Feedback stabilization of a switched linear system with an unknown disturbance under data-rate constraints

Guosong Yang, *Student Member, IEEE,* and Daniel Liberzon, *Fellow, IEEE*

***Abstract*—We study the problem of stabilizing a switched linear system with a completely unknown disturbance using sampled and quantized state feedback. The switching is assumed to be slow enough in the sense of combined dwell-time and average dwell-time, each individual mode is assumed to be stabilizable, and the data-rate is assumed to be large enough but finite. Extending the approach of reachable-set approximation and propagation from an earlier result on the disturbance-free case, we develop a communication and control strategy that achieves a variant of input-to-state stability with exponential decay. An estimate of the disturbance bound is introduced to counteract the unknown disturbance, and a novel algorithm is designed to properly adjust the estimate and recover the state when it escapes the range of quantization.**

I. Introduction

Feedback control under data-rate constraints has been an active research area for years, as surveyed in [1], [2]. In many application-related scenarios, it is important to limit the information flow in feedback loops due to bandwidth constraints, cost concerns, physical restrictions, security considerations, etc. Besides the practical motivations above, the question of how much information is needed to achieve a certain control objective is quite fundamental and intriguing from the theoretical viewpoint. In this work a finite data transmission rate is achieved by generating control inputs based on sampled and quantized state measurements, which is a standard modeling framework in the literature (see, e.g., [3], [4] and [5, Ch. 5]).

This paper studies the problem of feedback stabilization under data-rate constraints in the presence of an external disturbance. In this context, [3], [4] assumed known bounds on the disturbances and addressed asymptotic stabilization with minimum data-rate, while [6], [7] avoided such assumptions by switching repeatedly between the "zooming-out" and "zooming-in" stages and achieved input-to-state stability (ISS) [8]. See also [9], [10] for related results in a stochastic setting.

The study of switched and hybrid systems has attracted a lot of attention lately (particularly relevant results are discussed in [5], [11], [12] and many references therein). In stability and stabilization of switched systems, it is a standard technique to impose suitable slow-switching conditions, especially in the sense of dwell-time [13] and average dwell-time (ADT) [14]. This approach plays a crucial role in our analysis as well.

Towards stabilizing a switched system with a disturbance (but without data-rate constraints), [14] showed that ISS can be achieved under the same ADT condition as the one for stability in the disturbance-free case. Their result was made explicit only for the case of switched linear systems, and many similar results for switched nonlinear systems have been established since then (see, e.g., [15] for ISS with a dwell-time, [16] for ISS and integral-ISS with an average dwell-time, and [17] for input/output-to-state stability with an average dwell-time).

Early works on control under data-rate constraints in the context of switched systems were devoted to quantized control of Markov jump linear systems [18], [19], [20]. However, the discrete modes in the results above were always available to the controller, which would remove most of the difficulties in our problem setup. The problem of asymptotically stabilizing a switched linear system using sampled and quantized state feedback was studied in [21], which also serves as the basis for this work. In [21], the controller was assumed to have a partial knowledge of the switching, that is, the active mode was unknown except at the sampling times, and the switching was subject to a mild slow-switching condition characterized by a dwell-time and an average dwell-time. Assuming that the data rate was large enough but finite, asymptotic stability was achieved by propagating over-approximations of reachable sets of the state over sampling intervals. See [22] for a related result using output feedback.

This work generalizes the main result of [21] in the presence of a completely unknown disturbance. Extending the approach of reachable-set approximation and propagation from [21], we develop a communication and control strategy that achieves a variant of ISS with exponential decay. Due to the unknown disturbance, the state may be forced outside the approximation of the reachable set at a sampling time after it has already been inside an earlier one (i.e., the state escapes the range of quantization). Consequently, the closed-loop system alternates multiple times between the stabilizing and searching stages. An estimate of the disturbance bound is introduced in approximating reachable sets so that the state cannot escape unless the disturbance is larger than the estimate. A novel algorithm is designed to properly adjust the estimate and recover the state whenever it escapes so that the total length of searching stages is finite and the system eventually stays in a stabilizing stage, provided that the disturbance is globally essentially bounded (by an unknown value).

To the best of our knowledge, this work is the first result for the case that combines switching, disturbance, and date-rate constraints, with the exception of its preliminary version [23], and our earlier result [24] for the easier case of disturbances with known bounds. This paper improves [23] by establishing continuity of gain functions and removing unnecessary conditions in the main theorem, substantiating the results with complete proofs and clarifying remarks, and providing a detailed simulation study.

The authors are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S. Email: {yang150, liberzon}@illinois.edu. This work was supported by the NSF grants CNS-1217811 and ECCS-1231196.

This paper is organized as follows: Section II introduces the system definition, the information structure, and the basic assumptions. Our main result is stated in Section III. Section IV explains the communication and control strategy, assuming that suitable approximations of reachable sets are available. Such approximations are constructed in Section V. Section VI details the stability analysis with major steps summarized as technical lemmas. The simulation example is provided in Section VII. Section VIII concludes the paper with a brief summary and an outlook on future research topics.

II. Problem formulation

A. System definition

We are interested in stabilizing a switched linear control system with state $x \in \mathbb{R}^{n_x}$, control $u \in \mathbb{R}^{n_u}$ and disturbance $d \in \mathbb{R}^{n_d}$ modeled by

$$\dot{x} = A_\sigma x + B_\sigma u + D_\sigma d, \qquad x(0) = x_0, \tag{1}$$

where $\{(A_p, B_p, D_p) : p \in \mathcal{P}\}$ denotes a collection of matrix triples defining the modes (subsystems), $\mathcal{P}$ is a finite *index set*, and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ is a right-continuous, piecewise constant *switching signal* that specifies the active mode $\sigma(t)$ at each time t. The solution $x(\cdot)$ is absolutely continuous and satisfies the differential equation (1) away from discontinuities of σ (in particular, there are no state jumps). An admissible disturbance $d(\cdot)$ is a measurable and locally essentially bounded function. The switching signal σ is fixed but unknown to the sensor and the controller a priori. Discontinuities of σ are called *switching times*, or simply *switches*. The number of switches on a time interval $(\tau, t]$ is denoted by $N_\sigma(t, \tau)$.

Our first basic assumption is that the switching is slow in the sense of combined dwell-time and average dwell-time:

Assumption 1 (Switching)**.** The switching admits

1. a *dwell-time* τ_d such that $N_\sigma(t, \tau) \leq 1$ for all $\tau \geq 0$ and $t \in (\tau, \tau + \tau_d]$; and
2. an *average dwell-time* (ADT) $\tau_a > \tau_d$ such that
$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a} \qquad \forall\, t > \tau \geq 0 \tag{2}$$
with an integer $N_0 \geq 1$.

The notions of dwell-time [13] and average dwell-time [14] have become standard in the literature on switched systems. In Assumption 1, the dwell-time condition (item 1) can be written in the form of (2) with $\tau_a = \tau_d$ and $N_0 = 1$; meanwhile, the average dwell-time condition (item 2) would be implied by the dwell-time condition if the constraint $\tau_a > \tau_d$ is violated. Switching signals satisfying Assumption 1 were referred to as "hybrid dwell-time" signals in [25].

Our second basic assumption is that every individual mode is stabilizable:

Assumption 2 (Stabilizability)**.** For each $p \in \mathcal{P}$, the pair (A_p, B_p) is stabilizable, that is, there exists a state feedback gain matrix K_p such that $A_p + B_p K_p$ is Hurwitz.

In the following analysis, it is assumed that such a collection of stabilizing gain matrices $\{K_p : p \in \mathcal{P}\}$ has been selected and fixed. However, even in the disturbance-free case, and when all individual modes are stabilized via state feedback (or stable without feedback), stability of the switched system is not necessarily guaranteed (see, e.g., [5, p. 19]).

Throughout this work, $\|\cdot\|$ denotes the ∞-norm of a vector or the (induced) ∞-norm of a matrix, that is,

$$\|v\| := \|v\|_\infty := \max_{1 \leq i \leq n} |v_i|$$

for $v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$, and

$$\|M\| := \|M\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |M_{ij}|$$

for $M = (M_{ij}) \in \mathbb{R}^{n \times n}$. The left-sided limit of a piecewise absolutely continuous function z approaching t is denoted by $z(t^-) := \lim_{s \nearrow t} z(s)$.

We let δ_d denote the essential supremum ∞-norm of the disturbance d, that is,

$$\delta_d := \|d\|_\infty := \operatorname*{ess\,sup}_{s \geq 0} \|d(s)\| \leq \infty, \tag{3}$$

and call it the *disturbance bound*. In the following analysis, it is assumed that δ_d is finite (as the inequality (6) in our main result holds trivially when $\delta_d = \infty$). However, its value is *unknown* to the sensor and the controller.

B. Information structure

The feedback loop consists of a sensor and a controller. The sensor measures two sequences of data—indices of the active modes $\sigma(t_k)$ and quantized measurements (samples) of the state $x(t_k)$—and transmits them to the controller at *sampling times* $t_k = k\tau_s$, where k is a nonnegative integer and $\tau_s > 0$ is the *sampling period*. Each sample is encoded by an integer i_k from 0 to N^{n_x}, where N is an odd integer (so that the equilibrium at the origin is preserved). The controller generates the control input u to the switched linear system (1) based on the decoded data. As $\sigma(t_k) \in \mathcal{P}$ and $i_k \in \{0, 1, \dots, N^{n_x}\}$, the data transmission rate between the encoder and the decoder is given by

$$R = \frac{\log_2 |N^{n_x} + 1| + \log_2 |\mathcal{P}|}{\tau_s}$$

bits per unit of time, where $|\mathcal{P}|$ denotes the cardinality of the index set $\mathcal{P}$ (i.e., the number of modes). As illustrated in Fig. 1, this information structure allows us to separate the sensing and control tasks in the following sense: the sensor does not have access to the exact control objective, and the controller does not have access to the exact state. The communication and control strategy is explained in detail in Section IV.

The sampling period τ_s is assumed to be no larger than the dwell-time τ_d in Assumption 1, that is,

$$\tau_s \leq \tau_d, \tag{4}$$

so that there is at most one switch in each *sampling interval* $(t_k, t_{k+1}]$. Since the average dwell-time $\tau_a > \tau_d$ in Assumption 1, switches actually occur less often than once per sampling period.

Our last basic assumption imposes a lower bound on the data rate R:

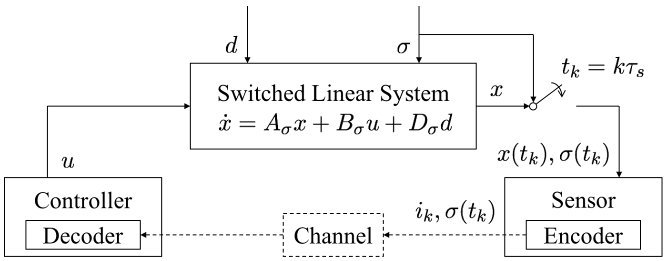

Fig. 1. Information structure

Assumption 3 (Data rate)**.** The sampling period τ_s satisfies

$$\Lambda_p := \|e^{A_p \tau_s}\| < N \qquad \forall\, p \in \mathcal{P}. \tag{5}$$

The inequality in (5) can be interpreted as a lower bound on the data rate R as it requires τ_s to be sufficiently small with respect to N. This bound is the same as the one for the case without disturbance [21, Assumption 3], and similar data-rate bounds appeared in [3], [26], [4] for stabilizing non-switched linear systems; see [7, Sec. V] and [21, Sec. 2.2] for more discussions on their relation.

III. Main result

The control objective is to robustly stabilize the system defined in Section II-A under the data-rate constraint described in Section II-B. More precisely, we intend to establish the following ISS-like property:

Theorem 1 (Exponential decay)**.** *Consider the switched linear control system* (1)*. Suppose that Assumptions 1–3 and the inequality* (4) *hold. Provided that the average dwell-time τ_a is sufficiently large, there exists a communication and control strategy that achieves*

$$\|x(t)\| \le e^{-\lambda t} g(\|x_0\|) + h(\|d\|_\infty) \qquad \forall\, t \ge 0 \tag{6}$$

for all initial states $x_0 \in \mathbb{R}^{n_x}$ and disturbances $d : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n_d}$, with a constant $\lambda > 0$ and functions $g, h : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$.

The lower bound on τ_a is given by (47) in Section VI-A3. The *exponential decay rate* λ is given by (62) and the nonlinear gain functions g and h by (63) in Section VI-C3. From the proof it will be clear that both g and h can be made continuous and strictly increasing. However, $g(0) > 0$ due to the sampling and quantization, $h(0) > 0$ due to the unknown disturbance, and both $g(s)$ and $h(s)$ have superlinear growth rates as $s \to \infty$, which is consistent with [27, Cor. 2.3]. Consequently, the state bound (6) does not give the standard *input-to-state stability* (ISS) [8], but rather the *input-to-state practical stability* (ISpS) [28]:

$$\|x(t)\| \le e^{-\lambda t}\gamma_x(\|x_0\|) + \gamma_d(\|d\|_\infty) + C \qquad \forall\, t \ge 0$$

with $\gamma_x, \gamma_d \in \mathcal{K}_\infty$ defined by $\gamma_x(s) := g(s) - g(0)$, $\gamma_d(s) := h(s) - h(0)$ and $C = g(0) + h(0) > 0$.

