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Robust observers and Pecora-Carroll synchronization with limited information

Boris Andrievsky Alexander L. Fradkov Daniel Liberzon

Abstract— **We study synchronization of nonlinear systems with robustness to disturbances that arise when measurements sent from the master system to the slave system are affected by quantization and time sampling. Viewing the synchronization problem as an observer design problem, we invoke a recently developed theory of nonlinear observers robust to output measurement disturbances and formulate a sufficient condition for robust synchronization. The approach is illustrated by a detailed analysis of the Pecora-Carroll synchronization scheme for the Lorenz system, for which an explicit bound on the synchronization error depending on the quantizer range and sampling period is derived.**

I. Introduction

The synchronization problem has attracted tremendous attention from several scientific communities after the publication of the seminal paper by Pecora & Carroll [1] over a quarter of century ago. In mid-2017 the paper [1] had more than 6000 citations (on Web of Science). Quite a number of monographs, special issues of journals, and surveys on synchronization have been published, see [2]–[11] and the references therein. The history of the ideas introduced in [1] can be found in the recent survey [12].

In most of the aforementioned works the authors study only idealized, disturbance-free versions of the synchronization problem. However, taking into account disturbances is important both for theoretical study and for practical implementation of the proposed methods. In particular, the issue of robustness to disturbances arises when the slave system has to rely on imprecise measurements of the master system's behavior. This can be due, for example, to time sampling and signal quantization. Robust synchronization in the presence of such effects is the subject of this paper.

The problem of synchronization under communication constraints and bounded disturbances has been considered in [13]–[15] for passifiable systems. These papers treated control systems in Lurie form satisfying a hyper-minimum-phase assumption on the linear part. The synchronization error was characterized in terms of the transmission rate and upper bounds on the disturbances. Among recent papers devoted to controlled synchronization of nonlinear systems under disturbances one can also mention [16], where an adaptive H^∞ solution is sought, and [17], where sliding mode control is proposed. However, very few rigorous quantitative results on robustness of the Pecora-Carroll scheme under bounded disturbances seem to be available. (As discussed in [18], most known synchronization schemes are quite sensitive to even small random noise.) Perhaps the closest existing work is [19] which establishes robustness to uncertainties satisfying inequality constraints and relies, like we do here, on Lyapunov-based observer design.

In this paper we study synchronization under information constraints which manifest themselves as errors corrupting the output measurements. The robust synchronization problem is formulated as that of obtaining a bounded (nonlinear) gain from this output measurement error to the synchronization error. Our approach is to cast the synchronization problem as an observer design problem and invoke recent results from [20] on nonlinear observers robust to output measurement disturbances in an input-to-state stability (ISS) sense [21]. This allows us to move beyond the passification method of [13]–[15] and potentially treat a more general class of systems. While the general idea of relating (robust) synchronization to observer design is not new (see, e.g., [19], [22]), it appears that this link has not been previously explored with robustness defined in an ISS sense; the new results on ISS observer design from [20] now make this possible, as we demonstrate here.

As an illustration, we study the Pecora-Carroll synchronization scheme for the Lorenz system with time sampling and quantization. We are able to work out explicit bounds on the ISS gain and the synchronization error for this example. Providing such precise quantitative characterizations of robustness is the main contribution of this work.

II. Robust synchronization and qDES observers

We consider the well-known Pecora-Carroll synchronization scheme [1]. The *master* system (also known as the *leader* or *drive* system) has the form

$$\begin{aligned} \dot{x} &= F(x,y) \\ \dot{y} &= G(x,y) \end{aligned} \tag{1}$$

where y represents the measured variables and x the rest of the state variables. The *slave* system (also known as the *follower* or *response* system) nominally has the form

$$\dot{\hat{x}} = F(\hat{x}, y) \tag{2}$$

As discussed, e.g., in [23], (2) can be viewed as the limiting case of the system

$$\dot{\hat{x}} = F(\hat{x}, \hat{y}), \qquad \dot{\hat{y}} = G(\hat{x}, \hat{y}) - k(\hat{y} - y) \tag{3}$$

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B. Andrievsky and A. L. Fradkov are with the Institute for Problems of Mechanical Engineering of RAS, 199178; with the Saint Petersburg State University, and with the ITMO University, Saint Petersburg, Russia boris.andrievsky@gmail.com, fradkov@mail.ru

D. Liberzon is with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A. liberzon@illinois.edu

which is a copy of (1) under the action of high-gain feedback control. Indeed, for large $k > 0$ this controlled system can be viewed as a singularly perturbed system, and its reduced system as $k \to \infty$ is precisely (2). We choose not to include y-dynamics and work with (2) rather than (3) because the y-variables are directly measured.

