then, the time between two successive switches is $1/2i$. By (18), condition (ii) in Corollary 2 is satisfied with $\alpha = \mu$. Hence, by Corollary 2 the switched linear system is stable. On the other hand, the number of switches during time interval $[i-1, i]$ is $2i$, which is unbounded as i increases. Therefore, there does not exist $\tau_a > 0$ and $N_0 > 0$ such that (14) holds, which means that the switching signal does not have an average dwell time. Thus, we cannot apply Theorem 4 to draw any conclusions on the stability of the switched linear system.

Although the earliest result on the stability of switched systems (Theorem 4) is not applicable to the example above, a more recent result along this line of research can be applied to address the example. We state a simplified version of this result in Theorem 5 and show that it is the same as Corollary 2 when specialized to switched linear systems with stable subsystems.

Theorem 5 (*Theorem 5 in Kundu & Chatterjee, 2015*). *The switched system (16) is globally asymptotically stable if:*

(i) *There exist continuously differentiable functions $V_p(\cdot) : \mathbb{R}^n \to \mathbb{R},\ p \in \mathcal{P}$ and two class $\mathcal{K}_\infty$ functions α_1 and α_2 such that*

$$\alpha_1(\|x\|) \le V_p(x) \le \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

(ii) *There exists a constant $\gamma > 0$ such that*

$$\frac{\partial V_p}{\partial x} f_p(x) \le -\gamma V_p(x) \quad \forall\, x \in \mathbb{R}^n,\ p \in \mathcal{P}.$$

(iii) *For any pair of p and q such that $p, q \in \mathcal{P}, p \ne q$, there exists a constant $\nu_{pq} > 0$ such that*

$$V_p(x) \le \nu_{pq} V_q(x) \quad \forall\, x \in \mathbb{R}^n.$$

(iv) *There exists a constant $\eta > 0$ such that*

$$\sum_{p,q \in \mathcal{P}, p \ne q} N_\sigma^{pq}(t, t+T) \ln \nu_{pq} < \gamma T + \eta$$

for all $t \ge 0$, $T \ge 0$, where $N_\sigma^{pq}(t, t+T)$ is the number of transitions from subsystem p to subsystem q during $(t, t+T]$.

Remark 3. In Kundu and Chatterjee (2015), the authors considered a more general case and derived a set of stability conditions when both stable and unstable subsystems are involved. In this paper, we consider only switched linear systems with stable subsystems. Hence, we have stated Theorem 5 for only such systems. Also, if individual decay factors for each $V_p,\ p \in \mathcal{P}$ are given, instead of a uniform factor γ in condition (ii), then the inequality in condition (iv) could be refined. However, this requires more information about each subsystem, which is beyond the scope considered in this paper.

We apply Theorem 5 to switched linear system (13) satisfying Assumption 1. We choose $V_p(x) := x^T P_p x$ as the Lyapunov function, where P_p is the solution to the Lyapunov equation

$$P_p A_p + A_p^T P_p = -I.$$

Then, conditions (i) and (ii) in Theorem 5 hold with $\alpha_1(\|x\|) = \beta_1\|x\|^2$, $\alpha_2(\|x\|) = \beta_2\|x\|^2$, and $\gamma = \beta_2^{-1}$. Furthermore, by inequality (4) in Proposition 1, we have

$$V_p(x) \le e^{2\beta_2^2\beta_1^{-1}\|A_p - A_q\|} V_q(x).$$

Hence, condition (iii) in Theorem 5 holds with $\nu_{pq} = e^{2\beta_2^2\beta_1^{-1}\|A_p - A_q\|}$. Then, condition (iv) in Theorem 5 becomes

$$\sum_{p,q \in \mathcal{P}, p \ne q} N_\sigma^{pq}(t, t+T) 2\beta_2^2\beta_1^{-1}\|A_p - A_q\| < \beta_2^{-1} T + \eta,$$

which is equivalent to

$$\sum_{p,q \in \mathcal{P}, p \ne q} N_\sigma^{pq}(t, t+T) \|A_p - A_q\| < \frac{\beta_1}{2\beta_2^3} T + \frac{\beta_1}{2\beta_2^2}\eta.$$

The inequality above is the same as condition (ii) in Corollary 2.