Remark 1. Following similar arguments as in [29, Sec. VI], the state bound (6) can be restated in terms of the ISS property with respect to a set. From (6) it follows that the *uniform asymptotic gain* (UAG) property [29] holds for the set $\mathcal{A} := \{v \in \mathbb{R}^{n_x} : \|v\| \le h(0)\}$, that is, for each pair of $\varepsilon, \delta > 0$ there exists a time $T_{\varepsilon,\delta} := \max\{\ln(g(h(0)+\delta)/\varepsilon)/\lambda, 0\}$ such that if $\|x_0\|_{\mathcal{A}} \le \delta$ then $\|x(t)\|_{\mathcal{A}} \le \gamma_d(\|d\|_\infty) + \varepsilon$ for all $t \ge T_{\varepsilon,\delta}$ and disturbances $d(\cdot)$, where $\gamma_d \in \mathcal{K}_\infty$ is defined by $\gamma_d(s) := h(s) - h(0)$, and $\|v\|_{\mathcal{A}} := \inf_{v' \in \mathcal{A}} \|v - v'\|$ denotes the (Chebyshev) distance from a point v to $\mathcal{A}$. In the context of non-switched systems, it has been shown that if UAG holds for $\mathcal{A}$ then the system is ISS with respect to the closure of the reachable set from $\mathcal{A}$ with $d \equiv 0$ [29, Lemma VI.2].

The state bound (6) also implies the following *practical stability* property: for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x_0\|, \|d\|_\infty \le \delta \quad \Rightarrow \quad \sup_{t \ge 0} \|x(t)\| \le \varepsilon + C \tag{7}$$

with the constant $C = g(0) + h(0) > 0$. In Section VI-D we establish practical stability with a smaller constant C following similar arguments as in [21, Sec. 5.5].

IV. Communication and control strategy

In this section we describe the communication and control strategy in detail, assuming that suitable approximations of reachable sets of the state are available at all sampling times. (Such approximations are derived in the next section.) More specifically, at each sampling time t_k, the reachable set is approximated by a hypercube $\mathcal{S}_k$ of radius E_k centered at x_k^*, that is, $\mathcal{S}_k := \{v \in \mathbb{R}^{n_x} : \|v - x_k^*\| \le E_k\}$. When constructing $\mathcal{S}_k$, we introduce an estimate δ_k of the disturbance bound δ_d in (3). Unlike the disturbance-free case, if $\delta_k < \delta_d$ then the state $x(t_k)$ may be forced outside the hypercube $\mathcal{S}_k$ at a sampling time t_k, even after $x(t_i) \in \mathcal{S}_i$ for some $i \le k$. Thus the system alternates multiple times between the stabilizing and searching stages described in Sections IV-B and IV-C, respectively. The rules of adjusting the estimate δ_k are explained in Section IV-D.

A. Terminology

The initial state x_0 is unknown. At $t = 0$, the sensor and the controller both receive $x_0^* = 0$ and arbitrary positive initial values E_0 and δ_0 (approximating $\|x_0\|$ and δ_d, respectively). Starting from $k = 0$, at each sampling time t_k the sensor first determines if

$$\|x(t_k) - x_k^*\| \le E_k, \tag{8}$$

that is, if the state $x(t_k) \in \mathcal{S}_k$. If so, we say the state is *visible* and the system is in a *stabilizing stage*; otherwise the state is *lost* and the system is in a *searching stage*. Once the state is visible, the system is in a stabilizing stage until a sampling time t_j at which (8) does not hold with $k = j$ (i.e., the state is visible at t_{j-1} and lost at t_j), in which case we say the state *escapes* at t_j; likewise, once the state is lost, the system is in a searching stage until a sampling time t_i at which (8) holds with $k = i$ (i.e., the state is lost at t_{i-1} and visible at t_i), in which case we say the state *is recovered* at t_i.

B. Stabilizing stage

At each sampling time t_k in a stabilizing stage, the encoder divides the hypercube $\mathcal{S}_k$ into N^{n_x} equal hypercubic boxes, N per dimension, encodes each box by a unique index from 1 to N^{n_x}, and transmits the index i_k of the box containing $x(t_k)$ along with the active mode $\sigma(t_k)$ to the decoder. The controller learns that (8) holds upon receiving $i_k \in \{1, \dots, N^{n_x}\}$. The decoder follows the same pre-defined indexing protocol as the encoder so that it is able to reconstruct the center c_k of the hypercubic box containing $x(t_k)$ from i_k. Simple calculation shows that

$$\|x(t_k) - c_k\| \le \frac{1}{N} E_k, \qquad \|c_k - x_k^*\| \le \frac{N-1}{N} E_k. \tag{9}$$

The controller then generates the control input $u(t) = K_{\sigma(t_k)} \hat{x}(t)$ for $t \in [t_k, t_{k+1})$, where $K_{\sigma(t_k)}$ is the state feedback gain matrix in Assumption 2, and $\hat{x}$ is the state of the auxiliary system

$$\dot{\hat{x}} = A_{\sigma(t_k)} \hat{x} + B_{\sigma(t_k)} u = (A_{\sigma(t_k)} + B_{\sigma(t_k)} K_{\sigma(t_k)}) \hat{x} \tag{10}$$

with the boundary condition

$$\hat{x}(t_k) = c_k. \tag{11}$$

(Note that $\hat{x}$ is reset to c_k at each sampling time t_k in a stabilizing stage.) Both the sensor and the controller have access to functions F and G so that they calculate

$$\begin{aligned} x_{k+1}^* &:= F(\sigma(t_k), \sigma(t_{k+1}), c_k), \\ E_{k+1} &:= G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_k) \end{aligned} \tag{12}$$

for the next sampling time t_{k+1} without further communication. The functions F and G are designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \le G(\sigma(t_k), \sigma(t_{k+1}), x_k^*, E_k, \delta_d), \tag{13}$$

and G is strictly increasing in the last argument, which is δ_k in (12) and δ_d in (13). Hence only if $\delta_k < \delta_d$ can the state escape at t_{k+1}. (Meanwhile, $x(t_{k+1}) \in S_{k+1}$ does not imply $\delta_k \ge \delta_d$.) The formulas for F and G are derived in Section V-A.

C. Searching stage

At each sampling time t_k in a searching stage, there is an unknown $\hat{D}_k$ such that

$$E_k < \|x(t_k) - x_k^*\| \le \hat{D}_k. \tag{14}$$

For example, when the state escapes at t_j we have $\hat{D}_j = G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d)$ from (13), while if it is lost at $t_0 = 0$ then $\hat{D}_0 = \|x_0\|$. The encoder sends $i_k = 0$, the "overflow symbol", to the decoder. Upon receiving $i_k = 0$, the controller learns the state is lost and then sets the control input $u = 0$ on $[t_k, t_{k+1})$. Both the sensor and the controller have access to a function $\hat{G}$ so that they calculate

$$\begin{aligned} x_{k+1}^* &:= x_k^*, \\ E_{k+1} &:= \hat{G}(x_k^*, (1+\varepsilon_E) E_k, \delta_k) \end{aligned} \tag{15}$$

without further communication, where $\varepsilon_E > 0$ is an arbitrary design parameter. The function $\hat{G}$ is designed so that

$$\|x(t_{k+1}) - x_{k+1}^*\| \le \hat{G}(x_k^*, \hat{D}_k, \delta_d), \tag{16}$$

and it is strictly increasing in the last two arguments. Note that the second argument of $\hat{G}$ in (16) is $\hat{D}_k$, whereas the one in (15) is $(1+\varepsilon_E)E_k$. Because of the additional coefficient $1+\varepsilon_E$, it is ensured that the growth rate of E_k dominates that of $\hat{D}_k$, and hence the state is recovered in a finite time, as shown in Section V-B following the derivation of $\hat{G}$.

D. Adjusting the estimate of the disturbance bound

When the state escapes at a sampling time t_j, the sensor and the controller learn that $\delta_{j-1} < \delta_d$ and adjust the estimate by enlarging it to $\delta_j = (1+\varepsilon_\delta)\delta_{j-1}$, where $\varepsilon_\delta > 0$ is a fixed arbitrary design parameter. The estimate remains unchanged in all other cases (in particular, it is adjusted only once per searching stage). Thus it is ensured that there is a finite number of searching stages in total, as the estimate becomes greater or equal to the disturbance bound δ_d after finitely many adjustments and the state cannot escape after that.

V. Approximations of reachable sets

In this section we derive the recursive formulas required to implement the communication and control strategy. In Section V-A we consider stabilizing stages and construct the functions F and G in (12) such that (13) holds. In Section V-B we consider searching stages, construct the function $\hat{G}$ in (15) such that (16) holds, and prove that the state is ensured to be recovered in a finite time.

A. Stabilizing stage

Suppose that the state is visible at a sampling time t_k, that is, (8) holds.

1) Sampling interval with no switch: When

$$\sigma(t_k) = p = \sigma(t_{k+1}) \tag{17}$$

for some $p \in \mathcal{P}$, there is no switch on $(t_k, t_{k+1}]$ due to (4). Combining the switched linear system (1) and the auxiliary system (10), we obtain

$$\begin{aligned} \dot{x} &= A_p x + B_p u + D_p d, \\ \dot{\hat{x}} &= A_p \hat{x} + B_p u. \end{aligned}$$

The error $e := x - \hat{x}$ satisfies

$$\dot{e} = A_p e + D_p d, \qquad \|e(t_k)\| = \|x(t_k) - c_k\| \le \frac{1}{N} E_k$$

on $[t_k, t_{k+1})$, where the boundary condition follows from (9) and (11). Hence

$$\begin{aligned} \|e(t_{k+1}^-)\| &= \left\| e^{A_p \tau_s} e(t_k) + \int_{t_k}^{t_{k+1}} e^{A_p (t_{k+1} - \tau)} D_p d(\tau) \mathrm{d}\tau \right\| \\ &\le \|e^{A_p \tau_s}\| \|e(t_k)\| + \left(\int_0^{\tau_s} \|e^{A_p s} D_p\| \mathrm{d}s \right) \delta_d \\ &\le \frac{\Lambda_p}{N} E_k + \Phi_p(\tau_s) \delta_d =: \hat{D}_{k+1} \end{aligned}$$

with Λ_p in (5) and the increasing function $\Phi_p : [0, \tau_s] \to \mathbb{R}$ defined by

$$\Phi_p(t) := \int_0^t \|e^{A_p s} D_p\| \mathrm{d}s. \tag{18}$$

Therefore, we set

$$E_{k+1} = G(p,p,x_k^*,E_k,\delta_k) := \frac{\Lambda_p}{N}E_k + \Phi_p(\tau_s)\delta_k. \quad (19)$$

As x is continuous, (13) holds with x_{k+1}^* set as the auxiliary state $\hat{x}$ approaching t_{k+1}, that is,

$$x_{k+1}^* = F(p,p,c_k) := \hat{x}(t_{k+1}^-) = S_p c_k \quad (20)$$

with $S_p := e^{(A_p+B_pK_p)\tau_s}$.

2) Sampling interval with a switch: When

$$\sigma(t_k) = p \neq q = \sigma(t_{k+1}) \quad (21)$$

for some $p,q \in \mathcal{P}$, there is exactly one switch on $(t_k,t_{k+1}]$ due to (4). Let $t_k+\bar{t}$ with $\bar{t} \in (0,\tau_s]$ denote the unknown switching time. Then

$$\sigma(t) = \begin{cases} p, & t \in [t_k, t_k+\bar{t}), \\ q, & t \in [t_k+\bar{t}, t_{k+1}]. \end{cases}$$

Before the switch, mode p is active on $[t_k,t_k+\bar{t})$. Following similar arguments as in the previous case, we see that the error $e = x - \hat{x}$ satisfies

$$\|e(t_k+\bar{t})\| \leq \frac{\|e^{A_p\bar{t}}\|}{N}E_k + \Phi_p(\bar{t})\delta_d$$

with Φ_p defined by (18). As the switching time $t_k+\bar{t}$ is unknown, we replace $x(t_k+\bar{t})$ with $\hat{x}(t_k+t') = e^{(A_p+B_pK_p)t'}c_k$ of the auxiliary system (10) at an arbitrarily selected time $t_k+t' \in [t_k,t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned} &\|\hat{x}(t_k+\bar{t}) - \hat{x}(t_k+t')\| \\ &\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\|c_k\| \\ &\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right), \end{aligned}$$

where the last inequality follows partially from (9). Then

$$\begin{aligned} &\|x(t_k+\bar{t}) - \hat{x}(t_k+t')\| \\ &\leq \|\hat{x}(t_k+\bar{t}) - \hat{x}(t_k+t')\| + \|e(t_k+\bar{t})\| \\ &\leq \|e^{(A_p+B_pK_p)\bar{t}} - e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right) \\ &\quad + \frac{\|e^{A_p\bar{t}}\|}{N}E_k + \Phi_p(\bar{t})\delta_d \\ &=: \hat{D}'_{k+1}(t',\bar{t}). \end{aligned} \quad (22)$$

After the switch, mode q is active on $[t_k+\bar{t},t_{k+1}]$. Combining the switched linear system (1) and the auxiliary system (10) with $u = K_p\hat{x}$, we obtain

$$\dot{z} = \bar{A}_{pq}z + \bar{D}_q d$$

for $z := (x^\top,\hat{x}^\top)^\top \in \mathbb{R}^{2n_x}$ with

$$\bar{A}_{pq} := \begin{bmatrix} A_q & B_qK_p \\ 0_{n_x\times n_x} & A_p+B_pK_p \end{bmatrix}, \qquad \bar{D}_q = \begin{bmatrix} D_q \\ 0_{n_x\times n_d} \end{bmatrix}.$$