In this paper we are interested in the situation where the output measurements received by the slave system are corrupted by additive disturbances. Such disturbances arise from information constraints; specifically, they can be caused by quantization effects, time sampling, time delays, or combinations thereof. As a result, the slave system becomes

$$\dot{\hat{x}} = F(\hat{x}, y + d) \tag{4}$$

The $\hat{y}$ dynamics are still not necessary, since $y+d$ can serve as a static estimate of y.

To define the synchronization objective, let us introduce the synchronization error

$$e := \hat{x} - x \tag{5}$$

In the absence of disturbances, when the slave system (2) is used, it is desired that $e(t) \to 0$ as $t \to \infty$. In the presence of disturbances, when the slave system (4) is used, this objective is no longer realistic. It is more reasonable to require that the interconnection of (1) and (4) have an ISS property from d to e; we refer to this as *robust synchronization*. Particularly suitable in the present context is a weaker variant of ISS considered in [20], called *quasi-ISS*. To define it, we need the following notation. A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$. A function β : $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$. We will denote by $\|\cdot\|_I$ the essential supremum norm of a signal on a time interval I; when no interval is specified, $[0, \infty)$ is the default.

The quasi-ISS property from d to e says that for each $K > 0$, there should exist a class $\mathcal{KL}$ function β_K and a class $\mathcal{K}_\infty$ function γ_K such that all solutions of (1), (4) satisfy

$$|e(t)| \leq \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]}) \tag{6}$$

whenever $\|x\|_{[0,t]} \leq K$ and $\|y\|_{[0,t]} \leq K$. The function γ_K is called a *quasi-ISS gain function*; when it is linear, i.e., $\gamma_K(r) = cr$ for some constant $c > 0$, we refer to c simply as a *quasi-ISS gain*. When the state of the system (1) is known to be globally bounded, quasi-ISS becomes standard ISS (this will be the case for the Lorenz system considered below). We view (6) as encoding the robust synchronization objective of interest in this paper. It means, in particular, that if the disturbance d is bounded or asymptotically vanishing, then so is the synchronization error e.

The slave system (4) can be regarded as a *reduced-order observer* for the master system (1). (If $\hat{y}$-dynamics were included, as in (3), it would be a full-order observer.) Observers achieving the above quasi-ISS property (6) were studied in [20] under the name of *quasi-Disturbance-to-Error Stable (qDES)* observers. A set of Lyapunov-based sufficient conditions for the qDES property was developed in [20]. It is then clear that in the present set-up, the results of [20] can be directly used to study robust synchronization.

In particular, adopting Corollary 3 of [20] to the master/plant (1) and slave/observer (4), with the synchronization error defined by (5), we arrive at the following (the proof is omitted due to space constraints).

Proposition 1 *Suppose there exists a C^1 function $V = V(e)$ and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all values of e, x, y we have:*

$$\alpha_1(|e|) \leq V(e) \leq \alpha_2(|e|), \quad \left|\frac{\partial V}{\partial e}(e)\right| \leq \alpha_4(|e|),$$
$$\frac{\partial V}{\partial e}(e)\big(F(e+x, y) - F(x, y)\big) \leq -\alpha_3(|e|), \tag{7}$$

and the "asymptotic ratio" condition

$$\limsup_{\xi \to \infty} \frac{\alpha_4(\xi)}{\alpha_3(\xi)} = 0 \tag{8}$$

holds. Then (4) *is a qDES observer for* (1)*, i.e., property* (6) *holds under the specified conditions.*

We now demonstrate this approach on a detailed example.

III. Lorenz system

Consider the following Lorenz system [24]:

$$\begin{aligned} \dot{x}_1 &= \sigma x_2 - \sigma x_1 \\ \dot{x}_2 &= -x_2 - x_1 x_3 + \theta x_1 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \\ y &= x_1 \end{aligned} \tag{9}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ are the state variables, β, σ, θ are constant parameters. It is known that for certain parameter values (e.g. for $\beta = 8/3$, $\sigma = 10$ and $\theta = 97$), system (9) exhibits chaotic behavior [24], [25]. This system fits into the form (1) with $x = (x_2, x_3)$.