Remark 4. As discussed in Section 1, Theorem 5 is an extension of Theorem 4 in terms of average dwell-time for each transition pair. Another extension of Theorem 4, described in Zhao et al. (2012, Lemma 3), is in terms of average dwell-time for each subsystem. Our results (Theorem 3) can be applied to derive a set of stability conditions, which can be shown to match (Zhao et al., 2012 Lemma 3) specified to switched linear systems. The derivation is similar to that when showing the equivalence between Corollary 2 and Theorem 5 specified to switched linear systems, and hence is not included here.

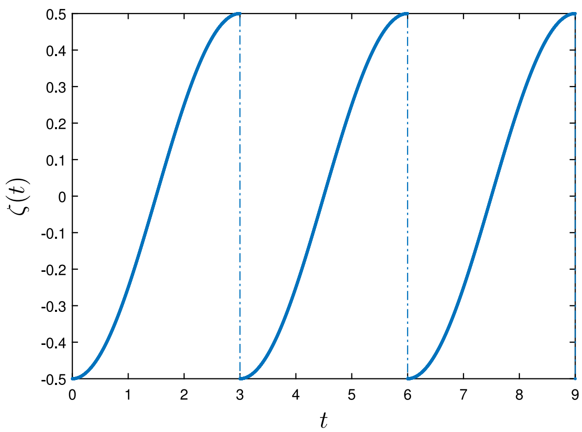

Fig. 1. $\zeta(t)$ vs. t.

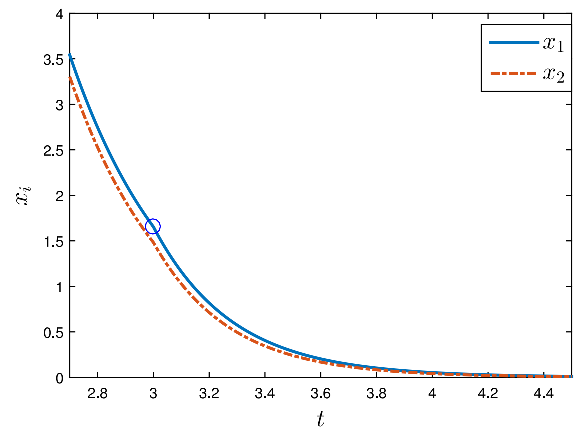

Fig. 2. Evolution of system states $x_1(t)$ and $x_2(t)$.

4. A numerical example

We present an example demonstrating how Theorem 3 can be applied to a general linear time-varying system. Consider a linear time-varying system (1) with system matrix

$$A(t) = A + \zeta(t)I, \quad t \geq 0,$$

where

$$A = \begin{bmatrix} -2.99 & -0.04 \\ 0.04 & -3.21 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\zeta(t) = 0.5 \sin\left(\frac{\pi}{3}(t \bmod 3) - 0.5\pi\right).$$

As shown in Fig. 1, the real-valued function $\zeta(\cdot)$ is a periodic function with period $\bar{T} = 3$. In addition, $\zeta(\cdot)$ is continuously differentiable everywhere except at $t = 3k, k = 1, 2, \ldots,$ where it has discontinuities.

Note that neither Theorem 2 nor Theorem 5 can be applied to this case, yet we can apply Theorem 3.

The total variation of $A(\cdot)$ over one period $[t, t + \bar{T}]$ can be computed as

$$\begin{aligned} \int_t^{t+\bar{T}} \|dA\| &= \int_0^{\bar{T}} \|dA\| \\ &= \int_0^{\bar{T}} \|\dot{A}(t)\|dt + \|A(\bar{T}) - A(\bar{T}^-)\| \\ &= \int_0^{\bar{T}} |\dot{\zeta}(t)|dt + |\zeta(\bar{T}) - \zeta(\bar{T}^-)| \\ &= 2, \end{aligned}$$

where the first equality holds due to the periodicity of $A(\cdot)$. Then, given any interval $[t, t+T]$, the total variation of $A(\cdot)$ over $[t, t+T]$ satisfies

$$\begin{aligned} \int_t^{t+T} \|dA\| &\leq T/\bar{T} \cdot \int_0^{\bar{T}} \|dA\| + \int_0^{\bar{T}} \|dA\| \\ &= \frac{2}{3}T + 2 \\ &=: \mu T + \alpha, \end{aligned}$$

where the inequality is due to the periodicity of $A(\cdot)$ and the fact that the total variation of $A(\cdot)$ over $[0, \bar{T}]$ is no less than that over any subinterval of $[0, \bar{T}]$.