Combining it with a second auxiliary system

$$\dot{\hat{z}} = \bar{A}_{pq}\hat{z}, \quad \hat{z}(t_k+t') = (\hat{x}(t_k+t')^\top, \hat{x}(t_k+t')^\top)^\top, \quad (23)$$

we obtain

$$\begin{aligned} \dot{z} &= \bar{A}_{pq}z + \bar{D}_q d, \\ \dot{\hat{z}} &= \bar{A}_{pq}\hat{z} \end{aligned}$$

with the boundary condition

$$\|z(t_k+\bar{t}) - \hat{z}(t_k+t')\| \leq \|x(t_k+\bar{t}) - \hat{x}(t_k+t')\| \leq \hat{D}'_{k+1}(t',\bar{t}),$$

where the first inequality follows from the property that the ∞-norms of two vectors v,w and their concatenation satisfy

$$\|(v^\top,w^\top)^\top\| = \max\{\|v\|,\|w\|\}. \quad (24)$$

Hence

$$\begin{aligned} &\|z(t_{k+1}^-) - \hat{z}(t_{k+1}-\bar{t}+t')\| \\ &= \left\| e^{\bar{A}_{pq}(\tau_s-\bar{t})}z(t_k+\bar{t}) + \int_{t_k+\bar{t}}^{t_{k+1}} e^{\bar{A}_{pq}(t_{k+1}-\tau)}\bar{D}_q d(\tau)\mathrm{d}\tau \right. \\ &\quad \left. - e^{\bar{A}_{pq}(\tau_s-\bar{t})}\hat{z}(t_k+t') \right\| \\ &\leq \|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|\|z(t_k+\bar{t}) - \hat{z}(t_k+t')\| \\ &\quad + \left(\int_0^{\tau_s-\bar{t}} \|e^{\bar{A}_{pq}s}\bar{D}_q\|\mathrm{d}s\right)\delta_d \\ &\leq \|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|\hat{D}'_{k+1}(t',\bar{t}) + \bar{\Phi}_{pq}(\tau_s-\bar{t})\delta_d \end{aligned}$$

with the increasing function $\bar{\Phi}_{pq} : [0,\tau_s] \to \mathbb{R}$ defined by

$$\bar{\Phi}_{pq}(t) := \int_0^t \|e^{\bar{A}_{pq}s}\bar{D}_q\|\mathrm{d}s.$$

Again, we replace $z(t_{k+1}^-)$ with $\hat{z}(t_k+t'') = e^{\bar{A}_{pq}(t''-t')}\hat{z}(t_k+t')$ of the second auxiliary system (23) at an arbitrarily selected time $t_k+t'' \in [t_k,t_{k+1}]$ via the triangle inequality. First,

$$\begin{aligned} &\|\hat{z}(t_{k+1}-\bar{t}+t') - \hat{z}(t_k+t'')\| \\ &\leq \|e^{\bar{A}_{pq}(\tau_s-\bar{t})} - e^{\bar{A}_{pq}(t''-t')}\|\|\hat{z}(t_k+t')\| \\ &= \|e^{\bar{A}_{pq}(\tau_s-\bar{t})} - e^{\bar{A}_{pq}(t''-t')}\|\|\hat{x}(t_k+t')\| \\ &\leq \|e^{\bar{A}_{pq}(\tau_s-\bar{t})} - e^{\bar{A}_{pq}(t''-t')}\|\|e^{(A_p+B_pK_p)t'}\| \\ &\quad \times \left(\|x_k^*\| + \frac{N-1}{N}E_k\right), \end{aligned}$$

where the equality follows from (24) and the last inequality partially from (9). Then

$$\begin{aligned} &\|z(t_{k+1}^-) - \hat{z}(t_k+t'')\| \\ &\leq \|z(t_{k+1}^-) - \hat{z}(t_{k+1}-\bar{t}+t')\| \\ &\quad + \|\hat{z}(t_{k+1}-\bar{t}+t') - \hat{z}(t_k+t'')\| \\ &\leq \|e^{\bar{A}_{pq}(\tau_s-\bar{t})}\|\hat{D}'_{k+1}(t',\bar{t}) + \|e^{\bar{A}_{pq}(\tau_s-\bar{t})} - e^{\bar{A}_{pq}(t''-t')}\| \\ &\quad \times \|e^{(A_p+B_pK_p)t'}\|\left(\|x_k^*\| + \frac{N-1}{N}E_k\right) + \bar{\Phi}_{pq}(\tau_s-\bar{t})\delta_d \\ &=: \hat{D}''_{k+1}(t',t'',\bar{t}). \end{aligned} \quad (25)$$

To remove the dependence on the unknown $\bar{t}$, we take the supremum over $\bar{t}$ (with fixed t' and t'') and obtain

$$\|z(t_{k+1}^-) - \hat{z}(t_k+t'')\| \leq \sup_{\bar{t}\in(0,\tau_s]} \hat{D}''_{k+1}(t',t'',\bar{t}) =: \hat{D}_{k+1}.$$

Therefore, we set E_{k+1} by first replacing the disturbance bound δ_d in $\hat{D}''_{k+1}(t',t'',\bar{t})$ with the estimate δ_k, and then taking the maximum over $\bar t$ (with the same fixed t' and t''), that is,

$$\begin{aligned}
&E_{k+1} = G(p,q,x_k^*,E_k,\delta_k)\\
&:= \sup_{\bar t\in(0,\tau_s]} \Bigg\{ \Big(\|e^{\bar A_{pq}(\tau_s-\bar t)} - e^{\bar A_{pq}(t''-t')}\|\|e^{(A_p+B_pK_p)t'}\| \\
&\quad + \|e^{\bar A_{pq}(\tau_s-\bar t)}\|\|e^{(A_p+B_pK_p)\bar t} - e^{(A_p+B_pK_p)t'}\|\Big)\|x_k^*\| \\
&\quad + \Bigg(\frac{N-1}{N}\Big(\|e^{\bar A_{pq}(\tau_s-\bar t)} - e^{\bar A_{pq}(t''-t')}\|\|e^{(A_p+B_pK_p)t'}\| \\
&\quad + \|e^{\bar A_{pq}(\tau_s-\bar t)}\|\|e^{(A_p+B_pK_p)\bar t} - e^{(A_p+B_pK_p)t'}\|\Big) \\
&\quad + \frac{1}{N}\|e^{\bar A_{pq}(\tau_s-\bar t)}\|\|e^{A_p\bar t}\|\Bigg)E_k \\
&\quad + \Big(\|e^{\bar A_{pq}(\tau_s-\bar t)}\|\Phi_p(\bar t) + \bar\Phi_{pq}(\tau_s-\bar t)\Big)\delta_k \Bigg\}.
\end{aligned} \tag{26}$$

(Clearly, the design parameters t', t'' should be selected so that E_{k+1} is minimized. However, their optimal values cannot be determined without imposing further constraints on the matrices $\{A_p, B_p, D_p, K_p : p \in \mathcal{P}\}$.) As x is continuous, (13) holds with x_{k+1}^* set as the projection of the second auxiliary state $\hat z$ approaching $t_k + t''$ onto the x-component, that is,

$$x_{k+1}^* = F(p,q,c_k) := (I_{n_x\times n_x}\ 0_{n_x\times n_x})\hat z(t_k+t'') = H_{pq}c_k \tag{27}$$

with

$$H_{pq} := (I_{n_x\times n_x}\ 0_{n_x\times n_x}) e^{\bar A_{pq}(t''-t')} \begin{pmatrix} I_{n_x\times n_x} \\ I_{n_x\times n_x}\end{pmatrix} e^{(A_p+B_pK_p)t'}.$$

In the remaining of this subsection, we derive a relatively simpler but more conservative bound of E_{k+1}, which is more useful for computations. First, the norm of the difference of two matrix exponentials can be simplified via the following lemma[1]:

Lemma 1. *Consider two square matrices X, Y. Then*

$$\|e^{X+Y} - e^X\| \le e^{\|X\|+\|Y\|}\|Y\|.$$

Proof. See Appendix A. □

Following Lemma 1 and the property that

$$\|e^{Ms}\| \le e^{\|M\||s|} \qquad \forall\, M \in \mathbb{R}^{n\times n}, \forall\, s \in \mathbb{R},$$

we obtain

$$E_{k+1} \le \alpha_{pq}\|x_k^*\| + \beta_{pq}E_k + \gamma_{pq}\delta_k \tag{28}$$

with

$$\begin{aligned}
\alpha_{pq} &:= e^{\|\bar A_{pq}\|\max\{\tau_s, 2(t''-t'), \tau_s+2(t'-t'')\}}\|\bar A_{pq}\| \\
&\quad\times \max\{t''-t', \tau_s+t'-t''\}\|e^{(A_p+B_pK_p)t'}\| \\
&\quad + e^{\|\bar A_{pq}\|\tau_s}e^{\|A_p+B_pK_p\|\max\{\tau_s,2t'\}} \\
&\quad\times \|A_p+B_pK_p\|\max\{\tau_s-t', t'\},\\
\beta_{pq} &:= \frac{N-1}{N}\alpha_{pq} + \frac{1}{N}e^{(\|\bar A_{pq}\|+\|A_p\|)\tau_s},\\
\gamma_{pq} &:= e^{\|\bar A_{pq}\|\tau_s}\Phi_p(\tau_s) + \bar\Phi_{pq}(\tau_s).
\end{aligned} \tag{29}$$

[1] Using Lemma 1 instead of the inequality that $\|M - I\| \le \|M\| + 1$ for all square matrices M in [21, eq. (20)] ensures that $\alpha_{pq} \to 0$ as $\tau_s \to 0$, a property we will use in the comparison to [14] in Remark 4. However, for a sufficiently large τ_s it is possible that the bound in Lemma 1 is worse.

Remark 2. If we set $t'' = t' = 0$ then (28) becomes

$$E_{k+1} \le \alpha_{pq}^0\|x_k^*\| + \beta_{pq}^0 E_k + \gamma_{pq}\delta_k$$

with

$$\begin{aligned}
\alpha_{pq}^0 &:= e^{\|\bar A_{pq}\|\tau_s}\|\bar A_{pq}\|\tau_s + e^{\|\bar A_{pq}\|\tau_s} \\
&\quad\times e^{\|A_p+B_pK_p\|\tau_s}\|A_p+B_pK_p\|\tau_s,\\
\beta_{pq}^0 &:= \frac{N-1}{N}\alpha_{pq}^0 + \frac{1}{N}e^{(\|\bar A_{pq}\|+\|A_p\|)\tau_s}.
\end{aligned} \tag{30}$$

While this choice of t', t'' considerably simplifies the formula of the bound, it does not necessarily minimize E_{k+1}.

B. Searching stage

Suppose that the state is lost at a sampling time t_k, that is, (14) holds.

1) Reachable-set approximation: Let $p = \sigma(t_k)$ and consider an arbitrary $t \in (t_k, t_{k+1}]$. If $\sigma(t) = p$ then there is no switch on $(t_k, t]$ due to (4), and hence

$$\begin{aligned}
&\|x(t) - x_k^*\| \\
&= \left\| e^{A_p(t-t_k)}x(t_k) + \int_{t_k}^t e^{A_p(t-\tau)}D_p d(\tau)\mathrm{d}\tau - x_k^* \right\| \\
&\le \|e^{A_p(t-t_k)} - I\|\|x_k^*\| + \|e^{A_p(t-t_k)}\|\|x(t_k) - x_k^*\| \\
&\quad + \left(\int_0^{t-t_k}\|e^{A_ps}D_p\|\mathrm{d}s\right)\delta_d \\
&\le \bar\Gamma\|x_k^*\| + \bar\Lambda\hat D_k + \bar\Phi\delta_d
\end{aligned}$$

with

$$\begin{aligned}
\bar\Gamma &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt} - I\|,\\
\bar\Lambda &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \|e^{A_pt}\| \ge 1,\\
\bar\Phi &:= \max_{t\in[0,\tau_s],\ p\in\mathcal{P}} \Phi_p(t) = \max_{p\in\mathcal{P}}\Phi_p(\tau_s).
\end{aligned} \tag{31}$$

If $\sigma(t) = q \ne p$ then there is exactly one switch on $(t_k, t]$ due to (4), and hence

$$\begin{aligned}
&\|x(t) - x_k^*\| \\
&= \left\| e^{A_q(t-t_k-\bar t)}x(t_k+\bar t) + \int_{t_k+\bar t}^t e^{A_q(t-\tau)}D_q d(\tau)\mathrm{d}\tau - x_k^*\right\| \\
&\le \|e^{A_q(t-t_k-\bar t)} - I\|\|x_k^*\| + \|e^{A_q(t-t_k-\bar t)}\|\|x(t_k+\bar t) - x_k^*\| \\
&\quad + \left(\int_0^{t-t_k-\bar t}\|e^{A_qs}D_q\|\mathrm{d}s\right)\delta_d \\
&\le \bar\Gamma\|x_k^*\| + \bar\Lambda\|x(t_k+\bar t) - x_k^*\| + \bar\Phi\delta_d \\
&\le \bar\Gamma\|x_k^*\| + \bar\Lambda(\bar\Gamma\|x_k^*\| + \bar\Lambda\hat D_k + \bar\Phi\delta_d) + \bar\Phi\delta_d \\
&\le (\bar\Lambda+1)\bar\Gamma\|x_k^*\| + \bar\Lambda^2\hat D_k + (\bar\Lambda+1)\bar\Phi\delta_d,
\end{aligned}$$

where $t_k + \bar t$ denotes the unknown switching time. As $\bar\Lambda \ge 1$, the bound for the second case holds for both cases, that is,

$$\|x(t) - x_k^*\| \le \bar\alpha\|x_k^*\| + \bar\beta\hat D_k + \bar\gamma\delta_d =: \hat D_{k+1} \tag{32}$$

for all $t \in (t_k, t_{k+1}]$ with

$$\bar\alpha := (\bar\Lambda+1)\bar\Gamma, \qquad \bar\beta := \bar\Lambda^2, \qquad \bar\gamma := (\bar\Lambda+1)\bar\Phi.$$

From $\bar\beta = \bar\Lambda^2 \ge 1$ it follows that $\hat D_{k+1} \ge \hat D_k$. In order to dominate the growth rate of $\hat D_{k+1}$, we set

$$\begin{aligned}
E_{k+1} &= \hat G(x_k^*, (1+\varepsilon_E)E_k, \delta_k)\\
&:= \bar\alpha\|x_k^*\| + (1+\varepsilon_E)\bar\beta E_k + \bar\gamma\delta_k
\end{aligned} \tag{33}$$

with the arbitrary design parameter $\varepsilon_E > 0$.