A. Boundedness of solutions

We claim that all solutions of the Lorenz system (9) are bounded and eventually enter the ball

$$\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - \sigma - \theta)^2 \leq \rho^2\} \tag{10}$$

whose radius $\rho = \rho(\beta, \theta, \sigma)$ is specified below. Such results are well known; see, e.g., [2], [26] and the references therein. Consider the candidate Lyapunov function

$$V(x_1, x_2, x_3) = \frac{1}{2}\Big(x_1^2 + x_2^2 + (x_3 - \sigma - \theta)^2\Big)$$

Its derivative along solutions of (9) is

$$\dot{V} = -\sigma x_1^2 - x_2^2 - \beta\Big(x_3 - \frac{\sigma+\theta}{2}\Big)^2 + \beta\frac{(\sigma+\theta)^2}{4}$$

which is negative outside an ellipsoid contained in the ball of radius $\frac{1}{2}(\sigma+\theta)\sqrt{1 + \beta\max\{1, \frac{1}{\sigma}\}}$ centered at $(0, 0, (\sigma+\theta)/2)$. This ball is in turn contained in the ball of radius

$$\rho := \frac{1}{2}(\sigma+\theta)\big(1 + \sqrt{1 + \beta\max\{1, 1/\sigma\}}\big) \tag{11}$$

centered at $(0, 0, \sigma+\theta)$, which serves as an invariant attractor proving the claim.

The actual upper bound on solutions of course depends on initial conditions. Namely, the solutions will remain in the smallest ball centered at $(0, 0, \sigma + \theta)$ which contains the initial state and whose radius is at least ρ. For future reference, we formally define this ball as

$$\Omega := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - \sigma - \theta)^2 \leq D^2\} \quad (12)$$

where

$$D := \max\{\rho, |(x_1(0), x_2(0), x_3(0) - \sigma - \theta)|\} \quad (13)$$

and ρ is defined in (11).

B. Time-sampling error

Suppose that the output is sampled at the times $t_k := kT$, $k = 0, 1, \ldots$, where $T > 0$ is the sampling period, so that the actual measured output received by the slave system is given by

$$x_1(t) + d(t) = x_1(t_k), \qquad t \in [t_k, t_{k+1})$$

where $d(t) := x_1(t_k) - x_1(t)$ is the disturbance that represents the time-sampling error. The above boundedness claim allows us to obtain an upper bound on this time-sampling error. For $t \in [t_k, t_{k+1})$ we have

$$x_1(t) = x_1(t_k) + \dot{x}_1(s)(t - t_k)$$

for some $s \in [t_k, t]$. Using the first differential equation in (9) and the fact that $t - t_k < T$, we have

$$|x_1(t) - x_1(t_k)| \leq \sigma |x_2(s) - x_1(s)| T$$

We know that $x(t)$ belongs to the ball Ω defined in (12), with radius D defined in (13), for all $t \geq 0$. The maximum of $|x_2 - x_1|$ over this ball is easily seen to be $\sqrt{2}D$. We can thus write

$$|x_1(t) - x_1(t_k)| \leq \sigma\sqrt{2}DT$$

and this gives a bound on the norm of the disturbance that arises from time sampling:

$$|d(t)| \leq \sigma\sqrt{2}DT \quad (14)$$

The bound (14) will be used below for evaluation of the synchronization error.

C. Binary digital quantization error

1) Data transmission procedure: Consider transmission of the signal over the digital communication channel, where both time-sampling and level quantization are present. Let us consider the binary coder with memory, cf. [27]–[29].

Let signal $y(t)$ be transmitted over the digital communication channel at sampling instants $t_k = kT$, where $T > 0$ is a constant sampling period, $k = 0, 1, \ldots$ are integers. At each k, the *deviation signal* $\delta[k]$ between transmitted signal $y(t_k)$ and a certain *centroid* $c[k]$ (defined below in the text) is calculated as $\delta[k] := y(t_k) - c[k]$. Signal $\delta[k]$ is subjected to the following binary quantization scheme:

$$\bar{\delta}[k] := M \operatorname{sign}(\delta[k]) \quad (15)$$

where $\operatorname{sign}(\cdot)$ is *signum function*, $M > 0$ may be referred to as a *quantizer range*. Then the quantizer output $\bar{\delta}[k] \in \{-M, M\}$ is transmitted over the communication channel to the decoder. The sequence of centroids $c[k]$ is recursively defined by the following algorithm:

$$c[k+1] = c[k] + \bar{\delta}[k], \quad c[0] = 0. \quad (16)$$

Equations (15), (16) describe the coder algorithm.