To apply Theorem 3, we need to compute the constants c, λ, and L, which are from Assumption 1 and (2). We first have

$$\begin{aligned} \|A(t)\| &\leq \|A\| + \|\zeta(t)I\| \\ &= \|A\| + |\zeta(t)| \\ &\leq \|A\| + 0.5 \\ &= 3.7086 \\ &= L. \end{aligned}$$

Note that $A = U\Lambda U^{-1}$, where

$$U = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -3.2 \end{bmatrix}.$$

Therefore, we have

$$\|e^{As}\| = \|Ue^{\Lambda s}U^{-1}\| \leq \|U\|\|U^{-1}\|\|e^{\Lambda s}\| = 1.5e^{-3s},$$

and

$$\begin{aligned} \|e^{A(t)s}\| &= \|e^{As+\zeta(t)Is}\| \\ &= \|e^{As}e^{\zeta(t)Is}\| \\ &\leq \|e^{As}\| \cdot e^{\zeta(t)s} \\ &\leq 1.5e^{-2.5s} \\ &= ce^{-\lambda s}. \end{aligned}$$

The second equality above holds since A and I commute. Based on c, λ, and L, it can be computed that $\frac{\beta_1}{2\beta_2^3} = \frac{2\lambda^3}{Lc^6} = 0.7398 > 2/3 = \mu$. Then, condition (iii) in Theorem 3 is satisfied. Hence, the linear time-varying system is stable.

We select the initial conditions as $x_1(0) = 1.5 \times 10^4$ and $x_2(0) = 2 \times 10^4$. The evolutions of system states are shown in Fig. 2, which are consistent with the conclusion reached by Theorem 3. Furthermore, it can be seen from Fig. 2 that the system matrix "jumps" at $t = 3$.

5. Conclusions

In this paper, we have derived a set of generalized stability conditions for slowly time-varying linear systems and applied it to derive two sets of stability conditions for switched linear systems. By doing so we have unified stability conditions for slowly time-varying linear systems and stability conditions for switched linear systems.

Several issues remain open for future research. First, there is the need to build relationships between stability conditions for slowly time-varying linear systems (see, e.g., Solo, 1994) and switched linear systems (see, e.g., Zhai et al., 2001; Zhao et al., 2015) in the case when the system matrix is unstable for some or even all time. Second, there is the need to establish relationships between stability conditions for nonlinear time-varying systems (see, e.g., Section 9.6 in Khalil (2002)) and switched nonlinear systems (see, e.g., Kundu & Chatterjee, 2015; Müller & Liberzon, 2012), including the case when unstable subsystems are present. We have obtained some preliminary results in this direction (Gao, Liberzon, & Başar, under review). Third, one has to quantitatively improve the result in Corollary 1 so as to better match the result in Theorem 4. Finally, one could apply the total variation approach to study the robustness of the exponential stability of slowly time-varying linear systems with respect to perturbations (related works can be found in Ilchmann and Mareels (2001) and the references therein).

Appendix. Total variation of a matrix-valued function

Proof of Lemma 1. We consider the case where $A(\cdot)$ has one discontinuity at d over $[a, b]$, and $a < d < b$. We show that

$$\int_a^b \|dA\| := \sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| = \int_a^d \|\dot{A}(t)\|dt + \int_d^b \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\|.$$

The proof can easily be adjusted to the case where f has no discontinuity or has finitely many discontinuities. We first show that

$$\sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| \le \int_a^d \|\dot{A}(t)\|dt + \int_d^b \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\|. \tag{A.1}$$

Given any partition $P = \{t_i \mid i = 0, 1, \ldots, k\}$ of the interval $[a, b]$ (recall that $a = t_0 < t_1 < \cdots < t_k = b$), there exists $t_{i^*} \in P$ such that $t_{i^*-1} < d \le t_{i^*}$. Then, we have

$$\sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| = \sum_{i=1}^{i^*-1} \|A(t_i) - A(t_{i-1})\| + \|A(t_{i^*}) - A(t_{i^*-1})\| + \sum_{i=i^*+1}^{k} \|A(t_i) - A(t_{i-1})\|.$$

Note that

$$\|A(t_i) - A(t_{i-1})\| = \left\| \int_{t_{i-1}}^{t_i} \dot{A}(t)dt \right\| \le \int_{t_{i-1}}^{t_i} \|\dot{A}(t)\|dt,$$

and thus we have

$$\begin{aligned}\sum_{i=1}^{i^*-1} \|A(t_i) - A(t_{i-1})\| &\le \sum_{i=1}^{i^*-1} \int_{t_{i-1}}^{t_i} \|\dot{A}(t)\|dt = \int_a^{t_{i^*-1}} \|\dot{A}(t)\|dt,\\ \sum_{i=i^*+1}^{k} \|A(t_i) - A(t_{i-1})\| &\le \sum_{i=i^*+1}^{k} \int_{t_{i-1}}^{t_i} \|\dot{A}(t)\|dt = \int_{t_{i^*}}^{b} \|\dot{A}(t)\|dt.\end{aligned} \tag{A.2}$$