2) Recovery in a finite time: Suppose that the state escapes at a sampling time t_j (or it is lost at $t_j = t_0 = 0$) and is lost at $t_{j+1}, \dots, t_{k-1}$. From the recursive formulas (32) and (33) it follows that

$$\begin{aligned} \hat{D}_k &= \bar{\beta}^{k-j}\hat{D}_j + \frac{\bar{\beta}^{k-j}-1}{\bar{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_d), \\ E_k &= \hat{\beta}^{k-j}E_j + \frac{\hat{\beta}^{k-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j) \end{aligned} \tag{34}$$

with[2]

$$\hat{\beta} := (1+\varepsilon_E)\bar{\beta} > \bar{\beta}.$$

Let $c_\beta := (\hat{\beta}-1)/(\bar{\beta}-1)$ and consider the integer-valued functions $\eta_E, \eta_\delta : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$\begin{aligned} \eta_E(s) &:= \begin{cases} \lceil \log_{1+\varepsilon_E} s \rceil, & s > 1; \\ 0, & 0 \leq s \leq 1, \end{cases} \\ \eta_\delta(s) &:= \begin{cases} \lceil \log_{1+\varepsilon_E} (c_\beta s) \rceil, & s > 1; \\ 0, & 0 \leq s \leq 1, \end{cases} \end{aligned} \tag{35}$$

where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function, that is, $\lceil s \rceil := \min\{m \in \mathbb{Z} : m \geq s\}$. Consider the integer

$$k' := j + \max\{\eta_E(\hat{D}_j/E_j), \eta_\delta(\delta_d/\delta_j)\}.$$

First, it holds that

$$\hat{\beta}^{k'-j}E_j \geq \bar{\beta}^{k'-j}(1+\varepsilon_E)^{\eta_E(\hat{D}_j/E_j)}E_j \geq \bar{\beta}^{k'-j}\hat{D}_j.$$

Second, if $\delta_d \leq \delta_j$ then

$$\frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j \geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d$$

due to $\hat{\beta} > \bar{\beta}$ and $k' \geq j$; otherwise

$$\begin{aligned} \frac{\hat{\beta}^{k'-j}-1}{\hat{\beta}-1}\delta_j &= \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}\frac{(1+\varepsilon_E)^{k'-j}\bar{\beta}^{k'-j}-1}{\bar{\beta}^{k'-j}-1}\delta_j \\ &> \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}(1+\varepsilon_E)^{k'-j}\delta_j \\ &\geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\frac{\bar{\beta}-1}{\hat{\beta}-1}(1+\varepsilon_E)^{\eta_\delta(\delta_d/\delta_j)}\delta_j \\ &\geq \frac{\bar{\beta}^{k'-j}-1}{\bar{\beta}-1}\delta_d. \end{aligned}$$

Hence $E_{k'} \geq \hat{D}_{k'}$, that is, the state is recovered no later than $t_{k'}$. Let t_i denote the sampling time of recovery. Then[3]

$$i - j \leq \max\{\eta_E(\hat{D}_j/E_j), \eta_\delta(\delta_d/\delta_j)\}. \tag{36}$$

(However, δ_d being unknown implies that neither the sensor nor the controller is able to predict how long it will take to recover the state.)

VI. Stability analysis

In this section we show that the communication and control strategy fulfills the claim of Theorem 1. We first derive a bound with exponential decay for each stabilizing stage in Section VI-A, and then a bound with exponential growth for each searching stage in Section VI-B. In Section VI-C we calculate the maximum number of searching stages and establish the exponential decay property in Theorem 1. A practical stability property is established in Section VI-D.

A. Stabilizing stage

1) Sampling interval with no switch: Consider a sampling interval $[t_k, t_{k+1}]$ such that (17) holds, as in Section V-A1. As $A_p + B_pK_p$ is Hurwitz, for S_p in (20) there exist positive definite matrices $P_p, Q_p \in \mathbb{R}^{n_x \times n_x}$ such that

$$S_p^\top P_p S_p - P_p = -Q_p < 0. \tag{37}$$

Let $\overline{\lambda}(M)$ and $\underline{\lambda}(M)$ denote the largest and smallest eigenvalues of any matrix M, respectively, and define

$$\chi_p := \frac{2n_x^2\|S_p^\top P_p S_p\|^2}{\underline{\lambda}(Q_p)} + n_x\|S_p^\top P_p S_p\|. \tag{38}$$

Due to the inequality in (5), there exists a sufficiently small $\phi_1 > 0$ such that $(1+\phi_1)\Lambda_{p'}^2 < N^2$ for all $p' \in \mathcal{P}$. Then for each p' there exists a sufficiently large $\rho_{p'} > 0$ such that

$$\frac{(N-1)^2}{N^2}\frac{\chi_{p'}}{\rho_{p'}} + \frac{(1+\phi_1)\Lambda_{p'}^2}{N^2} < 1. \tag{39}$$

Consider a family of positive definite functions $V_{p'} : \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$V_{p'}(x, E) := x^\top P_{p'} x + \rho_{p'} E^2, \qquad p' \in \mathcal{P}. \tag{40}$$

For the sampling interval $[t_k, t_{k+1}]$ with no switch, the following lemma provides a bound for $V_{\sigma(t_{k+1})}(x_{k+1}^*, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x_k^*, E_k)$ and δ_k:

Lemma 2. *Consider a sampling interval $[t_k, t_{k+1}]$ such that (8) and (17) hold. Then*

$$V_p(x_{k+1}^*, E_{k+1}) \leq \nu V_p(x_k^*, E_k) + \nu_d \delta_k^2 \tag{41}$$

with[4]

$$\begin{aligned} \nu &:= \max_{p \in \mathcal{P}} \nu_p, \\ \nu_p &:= \max\left\{\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p} + \frac{(1+\phi_1)\Lambda_p^2}{N^2}, 1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right\}, \\ \nu_d &:= \max_{p \in \mathcal{P}} \left(1 + \frac{1}{\phi_1}\right)\rho_p \Phi_p(\tau_s)^2. \end{aligned} \tag{42}$$

Proof. See Appendix B. □

[2]From (31) it follows that $\bar{\beta} = \bar{\Lambda}^2 \geq 1$, and $\bar{\Lambda} = 1$ only if all eigenvalues of all A_p have nonpositive real parts. In the following analysis we assume that $\bar{\beta} > 1$ (so that the first formula in (34) is well-defined), which can be achieved by letting $\bar{\beta} = \max\{\bar{\Lambda}^2, 1+\varepsilon\}$ for any $\varepsilon > 0$ if necessary. The special case where $\bar{\beta} = 1$ can be treated via similar arguments and is omitted here for brevity.

[3]The function η_δ is piecewise-defined since if $\delta_d \leq \delta_j$ in (36)—which is possible as the escape only implies $\delta_d > \delta_{j-1} = \delta_j/(1+\varepsilon_\delta)$—then the second term on the right-hand side of the second formula in (34) is larger than or equal to that of the first formula for all $k \geq j$. Similarly, the function η_E is piecewise-defined since if $\hat{D}_0 = \|x_0\| \leq E_0$ in (55) below then there is no searching stage at the beginning.

[4]The denominator in the second term of the maximum in the definition of ν_p in (42) is reduced to $1/n_x$ of the corresponding term in [21, eq. (34)]. This improvement is due to the more suitable inequalities (65) and (66) from linear algebra. The first numerator in the first term of the maximum in the definition of μ_{pq} in (44) is reduced to $1/n_x$ of the corresponding term in [21, eq. (37)] for the same reason.

2) Sampling interval with a switch: Consider a sampling interval $[t_k, t_{k+1}]$ such that (21) holds, as in Section V-A2. Let h_{pq} be the largest singular value of H_{pq} in (27), that is,

$$h_{pq} := \sqrt{\overline{\lambda}(H_{pq}^\top H_{pq})}.$$

Consider the functions V_p, V_q defined by (40). For the sampling interval $[t_k, t_{k+1}]$ with a switch, the following lemma provides a bound for $V_{\sigma(t_{k+1})}(x_{k+1}^*, E_{k+1})$ in terms of $V_{\sigma(t_k)}(x_k^*, E_k)$ and δ_k:

Lemma 3. *Consider a sampling interval $[t_k, t_{k+1}]$ such that* (8) *and* (21) *hold. Then*

$$V_q(x_{k+1}^*, E_{k+1}) \leq \mu V_p(x_k^*, E_k) + \mu_d \delta_k^2 \tag{43}$$

with

$$\begin{aligned} \mu &:= \max_{p,q\in\mathcal{P}} \mu_{pq}, \\ \mu_{pq} &:= \max\left\{ \frac{2\overline{\lambda}(P_q)h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2\rho_q}{\underline{\lambda}(P_p)}, \right. \\ &\qquad \left. \frac{(N-1)^2}{N^2}\frac{2n_x\overline{\lambda}(P_q)h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2\rho_q}{\rho_p} \right\}, \\ \mu_d &:= \max_{p,q\in\mathcal{P}} \left(1 + \frac{2}{\phi_2}\right)\rho_q\gamma_{pq}^2, \end{aligned} \tag{44}$$

where $\phi_2 > 0$ is an arbitrary design parameter.

Proof. See Appendix C. □

Remark 3. From the definition of ν in (42) and the inequality (39) it follows that $\nu < 1$. Meanwhile, if we set $t' = t'' = 0$ in (27) then $h_{pq} = 1$ for all $p, q \in \mathcal{P}$, and hence from the definition of μ in (44) it follows that

$$\mu = \max_{p,q\in\mathcal{P}} \mu_{pq} > \max_{p,q\in\mathcal{P}} \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} \geq 1 > \nu.$$

While this may not hold for general $t', t'' \in [0, \tau_s]$, we are able to ensure

$$\mu > \nu \tag{45}$$

by letting $\mu = 1$ if all $\mu_{pq} < 1$. Meanwhile, the relation between μ_d and ν_d depends on the values of ϕ_1 and ϕ_2. Since (43) holds for all $\phi_2 > 0$, for each ϕ_1 a small enough ϕ_2 (e.g., $\phi_2 = \phi_1$) can be selected so that

$$\mu_d \geq \nu_d. \tag{46}$$

(Alternatively, we can simply replace μ_d with $\max\{\mu_d, \nu_d\}$ if necessary.) In the following analysis we assume the inequalities (45) and (46) hold. Consequently, the bound in (43) holds for all sampling intervals in stabilizing stages regardless of whether there is a switch.

3) Combined bound at sampling times: Now we derive a lower bound on the average dwell-time τ_a in (2) that ensures a bound of $V_{\sigma(t_k)}(x_k^*, E_k)$ with exponential decay at each sampling times t_k in a stabilizing stage.

Lemma 4. *Consider a sequence of consecutive sampling times $t_i, \ldots, t_{k-1}$ in a stabilizing stage. Suppose that the average dwell-time τ_a and the sampling period τ_s satisfy*

$$\tau_a > \left(1 + \frac{\ln \mu}{\ln(1/\nu)}\right)\tau_s. \tag{47}$$

Then there exists a sufficiently small $\phi_3 \in (0, 1)$ such that

$$V_{\sigma(t_k)}(x_k^*, E_k) < \Theta^{N_0}\left(\theta^{k-i}V_{\sigma(t_i)}(x_i^*, E_i) + \Theta_d\delta_i^2\right) \tag{48}$$

with the integer N_0 in (2) *and*

$$\begin{aligned} \theta &:= \frac{(\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{\tau_s/\tau_a}}{(\nu + \phi_3(1-\nu))^{\tau_s/\tau_a - 1}} < 1, \\ \Theta &:= \frac{\mu + \phi_3(1-\nu)\mu_d/\nu_d}{\nu + \phi_3(1-\nu)} > 1, \\ \Theta_d &:= \frac{\mu}{\phi_3(1-\nu)}\nu_d + \mu_d. \end{aligned} \tag{49}$$

Proof. See Appendix D. □

Remark 4. In [14], the authors considered a switched linear system with input (disturbance) and derived a lower bound on the average dwell-time that ensured a variant of ISS with exponential decay.[5] The lower bound (47) on the average dwell-time τ_a in Lemma 4, in the absence of sampling and quantization, is consistent with the one in [14, Th. 2]. More specifically, the case without sampling and quantization can be approximated by letting $\tau_s \to 0$ and $N \to \infty$. Consequently, $S_p \to I + (A_p + B_pK_p)\tau_s$ in (20), $H_{pq} \to I$ in (27), $\alpha_{pq}, \beta_{pq} \to 0$ in (29), and hence

$$\nu \to 1 - \min_{p\in\mathcal{P}} \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}, \qquad \mu \to \max_{p,q\in\mathcal{P}} \frac{2\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)}$$

in (42) and (44) with large enough ρ_p for all $p \in \mathcal{P}$. Moreover, the first order approximation in τ_s of the Lyapunov equation (37) is given by

$$\big((A_p + B_pK_p)^\top P_p + P_p(A_p + B_pK_p)\big)\tau_s = -Q_p.$$

As the index set $\mathcal{P}$ is finite, from Assumption 2 it follows that there exists a constant $\lambda_0 > 0$ such that all $A_p + B_pK_p + \lambda_0 I$ are Hurwitz, and hence the (approximated) Lyapunov equation above holds with P_p satisfying

$$(A_p + B_pK_p + \lambda_0 I)^\top P_p + P_p(A_p + B_pK_p + \lambda_0 I) = -I,$$

and $Q_p = (2\lambda_0 P_p + I)\tau_s$. Then (47) can be approximated by

$$\tau_a > \frac{\ln(2\mu^*)}{\min_{p\in\mathcal{P}}\left(\frac{\underline{\lambda}(P_p)}{2\overline{\lambda}(P_p)}2\lambda_0 + \frac{1}{2\overline{\lambda}(P_p)}\right)}$$

with

$$\mu^* := \max_{p,q\in\mathcal{P},\ p\neq q} \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)},$$

which is in a similar form as the lower bound

$$\tau_a \geq \tau_a^* > \frac{\ln \mu^*}{2\lambda_0}$$

in [14] (see [5, p. 59, eq. (3.10)] for an explicit bound on τ_a^*). The additional terms result from the more complex Lyapunov functions (40) we used due to the sampling and quantization. In particular, the additional coefficients in the numerator and the first term of the denominator are generated when completing the squares. Meanwhile, we can made $\overline{\lambda}(P_p)$ arbitrarily large (and hence the second term of the denominator arbitrarily small) by selecting a sufficiently small λ_0.