A similar algorithm is implemented by the decoder: the decoder output $\bar{y}[k]$ is defined as

$$\bar{y}[k] := \bar{c}[k] + \bar{\delta}[k], \quad (17)$$

where the centroid $\bar{c}[k]$ is found in the decoder in accordance with (16):

$$\bar{c}[k+1] = \bar{c}[k] + \bar{\delta}[k], \quad \bar{c}[0] = 0. \quad (18)$$

(Thus the centroid sequence $\bar{c}[k]$ is the same as $c[k]$, and in what follows we make no distinction between the two.) Note that the considered coding scheme corresponds to the channel data rate of $R = T^{-1}$ bits per second. In between transmission times we define

$$\bar{y}(t) := \bar{y}[k], \qquad t \in [t_k, t_{k+1}).$$

2) Data transmission error: Let us evaluate an upper bound of data transmission error $d(t)$ given that the growth rate of $y(t)$ is uniformly bounded. A bound for $\dot{y}(t)$ was already found in Section III-B and it is

$$L_y := \sigma\sqrt{2}D. \quad (19)$$

To analyze the coder–decoder accuracy, let us derive an upper bound Δ on the transmission error $d(t) := \bar{y}(t) - y(t)$, defined as $\Delta = \sup_t |d(t)|$.

From the coding-decoding scheme (15)–(18) it is clear that for each time interval $t \in [t_k, t_{k+1})$, the transmission error may be represented as $d(t) = y(t) - \bar{y}[k] = y(t) - c[k+1]$. Due to the above bound on the rate of $y(t)$, over each time interval $t \in [t_k, t_{k+1})$ the magnitude of $d(t)$ is bounded by $|y(t_k) - c[k+1]| + L_y T$.

To evaluate $|y(t_k) - c[k+1]|$, assume that for a certain k it is valid that

$$|y(t_k) - c[k]| \leq 2M. \quad (20)$$

Then, after renovation of c by means of (16), the magnitude of $y(t_k) - c[k+1]$ does not exceed M. Therefore, during the interval $t \in [t_k, t_{k+1})$ the following inequality

$$|d(t)| < M + L_y T \quad (21)$$

holds. If M is chosen satisfying the condition

$$M > L_y T, \quad (22)$$

then at instant t_{k+1} inequality (20) will be fulfilled with $k+1$ instead of k and, using the induction argument, the same relation (20) will be valid for all subsequent steps.

Representing M in the form $M = \alpha L_y T$ for some $\alpha > 1$ one obtains the following relation for the upper bound of transmission error:

$$\Delta = (1 + \alpha) L_y T. \tag{23}$$

Inequality (22) imposes restrictions on sampling period T and quantizer range M for a given growth rate L_y of $y(t)$. If (22) is fulfilled, then magnitude $|d(t)|$ of data transmission error $d(t)$ does not exceed Δ. Otherwise the data transmission scheme based on (15)–(17) may fail.

Remark 1 The assumption has been made that (20) is valid for some (finite) k. This condition may be violated at the beginning of the process. But in view of boundedness and continuity of $y(t)$, it may be easily proven that for any $M > 0$ procedure (16) ensures that such a k exists. Alternatively, for speeding up convergence of $c[k]$ to the vicinity of $y(t_k)$, a zooming strategy may be employed (see [30], [31]).

Remark 2 Violation of (22) may eventually lead to loss of tracking by $c[k]$ of the values of $y(t_k)$ at some instant. After several steps, tracking may be restored (based on the aforementioned arguments), and time intervals when (20) is fulfilled or violated may alternate. Nevertheless, we consider this situation as data transmission failure.

D. qDES observer

Consider the reduced-order observer:

$$\begin{aligned} \dot{\hat{x}}_2 &= -\hat{x}_2 - \bar{y}\hat{x}_3 + \theta\bar{y} \\ \dot{\hat{x}}_3 &= -\beta\hat{x}_3 + \bar{y}\hat{x}_2 \end{aligned}$$

where we use the shorthand notation

$$\bar{y} = y + d$$

for the output signal received by the observer, to reflect the fact that the observer acts on sampled and quantized measurements of the system; here d is the transmission error as defined earlier. This observer is consistent with the form (4) for the slave system, and without disturbance it reduces to the classical Pecora-Carroll scheme. If desired, it can be completed by the static estimate $\hat{x}_1 = \bar{y}$ for x_1.