Moreover,

$$\begin{aligned}&\|A(t_{i^*}) - A(t_{i^*-1})\|\\ &= \left\|A(t_{i^*}) - A(d) + A(d) - A(d^-) + A(d^-) - A(t_{i^*-1})\right\|\\ &\le \|A(t_{i^*}) - A(d)\| + \left\|A(d) - A(d^-)\right\| + \left\|A(d^-) - A(t_{i^*-1})\right\|\\ &\le \int_d^{t_{i^*}} \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\| + \int_{t_{i^*-1}}^{d} \|\dot{A}(t)\|dt.\end{aligned} \tag{A.3}$$

Combining (A.2) and (A.3), we reach (A.1).

Next, we prove that

$$\sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| \ge \int_a^d \|\dot{A}(t)\|dt + \int_d^b \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\|, \tag{A.4}$$

which will complete the proof of Lemma 1.

Let $\epsilon > 0$ be given. By the uniform continuity of $\int_a^x \|\dot{A}(t)\|dt$ in x on $[a, d]$ (Fundamental Theorem of Calculus), there exists $\delta_1 > 0$ such that for any $a_r \in (a, a + \delta_1)$ and any $d_l \in (d - \delta_1, d)$, we have

$$\left| \int_a^{a_r} \|\dot{A}(t)\|dt \right| \le \epsilon/12, \quad \left| \int_{d_l}^{d} \|\dot{A}(t)\|dt \right| \le \epsilon/12,$$

which implies that

$$\left| \int_a^d \|\dot{A}(t)\|dt - \int_{a_r}^{d_l} \|\dot{A}(t)\|dt \right| = \left| \int_a^{a_r} \|\dot{A}(t)\|dt + \int_{d_l}^{d} \|\dot{A}(t)\|dt \right| \le \epsilon/6,$$

and hence

$$\int_a^d \|\dot{A}(t)\|dt \le \int_{a_r}^{d_l} \|\dot{A}(t)\|dt + \epsilon/6. \tag{A.5}$$

Similarly, there exists $\delta_2 > 0$ such that for any $d_r \in (d, d + \delta_2)$ and any $b_l \in (b - \delta_2, b)$, we have

$$\int_d^b \|\dot{A}(t)\|dt \le \int_{d_r}^{b_l} \|\dot{A}(t)\|dt + \epsilon/6. \tag{A.6}$$

On the other hand, since $A(\cdot)$ is continuous from the right and has left limit at d, there exist $\delta_3 > 0$, $\delta_4 > 0$ such that for any $d_l \in (d - \delta_3, d)$ and any $d_r \in (d, d + \delta_4)$, we have[4]

$$\|A(d_l) - A(d^-)\| \le \epsilon/12, \quad \|A(d) - A(d_r)\| \le \epsilon/12.$$

Then,

$$\begin{aligned}&\|A(d) - A(d^-)\|\\ &= \|A(d) - A(d_r) + A(d_r) - A(d_l) + A(d_l) - A(d^-)\|\\ &\le \|A(d_r) - A(d_l)\| + \epsilon/6\end{aligned} \tag{A.7}$$

[4] As mentioned before, limit and continuity of matrix-valued functions are defined elementwise. To obtain the stated inequalities, one needs to apply the following property of matrix norm (Golub & Loan, 2012, Eq. 2.3.8): $\|A\| \le \sqrt{mn} \max_{i,j} |a_{ij}|$, where a_{ij} is the ijth element in A, $1 \le i \le m$, and $1 \le j \le n$.