[5] More precisely, the result in [14] is stated in terms of "input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_\infty$-induced norm", which ensures an exponential decay rate.

B. Searching stage

1) Recovery: Suppose that the state escapes at a sampling time t_j and is recovered at t_i, as in Section V-B. At t_j, from (13) and (14) it follows that

$$E_j < \|x(t_j) - x_j^*\| \le \hat{D}_j$$

with

$$\begin{aligned}\hat{D}_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_d),\\ E_j &= G(\sigma(t_{j-1}), \sigma(t_j), x_{j-1}^*, E_{j-1}, \delta_{j-1}).\end{aligned}$$

From the formulas (19) and (26) of G we see that

$$\hat{D}_j/E_j < \delta_d/\delta_{j-1} = (1+\varepsilon_\delta)\delta_d/\delta_j.$$

Let $c_\varepsilon := \max\{1+\varepsilon_\delta, (\hat{\beta}-1)/(\bar{\beta}-1)\}$ and define an integer-valued function $\eta : \mathbb{R}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ by

$$\begin{aligned}\eta(s) &:= \max\{\eta_E((1+\varepsilon_\delta)s), \eta_\delta(s)\}\\ &= \begin{cases} \lceil \log_{1+\varepsilon_E}(c_\varepsilon s)\rceil, & s > 1;\\ \lceil \log_{1+\varepsilon_E}((1+\varepsilon_\delta)s)\rceil, & (1+\varepsilon_\delta)^{-1} < s \le 1;\\ 0, & 0 \le s \le (1+\varepsilon_\delta)^{-1}.\end{cases}\end{aligned} \tag{50}$$

Then from (36) it follows that

$$i - j \le \eta(\delta_d/\delta_j), \tag{51}$$

which, combined with (34), implies that

$$\begin{aligned}E_i &= \hat{\beta}^{i-j}E_j + \frac{\hat{\beta}^{i-j}-1}{\hat{\beta}-1}(\bar{\alpha}\|x_j^*\| + \bar{\gamma}\delta_j)\\ &< \hat{\beta}^{i-j}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right)\\ &< \hat{\beta}^{\eta(\delta_d/\delta_j)}\left(\frac{\bar{\alpha}}{\hat{\beta}-1}\|x_j^*\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_j\right).\end{aligned} \tag{52}$$

For the searching stage $[t_j, t_i)$, the following lemma provides a bound for $V_{\sigma(t_i)}(x_i^*, E_i)$ at the recovery in terms of $V_{\sigma(t_j)}(x_j^*, E_j)$ at the escape and δ_d, δ_j:

Lemma 5. *Suppose that the state escapes at a sampling time t_j and is recovered at t_i. Then*

$$V_{\sigma(t_i)}(x_i^*, E_i) \le \hat{\beta}^{2\eta(\delta_d/\delta_j)}\big(\omega V_{\sigma(t_j)}(x_j^*, E_j) + \omega_d \delta_j^2\big) \tag{53}$$

with

$$\begin{aligned}\omega &:= \max_{p,q\in\mathcal{P}} \omega_{pq},\\ \omega_{pq} &:= \max\left\{\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2\rho_q}{(\hat{\beta}-1)^2\underline{\lambda}(P_p)}, \frac{(2+\phi_4)\rho_q}{\rho_p}\right\},\\ \omega_d &:= \max_{q\in\mathcal{P}}\left(1+\frac{2}{\phi_4}\right)\frac{\bar{\gamma}^2\rho_q}{(\hat{\beta}-1)^2},\end{aligned} \tag{54}$$

where $\phi_4 > 0$ is an arbitrary design parameter.

Proof. See Appendix E. □

2) Initial capture: The case where the state is lost at $t_0 = 0$ and is recovered at t_{i_0} for the first time can be analyzed in a similar manner. From (36) with $j = 0$ and $\hat{D}_0 = \|x_0\|$ it follows that

$$i_0 \le \eta_E(\|x_0\|/E_0) + \eta_\delta(\delta_d/\delta_0), \tag{55}$$

which, combined with (34) and $x_0^* = 0$, implies that

$$E_{i_0} < \hat{\beta}^{\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0)}\left(E_0 + \frac{\bar{\gamma}}{\hat{\beta}-1}\delta_0\right).$$

For the searching stage $[0, t_{i_0})$, the following lemma provides a bound for $V_{\sigma(t_{i_0})}(0, E_{i_0})$ at the first recovery in terms of $V_{\sigma(0)}(0, E_0) = \rho_{\sigma(0)}E_0^2$ at $t = 0$ and $\|x_0\|, E_0, \delta_d, \delta_0$:

Lemma 6. *Suppose that the state is lost at $t_0 = 0$ and is recovered at t_{i_0}. Then*

$$\begin{aligned}V_{\sigma(t_{i_0})}(0, E_{i_0}) \le{}& \hat{\beta}^{2(\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0))}\\ &\times \big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d\delta_0^2\big)\end{aligned} \tag{56}$$

with

$$\omega_0 := \max_{q\in\mathcal{P}}\left(1+\frac{\phi_4}{2}\right)\frac{\rho_q}{\rho_{\sigma(0)}} \le \frac{1}{2}\omega,$$

where ϕ_4 is the design parameter in (54).

Proof. The proof is essentially the same as the one of Lemma 5 and is omitted here. □

C. Exponential decay

1) Number of searching stages: As explained in Section IV-D, the closed-loop system alternates between a finite number of searching and stabilizing stages, and eventually stays in a stabilizing stage. Let $0 = j_0 \le i_0 < j_1 < i_1 < \cdots < j_{N_s} < i_{N_s}$ be such that $[t_{j_m}, t_{i_m})$ denotes a searching stage and $[t_{i_m}, t_{j_{m+1}})$ a stabilizing stage for each $m \in \{0, \dots, N_s\}$.[6] As the estimate is enlarged by a factor of $1+\varepsilon_\delta$ every time the state escapes, it satisfies

$$\delta_k = (1+\varepsilon_\delta)^m\delta_0 \qquad \forall\, k \in \{j_m, \dots, j_{m+1}-1\}.$$

Hence $N_s \le N_d(\delta_d)$ with the integer-valued function $N_d : \mathbb{R}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ defined by

$$N_d(s) := \begin{cases}\lceil \log_{1+\varepsilon_\delta}(s/\delta_0)\rceil, & s > \delta_0;\\ 0, & 0 \le s \le \delta_0.\end{cases} \tag{57}$$

2) Global bound at sampling times: Combining the bound in Lemma 4 for stabilizing stages and the ones in Lemmas 5, 6 for searching stages, we establish a global bound for $V_{\sigma(t_k)}(x_k^*, E_k)$ in stabilizing stages in terms of $\rho_{\sigma(0)}$ at $t = 0$ and $\|x_0\|, E_0, \delta_d, \delta_0$:

Lemma 7. *Consider a sampling time t_k such that* (8) *holds. Then*

$$\begin{aligned}V_{\sigma(t_k)}(x_k^*, E_k) \le{}& \Theta^{N_0}\Psi^{N_d(\delta_d)}\psi^{2L_d(\delta_d)}\big(\theta^k\psi^{2L_x(\|x_0\|)}\\ &\times(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2) + C_d(\delta_d)\delta_0^2\big)\end{aligned}$$

[6] There is a searching stage at the beginning (i.e., $i_0 > 0$) if and only if $\|x_0\| > E_0$; for the final stabilizing stage to be well-defined we let $j_{N_s+1} := \infty$ and $t_{j_{N_s+1}} := \infty$.

with the functions $L_x, L_d : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ *and* $C_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ *defined by*[7]

$$L_x(s) := \eta_E(s/E_0),$$
$$L_d(s) := \eta_\delta(s/\delta_0) + \sum\nolimits_{l=1}^{N_d(s)} \eta((1+\varepsilon_\delta)^{-l} s/\delta_0),$$
$$C_d(s) := \Theta_d + (\Theta_d + \omega_d) \sum\nolimits_{l=1}^{N_d(s)} \psi_d^l,$$

and the constants

$$\psi := \hat{\beta}\theta^{-1/2}, \qquad \Psi := \omega\Theta^{N_0}, \qquad \psi_d := (1+\varepsilon_\delta)^2/\Psi,$$

where η_E, η_δ *are defined by* (35), η *by* (50) *and* N_d *by* (57).

Proof. See Appendix F. □

Remark 5. The gain functions N_d, L_x, L_d, C_d in Lemma 7 are piecewise constant, and satisfy $L_x(s) = 0$ for all $0 \leq s \leq E_0$ and $N_d(s) = L_d(s) = 0$ for all $0 \leq s \leq \delta_0$. A more conservative bound that depends continuously on $\|x_0\|$ and δ_d can be established by replacing them with continuous, strictly increasing gain functions. First, $N_d(s) \leq \bar{N}_d(s)$ for all $s \geq 0$ with $\bar{N}_d \in \mathcal{K}_\infty$ defined by

$$\bar{N}_d(s) := \begin{cases} 1 + \log_{1+\varepsilon_\delta}(s/\delta_0), & s > \delta_0; \\ s/\delta_0, & 0 \leq s \leq \delta_0. \end{cases}$$

Second, $L_x(s) \leq \bar{L}_x(s)$ for all $s \geq 0$ with $\bar{L}_x \in \mathcal{K}_\infty$ defined by

$$\bar{L}_x(s) := \begin{cases} 1 + \log_{1+\varepsilon_E}(s/E_0), & s > E_0; \\ s/E_0, & 0 \leq s \leq E_0. \end{cases}$$

Third, $L_d(s) \leq \bar{L}_d(s)$ for all $s \geq 0$ with $\bar{L}_d \in \mathcal{K}_\infty$ defined by

$$\begin{aligned}\bar{L}_d(s) := &\log_{1+\varepsilon_E}(c_\beta s/\delta_0) + (\bar{N}_d(s) - 1)\log_{1+\varepsilon_E}(c_\varepsilon s/\delta_0) \\ &+ \log_{1+\varepsilon_E}(s/\delta_0) + \bar{N}_d(s) + 1 \\ &- \big(\bar{N}_d(s)(\bar{N}_d(s)+1)/2 - 1\big)\log_{1+\varepsilon_E}(1+\varepsilon_\delta)\end{aligned}$$

for $\delta_d > \delta_0$; and

$$\bar{L}_d(s) := (2 + \log_{1+\varepsilon_E} c_\beta)s/\delta_0$$

for $0 \leq \delta_d \leq \delta_0$. Finally, $C_d(s) \leq \bar{C}_d(s)$ for all $s \geq 0$ with the continuous, strictly increasing function $\bar{C}_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ defined by[8]

$$\bar{C}_d(s) := \Theta_d + \frac{1 - \psi_d^{\bar{N}_d(s)}}{1 - \psi_d}\psi_d(\Theta_d + \omega_d).$$

3) Intersample bound: First, consider an arbitrary t in a stabilizing stage, that is, a $t \in [t_k, t_{k+1}]$ such that (8) holds. Following similar arguments as in Section V-A2 with $t' = t'' = 0$, we replace $x(t)$ with c_k in (9), the center of the hypercubic box containing $x(t_k)$, via the triangle inequality. If there is no switch on $(t_k, t]$ then (22) holds with $t - t_k$ in place of $\bar{t}$, and hence

$$\|x(t) - c_k\| = \|x(t) - \hat{x}(t_k)\| \leq \hat{D}'_{k+1}(0, t - t_k);$$

otherwise, there is exactly one switch on $(t_k, t]$ due to (4), then (25) holds with t in place of t_{k+1}^- (and $\bar{t} \in (0, t - t_k]$ denoting the unknown switching time), and hence

$$\|x(t) - c_k\| \leq \|z(t) - \hat{z}(t_k)\| \leq \hat{D}''_{k+1}(0, 0, t - t_k),$$

where the first inequality follows from (24). Comparing the corresponding coefficients in (22), (25), (29) and (30), we see straightforwardly that in both cases

$$\|x(t) - c_k\| \leq \alpha^0\|x_k^*\| + \beta^0 E_k + \gamma\delta_d$$

with

$$\alpha^0 := \max_{p,q\in\mathcal{P}} \alpha_{pq}^0, \ \beta^0 := \max_{p,q\in\mathcal{P}} \beta_{pq}^0, \ \gamma := \max_{p,q\in\mathcal{P}} \gamma_{pq}. \tag{58}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}\|x(t)\| &\leq \|c_k\| + \|x(t) - c_k\| \\ &\leq (\alpha^0 + 1)\|x_k^*\| + \left(\beta^0 + \frac{N-1}{N}\right)E_k + \gamma\delta_d,\end{aligned}$$

where the second inequality follows from (9). Let

$$\begin{aligned}\lambda_{\min}^P &:= \min_{p\in\mathcal{P}} \underline{\lambda}(P_p), & \lambda^P &:= \max_{p\in\mathcal{P}} \overline{\lambda}(P_p), \\ \rho_{\min} &:= \min_{p\in\mathcal{P}} \rho_p, & \rho &:= \max_{p\in\mathcal{P}} \rho_p,\end{aligned} \tag{59}$$

and define

$$\xi := \frac{\lambda_{\min}^P}{\rho_{\min}} \frac{(\beta^0 + 1 - 1/N)^2}{(\alpha^0 + 1)^2},$$
$$\Xi := \sqrt{\frac{(\alpha^0+1)^2}{\lambda_{\min}^P} + \frac{(\beta^0 + 1 - 1/N)^2}{\rho_{\min}}}.$$

Then from Young's inequality with ξ it follows that[9]

$$\begin{aligned}&\left((\alpha^0+1)\|x_k^*\| + \left(\beta^0 + \frac{N-1}{N}\right)E_k\right)^2 \\ &\leq (1+\xi)(\alpha^0+1)^2\|x_k^*\|^2 + \left(1 + \frac{1}{\xi}\right)\left(\beta^0 + \frac{N-1}{N}\right)^2 E_k^2 \\ &= \Xi^2(\lambda_{\min}^P\|x_k^*\|^2 + \rho_{\min}E_k^2) \\ &\leq \Xi^2 V_{\sigma(t_k)}(x_k^*, E_k).\end{aligned}$$

Hence

$$\|x(t)\| \leq \Xi\sqrt{V_{\sigma(t_k)}(x_k^*, E_k)} + \gamma\delta_d$$

which, combined with Lemma 7 and Remark 5, implies that

$$\begin{aligned}\|x(t)\| \leq \Xi\Theta^{N_0/2}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(&\theta^{k/2}\psi^{\bar{L}_x(\|x_0\|)} \\ &\times(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)}\delta_0\Big) + \gamma\delta_d,\end{aligned}$$

where the last inequality follows partially from the property

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \qquad \forall\, a, \forall\, b \geq 0. \tag{60}$$

Moreover, due to $t \in [t_k, t_{k+1}]$ and $\theta < 1$ it holds that

$$\theta^{k/2} \leq \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

[7] The sum $L_x(\|x_0\|) + L_d(\delta_d)$ gives a bound for the total length of all searching stages (in terms of sampling intervals).