Define the state estimation error vector

$$e = \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} := \begin{pmatrix} \hat{x}_2 - x_2 \\ \hat{x}_3 - x_3 \end{pmatrix}$$

The error dynamics are

$$\begin{aligned} \dot{e}_2 &= (-\hat{x}_2 - \bar{y}\hat{x}_3 + \theta\bar{y}) - (-x_2 - x_1 x_3 + \theta x_1) \\ \dot{e}_3 &= (-\beta\hat{x}_3 + \bar{y}\hat{x}_2) - (-\beta x_3 + x_1 x_2) \end{aligned}$$

which we can rewrite as

$$\begin{aligned} \dot{e}_2 &= (-\hat{x}_2 - \bar{y}\hat{x}_3 + \cancel{\theta\bar{y}}) - (-x_2 - \bar{y}x_3 + \cancel{\theta\bar{y}}) \\ &\quad + (\cancel{-x_2} - \bar{y}x_3 + \theta\bar{y}) - (\cancel{-x_2} - x_1 x_3 + \theta x_1) \\ \dot{e}_3 &= (-\beta\hat{x}_3 + \bar{y}\hat{x}_2) - (-\beta x_3 + \bar{y}x_2) \\ &\quad + (\cancel{-\beta x_3} + \bar{y}x_2) - (\cancel{-\beta x_3} + x_1 x_2) \end{aligned} \tag{24}$$

Consider the candidate Lyapunov function

$$V(e_2, e_3) := \frac{1}{2}(e_2^2 + e_3^2)$$

Its derivative along the error dynamics can be written as the sum of two terms. The first term is obtained by considering only the terms in the first and second parentheses on the right-hand sides of (24), and it is

$$\begin{aligned} &e_2(-\hat{x}_2 - \bar{y}\hat{x}_3) - e_2(-x_2 - \bar{y}x_3) \\ &+ e_3(-\beta\hat{x}_3 + \bar{y}\hat{x}_2) - e_3(-\beta x_3 + \bar{y}x_2) \\ &= -e_2^2 - \cancel{e_2\bar{y}e_3} - \beta e_3^2 + \cancel{e_3\bar{y}e_2} = -e_2^2 - \beta e_3^2 \\ &\le -\min\{1, \beta\}|e|^2 \end{aligned}$$

This is precisely the inequality (7) for the specific example at hand.

The second term in $\dot{V}$ comes from considering the remaining terms on the right-hand side of (24), and it is

$$\begin{aligned} &e_2(-\bar{y}x_3 + \theta\bar{y}) - e_2(-x_1 x_3 + \theta x_1) + e_3\bar{y}x_2 - e_3 x_1 x_2 \\ &= e_2 x_3(x_1 - \bar{y}) + e_2\theta(\bar{y} - x_1) + e_3 x_2(\bar{y} - x_1) \\ &= \big(e_2(\theta - x_3) + e_3 x_2\big)d \le C|e||d| \end{aligned}$$

where

$$C := \sup_{t \ge 0} \left| \begin{pmatrix} \theta - x_3 \\ x_2 \end{pmatrix} \right|$$

is finite because $x(t)$ evolves in the compact set Ω defined in (12) by the analysis of Section III-A. In fact, since Ω is the ball of radius D centered at $(0, 0, \sigma + \theta)$, it is easy to see that

$$C = \sup_{x \in \Omega} \left| \begin{pmatrix} \theta - x_3 \\ x_2 \end{pmatrix} \right| = D + \sigma. \tag{25}$$

Since $|e|/|e|^2 \to 0$ as $|e| \to \infty$, the observer is qDES by Corollary 3 of [20]. This corresponds to the asymptotic ratio condition (8) with α_4 linear and α_3 quadratic. Therefore, the ISS property from the measurement disturbance d to the state estimation error e is guaranteed (because we know that the state of the plant remains bounded). In fact, from the above calculation of $\dot{V}$ it is easy to see that $C/\min\{1, \beta\}$ is an upper bound on the ISS gain.

This means, in particular, that

$$\limsup_{t \to \infty} |e(t)| \le \frac{C}{\min\{1, \beta\}} \|d\| \le \frac{(1 + \alpha)\sigma\sqrt{2}DTC}{\min\{1, \beta\}} \tag{26}$$

where to arrive at the last inequality we used (19) and (23). Recall that in the case of time-sampling only (Section III-B) we can set $\alpha = 0$, while in the presence of binary quantization (Section III-C) we have $\alpha > 1$ and the quantizer range must be chosen as $M = \alpha L_y T$. We see that the achievable synchronization error is inversely proportional to the information transmission rate $R = T^{-1}$.