We select $a_r \in (a, a+\delta_1)$, $b_l \in (b-\delta_2, b)$, $d_l \in (d-\min\{\delta_1, \delta_3\}, d)$, and $d_r \in (d, d+\min\{\delta_2, \delta_4\})$ such that (A.5)–(A.7) hold. Consequently, we have

$$\begin{aligned} &\int_a^d \|\dot{A}(t)\|dt + \int_d^b \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\| \\ \leq &\int_{a_r}^{d_l} \|\dot{A}(t)\|dt + \int_{d_r}^{b_l} \|\dot{A}(t)\|dt + \|A(d_r) - A(d_l)\| \\ &+ \epsilon/2. \end{aligned} \tag{A.8}$$

We consider the interval $[a_r, d_l]$. Given any partition $P_1 = \{t_s^1 \mid s = 0, 1, \dots, k_1\}$ of $[a_r, d_l]$, define *mesh* of P_1 as

$$\text{mesh}(P_1) := \max_{1\leq s\leq k_1} (t_s^1 - t_{s-1}^1).$$

By the definition of Riemann integral, there exists $\eta_1 > 0$ such that for any partition P_1 of $[a_r, d_l]$ satisfying mesh$(P_1) < \eta_1$, we have

$$\left| \int_{a_r}^{d_l} \|\dot{A}(t)\|dt - \sum_{s=1}^{k_1} \|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) \right| \leq \epsilon/8,$$

which implies that

$$\int_{a_r}^{d_l} \|\dot{A}(t)\|dt \leq \sum_{s=1}^{k_1} \|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) + \epsilon/8. \tag{A.9}$$

Recall that $\dot{A}(\cdot)$ is continuous on (a, d), and $[a_r, d_l] \subseteq (a, d)$. Hence, $\dot{A}(\cdot)$ is uniformly continuous on $[a_r, d_l]$. That is, each element in $\dot{A}(\cdot)$ is uniformly continuous on $[a_r, d_l]$. Denote by $\dot{a}_{ij}(\cdot)$ the ijth element in $\dot{A}(\cdot)$. Then, there exists $\delta_{ij} > 0$ such that for any $x, y \in [a_r, d_l]$ satisfying $|x - y| < \delta_{ij}$, we have

$$|\dot{a}_{ij}(x) - \dot{a}_{ij}(y)| < \frac{\epsilon}{8\sqrt{mn}(d_l - a_r)}. \tag{A.10}$$

Taking $\eta_2 = \min_{1\leq i\leq m, 1\leq j\leq n} \delta_{ij}$, we have that (A.10) holds for all $\dot{a}_{ij}(\cdot)$ given that $x, y \in [a_r, d_l]$ and $|x - y| < \eta_2$.

Based on η_1 and η_2 defined above, we are able to construct a partition P_1 of $[a_r, d_l]$ such that mesh$(P_1) < \min\{\eta_1, \eta_2\}$. Consider any sub-interval $[t_{s-1}^1, t_s^1]$ induced by the partition P_1, and any element $a_{ij}(\cdot)$ in $A(\cdot)$. By the Mean Value Theorem, there exists $\xi_{ij} \in (t_{s-1}^1, t_s^1)$ such that

$$\begin{aligned} &a_{ij}(t_s^1) - a_{ij}(t_{s-1}^1) \\ = &\dot{a}_{ij}(\xi_{ij})(t_s^1 - t_{s-1}^1) \\ = &\dot{a}_{ij}(t_s^1)(t_s^1 - t_{s-1}^1) + \big(\dot{a}_{ij}(\xi_{ij}) - \dot{a}(t_s^1)\big)(t_s^1 - t_{s-1}^1). \end{aligned}$$

Then, we have

$$A(t_s^1) - A(t_{s-1}^1) = \dot{A}(t_s^1)(t_s^1 - t_{s-1}^1) + \tilde{A} \cdot (t_s^1 - t_{s-1}^1),$$

where $\tilde{A}$ is an $m \times n$ matrix whose ijth element, denoted by $\tilde{a}_{ij}$, equals $\dot{a}_{ij}(\xi_{ij}) - \dot{a}(t_s^1)$. By the triangle inequality, we have

$$\|A(t_s^1) - A(t_{s-1}^1)\| \geq \|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) - \|\tilde{A}\|(t_s^1 - t_{s-1}^1),$$

or equivalently

$$\|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) \leq \|A(t_s^1) - A(t_{s-1}^1)\| + \|\tilde{A}\|(t_s^1 - t_{s-1}^1). \tag{A.11}$$

Furthermore,

$$\begin{aligned} \|\tilde{A}\| &\leq \sqrt{mn} \max_{i,j} |\tilde{a}_{ij}| \\ &= \sqrt{mn} \max_{i,j} |\dot{a}_{ij}(\xi_{ij}) - \dot{a}_{ij}(t_s^1)| \\ &\leq \frac{\epsilon}{8(d_l - a_r)}. \end{aligned} \tag{A.12}$$