[8] For $\bar{C}_d$ to be well-defined the design parameter ε_δ should be selected so that $\psi_d \neq 1$. The special case where $\psi_d = 1$ can be treated via similar arguments and is omitted here for brevity (cf. footnote 2).

[9] For an $\varepsilon > 0$, Young's inequality with ε states that $ab \leq \varepsilon a^2/2 + b^2/(2\varepsilon)$ for all $a, b \in \mathbb{R}$. When $\varepsilon = 1$, the term "with ε" is omitted for brevity.

Second, consider an arbitrary $t \in [t_{j_m}, t_{i_m})$. From (32) it follows that

$$\|x(t) - x^*_{j_m}\| \le \hat{D}_{i_m} \le E_{i_m}.$$

Following similar arguments as in the first case, we obtain

$$\begin{aligned}\|x(t)\| &\le \|x^*_{j_m}\| + E_{i_m} \\ &\le (\alpha^0 + 1)\|x^*_{i_m}\| + \left(\beta^0 + \frac{N-1}{N}\right)E_{i_m} \\ &\le \Xi\sqrt{V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m})} \\ &\le \Xi\Theta^{N_0/2}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(\theta^{i_m/2}\psi^{\bar{L}_x(\|x_0\|)} \\ &\quad \times (\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)}\delta_0\Big) + \gamma\delta_d,\end{aligned}$$

in which

$$\theta^{i_m/2} \le \theta^{t/(2\tau_s)} < \theta^{-1/2}\theta^{t/(2\tau_s)}.$$

Combining the results above, we obtain

$$\begin{aligned}\|x(t)\| \le{}& \frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\Big(\theta^{t/(2\tau_s)}\psi^{\bar{L}_x(\|x_0\|)} \\ & \times (\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) + \sqrt{\bar{C}_d(\delta_d)}\delta_0\Big) + \gamma\delta_d\end{aligned} \tag{61}$$

for all $t \ge 0$. From Young's inequality with an arbitrary design parameter $\phi > 0$ it follows that[10]

$$\begin{aligned}\|x(t)\| \le{}& \left(\frac{1}{2\phi}\psi^{2\bar{L}_x(\|x_0\|)} + \frac{\phi}{2}\Psi^{\bar{N}_d(\delta_d)}\psi^{2\bar{L}_d(\delta_d)}\right)\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}} \\ & \times \theta^{t/(2\tau_s)}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0) \\ & + \Xi\Theta^{N_0/2}\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)}\sqrt{\bar{C}_d(\delta_d)}\delta_0 + \gamma\delta_d \\ ={}& \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar{L}_x(\|x_0\|)}\theta^{t/(2\tau_s)} \\ & + \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar{N}_d(\delta_d)}\psi^{2\bar{L}_d(\delta_d)} \\ & + \Xi\Theta^{N_0/2}\sqrt{\bar{C}_d(\delta_d)}\delta_0\Psi^{\bar{N}_d(\delta_d)/2}\psi^{\bar{L}_d(\delta_d)} + \gamma\delta_d\end{aligned}$$

for all $t \ge 0$. Hence (6) holds with the exponential decay rate

$$\lambda := -\frac{\ln\theta}{2\tau_s} > 0, \tag{62}$$

and the gain functions $g, h : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ defined by

$$\begin{aligned}g(s) &:= \frac{1}{2\phi}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\psi^{2\bar{L}_x(s)}, \\ h(s) &:= \frac{\phi}{2}\frac{\Xi\Theta^{N_0/2}}{\sqrt{\theta}}(\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0)\Psi^{\bar{N}_d(s)}\psi^{2\bar{L}_d(s)} \\ &\quad + \Xi\Theta^{N_0/2}\sqrt{\bar{C}_d(s)}\delta_0\Psi^{\bar{N}_d(s)/2}\psi^{\bar{L}_d(s)} + \gamma s.\end{aligned} \tag{63}$$

[10]Here Young's inequality is applied to restate the state bound (61) in the standard form (6) of ISS-type properties. However, besides increasing the value of the state bound, it also has the following consequence: If there is no disturbance and the sensor and the controller know that, then $\delta_d = \delta_0 = 0$, and hence (61) becomes $\|x(t)\| \le \Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0\theta^{t/(2\tau_s)}\psi^{\bar{L}_x(\|x_0\|)}$, that is, it reduces to a similar form as the one for the disturbance-free case [21, eq. (5)]. Meanwhile, (6) cannot be reduced to the same form since $h(\delta_d) = \phi\Xi\Theta^{N_0/2}\sqrt{\omega_0\rho/\theta}E_0/2 > 0$ even if $\delta_d = \delta_0 = 0$.

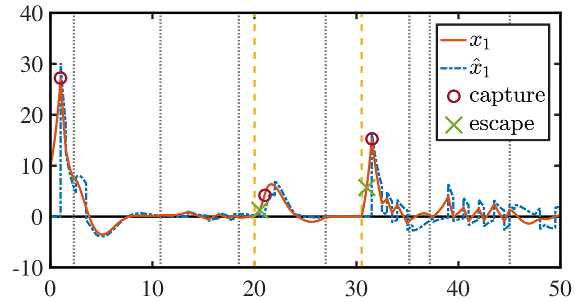

Fig. 2. Simulation example

D. Practical stability

As discussed in Section III, the bound (6) implies that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that (7) holds with

$$\begin{aligned}C &= g(0) + h(0) \\ &= \Xi\Theta^{N_0/2}\left(\left(\frac{1}{2\phi} + \frac{\phi}{2}\right)\frac{\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0}{\sqrt{\theta}} + \sqrt{\Theta_d}\delta_0\right) \\ &\ge \Xi\Theta^{N_0/2}\left(\frac{\sqrt{\omega_0\rho}E_0 + \sqrt{\omega_d}\delta_0}{\sqrt{\theta}} + \sqrt{\Theta_d}\delta_0\right).\end{aligned}$$

Based on similar arguments as in Sections V-A, VI-A3, VI-C3 and [21, Sec. 5.5], the state bound (7) can be established with a smaller C:

Proposition 2 (Practical stability)**.** *Suppose that the average dwell-time τ_a and the sampling period τ_s satisfy* (47)*. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that* (7) *holds with*

$$C = \Xi\Theta^{N_0/2}\sqrt{\Theta_d}\delta_0. \tag{64}$$

Proof. See Appendix G. □

Proposition 2 also improves the practical stability result in [23, Th. 1]. Moreover, from the proof it will be clear that the additional bound [23, eq. (39)] on the average dwell-time τ_a is actually not necessary for establishing practical stability.

VII. Simulation Example

Our communication and control strategy is simulated with the following data: $\mathcal{P} = \{1, 2\}$,

$$\begin{aligned}A_1 &:= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, & B_1 &= \begin{bmatrix}1\\0\end{bmatrix}, & K_1 &= \begin{bmatrix}-2 & 0\end{bmatrix}; \\ A_2 &:= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, & B_2 &= \begin{bmatrix}0\\1\end{bmatrix}, & K_2 &= \begin{bmatrix}0 & -1\end{bmatrix}.\end{aligned}$$

and $\tau_s = 0.5, N = 5, \tau_d = 1.05, \tau_a = 7.55, N_0 = 5$ so that the basic Assumptions 1–3 hold. We set $t' = t'' = 0$ in (26), $\varepsilon_E = 0.8$ in (15) and $\varepsilon_\delta = 1$ in (57). The disturbance d is kept 0 most of the time and turned on for two sampling intervals with the constant value 9 when the state becomes small. The initial estimate is $\delta_0 = 2$. Fig. 2 plots a typical behavior of the first component x_1 of the continuous state (in orange solid line) and the corresponding component $\hat{x}_1$ of the auxiliary state (in blue dash-dot line). Switching times are denoted by vertical gray dotted lines, and sampling times when the disturbance is turned on by vertical yellow dashed

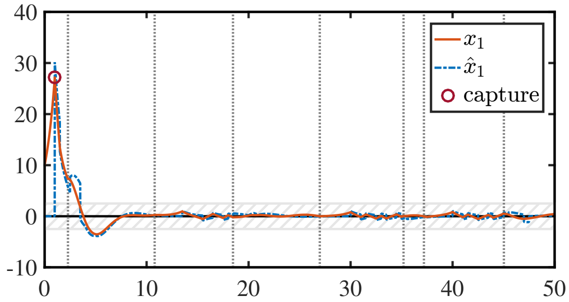

Fig. 3. With a constant estimate $\delta_k \equiv \delta_0$, the state $x \nrightarrow 0$ even if $d \equiv 0$.

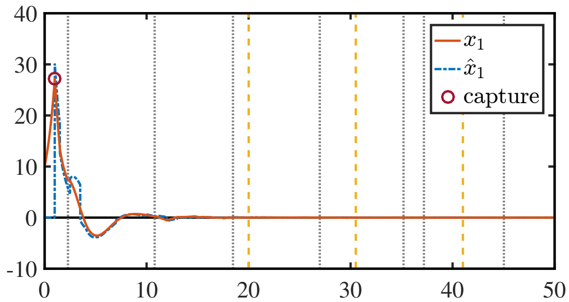

Fig. 4. With a converging estimate $\delta_k \to 0$, the state $x \to 0$ when $d \to 0$.

lines; captures are marked by red circles, and escapes by green crosses. Observe the searching stages at $t = 0$ (the state is lost due to $\|x_0\| > E_0$) and $t = 20.5$ and 31 (the state escapes due to the disturbance), and the nonsmooth behavior of x when $\hat{x}$ experiences a jump. The value of τ_a is empirically selected to be large enough to provide consistent convergence in simulations. For this example, the theoretical lower bound (47) on the average dwell-time τ_a is approximately 28.13, which is rather conservative.

Fig. 3 exhibits the cases where the unknown disturbance d is transient or $d \equiv 0$, so that once the state is captured it will never escape. Due to the nonzero initial estimate δ_0, the state x will converge to the set $\mathcal{A} = \{v \in \mathbb{R}^{n_x} : \|v\| \le h(0)\}$ (visualized by the shaded area) instead of the origin. Following essentially the idea of "zooming-in" from [6], we are able to make the state converge to the origin by halving the estimate δ_k every ten sampling intervals, as shown in Fig. 4. We conjecture that, for general disturbances, a similar modification to our communication and control strategy can be made to establish ISS with respect to the origin.

VIII. Concluding Remarks

A. Summary

In this paper, we studied the feedback stabilization of a switched linear system with a completely unknown disturbance under data-rate constraints. A finite data transmission rate was achieved via the sampling and quantization of state measurements. We extended the approach of reachable-set approximation and propagation from [21] by introducing an estimate of the disturbance bound to compensate the disturbance. A communication and control strategy was designed to achieve a variant of input-to-state stability with exponential decay, based on a novel algorithm for adjusting the estimate and recovering the state when it escapes the range of quantization.

B. Future research topics

As discussed in Section VII, we intend to advance our result via the "zooming-in" technique from [6] to establish ISS with respect to the origin. However, reducing the estimate in stabilizing stages can lead to an unbounded number of searching stages, and further work is required to establish convergence for the communication and control strategy.

For a non-switched linear control system, the minimum data rate for feedback stabilization [3], [4] is equal to the topological entropy [30] of the open-loop system. In the context of switched systems, neither the topological entropy nor the minimum data rate for feedback stabilization has been well-established. (See [31] for some initial results on entropy bounds of switched linear systems with Lie structure.) These two notions and their relation could become intriguing topics for future research.

Appendix A
Proof of Lemma 1

For all square matrices X, Y it holds that

$$\begin{aligned}
&\|e^{X+Y} - e^X\| \\
&= \left\| \sum_{m=0}^{\infty} \frac{1}{m!}((X+Y)^m - X^m) \right\| \\
&= \left\| \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{i=0}^{m} \binom{m}{i} X^{m-i} Y^i - X^m \right) \right\| \\
&= \left\| \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i=1}^{m} \binom{m}{i} X^{m-i} Y^i \right\| \\
&= \left\| \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{X^{m-1-j} Y^{j+1}}{j+1} \right\| \\
&\le \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \|X\|^{m-1-j} \|Y\|^{j+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} (\|X\| + \|Y\|)^l \|Y\| \\
&= e^{\|X\|+\|Y\|} \|Y\|.
\end{aligned}$$

Appendix B
Proof of Lemma 2

We first recall the following useful facts from linear algebra. From the definition of ∞-norm it follows that

$$\|v\|^2 \le v^\top v, \qquad v_1^\top v_2 \le n \|v_1\| \|v_2\| \tag{65}$$

for all vectors $v, v_1, v_2 \in \mathbb{R}^n$. Also,

$$\underline{\lambda}(S) \le \frac{v^\top S v}{v^\top v} \le \overline{\lambda}(S) \qquad \forall\, v \in \mathbb{R}^n \backslash \{0\} \tag{66}$$

for all symmetric matrices $S \in \mathbb{R}^{n \times n}$ (i.e., $S^\top = S$).