E. Simulations

1) Simulation parameters: The classic parameter values of Lorenz system (9) are taken for the simulations: $\beta = 8/3$, $\theta = 97$, $\sigma = 10$. For the chosen parameters, relation (11) yields $\rho = 156$. For simplicity we assume that the initial

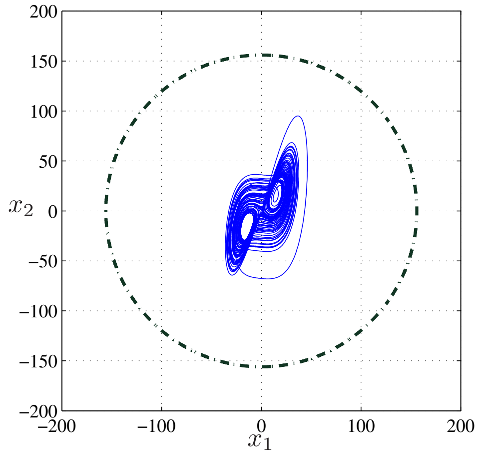

Fig. 1. ρ-circle and the Lorenz attractor (projection onto the (x_1, x_2)-plane).

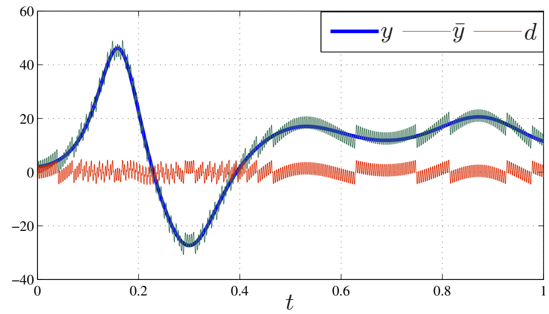

Fig. 2. Time histories of $y(t)$, $\bar{y}(t)$, $d(t)$ for $T = 2 \cdot 10^{-3}$ s.

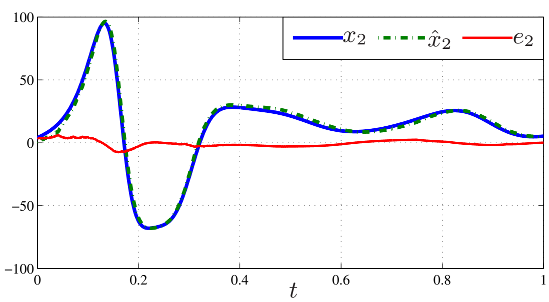

Fig. 3. Time histories of $x_2(t)$, $\hat{x}_2(t)$, $e_2(t) = x_2(t) - \hat{x}_2(t)$ for $T = 2 \cdot 10^{-3}$ s.

state vector $x(0)$ belongs to the ball of radius ρ specified by (10) and (11). Projections onto the (x_1, x_2)-plane of this ball and the phase plot of the Lorenz system, starting from $x_1(0) = 2$, $x_2(0) = 4$, $x_3(0) = 2$, are depicted in Fig. 1.

Expression (19) gives $L_y = \sigma\sqrt{2}\rho \approx 2210$.

2) Data transmission error: Simulation results for $\alpha = 1.01$ and various T are depicted in Figs. 2–6.

Based on (22), for $T = 2 \cdot 10^{-3}$ s the value of M was taken as $M = 4.42$. In this case (23) gives $\Delta = 8.88$. Simulation run during $100\,$s gives the "measured" values $L_{y,\text{sim}} = 850$ and $\Delta_{\text{sim}} = 4.79$. The corresponding time histories are depicted in Figs. 2, 3.

For the case of $T = 5 \cdot 10^{-3}$ s the corresponding time histories are depicted in Figs. 4, 5. For $T = 5 \cdot 10^{-3}$ s, (23) gives $\Delta = 11.2$. The signal transmission bound found by the simulation is as $\Delta_{\text{sim}} = 14.3$.

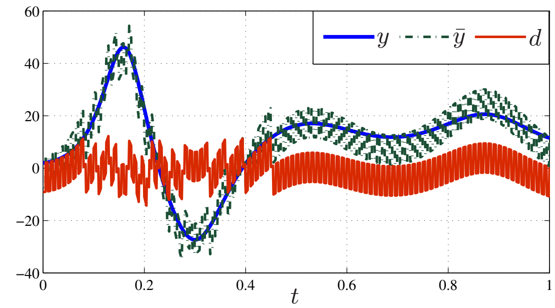

Fig. 4. Time histories of $y(t)$, $\bar{y}(t)$, $d(t)$ for $T = 5 \cdot 10^{-3}$ s.