The first inequality is a property of matrix norm described in footnote 4. The second inequality holds since $|t_s^1 - \xi_{ij}| \leq |t_s^1 - t_{s-1}^1| < \delta_2$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, which implies that (A.10) holds for all $1 \leq i \leq m$, $1 \leq j \leq n$. Combining (A.11) and (A.12), we have

$$\begin{aligned} &\|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) \\ \leq &\|A(t_s^1) - A(t_{s-1}^1)\| + \frac{\epsilon}{8(d_l - a_r)}(t_s^1 - t_{s-1}^1). \end{aligned}$$

Note that (A.12) holds for all $[t_{s-1}^1, t_s^1]$, $s = 1, \dots, k_1$, and thus the inequality above holds for all $[t_{s-1}^1, t_s^1]$. Summing up both sides over $s = 1, \dots, k_1$ and applying the fact that $\sum_{1\leq s\leq k_1} t_s^1 - t_{s-1}^1 = d_l - a_r$, we have

$$\sum_{s=1}^{k_1} \|\dot{A}(t_s^1)\|(t_s^1 - t_{s-1}^1) \leq \sum_{s=1}^{k_1} \|A(t_s^1) - A(t_{s-1}^1)\| + \frac{\epsilon}{8}.$$

Combining the inequality above with (A.9) (which holds since mesh$(P) < \delta_1$), we have

$$\int_{a_r}^{d_l} \|\dot{A}(t)\|dt \leq \sum_{s=1}^{k_1} \|A(t_s^1) - A(t_{s-1}^1)\| + \frac{\epsilon}{4}. \tag{A.13}$$

Similarly, we can construct a partition $P_2 = \{t_s^2 | s = 0, 1, \dots, k_2\}$ of $[d_r, b_l]$ such that

$$\int_{d_r}^{b_l} \|\dot{A}(t)\|dt \leq \sum_{s=1}^{k_2} \|A(t_s^2) - A(t_{s-1}^2)\| + \frac{\epsilon}{4}. \tag{A.14}$$

Based on P_1 and P_2, we can construct a partition P of interval $[a, b]$ as follows:

$$\begin{aligned} P &= \{t_i | i = 0, 1, \dots, k\} \\ &= \{a, t_0^1, t_1^1, \dots, t_{k_1}^1, t_0^2, t_1^2, \dots, t_{k_2}^2, b\}. \end{aligned}$$

Combining (A.8), (A.13), and (A.14), we have

$$\begin{aligned} &\int_a^d \|\dot{A}(t)\|dt + \int_d^b \|\dot{A}(t)\|dt + \left\|A(d) - A(d^-)\right\| \\ \leq &\int_{a_r}^{d_l} \|\dot{A}(t)\|dt + \int_{d_r}^{b_l} \|\dot{A}(t)\|dt + \|A(d_r) - A(d_l)\| \\ &+ \epsilon/2 \\ \leq &\sum_{s=1}^{k_1} \|A(t_s^1) - A(t_{s-1}^1)\| + \epsilon/4 \\ &+ \sum_{s=1}^{k_2} \|A(t_s^2) - A(t_{s-1}^2)\| + \epsilon/4 \\ &+ \|A(d_r) - A(d_l)\| + \epsilon/2 \\ = &\sum_{s=1}^{k_1} \|A(t_s^1) - A(t_{s-1}^1)\| + \sum_{s=1}^{k_2} \|A(t_s^2) - A(t_{s-1}^2)\| \\ &+ \left\|A(t_0^2) - A(t_{k_1}^1)\right\| + \epsilon \\ \leq &\sum_{s=1}^{k_1} \|A(t_s^1) - A(t_{s-1}^1)\| + \sum_{s=1}^{k_2} \|A(t_s^2) - A(t_{s-1}^2)\| \\ &+ \left\|A(t_0^2) - A(t_{k_1}^1)\right\| \\ &+ \|A(t_0^1) - A(a)\| + \|A(b) - A(t_{k_2}^2)\| + \epsilon \\ = &\sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| + \epsilon \\ \leq &\sup_{P\in\mathbb{P}} \sum_{i=1}^{k} \|A(t_i) - A(t_{i-1})\| + \epsilon. \end{aligned}$$

Since the above inequality holds for any $\epsilon > 0$, we conclude the validity of (A.4). □

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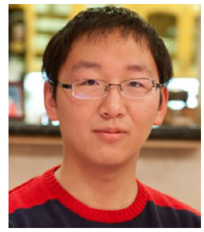

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