At t_{k+1}, following (17) we obtain

$$V_p(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_p x_{k+1}^* + \rho_p E_{k+1}^2$$

with x_{k+1}^* given by (20) and E_{k+1} by (19). First, (20) can be rewritten as

$$x_{k+1}^* = S_p c_k = S_p(x_k^* + \Delta_k)$$

with $\Delta_k := c_k - x_k^*$. Then

$$\begin{aligned}
&(x_{k+1}^*)^\top P_p x_{k+1}^* \\
&= (x_k^* + \Delta_k)^\top S_p^\top P_p S_p (x_k^* + \Delta_k) \\
&= (x_k^*)^\top S_p^\top P_p S_p x_k^* + 2(x_k^*)^\top S_p^\top P_p S_p \Delta_k + \Delta_k^\top S_p^\top P_p S_p \Delta_k \\
&\le (x_k^*)^\top (P_p - Q_p) x_k^* \\
&\quad + 2n_x \|x_k^*\| \|S_p^\top P_p S_p\| \|\Delta_k\| + n_x \|S_p^\top P_p S_p\| \|\Delta_k\|^2,
\end{aligned}$$

where the last inequality follows from (37) and (65). Moreover, (65) and (66) imply that

$$\begin{aligned}
(x_k^*)^\top Q_p x_k^* &\ge \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \ge \frac{\underline{\lambda}(Q_p)}{\overline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^*, \\
(x_k^*)^\top Q_p x_k^* &\ge \underline{\lambda}(Q_p)(x_k^*)^\top x_k^* \ge \underline{\lambda}(Q_p)\|x_k^*\|^2.
\end{aligned}$$

Combining the inequalities above and completing the square, we obtain

$$\begin{aligned}
&(x_{k+1}^*)^\top P_p x_{k+1}^* \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* - \frac{1}{2}\underline{\lambda}(Q_p)\|x_k^*\|^2 \\
&\quad + 2n_x \|x_k^*\| \|S_p^\top P_p S_p\| \|\Delta_k\| + n_x \|S_p^\top P_p S_p\| \|\Delta_k\|^2 \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \chi_p \|\Delta_k\|^2 \\
&\quad - \left(\sqrt{\frac{1}{2}\underline{\lambda}(Q_p)}\|x_k^*\| - \frac{\sqrt{2} n_x \|S_p^\top P_p S_p\|}{\sqrt{\underline{\lambda}(Q_p)}}\|\Delta_k\|\right)^2 \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2}\chi_p E_k^2,
\end{aligned}$$

where the last inequality follows partially from (9). Second, from (19) and Young's inequality with ϕ_1 it follows that

$$\begin{aligned}
E_{k+1}^2 &= \left(\frac{\Lambda_p}{N}E_k + \Phi_p(\tau_s)\delta_k\right)^2 \\
&\le \frac{(1+\phi_1)\Lambda_p^2}{N^2}E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\Phi_p(\tau_s)^2\delta_k^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V_p(x_{k+1}^*, E_{k+1}) \\
&\le \left(1 - \frac{\underline{\lambda}(Q_p)}{2\overline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* + \left(\frac{(N-1)^2}{N^2}\frac{\chi_p}{\rho_p}\right. \\
&\quad \left. + \frac{(1+\phi_1)\Lambda_p^2}{N^2}\right)\rho_p E_k^2 + \left(1 + \frac{1}{\phi_1}\right)\rho_p \Phi_p(\tau_s)^2 \delta_k^2,
\end{aligned}$$

which in turn implies (41).

Appendix C
Proof of Lemma 3

At t_{k+1}, following (21) we obtain

$$V_q(x_{k+1}^*, E_{k+1}) = (x_{k+1}^*)^\top P_q x_{k+1}^* + \rho_q E_{k+1}^2$$

with x_{k+1}^* given by (27) and E_{k+1} by (26). First, (27) can be rewritten as

$$x_{k+1}^* = H_{pq} c_k = H_{pq}(x_k^* + \Delta_k)$$

with $\Delta_k := c_k - x_k^*$. Then

$$\begin{aligned}
&(x_{k+1}^*)^\top P_q x_{k+1}^* \\
&= (x_k^* + \Delta_k)^\top H_{pq}^\top P_q H_{pq}(x_k^* + \Delta_k) \\
&\le \overline{\lambda}(P_q) h_{pq}^2 (x_k^* + \Delta_k)^\top (x_k^* + \Delta_k) \\
&\le \overline{\lambda}(P_q) h_{pq}^2 (2(x_k^*)^\top x_k^* + 2\Delta_k^\top \Delta_k) \\
&\le 2\overline{\lambda}(P_q) h_{pq}^2 (x_k^*)^\top x_k^* + 2n_x \overline{\lambda}(P_q) h_{pq}^2 \|\Delta_k\|^2 \\
&\le \frac{2\overline{\lambda}(P_q) h_{pq}^2}{\underline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^* + \frac{(N-1)^2}{N^2} 2n_x \overline{\lambda}(P_q) h_{pq}^2 E_k^2,
\end{aligned}$$

where the inequalities follows from (9), (65), (66) and Young's inequality. Second, from (28) and Young's inequality with ϕ_2 it follows that

$$\begin{aligned}
E_{k+1}^2 &\le (\alpha_{pq}\|x_k^*\| + \beta_{pq}E_k + \gamma_{pq}\delta_k)^2 \\
&\le (2+\phi_2)(\alpha_{pq}^2\|x_k^*\|^2 + \beta_{pq}^2 E_k^2) + \left(1 + \frac{2}{\phi_2}\right)\gamma_{pq}^2\delta_k^2
\end{aligned}$$

for every $\phi_2 > 0$, in which

$$\|x_k^*\|^2 \le (x_k^*)^\top x_k^* \le \frac{1}{\underline{\lambda}(P_p)}(x_k^*)^\top P_p x_k^*$$

due to (65) and (66). Therefore,

$$\begin{aligned}
&V_q(x_{k+1}^*, E_{k+1}) \\
&\le \left(\frac{2\overline{\lambda}(P_q)h_{pq}^2}{\underline{\lambda}(P_p)} + \frac{(2+\phi_2)\alpha_{pq}^2\rho_q}{\underline{\lambda}(P_p)}\right)(x_k^*)^\top P_p x_k^* \\
&\quad + \left(\frac{(N-1)^2}{N^2}\frac{2n_x\overline{\lambda}(P_q)h_{pq}^2}{\rho_p} + \frac{(2+\phi_2)\beta_{pq}^2\rho_q}{\rho_p}\right)\rho_p E_k^2 \\
&\quad + \left(1 + \frac{2}{\phi_2}\right)\rho_q\gamma_{pq}^2\delta_k^2,
\end{aligned}$$

which in turn implies (43).

Appendix D
Proof of Lemma 4

First, consider a function $\zeta : [0,1) \to \mathbb{R}$ defined by

$$\zeta(s) = 1 + \frac{\ln(\mu + s(1-\nu)\mu_d/\nu_d)}{\ln(1/(\nu + s(1-\nu)))}.$$

From (45) and (46) it follows that $\Theta > 1$ in (49), and that ζ is continuous and increasing. Moreover, as

$$\zeta(0) = 1 + \frac{\ln\mu}{\ln(1/\nu)} < \frac{\tau_a}{\tau_s}$$

due to (47), there exists a sufficiently small $\phi_3 \in (0,1)$ such that $\zeta(\phi_3) < \tau_a/\tau_s$, and hence $\theta < 1$ in (49).

The remaining proof follows in principle from the arguments in [17], [32]. If there exists an $l \in \{i, \ldots, k-1\}$ such that

$$V_{\sigma(t_l)}(x_l^*, E_l) > \frac{1}{\phi_3(1-\nu)}\nu_d\delta_l^2, \tag{67}$$

then (41) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\nu + \phi_3(1-\nu))V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) = \sigma(t_l)$; whereas (43) implies that

$$V_{\sigma(t_{l+1})}(x_{l+1}^*, E_{l+1}) < (\mu + \phi_3(1-\nu)\mu_d/\nu_d)V_{\sigma(t_l)}(x_l^*, E_l)$$

if $\sigma(t_{l+1}) \neq \sigma(t_l)$. Hence for two integers l', l'' such that $i \le l' < l'' \le k$ and that (67) holds for all $l \in \{l', \dots, l''-1\}$,

$$\begin{aligned}
&V_{\sigma(t_{l''})}(x^*_{l''}, E_{l''}) \\
&< (\mu + \phi_3(1-\nu)\mu_d/\nu_d)^{N_\sigma(t_{l''}, t_{l'})} \\
&\quad \times (\nu + \phi_3(1-\nu))^{l''-l'-N_\sigma(t_{l''}, t_{l'})} V_{\sigma(t_{l'})}(x^*_{l'}, E_{l'}) \\
&= (\nu + \phi_3(1-\nu))^{l''-l'} \Theta^{N_\sigma(t_{l''}, t_{l'})} V_{\sigma(t_{l'})}(x^*_{l'}, E_{l'}), \\
&< (\nu + \phi_3(1-\nu))^{l''-l'} \Theta^{N_0 + (l''-l')\tau_s/\tau_a} V_{\sigma(t_{l'})}(x^*_{l'}, E_{l'}) \\
&= \theta^{l''-l'} \Theta^{N_0} V_{\sigma(t_{l'})}(x^*_{l'}, E_{l'}),
\end{aligned}$$

where $N_\sigma(t_{l''}, t_{l'})$ denotes the number of switches on $(t_{l'}, t_{l''}]$, and the last inequality follows from $\Theta > 1$ and the ADT condition (2). Therefore, if (67) holds for all $l \in \{i, \dots, k-1\}$ then

$$V_{\sigma(t_k)}(x^*_k, E_k) < \theta^{k-i} \Theta^{N_0} V_{\sigma(t_i)}(x^*_i, E_i);$$

otherwise, for

$$k' := \max\left\{ l \le k-1 : V_{\sigma(t_l)}(x^*_l, E_l) \le \frac{1}{\phi_3(1-\nu)} \nu_d \delta_l^2 \right\}$$

it holds that

$$\begin{aligned}
V_{\sigma(t_{k'+1})}(x^*_{k'+1}, E_{k'+1}) &\le \mu V_{\sigma(t_{k'})}(x^*_{k'}, E_{k'}) + \mu_d \delta^2_{k'} \\
&\le \frac{\mu}{\phi_3(1-\nu)} \nu_d \delta^2_{k'} + \mu_d \delta^2_{k'} = \Theta_d \delta^2_{k'}
\end{aligned}$$

(see also Remark 3), and hence

$$\begin{aligned}
V_{\sigma(t_k)}(x^*_k, E_k) &< \theta^{k-k'-1} \Theta^{N_0} V_{\sigma(t_{k'+1})}(x^*_{k'+1}, E_{k'+1}) \\
&\le \Theta^{N_0} \Theta_d \delta^2_{k'}
\end{aligned}$$

as (67) holds for all $l \in \{k'+1, \dots, k-1\}$. The proof of Lemma 4 is completed by combining the bounds for the two cases and noticing that $\delta_l = \delta_i$ for all $l \in \{i, \dots, k-1\}$.

Appendix E
Proof of Lemma 5

Let $p, q \in \mathcal{P}$ denote the active modes at sampling times t_j, t_i, respectively. At the sampling time t_i of recovery we have

$$V_q(x^*_i, E_i) = (x^*_i)^\top P_q x^*_i + \rho_q E_i^2$$

with $x^*_i = x^*_j$ and E_i bounded by (52). First, from (66) it follows that

$$(x^*_i)^\top P_q x^*_i \le \frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} (x^*_j)^\top P_p x^*_j.$$

Second, following (52) and Young's inequality with ϕ_4 we obtain

$$\begin{aligned}
E_i^2 &\le \hat{\beta}^{2\eta(\delta_d/\delta_j)} \left(\frac{\bar{\alpha}}{\hat{\beta}-1} \|x^*_j\| + E_j + \frac{\bar{\gamma}}{\hat{\beta}-1} \delta_j \right)^2 \\
&\le \hat{\beta}^{2\eta(\delta_d/\delta_j)} \Bigg((2+\phi_4) \left(\frac{\bar{\alpha}^2}{(\hat{\beta}-1)^2} \|x^*_j\|^2 + E_j^2 \right) \\
&\quad + \left(1 + \frac{2}{\phi_4}\right) \frac{\bar{\gamma}^2}{(\hat{\beta}-1)^2} \delta_j^2 \Bigg)
\end{aligned}$$

for every $\phi_4 > 0$, in which

$$\|x^*_j\|^2 \le (x^*_j)^\top x^*_j \le \frac{1}{\underline{\lambda}(P_p)} (x^*_j)^\top P_p x^*_j$$

due to (65) and (66). Therefore,

$$\begin{aligned}
&V_q(x^*_i, E_i) \\
&\le \hat{\beta}^{2\eta(\delta_d/\delta_j)} \Bigg(\left(\frac{\overline{\lambda}(P_q)}{\underline{\lambda}(P_p)} + \frac{(2+\phi_4)\bar{\alpha}^2 \rho_q}{(\hat{\beta}-1)^2 \underline{\lambda}(P_p)} \right) (x^*_j)^\top P_p x^*_j \\
&\quad + \frac{(2+\phi_4)\rho_q}{\rho_p} \rho_p E_k^2 + \left(1 + \frac{2}{\phi_4}\right) \frac{\bar{\gamma}^2 \rho_q}{(\hat{\beta}-1)^2} \delta_j^2 \Bigg),
\end{aligned}$$

which in turn implies (53).