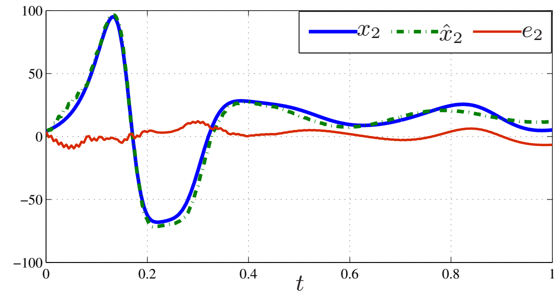

Fig. 5. Time histories of $x_2(t)$, $\hat{x}_2(t)$, $e_2(t) = x_2(t) - \hat{x}_2(t)$ for $T = 5 \cdot 10^{-3}$ s.

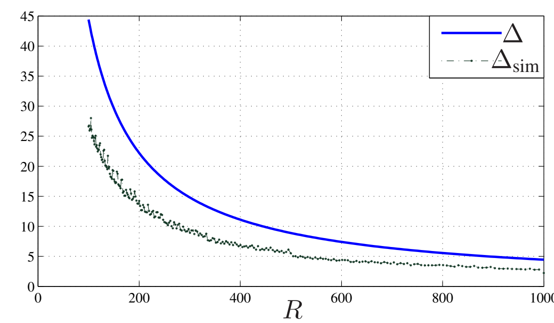

Fig. 6. Transmission error bounds vs data bit-rate R (bit/s). Δ – theoretical bound, Δ_{sim} – simulation result.

Dependencies of Δ, Δ_{sim} on data bit-rate $R \in [10^2, 10^3]$ bit/s are shown in Fig. 6.

It is worth mentioning that the coder range M in the aforementioned simulation has been found based on the theoretical bound $L_y = \sigma\sqrt{2}\rho$ as $L_y = 2210$. An actual bound L_y, obtained by intensive simulations, is about 850. Application of this value to $M = \alpha L_y T$ (for the same α) makes it possible to significantly reduce data transmission error Δ and reduces difference between theoretical and computational evaluation of $\Delta(R)$, which is seen on Fig. 7.

3) Synchronization error: To find an upper bound of the estimation error $|e(t)|$ theoretically, let us use relation (25). Under the aforementioned assumption that $x(0)$ belongs to the ball (10), (11) of radius ρ, we have that $x(t)$ remains in

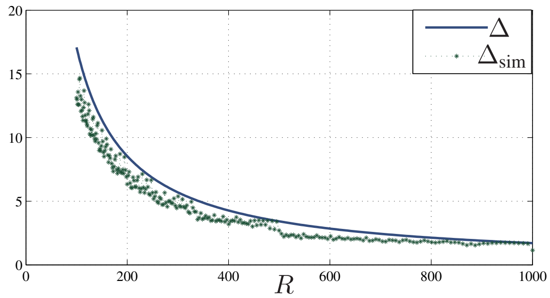

Fig. 7. Transmission error bounds vs data bit-rate R (bit/s) for $L_y = 850$. Δ – theoretical bound, Δ_{sim} – simulation result.

this ball for all t, and it follows that $C = \rho + \sigma$.

Therefore, by (26) we obtain

$$\limsup_{t\to\infty} |e(t)| \le \frac{\rho+\sigma}{\min\{1,\beta\}}\Delta, \tag{27}$$

where ρ and Δ are given by (11) and (23), respectively. For given parameter values, one obtains that $\limsup_{t\to\infty} |e(t)| \le 166\Delta$.

Computer experiments demonstrate that in practice the robust synchronization scheme works significantly better than the theory predicts. This difference between the theoretical and simulation-based evaluation may be explained by a "random" character of the data transmission error, which is not taken into account in the worst-case theoretical analysis.

IV. Conclusions

A recently introduced framework of qDES nonlinear observers was applied to characterize robustness of the Pecora-Carroll synchronization scheme to errors arising from time sampling and quantization of the measurements sent from the master to the slave system. For the Lorenz system example, a specific expression on the synchronization error in terms of the information transmission rate and system parameters was derived. Future studies will be directed at closing the gap between the derived theoretical results and the better ones observed in simulations, as well as at applying the approach to other system classes and other synchronization schemes.

References

[1] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, pp. 821–823, 1990.