Appendix F
Proof of Lemma 7

Let $[t_{i_m}, t_{j_{m+1}})$ denote the stabilizing stage containing t_k, that is, $i_m \le k \le j_{m+1} - 1$. Substituting (48) with $i = i_m$ and $k = j_{m+1}$ into (53) with $j = j_{m+1}$ and $i = i_{m+1}$, we obtain

$$\begin{aligned}
&V_{\sigma(t_{i_{m+1}})}(x^*_{i_{m+1}}, E_{i_{m+1}}) \\
&\le \hat{\beta}^{2\eta(\delta_d/\delta_{j_{m+1}})} \big(\omega V_{\sigma(t_{j_{m+1}})}(x^*_{j_{m+1}}, E_{j_{m+1}}) + \omega_d \delta^2_{j_{m+1}} \big) \\
&< \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\omega \Theta^{N_0} \big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + \Theta_d (1+\varepsilon_\delta)^{2m} \delta_0^2 \big) + \omega_d (1+\varepsilon_\delta)^{2(m+1)} \delta_0^2 \big) \\
&= \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{j_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big),
\end{aligned}$$

in which

$$\theta^{j_{m+1}-i_m} \le \theta^{i_{m+1}-i_m} \theta^{-\eta(\delta_d/\delta_{j_{m+1}})}$$

due to (51) and $\theta < 1$. Hence

$$\begin{aligned}
&V_{\sigma(t_{i_{m+1}})}(x^*_{i_{m+1}}, E_{i_{m+1}}) \\
&< \Psi \hat{\beta}^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{i_{m+1}-i_m} \theta^{-\eta(\delta_d/\delta_{j_m})} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big) \\
&\le \Psi \psi^{2\eta(\delta_d/\delta_{i_{m+1}})} \big(\theta^{i_{m+1}-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2m} \delta_0^2 \big).
\end{aligned}$$

Based on this recursive bound, it is straightforward to derive that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) \\
&< \Psi \psi^{2\eta(\delta_d/\delta_{i_m})} \big(\theta^{i_m - i_{m-1}} \Psi \psi^{2\eta(\delta_d/\delta_{i_{m-1}})} \\
&\quad \times \big(\theta^{i_{m-1}-i_{m-2}} V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2(m-2)} \delta_0^2 \big) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\varepsilon_\delta)^{2(m-1)} \delta_0^2 \big) \\
&\le \Psi^2 \psi^{2(\eta(\delta_d/\delta_{i_m}) + \eta(\delta_d/\delta_{i_{m-1}}))} \\
&\quad \times \big(\theta^{i_m - i_{m-2}} V_{\sigma(t_{i_{m-2}})}(x^*_{i_{m-2}}, E_{i_{m-2}}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1+\psi_d)(1+\varepsilon_\delta)^{2(m-2)} \delta_0^2 \big) \\
&< \cdots \\
&< \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})} \big(\theta^{i_m - i_0} V_{\sigma(t_{i_0})}(0, E_{i_0}) \\
&\quad + (\Theta_d + \psi_d \omega_d)(1 + \psi_d + \cdots + \psi_d^{m-1}) \delta_0^2 \big), \\
&< \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})} \Big(\theta^{i_m - i_0} V_{\sigma(t_{i_0})}(0, E_{i_0}) \\
&\quad + (\Theta_d + \psi_d \omega_d) \delta_0^2 \sum_{l=0}^{m-1} \psi_d^l \Big),
\end{aligned}$$

which, combined with (55) and (56), implies that

$$\begin{aligned}
&V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m})\\
&\leq \Psi^m \psi^{2\sum_{l=1}^m \eta(\delta_d/\delta_{i_l})}\Big(\theta^{i_m - i_0}\hat{\beta}^{2(\eta_E(\|x_0\|/E_0)+\eta_\delta(\delta_d/\delta_0))}\\
&\quad \times \big(\omega_0 V_{\sigma(0)}(0, E_0) + \omega_d \delta_0^2\big)\\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big)\\
&\leq \Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m}\psi^{2\eta_E(\|x_0\|/E_0)}\\
&\quad \times (\omega_0 \rho_{\sigma(0)} E_0^2 + \omega_d \delta_0^2) + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big).
\end{aligned}$$

Finally, substituting the previous bound into (48) with $i = i_m$, we obtain

$$\begin{aligned}
&V_{\sigma(t_k)}(x^*_k, E_k)\\
&< \Theta^{N_0}\big(\theta^{k-i_m} V_{\sigma(t_{i_m})}(x^*_{i_m}, E_{i_m}) + \Theta_d \delta_{i_m}^2\big)\\
&< \Theta^{N_0}\Big(\theta^{k-i_m}\Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\Big(\theta^{i_m}\\
&\quad \times \psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2)\\
&\quad + (\Theta_d + \psi_d\omega_d)\delta_0^2 \sum\nolimits_{l=0}^{m-1} \psi_d^l\Big) + \Theta_d(1+\varepsilon_\delta)^{2m}\delta_0^2\Big)\\
&\leq \Theta^{N_0}\Psi^m \psi^{2(\eta_\delta(\delta_d/\delta_0)+\sum_{l=1}^m \eta(\delta_d/\delta_{i_l}))}\\
&\quad \times \Big(\theta^k \psi^{2\eta_E(\|x_0\|/E_0)}(\omega_0\rho_{\sigma(0)}E_0^2 + \omega_d\delta_0^2)\\
&\quad + \Big(\Theta_d \sum\nolimits_{l=0}^{m} \psi_d^l + \omega_d \sum\nolimits_{l=1}^{m} \psi_d^l\Big)\delta_0^2\Big).
\end{aligned}$$

The proof of Lemma 7 is completed by replacing m with its upper bound $N_d(\delta_d)$.

APPENDIX G
PROOF OF PROPOSITION 2

First, suppose $\|x_0\| \leq \delta \leq E_0$ and $\delta_d \leq \delta \leq \delta_0$. Then the system is always in the stabilizing stage, and the estimate of the disturbance bound is always δ_0. Suppose also that there is an integer $k_1 \geq 1$ such that $c_k = 0$ (i.e., the state $x(t_k)$ is inside the central hypercubic box) for all $k \leq k_1 - 1$. Then similar arguments as in Sections V-A1 and V-A2 show that $u \equiv 0$ on $[0, t_{k_0})$, $x^*_k = 0$ for all $k \in \{0, \ldots, k_1\}$, and hence

$$E_{k+1} \geq \frac{\Lambda_{\min}}{N} E_k \qquad \forall\, k \in \{0, \ldots, k_1 - 1\} \tag{68}$$

with

$$\Lambda_{\min} := \min_{p \in \mathcal{P}} \Lambda_p$$

due to (19) and (26).

Second, following similar analysis on state bounds when $u \equiv 0$ as in Section V-B, for each $k \leq k_1$ we see that

$$\|x(t)\| \leq \bar{\beta}^k \|x_0\| + \frac{\bar{\beta}^k - 1}{\bar{\beta} - 1}\bar{\gamma}\delta_d \qquad \forall\, t \leq t_k. \tag{69}$$

Third, following similar arguments as in Section VI-C3, for each $k \geq k_1$ we obtain

$$\|x(t)\| \leq \Xi\sqrt{V_{\sigma(t_k)}(x^*_k, E_k)} + \gamma\delta_d \qquad \forall\, t \in [t_k, t_{k+1}],$$

which, combined with (48) for $i = 0$ and (60), implies that

$$\|x(t)\| \leq \Xi\Theta^{N_0/2}(\theta^{k_1}\sqrt{\rho}E_0 + \sqrt{\Theta_d}\delta_0) + \gamma\delta_d \qquad \forall\, t \geq t_{k_1} \tag{70}$$

with ρ in (59).

Finally, the proof of Lemma 2 is completed via the following three steps. First, given an arbitrary $\varepsilon > 0$, from (64) and (70) it follows that if

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 + \gamma\delta_d \leq \varepsilon$$

then $\|x(t)\| \leq \varepsilon + C$ for all $t \geq t_{k_1}$. Second, taking E_0 as fixed, calculate a sufficiently large k_1 such that

$$\Xi\Theta^{N_0/2}\theta^{k_1}\sqrt{\rho}E_0 \leq \varepsilon/2.$$

Finally, calculate a sufficiently small δ such that $\gamma\delta \leq \varepsilon/2$ and

$$\Big(\bar{\beta}^{k_1} + \frac{\bar{\beta}^{k_1} - 1}{\bar{\beta} - 1}\bar{\gamma}\Big)\delta \leq \varepsilon,$$

which, combined with (69), implies that $\|x(t)\| \leq \varepsilon$ for all $t \leq t_{k_1}$; and that $\delta \leq \delta_0$ and

$$\Big(\bar{\beta}^{k_1-1} + \frac{\bar{\beta}^{k_1-1} - 1}{\bar{\beta} - 1}\bar{\gamma}\Big)\delta \leq \Big(\frac{\Lambda_{\min}}{N}\Big)^{k_1-1}\frac{E_0}{N},$$

which, combined with (68) and (69), implies that $c_k = 0$ for all $k \leq k_1 - 1$ (and that the systems is always in the stabilizing stage), making the analysis above valid.

REFERENCES

[1] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[2] Z.-P. Jiang and T. Liu, "Quantized nonlinear control — a survey," *Acta Automatica Sinica*, vol. 39, no. 11, pp. 1820–1830, 2013.

[3] J. P. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *15th International Symposium on Mathematical Theory of Networks and Systems*, 2002, pp. 1–15.

[4] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[5] D. Liberzon, *Switching in Systems and Control*. Birkhäuser Boston, 2003.

[6] D. Liberzon and D. Nešić, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 767–781, 2007.

[7] Y. Sharon and D. Liberzon, "Input to state stabilizing controller for systems with coarse quantization," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 830–844, 2012.

[8] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.

[9] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[10] A. S. Matveev and A. V. Savkin, "Stabilization of stochastic linear plants via limited capacity stochastic communication channels," in *45th IEEE Conference on Decision and Control*, 2006, pp. 484–489.

[11] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*. Springer London, 2000.

[12] R. Shorten, F. R. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.

[13] A. S. Morse, "Supervisory control of families of linear set-point controllers—Part I. Exact matching," *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.

[14] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *38th IEEE Conference on Decision and Control*, vol. 3. IEEE, 1999, pp. 2655–2660.

[15] W. Xie, C. Wen, and Z. Li, "Input-to-state stabilization of switched nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1111–1116, 2001.

[16] L. Vu, D. Chatterjee, and D. Liberzon, "Input-to-state stability of switched systems and switching adaptive control," *Automatica*, vol. 43, no. 4, pp. 639–646, 2007.

[17] M. A. Müller and D. Liberzon, "Input/output-to-state stability and state-norm estimators for switched nonlinear systems," *Automatica*, vol. 48, no. 9, pp. 2029–2039, 2012.

[18] G. N. Nair, S. Dey, and R. J. Evans, "Infimum data rates for stabilising Markov jump linear systems," in *42nd IEEE Conference on Decision and Control*, vol. 2, 2003, pp. 1176–1181.

[19] C. Zhang, K. Chen, and G. E. Dullerud, "Stabilization of Markovian jump linear systems with limited information – a convex approach," in *2009 American Control Conference*, 2009, pp. 4013–4019.

[20] Q. Ling and H. Lin, "Necessary and sufficient bit rate conditions to stabilize quantized Markov jump linear systems," in *2010 American Control Conference*, 2010, pp. 236–240.

[21] D. Liberzon, "Finite data-rate feedback stabilization of switched and hybrid linear systems," *Automatica*, vol. 50, no. 2, pp. 409–420, 2014.

[22] M. Wakaiki and Y. Yamamoto, "Output feedback stabilization of switched linear systems with limited information," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 3892–3897.

[23] G. Yang and D. Liberzon, "Finite data-rate stabilization of a switched linear system with unknown disturbance," in *10th IFAC Symposium on Nonlinear Control Systems*, 2016, pp. 1103–1108.

[24] ——, "Stabilizing a switched linear system with disturbance by sampled-data quantized feedback," in *2015 American Control Conference*, 2015, pp. 2193–2198.

[25] L. Vu and D. Liberzon, "Supervisory control of uncertain linear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 27–42, 2011.

[26] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 48, no. 2, pp. 304–307, 2003.

[27] N. C. Martins, "Finite gain stabilization requires analog control," *Systems & Control Letters*, vol. 55, no. 11, pp. 949–954, 2006.

[28] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals, and Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[29] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[30] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[31] A. J. Schmidt, "Topological entropy bounds for switched linear systems with Lie structure," Master Thesis, University of Illinois at Urbana-Champaign, 2016.

[32] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

Guosong Yang (S'11) was born in Changchun, China, in 1989. He received the B.Eng. degree in Electronic Engineering from Hong Kong University of Science and Technology, Kowloon, Hong Kong, in 2011, and the M.S. degree in Electrical and Computer Engineering from University of Illinois at Urbana-Champaign, Urbana, IL, where he is currently pursuing the Ph.D. degree in Electrical and Computer Engineering. His research interests include switched and hybrid systems, control with limited information, and nonlinear control theory.

Daniel Liberzon (M'98–SM'04–F'13) was born in the former Soviet Union in 1973. He did his undergraduate studies in the Department of Mechanics and Mathematics at Moscow State University, Moscow, Russia and received the Ph.D. degree in mathematics from Brandeis University, Waltham, MA in 1998 (under Prof. Roger W. Brockett of Harvard University).

Following a postdoctoral position in the Department of Electrical Engineering at Yale University from 1998 to 2000, he joined the University of Illinois at Urbana-Champaign, where he is now a professor in the Electrical and Computer Engineering Department and the Coordinated Science Laboratory.

His research interests include nonlinear control theory, switched and hybrid dynamical systems, control with limited information, and uncertain and stochastic systems.

He is the author of the books *Switching in Systems and Control* (Birkhauser, 2003) and *Calculus of Variations and Optimal Control Theory: A Concise Introduction* (Princeton Univ. Press, 2012).

His work has received several recognitions, including the 2002 IFAC Young Author Prize and the 2007 Donald P. Eckman Award. He delivered a plenary lecture at the 2008 American Control Conference. He has served as Associate Editor for the journals IEEE TRANSACTIONS ON AUTOMATIC CONTROL and MATHEMATICS OF CONTROL, SIGNALS, AND SYSTEMS.