[2] A. L. Fradkov and A. Y. Pogromsky, *Introduction to Control of Oscillations and Chaos*. Singapore: World Scientific, 1998.

[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences*. Cambridge, UK: Cambridge University Press, 2001.

[4] J. M. Gonzàlez-Miranda, *Synchronization and Control of Chaos. An Introduction for Scientists and Engineers*. Singapore: World Scientific, 2004.

[5] M. Kennedy and M. Ogorzalek, Eds., *IEEE Trans. Circuits Syst. I, Special issue "Chaos control and synchronization"*, vol. 44, no. 10, 1997.

[6] H. Nijmeijer, Ed., *Systems and Control Letters, Special issue on Control and Synchronization of Chaos*, vol. 34, no. 5, 1997.

[7] V. Anishchenko, Ed., *Discrete Dynamics in Nature and Society, Special issue on the problem of synchronization of nonlinear processes in dynamical systems of different nature and the degree of complexity*, vol. 4, no. 3, 2000. [Online]. Available: http://eudml.org/doc/121221

[8] *Int. J. Systems Science, Special Issue on Synchronization and Control of Complex Networks*, vol. 40, no. 9, 2009.

[9] L. Glass, "Synchronization and rhythmic processes in physiology," *Nature*, vol. 410, pp. 277–284, 2001.

[10] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, pp. 1–101, 2002.

[11] F. Blaabjerg, R. Teodorescu, M. Liserre, and A. V. Timbus, "Overview of control and grid synchronization for distributed power generation systems," *IEEE Trans. Ind. Electron.*, vol. 53, pp. 1398–1409, 2006.

[12] L. M. Pecora and T. L. Carroll, "Synchronization of chaotic systems," *Chaos*, vol. 25, 2015, article 097611.

[13] B. Andrievskii and A. L. Fradkov, "Method of passification in adaptive control, estimation, and synchronization," *Autom. Remote Control*, vol. 67, pp. 1699–1731, 2006.

[14] A. L. Fradkov, B. Andrievsky, and R. J. Evans, "Controlled synchronization under information constraints," *Physical Review E*, vol. 78, pp. 036 210 1–6, 2008.

[15] A. L. Fradkov, B. Andrievsky, and M. S. Ananyevskiy, "Passification based synchronization of nonlinear systems under communication constraints and bounded disturbances," *Automatica*, vol. 55, pp. 287–293, 2015.

[16] C. K. Ahn, S.-T. Jung, S.-K. Kang, and S.-C. Joo, "Adaptive H_∞ synchronization for uncertain chaotic systems with external disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, pp. 2168–2177, 2010.

[17] Q. Zhang, "Robust synchronization of FitzHugh-Nagumo network with parameter disturbances by sliding mode control," *Chaos Solitons & Fractals*, vol. 58, pp. 22–26, 2014.

[18] T. L. Carroll, "Chaotic systems that are robust to added noise," *Chaos*, vol. 15, 2005, article 013901.

[19] A. Pogromsky and H. Nijmeijer, "Observer-based robust synchronization of dynamical systems," *Int. J. Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 8, pp. 2243–2254, 1998.

[20] H. Shim and D. Liberzon, "Nonlinear observers robust to measurement disturbances in an ISS sense," *IEEE Trans. Automat. Contr.*, vol. 61, pp. 48–61, 2016.

[21] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, pp. 435–443, 1989.

[22] H. Nijmeijer and I. Mareels, "An observer looks at synchronization," *IEEE Trans. Circuits Syst. I*, vol. 44, pp. 882–890, 1997.

[23] A. L. Fradkov, *Cybernetical Physics: From Control of Chaos to Quantum Control*. Berlin: Springer, 2007.

[24] E. N. Lorenz, "Deterministic nonperiodic flow," *J. Atmospheric Sciences*, vol. 20, pp. 130–141, 1963.

[25] K. Cuomo, A. Oppenheim, and S. Strogatz, "Synchronization of Lorenz-based chaotic circuits with application to communications," *IEEE Trans. Circuits Syst. II*, vol. 40, pp. 626–633, 1993.

[26] A. Pogromsky, G. Santoboni, and H. Nijmeijer, "Partial synchronization: from symmetry towards stability," *Physica D*, vol. 172, pp. 65–87, 2002.

[27] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1056–1068, 2004.

[28] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[29] A. L. Fradkov, B. Andrievsky, and R. J. Evans, "Chaotic observer-based synchronization under information constraints," *Physical Review E*, vol. 73, p. 066209, 2006.

[30] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1279–1289, 2000.

[31] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.